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Ball Convergence of a Parametric Efficient Family of Iterative Methods for Solving Nonlinear Equations

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Abstract: The goal is to extend the applicability of Newton-Traub-like methods in cases not covered in earlier articles requiring the usage of derivatives up to order seven that do not appear in the methods. The price we pay by using conditions on the first derivative that actually appear in the method is that we show only linear convergence. To find the convergence order is not our intention, however, since this is already known in the case where the spaces coincide with the multidimensional Euclidean space. Note that the order is rediscovered by using ACOC or COC, which require only the first derivative. Moreover, in earlier studies using Taylor series, no computable error distances were available based on generalized Lipschitz conditions. Therefore, we do not know, for example, in advance, how many iterates are needed to achieve a predetermined error tolerance. Furthermore, no uniqueness of the solution results is available in the aforementioned studies, but we also provide such results. Our technique can be used to extend the applicability of other methods in an analogous way, since it is so general. Finally note that local results of this type are important, since they demonstrate the difficulty in choosing initial points. Our approach also extends the applicability of this family of methods from the multi-dimensional Euclidean to the more general Banach space case. Numerical examples complement the theoretical results.

Keywords: Banach space valued mapping; parametric family of methods; ball convergence; Euclidean space

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1. Introduction

Let $\mathscr{B}_1$, $\mathscr{B}_2$ denote Banach spaces and $T \subseteq \mathscr{B}_1$ be a nonempty convex and open set. Set $\mathscr{L}B(\mathscr{B}_1, \mathscr{B}_2) = \{V : \mathscr{B}_1 \to \mathscr{B}_2 \quad \text{is by a bounded linear operator}\}$. Many problems in mechanics, biomechanics, physics, mathematical chemistry, economics, radiative transfer, biology, ecology, medicine, engineering, and other areas [1–25] are reduced to an equation (nonlinear)

$$F(x) = 0, \tag{1}$$

with $F : T \to \mathscr{B}_2$ being continuously differentiable in the Fréchet sense. Therefore solving Equation (1) is an extremely important and difficult problem in general. A solution ξ is very difficult to find, especially in closed or analytical form. This function forces practitioners and researchers to develop higher order and efficient methods converging to ξ by starting from a point $x_0 \in T$ sufficiently close to it [1–28]. Motivated by Traub-like and Newton's methods, the following methods were studied in [10]:

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1} F(x_k) \\ z_k &= y_k - [w_k, y_k; F]^{-1} F(y_k) \\ x_{k+1} &= z_k - [w_k, y_k; F]^{-1} F(z_k), \end{aligned} \tag{2}$$

where $[\cdot,\cdot;F] : T \times T \to \mathscr{L}B(\mathscr{B}_1, \mathscr{B}_2)$ is a divided difference of order one [18] and $w_n = w(x_n)$, $w : T \to T$ is a given iteration function. To be more precise, the special choice of w given by

$$w_n = y_n + \alpha F(y_n) + \beta F(y_n)^2, \tag{3}$$

was used in [10], for parameters α and β that are not zero at the same time, $f_i(x)$ the co-ordinate functions for F and

$$F(x)^2 = (f_1^2(x), f_2^2(x), \cdots, f_n^2(x))^T. \tag{4}$$

Moreover, they used method (2) when $\mathscr{B}_1 = \mathscr{B}_2 = \mathbb{R}^k$, and compared it favorably to other sixth-order methods.

However, in this article we do not necessarily assume that w is given by (3). The sixth convergence order has been verified for any value of α and β using Taylor series, but hypothesizes on the derivatives up to seventh order [10]. That simply means that convergence is not guaranteed for mappings that are not up to seventh-order differentiable.

For example, consider h on $T = [-\frac{1}{2}, \frac{3}{2})$ as

$$h(t) = \begin{cases} t^3 lnt^2 + t^5 - t^4, & t \neq 0 \\ 0, & t = 0. \end{cases} \tag{5}$$

Then, we calculate

$$h'(t) = 3t^2 lnt^2 + 5t^4 - 4t^3 + 2t^2,$$

$$h''(t) = 6tlnt^2 + 20t^3 + 12t^2 + 10t$$

and

$$h'''(t) = 6lnt^2 + 60t^2 - 24t + 22.$$

Notice that $h'''(t)$ is unbounded on T.

