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Analytical and Qualitative Study of Some Families of FODEs via Differential Transform Method

Neelma, Eiman and Kamal Shah *

Department of Mathematics, University of Malakand, Chakdara 18800, Khyber Pakhtunkhwa, Pakistan; nirmauop1@gmail.com (N.); ehuzaifa@gmail.com (E.)
* Correspondence: kamal@uom.edu.pk

Abstract: This current work is devoted to develop qualitative theory of existence of solution to some families of fractional order differential equations (FODEs). For this purposes we utilize fixed point theory due to Banach and Schauder. Further using differential transform method (DTM), we also compute analytical or semi-analytical results to the proposed problems. Also by some proper examples we demonstrate the results.

Keywords: qualitative theory; differential transform method; existence and uniqueness

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1. Introduction

In last few decades, great interest has been given to fractional calculus. Particularly, the area to deal FODEs has given much more importance. Since the said area has large number of applications in real world problems. Therefore in past few decades many researchers studied FODEs for various kinds of solutions including analytical, approximate and iterative solutions (see detail [1–4]). In this regards many theories have been formed to handle the aforesaid equations from different aspects. The area of the qualitative theory has been very well explored in last few years (see [5–8]). From analytical point of view researchers have tried to find exact solutions to these equations by exploiting various tools. But to compute exact or analytical solution to each and every FODE is very tough task. Therefore researchers have introduced many methods, tools to handle such problems for numerical or semi-analytical solutions. In this regard plenty of research articles, books have been written. Some of them we refer as [9–12] and the references there in. In first glance integral transforms have been used to find approximate or exact solutions to FODEs, see [13]. Later on perturbation techniques have also been greatly used during 2000 to 2010, see detail [14–16]. The decompositions techniques which increasingly used for treating classical differential equations have been extended to handle FODEs in number of ways, for detail we refer [17,18]. Also some authors have applied homotopy analysis transform methods in more accurate ways to handle many problems of the aforesaid area. The DTM has been utilized to solve various engineering problem like circuit problems, (detail can be seen in [19]). The aforementioned technique has been increasingly used to solve classical and partial differential equations as well as integral equations, (see [20–22]). Also the said transform has been used to solve some problems of the mentioned area ([23–26]). Here in this research work, we use the said method to handle some families of FODEs of different nature. For more detail work in recent times, we refer [27–31].

Motivated from the mentioned work as above, we establish some adequate results for existence and uniqueness of solutions to some families of nonlinear FODEs. In this regards, we use Banach and Schauder fixed point results to achieve the goals. Also for the intended analytical results, we use DTM to handle the proposed problems. In first case, we solve a nonlinear class of FODEs over the unbounded domain for approximate analytical solution given as

$$\begin{aligned} {}^{C}_{0}\mathrm{D}^{\alpha}_{t}u(t) &= f(t,u(t)),\ t\in[0,b],\ \alpha\in(0,1], \\ u(0) &= u_0,\ u_0\in\mathrm{R}, \end{aligned} \tag{1}$$

where $f:[0,b]\times\mathrm{R}\to\mathrm{R}$. Extending the procedure we then treat the following system of three components

$$\begin{cases} {}^{C}_{0}\mathrm{D}^{\alpha}_{t}u(t) = f_1(t,u(t),v(t),w(t)), \\ {}^{C}_{0}\mathrm{D}^{\alpha}_{t}v(t) = f_2(t,u(t),v(t),w(t)), \\ {}^{C}_{0}\mathrm{D}^{\alpha}_{t}w(t) = f_3(t,u(t),v(t),w(t)), \end{cases} \tag{2}$$

subject to initial conditions given as

$$u(0)=0,\ v(0)=0,\ w(0)=0,$$

where $f_i\ (i=1,2,3):[0,b]\times\mathrm{R}^3\to\mathrm{R}$ and $\alpha\in(0,1]$. Here we aim that the following three components system is a special case of system (2).

$$\begin{cases} {}^{C}_{0}\mathrm{D}^{\alpha}_{t}u(t) = -u(t), \\ {}^{C}_{0}\mathrm{D}^{\alpha}_{t}v(t) = u(t)-v^2(t), \\ {}^{C}_{0}\mathrm{D}^{\alpha}_{t}w(t) = v^2(t), \end{cases} \tag{3}$$

subject to initial conditions given as

$$u(0)=0,\ v(0)=0,\ w(0)=0.$$

where $\alpha\in(0,1]$. If we put $\alpha=1$ in (3), we get the system studied in [32].