Another problem is that there are no error bounds on $\|x_n - \xi\|$ or results on uniqueness of ξ or how close to ξ we should start, and the selection of x_0 is really " a shot in the dark". To address all these concerns about this very efficient and useful method, we only use conditions on the first derivative. Moreover, convergence radius, error estimates, and uniqueness results are computed based on these conditions. Furthermore, we rely on the computational order (COC) or approximated computational convergence order (ACOC) formulae to determine the order [8,25]. These formulae also only use the first derivative. That is how we extend the applicability of method (2). Our technique can also be used to study other methods in an analogous way.

It is worth noticing that the $\alpha = \beta = 0$, method (2) reduces to Newton-Secant-like methods, and if $\beta = 0$ to Newton-Steffensen-like methods.

The layout for the rest of the article includes the convergence analysis of method (2) in Section 2 and the numerical examples in Section 3.

2. Convergence Analysis of Method (2)

Let $D = [0, \infty)$. Let also $\varphi_0 : D \to \mathbb{R}$ be increasing and continuous, satisfying $\varphi_0(0) = 0$. Assume

$$\varphi_0(t) = 1 \tag{6}$$

having one positive zero (at least). Let r_0 be the minimal zero. Set $D_0 = [0, r_0)$. Assume there exists a function $\varphi : D_0 \to \mathbb{R}$ that is continuous and increasing satisfying $\varphi(0) = 0$. Consider g_1 and $\overline{g_1}$ on D_0 as $g_1(t) = \frac{\int_0^1 \varphi((1-\tau)t)\,d\tau}{1-\varphi_0(t)}$, and $\overline{g_1}(t) = g_1(t) - 1$. By these definitions $\overline{g_1}(0) = -1$ and $\overline{g_1}(t) \to \infty$ for $t \to r_0^-$. The application of the intermediate value theorem on function $\overline{g_1}$ assures the existence of at least one zero in $(0, r_0)$. Denote by ρ_1 the minimal such zero.

Assume

$$\varphi_0(g_1(t)\,t) = 1, \quad \varphi_2(\varphi_5(t), g_1(t)\,t) = 1 \tag{7}$$

have at least one positive zero, where φ_5 is as φ. Denote by r_1 the minimal such zero and let $D_1 = [0, \bar{r}_0]$, where $\bar{r}_0 = \min\{r_0, r_1\}$. Assume there exist functions $\varphi_1 : D_1 \to \mathbb{R}$, $\varphi_2 : D_1 \times D_1 \to \mathbb{R}$, and $\varphi_3 : D_1 \to \mathbb{R}$ that are continuous and increasing. Consider functions g_2 and $\overline{g_2}$ on D_1 as

$$g_2(t) = \left(\frac{\int_0^1 \varphi((1-\tau)\,g_1(t)\,t)\,d\tau}{1-\varphi_0(g_1(t)\,t)} + \frac{\varphi_1(\varphi_5(t) + g_1(t)\,t) \int_0^1 \varphi_3(\tau\, g_1(t)\,t)\,d\tau}{(1-\varphi_0(g_1(t)\,t))(1-\varphi_2(\varphi_5(t), g_1(t)\,t))} \right) g_1(t)$$

and

$$\overline{g_2}(t) = g_2(t) - 1.$$

By these definitions $\overline{g_2}(0) = -1$, and $\overline{g_2}(t) \to \infty$ for $t \to \bar{r}_0^-$. Denote by ρ_2 the minimal zero of function $\overline{g_2}$ on $(0, \bar{r}_0)$.

Assume

$$\varphi_0(g_2(t)\,t) = 1 \tag{8}$$

having at least one positive zero at least. Let r_2 be the minimal zero. Set $D_2 = [0, \bar{\bar{r}}_0)$, where $\bar{\bar{r}}_0 = \min\{\bar{r}_0, r_2\}$. Consider g_3 and $\overline{g_3}$ on D_2 as

$$g_3(t) = \left(\frac{\int_0^1 \varphi((1-\tau)\,g_2(t)\,t)\,d\tau}{1-\varphi_0(\lambda)} + \frac{\varphi_4(\varphi_5(t) + \lambda, g_1(t)\,t + \lambda) \int_0^1 \varphi_3(\tau\lambda)\,d\tau}{(1-\varphi_0(\lambda))(1-\varphi_2(\varphi_5(t), g_1(t)\,t))} \right) g_2(t)$$

and

$$\overline{g_3}(t) = g_3(t) - 1.$$

where

$$\lambda = g_2(t)\,t$$

By these definitions $\overline{g_3}(0) = -1$, and $\overline{g_3}(t) \to \infty$ for $t \to \bar{\bar{r}}_0^-$. Denote by ρ_3 the minimal zero of function $\overline{g_3}$ on $(0, \bar{\bar{r}}_0)$.

Define a radius of convergence ρ by

$$\rho = \min\{\rho_j\}, \quad j = 1, 2, 3. \tag{9}$$

By the preceding we have for all $t \in [0, \rho)$

$$0 \le \varphi_0(t) < 1, \tag{10}$$

$$0 \le \varphi_2(\varphi_5(t), g_1(t)\,t) < 1, \tag{11}$$

$$0 \le \varphi_0(g_1(t)\,t) < 1, \tag{12}$$

$$0 \le \varphi_0(g_2(t)\,t) < 1 \tag{13}$$

and

$$o \le g_j(t) < 1. \tag{14}$$

Define $S(x, \mu) = \{y \in T : \|y - x\| < \mu\}$ and let $\overline{S}(x, \mu)$ be the closure of $S(x, \mu)$. Next, we list the conditions (A) to be used in the convergence analysis:

(a_1) $F : T \to \mathscr{B}_2$ is continuous, differentiable, $[\cdot,\cdot;F] : T \times T \to \mathscr{L}B(\mathscr{B}_1, \mathscr{B}_2)$ is a divided difference of order one, $F(\xi) = 0$, and $F'(\xi)^{-1} \in \mathscr{L}B(\mathscr{B}_1, \mathscr{B}_2)$ for some $\xi \in T$.

(a_2) $\varphi_0 : D \to \mathbb{R}$ is increasing, continuous, $\varphi_0(0) = 0$ and for all $x \in T$

$$\|F'(\xi)^{-1}(F'(x) - F'(\xi))\| \le \varphi_0(\|x - \xi\|).$$

Define $T_0 = T \cap S(\xi, r_0)$, where r_0 is given in (6).

(a_3) $\varphi : D_0 \to \mathbb{R}$ is continuous, increasing, $\varphi(0) = 0$ and for $x, y \in T_0$

$$\|F'(\xi)^{-1}(F'(y) - F'(x))\| \le \varphi(\|y - x\|).$$

(a_4) $\varphi_1 : D_1 \to \mathbb{R}$, $\varphi_3 : D_1 \to \mathbb{R}$, $\varphi_2 : D_1 \times D_1 \to \mathbb{R}$, $\varphi_5 : D_1 \to \mathbb{R}$, are increasing, continuous, $w : T \to T$ is continuous and for all $x, y \in T_1 := T \cap S(\xi, \bar{r}_0)$

$$\|F'(\xi)^{-1}([w, y; F] - F'(y))\| \le \varphi_1(\|w - y\|),$$
$$\|F'(\xi)^{-1}([w, y; F] - F'(\xi))\| \le \varphi_2(\|w - \xi\|, \|y - \xi\|),$$
$$\|F'(\xi)^{-1} F'(x)\| \le \varphi_3(\|x - \xi\|),$$

and

$$\|w - \xi\| \le \varphi_5(\|x - \xi\|),$$

where $\bar{r}_0$ is given in (7).