For the justification, we provide several examples along with numerical plots.

2. Fundamental Concepts

Here we recall some basic concepts that are needed in this study.

Definition 1 ([1])**.** *Integral of fractional order$\alpha>0$ of a function $u:(0,\infty)\to R$ is recalled as*

$${}_{0}\mathrm{I}^{\alpha}_{t}u(t) = \int_0^t \frac{u(\theta)}{\Gamma(\alpha)(t-\theta)^{1-\alpha}}d\theta, \tag{4}$$

keeping in mind that integral at the right side is convergent. Further a simple and important property of ${}_{0}\mathrm{I}^{\alpha}_{t}$ is given by

$${}_{0}\mathrm{I}^{\alpha}_{t}t^{\delta},\ \textit{where}\ \delta>0 = \frac{\Gamma(\delta+1)}{\Gamma(\delta+\alpha+1)}t^{\alpha+\delta}. \tag{5}$$

Definition 2 ([1])**.** *Fractional derivative in Caputo sense with order $\alpha>0$ to a mapping $u\in C[0,1]$ is recalled as*

$$\begin{aligned} {}^{C}_{0}\mathrm{D}^{\alpha}_{t}u(t) &= \int_0^t \frac{(t-\theta)^{n-\alpha-1}}{\Gamma(n-\alpha)}\frac{d^n}{d\theta^n}u(\theta)d\theta,\ n-1<\alpha\le n, \\ \textit{such that}\ n &= [\alpha]+1, \end{aligned} \tag{6}$$

where integral on right is pointiest exists on R^+. Further a remarkable property is provided as

$${}^{C}_{0}\mathrm{D}^{\alpha}_{t}t^{p} = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}t^{p-\alpha}. \tag{7}$$

The following results need in the sequel.

Lemma 1 ([7])**.** *Let $\alpha > 0$, then*

$$
{}_0\mathrm{I}_t^{\alpha}[{}_0^C\mathrm{D}_t^{\alpha}u(t)] = u(t) + \sum_{j=0}^{n-1}\frac{u^{(j)}(0)}{\Gamma(j+1)}t^j. \tag{8}
$$

Definition 3 ([23])**.** *For a function $g(t) = {}_0^C\mathrm{D}_t^{\alpha}u(t)$, then differential transform DT is recalled as*

$$
G(k) = \frac{\Gamma(\alpha+1+\frac{k}{\alpha_1})}{\Gamma(k\alpha+1)}U(k+\alpha\alpha_1),
$$

where α_1 is the unknown value of the fraction. The inverse DT of $G(k)$ is given by

$$
g(t) = \sum_{k=0}^{\infty}\left[\frac{\Gamma(\alpha+1+\frac{k}{\alpha_1})}{\Gamma(k\alpha+1)}U(k+\alpha\alpha_1)\right]t^{\frac{k}{\alpha}}.
$$

Further detail properties of the mentioned transform are given in [25,26].

Definition 4. *Let $X = C[0,b]$, $b > 0$ be the space of all continues functions from $[0,b]$ to R, then it is complete normed space also called Banach space. Endowing a norm on X as*

$$
\|u\|_X = \sup_{t\in[0,b]}\{|u(t)| : u \in X\}.
$$

Remark 1. *The product space $X \times X \times X$ is also a Banach space whose norm is defined by*

$$
\|(u,v,w)\|_X = \|u\|_X + \|v\|_X + \|w\|_X = \sup_{t\in[0,T]}\{|u(t)| + |v(t)| + |w(t)| : (u,v,w) \in X \times X \times X\}.
$$

Theorem 1 ([33])**.** *Let X is Banach space and an operator $T : X \to X$ is a completely continuous operator and the set $\mathbf{B} = \{u \in X : u = \lambda Tu, \ \lambda \in [0,b]\}$ is bounded, then T has fixed point.*

3. Study of Problem I

In this section we study the family of scaler FODEs (1). This section is further spliced in to two subsections.