(a_5) $\varphi_4 : D_2 \times D_2 \to \mathbb{R}$ is increasing, continuous and for all $y, z \in T_2 := T \cap S(x_0, \bar{\bar{r}}_0)$

$$\|F'(\xi)^{-1}([w, y; F] - F'(z))\| \le \varphi_4(\|w - z\|, \|y - z\|),$$

(a_6) $\overline{S}(\xi, \rho) \subset T$, $r_0, \bar{r}_0, \bar{\bar{r}}_0$ given by (6), (7), (8) exist and ρ is defined in (9).

(a_7) There exist $\bar{\rho} \ge \rho$ such that

$$\int_0^1 \varphi_0(\tau \bar{\rho})\, d\tau < 1.$$

Define $T_3 = T \cap \overline{S}(\xi, \bar{\rho})$.

Theorem 1. *Assume conditions (A) hold. Then, for $x_0 \in S(\xi, \rho) - \{\xi\}$, $\{x_n\}$ produced by (2) is such that $\{x_n\} \subseteq S(\xi, \rho)$, $\lim_{n\to\infty} x_n = \xi$ so that*

$$\|y_n - \xi\| \le g_1(\|x_n - \xi\|)\, \|x_n - \xi\| \le \|x_n - \xi\| < \rho \quad (15)$$
$$\|z_n - \xi\| \le g_2(\|x_n - \xi\|)\, \|x_n - \xi\| \le \|x_n - \xi\|, \quad (16)$$

and

$$\|x_{n+1} - \xi\| \le g_3(\|x_n - \xi\|)\, \|x_n - \xi\| \le \|x_n - \xi\|, \quad (17)$$

where functions g_j were given previously. The only solution of Equation (1) is ξ, in T_3, where T_3 is given in (a_7).

Proof. If $v \in S(\xi, \rho)$, then (6), (9), (11) and (a_2) give

$$\|F'(\xi)^{-1}(F'(\xi) - F'(v))\| \le \varphi_0(\|\xi - v\|) \le \varphi_0(r_0) \le \varphi_0(\rho) < 1. \quad (18)$$

This estimation together with the lemma by Banach for operators that are invertible [5,22] assures $F'(v)^{-1} \in \mathscr{L}B(\mathscr{B}_1, \mathscr{B}_2)$ with

$$\|F'(v)^{-1} F'(\xi)\| \le \frac{1}{1 - \varphi_0(\|v - \xi\|)}. \quad (19)$$

It also follows that iterate y_0 exists by (19), and the first substep of method (2). Using (2) (the first substep) (9), (11), (14) (for $j = 1$), (19) (for $v = x_0$), and (a_3)

$$\begin{aligned} \|y_0 - \xi\| &= \|x_0 - \xi - F'(x_0)^{-1} F(x_0)\| \\ &\leq \|F'(x_0)^{-1} F'(\xi)\| \\ &\| \int_0^1 F'(\xi)^{-1} (F'(\xi + \tau (x_0 - \xi)) - F'(x_0)) \, d\tau \, (x_0 - \xi)\| \\ &\leq \frac{\int_0^1 \varphi((1-\tau)\|x_0-\xi\|) \, d\tau \, \|x_0-\xi\|}{1-\varphi_0(\|x_0-\xi\|)} = g_1(\|x_0 - \xi\|) \|x_0 - \xi\| \\ &\leq \|x_0 - \xi\| < \rho, \end{aligned} \tag{20}$$

showing that $y_0 \in S(\xi, \rho)$ and (15) is true for $n = 0$. By (a_1) and (a_4) we have

$$\begin{aligned} \|F'(\xi)^{-1} F(v)\| &= \|F'(\xi)^{-1} (F(v) - F(\xi))\| \\ &= \| \int_0^1 F'(\xi)^{-1} F'(\xi + \tau (v - \xi)) \, d\tau \, (v - \xi)\| \\ &\leq \int_0^1 \varphi_3(\tau \|v - \xi\|) \, d\tau \, \|v - \xi\|. \end{aligned} \tag{21}$$

We get the estimates by (9), (11), and (a_3)

$$\|F'(\xi)^{-1}([w_0, y_0; F] - F'(\xi))\| \leq \varphi_2(\|w_0 - \xi\|, \|y_0 - \xi\|) \leq \varphi_0(\rho) < 1 \tag{22}$$

leading to

$$\|[w_0, y_0; F]^{-1} F'(\xi)\| \leq \frac{1}{1 - \varphi_2(\|w_0 - \xi\|, \|y_0 - \xi\|)}, \tag{23}$$

so z_0 and x_1 exist. Then, by (19) (for $v = x_0, y_0$), (9), (14) (for $j = 2$), (20), (21), (a_3) method (2) (second substep) we obtain

$$\begin{aligned} \|z_0 - \xi\| &= \|(y_0 - \xi - F'(y_0)^{-1} F(y_0)) \\ &+ F'(y_0)^{-1} ([w_0, y_0; F] - F'(y_0)) [w_0, y_0; F]^{-1} F(y_0)\| \\ &\leq \Bigg\{ \frac{\int_0^1 \varphi((1-\tau)\|y_0-\xi\| \, d\tau}{1-\varphi_0(\|y_0-\xi\|)} \\ &+ \frac{\varphi_1(\|(w_0-\xi)+(\xi-y_0)\|) \int_0^1 \varphi_3(\tau \|y_0-\xi\|) \, d\tau}{(1-\varphi_0(\|y_0-\xi\|))(1-\varphi_2(\|w_0-\xi\|, \|y_0-\xi\|))} \Bigg\} \|y_0 - \xi\| \\ &\leq g_2(\|x_0 - \xi\|) \|x_0 - \xi\| \leq \|x_0 - \xi\| < \rho, \end{aligned} \tag{24}$$

showing that $z_0 \in S(\xi, \rho)$ and (16) is true for $n = 0$. In view of (9), (14), (19) (for $v = z_0$), (20)–(24), and the last substep of method (2), we obtain the estimations

$$\begin{aligned} \|x_1 - \xi\| &= \|(z_0 - \xi - F'(z_0)^{-1} F(z_0)) \\ &+ F'(z_0)^{-1} ([w_0, y_0; F] - F'(z_0)) [w_0, y_0; F]^{-1} F(z_0)\| \\ &\leq \Bigg\{ \frac{\int_0^1 \varphi((1-\tau)\|z_0-\xi\| \, d\tau}{1-\varphi_0(\|z_0-\xi\|)} \\ &+ \frac{\varphi_4(\|(w_0-\xi)+(\xi-z_0)\|, \|(y_0-\xi)+(\xi-z_0)\|) \int_0^1 \varphi_3(\tau \|z_0-\xi\|) \, d\tau}{(1-\varphi_0(\|z_0-\xi\|))(1-\varphi_2(\|w_0-\xi\|, \|z_0-\xi\|))} \Bigg\} \\ &\times \|z_0 - \xi\| \\ &\leq g_3(\|x_0 - \xi\|) \|x_0 - \xi\| \leq \|x_0 - \xi\| < \rho, \end{aligned} \tag{25}$$

which completes the induction for estimations (15)–(17) if $n = 0$ and $x_1 \in S(\xi, \rho)$. By repeating the previous estimations for x_m, y_m, z_m, x_{m+1} replacing x_0, y_0, z_0, x_1, respectively, the induction for items (15)–(17) is completed. Moreover, by the estimation

$$\|x_{m+1} - \xi\| \leq \gamma \|x_m - \xi\| \leq \rho, \quad \gamma = g_3(\|x_0 - \xi\|) \in [0, 1), \tag{26}$$

we arrive at $\lim_{m\to\infty} x_m = \xi$, and $x_{m+1} \in S(\xi, \rho)$. Let $G = \int_0^1 F'(\xi + \tau(\xi_0 - \xi))\, d\tau$ for $\xi_0 \in T_3$ and $F(\xi_0) = 0$. By (a_2), (a_7), we get that

$$\|F'(\xi)^{-1}(G - F'(\xi))\| \leq \int_0^1 \varphi_0((1-\tau)\|\xi_0 - \xi\|)\, d\tau \leq \int_0^1 \varphi_0(\tau\bar{\rho})\, d\tau < 1, \tag{27}$$

so $G^{-1} \in \mathscr{L}B(\mathscr{B}_1, \mathscr{B}_2)$, leading to $\xi = \xi_0$, where we also used the estimation $0 = F(\xi_0) - F(\xi) = G(\xi_0 - \xi)$. □