3.1. Qualitative Theory of Problem I

Here in this section before to establish the numerical techniques, first we establish the qualitative aspects of the consider problem (1). To proceeded further, we consider the following Theorem.

Theorem 2. *Let $y \in L[0,b]$, then he unique solution of the problem*

$$
\begin{aligned}
{}_0^C\mathrm{D}_t^{\alpha}u(t) &= y(t), \ t \in [0,b], \ 0 < \alpha \leq 1,\\
u(0) &= u_0, \ u_0 \in \mathrm{R},
\end{aligned} \tag{9}
$$

is expressed as

$$
u(t) = u_0 + \int_0^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)}y(\theta)d\theta, \ t \in [0,b]. \tag{10}
$$

Proof. Utilizing ${}_0\mathrm{I}_t^{\alpha}$ on both sides of (15) and plugging initial condition, one has the required results (11). □

Lemma 2. *Inview of Theorem 2, the unique solution of Problem I is given by*

$$u(t) = u_0 + \int_0^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} f(\theta, u(\theta))d\theta, \ t \in [0,b]. \tag{11}$$

Proof. Proof is same is given in Theorem 2. □

We need the given assumptions for onward analysis.

(C_1) For any $u, v \in \mathrm{R}$, $\exists$ a constant $L_f > 0$ with

$$|f(t,u) - f(t,v)| \leq L_f |u - v|, \ t \in [0,b].$$

(C_2) Let $\exists$ constants $M_f > 0$ and $C_f > 0$, then the given growth condition hold

$$|f(t,u(t))| \leq M_f + C_f|u|, \ t \in [0,b].$$

Let $T : X \to X$ be an operator defined by

$$Tu(t) = u_0 + \int_0^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} f(\theta, u(\theta))d\theta, \ t \in [0,b]. \tag{12}$$

Theorem 3. *Under the assumption (C_1), the operator given in (12) has unique fixed point under the condition*

$$\frac{L_f b^\alpha}{\Gamma(\alpha+1)} < 1, \ b > 0.$$

Proof. Suppose $u, v \in X$, then consider

$$\begin{aligned}
\|Tu - Tv\|_X &= \sup_{t\in[0,b]} |Tu(t) - Tv(t)| \\
&= \sup_{t\in[0,b]} \left| \int_0^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} [f(\theta,u(\theta)) - f(\theta,v(\theta))]d\theta \right| \\
&\leq \frac{L_f}{\Gamma(\alpha)} \sup_{t\in[0,b]} \int_0^t (t-\theta)^{\alpha-1} |u(\theta) - v(\theta)|d\theta \\
&\leq \frac{L_f}{\Gamma(\alpha+1)} t^\alpha \|u - v\|_X \\
&\leq \frac{L_f b^\alpha}{\Gamma(\alpha+1)} \|u - v\|_X.
\end{aligned}$$

Hence T is contraction, so T has at most one fixed point. Thus the considered problem I has at most one solution. □

Theorem 4. *Under Assumption (C_2), the considered Problem I has at least one solution.*

Proof. Let $\mathbf{B} = \{u \in C[0,b] : \ |u\|_X \leq r\}$. Clearly $\mathbf{B} \neq \emptyset$ closed and convex subset of X. Then to prove that T is continues, let $\{u_n\}$ be a sequence in $\mathbf{B}$ such that $u_n \to u$ as $n \to \infty$. Let for $t \in [0,b]$, we have

$$\begin{aligned}
\|Tu_n - Tu\|_X &= \sup_{t\in[0,b]} |Tu_n(t) - Tu(t)| \\
&= \sup_{t\in[0,b]} \left| \int_0^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} [f(\theta,u_n(\theta)) - f(\theta,u(\theta))]d\theta \right| \\
&\leq \frac{L_f}{\Gamma(\alpha)} \sup_{t\in[0,b]} \int_0^t (t-\theta)^{\alpha-1} |u_n(\theta) - u(\theta)|d\theta.
\end{aligned}$$