Remark 1. *(a)* *Let $\varphi_0(t) = K_0 t$ and $\varphi(t) = Kt$. The radius $r_1 = \frac{2}{2K_0+K}$ was obtained by Argyros in [1] as the convergence radius for Newton's method under conditions (17)–(19). Notice that the convergence radius for Newton's method given independently by Rheinboldt [23] and Traub [25] is given by*

$$\rho = \frac{2}{3K_1} < r_1,$$

where K_1 is the Lipschitz constant on T, so $K_0 \leq K_1$ and $K \leq K_1$. Define $f(x) = e^x - 1$ and $T = \bar{S}(0,1)$. Then, we find $K_0 = e - 1 < K = e^{\frac{1}{e-1}} < K = e$, so $\rho = 0.24252961 < r_1 = 0.3827$.

Moreover, the new error bounds [1–4] are:

$$\|x_{n+1} - x^*\| \leq \frac{L}{1 - L_0\|x_n - x^*\|}\|x_n - x^*\|^2,$$

whereas the old ones [12,14] are

$$\|x_{n+1} - x^*\| \leq \frac{L_1}{1 - L_1\|x_n - x^*\|}\|x_n - x^*\|^2.$$

Therefore the new bounds are tighter if $L_0 < L$ or $L < L_1$. Clearly, we do not expect the radius of convergence of method (2) given by r to be larger than r_1 (see (8)).

(b) *By (a2) and*

$$\begin{aligned} \|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\ &\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq 1 + \varphi_0(\|x - x^*\|), \end{aligned}$$

the condition on φ_3 can be dropped and we can set

$$\varphi_3(t) = 1 + \varphi_0(t).$$

3. Numerical Examples

We use $[x, y; F] = \int_0^1 F'(y + \tau(x - y))\, d\tau$ and w as given in (3) for $\alpha = \frac{1}{10}\|F'(\xi)\|$, and $\beta = \frac{1}{10}\|F'(\xi)\|^2$. In view of the definition of the divided difference, conditions (A) and the estimation (for $x \in S(\xi, \rho)$)

$$\begin{aligned} \|w(x) - \xi\| &\leq \|y(x) - \xi\| + \alpha\|F'(\xi)^{-1}F(y(x))\| + \beta\|F'(\xi)^{-1}F(y(x))\|^2 \\ &\leq g_1(t)\,t + \alpha\int_0^1 \varphi_3(\tau g_1(t)\,t)\, d\tau + \beta\Big(\int_0^1 \varphi_3(\tau g_1(t)\,t)\, d\tau\Big)^2 = \varphi_5(t). \end{aligned}$$

Then, we can choose functions φ_i, $i = 1, 2, 4$ in terms of functions φ_0, φ, φ_5 as follows

$$\varphi_1(t) = \frac{\varphi(\varphi_5(t)) + \varphi(g_1(t)\,t)}{2}$$

$$\varphi_2(s,t) = \frac{\varphi_0(\varphi_5(s)) + \varphi_0(g_1(t)\,t)}{2}$$

and
$$\varphi_4(s,t) = \varphi_2(s,t) + \varphi_0(t)$$
in all examples.