Due to continuity of f, we see that

$$\|Tu_n - Tu\|_X \to 0, \text{ as } n \to 0.$$

Hence T is continues. Further to prove that $T(\mathbf{B}) \subset \mathbf{B}$, we take

$$\begin{aligned}
\|Tu\|_X &= \sup_{t\in[0,b]} |Tu(t)| \\
&= \sup_{t\in[0,b]} \left| \int_0^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} [f(\theta, u(\theta))d\theta \right| \\
&\le \frac{b^\alpha}{\Gamma(\alpha+1)} [M_f + C_f \|u\|_X] \\
&\le r.
\end{aligned}$$

Hence $\|Tu\|_X \le r$ implies that T is bounded. To show that T is relatively compact, it is need to prove that T is equi-continues operator. To achieve the goal, let $t_1, t_2 \in [0, b]$ and considering

$$\begin{aligned}
|Tu(t_2) - Tu(t_1)| &= \int_0^{t_2} \frac{1}{\Gamma(\alpha)} (t_2-\theta)^{\alpha-1} f(\theta, u(\theta))d\theta - \int_0^{t_1} (t_1-\theta)^{\alpha-1} f(\theta, u(\theta))d\theta \\
&\le \frac{M_f + C_f r}{\Gamma(\alpha)} \left[\int_0^{t_2} \frac{1}{\Gamma(\alpha)} (t_2-\theta)^{\alpha-1} - \int_0^{t_1} \frac{1}{\Gamma(\alpha)} (t_1-\theta)^{\alpha-1} \right] \\
&\le \frac{M_f + C_f r}{\Gamma(\alpha+1)} (t_2^\alpha - t_1^\alpha).
\end{aligned} \tag{13}$$

□

Since as $t_2 \to t_1$ in right side of (13) leads to zero, so we have $\|Tu(t_2) - Tu(t_1)\|_X \to 0$ as $t_2 \to t_1$. Hence T is equi-continues operator. Thus By Arzelaá Theorem, T has at least one fixed point. Hence the considered Problem I has at least one solution.

3.2. Numerical Procedure for Problem I

In this section, we are going to find the analytical solutions of the following nonlinear FODEs as inview of DTM, we know that

$$y(t) = {}_0^C\mathrm{D}_t^\alpha u(t),$$

then

$$Y(k) = \frac{\Gamma(\alpha + 1 + \frac{k}{\alpha_1})}{\Gamma(1 + \frac{k}{\alpha_1})} G(k + \alpha_1 \alpha).$$

Hence inview of this (1) takes the form

$$\begin{aligned}
U(k + \alpha\alpha_1) &= \frac{\Gamma(1 + \frac{k}{\alpha_1})}{\Gamma(\alpha + 1 + \frac{k}{\alpha_1})} \left[F(k, U(0), U(1), U(2), \dots, U(k)) \right] \\
U(k) &= 0, \text{ for } k = 0, 1, 2, \dots, \alpha\alpha_1 - 1, \\
U(0) &= u_0.
\end{aligned} \tag{14}$$

Evaluating the above series (14), we get the solution as

$$u(t) = \sum_{k=0}^{\infty} U(k) t^{\frac{k}{\alpha}}.$$

3.3. Numerical Problems to Verify the Establish Analysis

In this section under various initial values, we provide some examples as bellow.

Example 1. *Consider the given FODE as*

$$\begin{aligned} {}^{C}_{0}\mathrm{D}^{\alpha}_{t}u(t) &= u(t),\ t \in [0, b],\ 0 < \alpha \leq 1, \\ u(0) &= 1. \end{aligned} \tag{15}$$

Thank to above procedure given in (14), we have

$$\begin{aligned} U(k + \alpha\alpha_1) &= \frac{\Gamma(1 + \frac{k}{\alpha_1})}{\Gamma(\alpha + 1 + \frac{k}{\alpha_1})}[U(k)] \\ U(k) &= 0,\ \textit{for}\ k = 0, 1, 2, \ldots, \alpha\alpha_1 - 1, \\ U(0) &= 1. \end{aligned} \tag{16}$$

Computing $U(k)$ up to $k = 100$, we have

$$U(0) = 1,\ U(1) = 1,\ U(2) = \frac{1}{2!}, \ldots, U(100) = \frac{1}{100!}$$

and so on. Thus we have the result

$$u(t) = U(0) + U(1)t^{\frac{1}{\alpha}} + U(2)t^{\frac{2}{\alpha}} + U(3)t^{\frac{3}{\alpha}} + U(4)t^{\frac{4}{\alpha}} + \cdots + U(100)t^{\frac{100}{\alpha}} + \ldots. \tag{17}$$

Putting $\alpha = 1$, we get

$$u(t) = 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^{100}}{100!} + \cdots = \exp(t). \tag{18}$$

Plots at various fractional order of solutions is given in Figure 1.