Example 1. *Consider $\mathscr{B}_1$ and $\mathscr{B}_2$ to be the space of continuous functions and defined on interval $[0,1]$ with $T = \bar{S}(0,1)$. Let G on T be*

$$G(\psi)(z) = \psi(z) - 5\int_0^1 z\tau\psi(\tau)^3 d\tau. \tag{28}$$

Then, we find

$$G'(\psi)((\xi(x)) = \xi(x) - 15\int_0^1 z\tau\psi(\tau)^2\xi(\tau)d\tau, \text{ for each } \xi \in T.$$

Notice that we can take $\xi = 0$, giving $\varphi(t) = 15t$, $\varphi_0(t) = 7.5t$, $\varphi_3(t) = 15$, $\alpha = \beta = \frac{1}{10}$. This way, we have that

$$\rho_1 = 0.066667, \rho_2 = 0.00097005, \rho = \rho_3 = 0.000510334.$$

Example 2. *By the motivational example, represented in Figure 1, we choose $\varphi_0(t) = \varphi(t) =$ $96.662907t$, $\varphi_3(t) = 1.0631$, $\alpha = \frac{3}{10}$, and $\beta = \frac{9}{10}$. Then, the parameters for method (2) are for $\xi = 1$*

$$\rho_1 = 0.00689682, \rho_2 = 0.00000221412, \rho = \rho_3 = 0.00000121412.$$

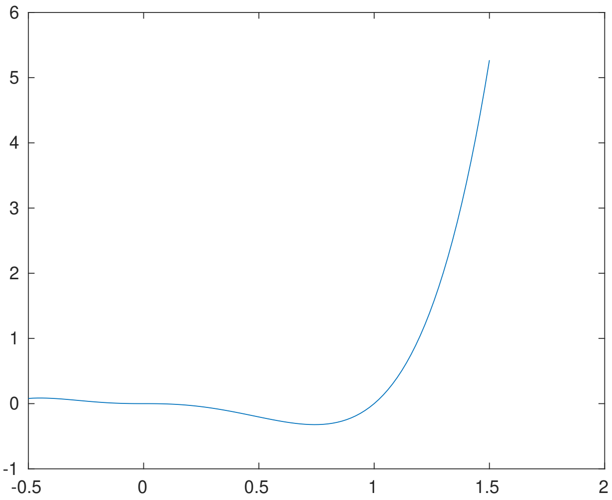

Figure 1. Plot of motivational function.

Example 3. *Let $\mathbb{B}_1 = \mathbb{B}_2 = \mathbb{R}^3$, $T = S(0,1)$, and let F on T be*

$$F(x) = F(x_1,x_2,x_3) = (e^{x_1} - 1, \frac{e-1}{2}x_2{}^2 + x_2, x_3)^T. \tag{29}$$

For the points $u = (u_1,u_2,u_3)^T$, we find

$$F'(u) = \begin{pmatrix} e^{u_1} & 0 & 0 \\ 0 & (e-1)u_2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, for $x^* = (0,0,0)^T$ we get $F'(\xi) = diag(1,1,1)$, $\varphi(t) = e^{\frac{1}{e-1}}t$, $\varphi_0(t) = (e-1)t$, $\varphi_3(t) = e^{\frac{1}{e-1}}$, $\alpha = \beta = \frac{1}{10}$.

Then, we obtain that

$$\rho_1 = 0.382692, \rho_2 = 0.96949, \rho = \rho_3 = 0.154419.$$

4. Conclusions

In this article we extended the applicability of Newton-Traub-like methods in cases not covered before requiring the usage of derivatives up to order seven that do not appear in the methods. The price we pay by using conditions on the first derivative that actually appears on the method is that we show only linear convergence. To find the convergence order is not, however, our intention, since this is already known in the case of spaces that coincide with the multidimensional Euclidean space. Notice that the order is rediscovered by using ACOC or COC, which require only the first derivative. Moreover, in earlier studies using Taylor series, no computable error distances were available based on generalized Lipschitz conditions. Therefore, we do not know, for example, in advance, how many iterates are needed to achieve a predetermined error tolerance. Furthermore, no uniqueness of the solution results is available in the aforementioned studies, but we also provide such results. Our technique can be used to extend the applicability of other methods in an analogous way, since it is so general. Finally notice that local results of this type are important, since they demonstrate the difficulty in choosing initial points.

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