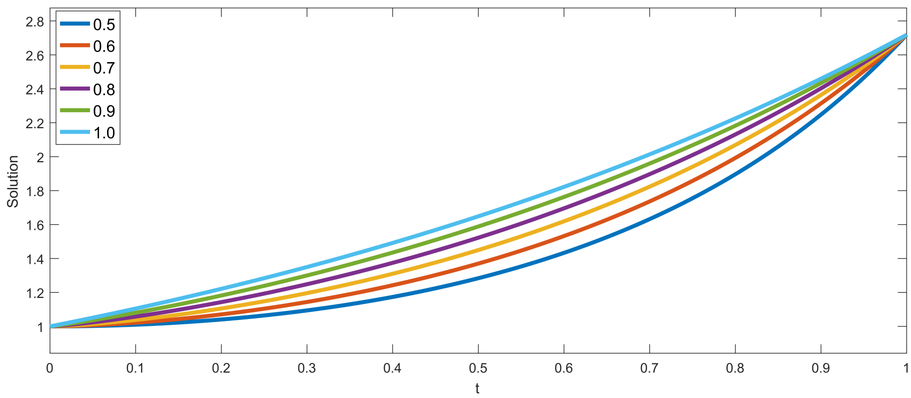

Figure 1. Plots of solution upto first 100 terms at various values of fractional order α of Example (1).

From the plot, we see as $\alpha \to 1$, the approximate solution tending near to the solution at integer order. Further the accuracy may be improved by involving more terms in the solution.

Example 2. *Consider the given FODE as*

$$\begin{aligned} {}^{C}_{0}\mathrm{D}^{0.9}_{t}u(t) &= u(t) + u^2(t),\ t \in [0, b], \\ u(0) &= 1. \end{aligned} \tag{19}$$

Thank to above procedure given in (14), we have

$$\begin{aligned} U(k + 9) &= \frac{\Gamma(1 + \frac{k}{10})}{\Gamma(\alpha + 1 + \frac{k}{10})}[U(k) + \sum_{l=0}^{k} U(l)U(k - l)] \\ U(k) &= 0,\ \textit{for}\ k = 1, 2, \ldots, \alpha\alpha_1 - 1, \\ U(0) &= 1. \end{aligned} \tag{20}$$

Computing $U(k)$ up to $k=100$, we have

$$U(1)=0,\ U(2)=0,\dots,U(8)=0,\ U(9)=1,\ U(10)=0,\ U(11)=0,\ \dots \tag{21}$$

and so on. Thus we have the result

$$\begin{aligned} u(t) &= U(0)+U(1)t^{\frac{1}{\alpha}}+U(2)t^{\frac{2}{\alpha}}+U(3)t^{\frac{3}{\alpha}}+U(4)t^{\frac{4}{\alpha}}+\dots \\ &+ U(100)t^{\frac{100}{\alpha}}+\dots. \end{aligned} \tag{22}$$

Putting values from (21), *we get*

$$u(t)=1+\frac{\Gamma(\alpha+0.9)}{\Gamma(\alpha+1.8)}t^{\frac{10}{\alpha}}+\frac{\Gamma(\alpha+1.9)}{\Gamma(\alpha+2.8)}t^{\frac{30}{\alpha}}+\dots. \tag{23}$$

In Figure 2, we plot approximate solutions for various fractional order up to first three terms as:

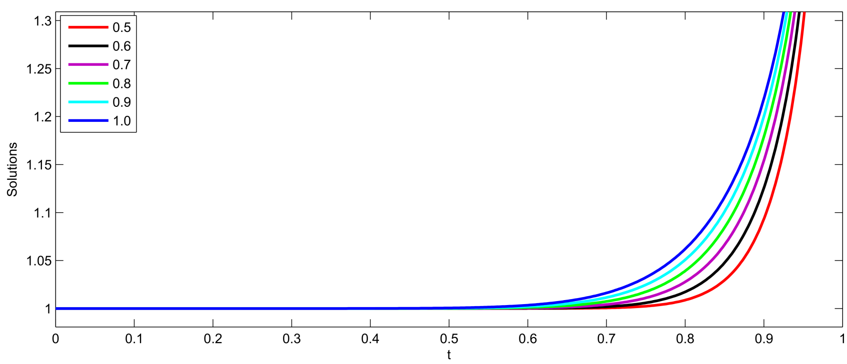

Figure 2. Plots of solution upto first three terms at various values of fractional order α of Example (2).

Example 3. *Consider non-homogenous nonlinear FODE as*

$$\begin{aligned} {}_0^C\mathrm{D}_t^{\alpha}u(t) &= u(t)+u^2(t)+\exp(t),\ t\in[0,1], \\ u(0) &= 1. \end{aligned} \tag{24}$$

Thank to DTM, we have

$$\begin{aligned} U(k+9) &= \frac{\Gamma(1+\frac{k}{10})}{\Gamma(\alpha+1+\frac{k}{10})}\left[U(k)+\sum_{l=0}^{k}U(l)U(k-l)+\frac{2^k}{\Gamma(k+1)}\right] \\ U(k) &= 0,\ \text{for } k=1,2,\dots,\alpha\alpha_1-1, \\ U(0) &= 1. \end{aligned} \tag{25}$$

Computing for $k=10$, we have

$$\begin{aligned} U(10) &= \frac{2}{\Gamma(\alpha+1)},\ U(11)=2\frac{\Gamma(1.1)}{\Gamma(\alpha+1.1)},\ U(12)=2\frac{\Gamma(1.2)}{\Gamma(\alpha+1.2)} \\ U(13) &= \frac{4\Gamma(1.3)}{3\Gamma(\alpha+1.3)}, \end{aligned} \tag{26}$$

and so on. Hence the solution is given by using (27) *as*

$$\begin{aligned} u(t) &= U(0)+U(1)t^{\frac{1}{\alpha}}+U(2)t^{\frac{2}{\alpha}}+U(3)t^{\frac{3}{\alpha}}+U(4)t^{\frac{4}{\alpha}}+\dots+U(100)t^{\frac{100}{\alpha}}+\dots \\ &= 1+t^{\frac{9}{\alpha}}+\frac{2}{\Gamma(\alpha+1)}t^{\frac{10}{\alpha}}+2\frac{\Gamma(1.1)}{\Gamma(\alpha+1.1)}t^{\frac{11}{\alpha}} \\ &+ 2\frac{\Gamma(1.2)}{\Gamma(\alpha+1.2)}t^{\frac{12}{\alpha}}+\frac{4\Gamma(1.3)}{3\Gamma(\alpha+1.3)}t^{\frac{13}{\alpha}}+\dots. \end{aligned} \tag{27}$$

In Figure 3, we plot approximate solutions for various fractional order up to first three terms as:

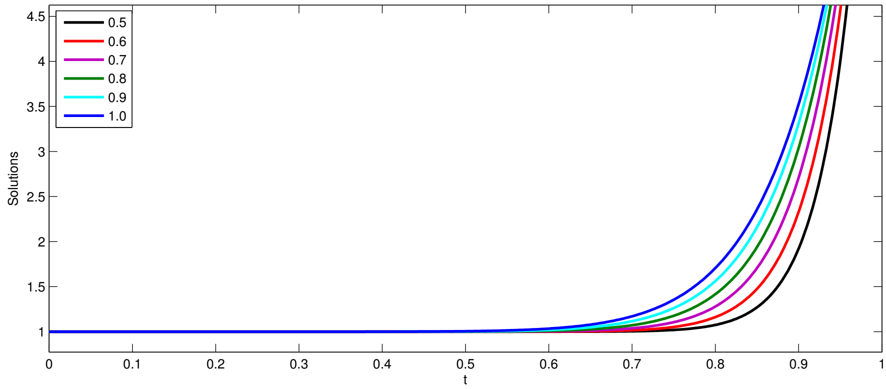

Figure 3. Plots of solution upto first three terms at various values of fractional order α of Example (3).

4. Study of Problem II

Here in this section, we study the system under consideration for the required two aspects. This portion is divided in to two sub portions.

4.1. Qualitative Theory of (2)

Here in this subsection, we provide qualitative analysis of the considered system (2) of nonlinear FODEs. Then we extend our study to derive the required results for the special system (3). To establish existence theory we provide some assumptions that to be hold:

(C_3) For any $u, v, w,\ \bar{u}, \bar{v}, \bar{w} \in$ R, there exist a constant $L_{f_i} > 0$ such that

$$|f_i(t, u, v, w) - f(t, \bar{u}, \bar{v}, \bar{w})| \leq L_{f_i}[|u - \bar{u}| + |v - \bar{v}| + |w - \bar{w}|],\ t \in [0, b],\ \text{for } i = 1, 2, 3.$$

(C_4) Let there exists constant $M_{f_i} > 0$ and $C_{f_i} > 0$; such that the given growth condition hold:

$$|f_i(t, u(t), v(t), w(t))| \leq M_{f_i} + C_{f_i}[|u| + |v| + |w|],\ t \in [0, b],$$

for $i = 1, 2, 3$.

Let $T = (T_1, T_2, T_3) : X \times X \times X \to X \times X \times X$ be an operator defined by

$$\begin{cases} T_1(u, v, w)(t) = \int_0^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} f_1(\theta, u(\theta), v(\theta), w(\theta))d\theta,\ t \in [0, b], \\ T_2(u, v, w)(t) = \int_0^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} f_2(\theta, u(\theta), v(\theta), w(\theta))d\theta,\ t \in [0, b], \\ T_3(u, v, w)(t) = \int_0^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} f_3(\theta, u(\theta), v(\theta), w(\theta))d\theta,\ t \in [0, b]. \end{cases} \tag{28}$$

For uniqueness of solution, we give the following result as

Theorem 5. *Under assumption (C_3), the system (28) has unique solution if and only if* $\frac{3b^{\alpha}L_{f_0}}{\Gamma(\alpha+1)} < 1$, *where* $L_{f_0} = \max\{L_{f_1}, L_{f_2}, L_{f_3}\}$.

Proof. Let $u, v, w,\ \bar{u}, \bar{v}, \bar{w} \in$ R, then considering

$$\begin{aligned}\|T_1(u,v,w)-T_1(\bar{u},\bar{v},\bar{w})\|_{X\times X\times X} &= \sup_{t\in[0,b]}|T_1(u,v,w)(t)-T_1(\bar{u},\bar{v},\bar{w})(t)| \\ &\le \max_{t\in[0,b]}\int_0^t\frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)}L_{f_1}[|u-\bar{u}|+|v-\bar{v}|+|w-\bar{w}|]d\theta \\ &\le \frac{b^\alpha L_{f_1}}{\Gamma(\alpha+1)}[\|u-\bar{u}\|_X+\|v-\bar{v}\|_X+\|w-\bar{w}\|_X]. \end{aligned} \tag{29}$$

Similarly

$$\|T_2(u,v,w)-T_2(\bar{u},\bar{v},\bar{w})\|_{X\times X\times X} \le \frac{b^\alpha L_{f_1}}{\Gamma(\alpha+1)}[\|u-\bar{u}\|_X+\|v-\bar{v}\|_X+\|w-\bar{w}\|_X] \tag{30}$$

and

$$\|T_3(u,v,w)-T_3(\bar{u},\bar{v},\bar{w})\|_{X\times X\times X} \le \frac{b^\alpha L_{f_1}}{\Gamma(\alpha+1)}[\|u-\bar{u}\|_X+\|v-\bar{v}\|_X+\|w-\bar{w}\|_X]. \tag{31}$$

Hence from (29), (30) and (31), we have

$$\|T(u,v,w)-T(\bar{u},\bar{v},\bar{w})\|_{X\times X\times X} \le \left[\frac{b^\alpha L_{f_1}}{\Gamma(\alpha+1)}+\frac{b^\alpha L_{f_2}}{\Gamma(\alpha+1)}\right. \tag{32}$$

$$\left. + \frac{b^\alpha L_{f_3}}{\Gamma(\alpha+1)}\right][\|u-\bar{u}\|_X+\|v-\bar{v}\|_X+\|w-\bar{w}\|_X]. \tag{33}$$

Let $L_{f_0}=\max\{L_{f_i},\text{ for } i=1,2,3\}$, we have from (32)

$$\begin{aligned}\|T(u,v,w)-T(\bar{u},\bar{v},\bar{w})\|_{X\times X\times X} &\le \frac{3b^\alpha L_{f_0}}{\Gamma(\alpha+1)}[\|u-\bar{u}\|_X+\|v-\bar{v}\|_X+\|w-\bar{w}\|_X] \\ &= \frac{3b^\alpha L_{f_0}}{\Gamma(\alpha+1)}\|(u,v,w)-(\bar{u},\bar{v},\bar{w})\|_{X\times X\times X}.\end{aligned}$$

Hence the considered system has unique solution. □

Theorem 6. *Under the Hypothesis (C_3) and (C_4), the considered system (2) has at least one solution.*

Proof. Let $\mathbf{S}=\{(u,v,w)\in X\times X\times X:\ |(u,v,w)\|_{X\times X\times X}\le r\}$. Clearly $\mathbf{S}\neq\emptyset$ closed and convex subset of $X\times X\times X$. Then to prove that $T=(T_1,T_2,T_3)$ is continues, let $\{(u_n,v_n,w_n)\}$ be a sequence in $\mathbf{S}$ such that $(u_n,v_n,w_n)\to(u,v,w)$ as $n\to\infty$. Let for $t\in[0,b]$, we have

$$\begin{aligned}\|T(u_n,v_n,w_n)-T(u,v,w)\|_{X\times X\times X} &= \sup_{t\in[0,b]}|T(u_n,v_n,w_n)(t)-T(u,v,w)(t)| \\ &= \sup_{t\in[0,b]}\left|\int_0^t\frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)}[f_1(\theta,u_n(\theta),v_n(\theta),w_n(\theta))\right. \\ &- \left. f(\theta,u(\theta),v(\theta),w(\theta))]d\theta\right| \\ &\le \frac{L_{f_1}}{\Gamma(\alpha)}\sup_{t\in[0,b]}\int_0^t(t-\theta)^{\alpha-1}|(u_n,v_n,w_n)(\theta)-(u,v,w)(\theta)|d\theta.\end{aligned}$$

Due to continuity of f_1, we see that

$$\|T(u_n, v_n, w_n) - T(u, v, w)\|_X \to 0, \text{ as } n \to 0.$$

Hence T_1 is continues. Similarly T_2, T_3 are also continues. So T is continues. Further to prove that $T(\mathbf{S}) \subset \mathbf{S}$, we take

$$\begin{aligned} \|T_1(u,v,w)\|_{X\times X\times X} &= \sup_{t\in[0,b]} |T_1(u,v,w)(t)| \\ &= \sup_{t\in[0,b]} \left| \int_0^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} [f(\theta, u(\theta), v(\theta), w(\theta)) d\theta \right| \\ &\le \frac{b^\alpha}{\Gamma(\alpha+1)} [M_{f_1} + C_{f_1} [\|u\|_X + \|v\|_X + \|w\|_X] \\ &\le \frac{r}{3}. \end{aligned}$$

Hence $\|T_1(u,v,w)\| \le \frac{r}{3}$ implies that T_1 is bounded. In same line $\|T_2(u,v,w)\|_{X\times X\times X} \le \frac{r}{3}$ and $\|T_3(u,v,w)\|_{X\times X\times X} \le \frac{r}{3}$. Thus we have

$$\begin{aligned} \|T(u,v,w)\|_{X\times X\times X} &= \|T_1(u,v,w)\|_X + \|T_2(u,v,w)\|_X + \|T_3(u,v,w)\|_X \\ &\le \frac{r}{3} + \frac{r}{3} + \frac{r}{3} = r \\ \text{which yields that } \|T_1(u,v,w)\|_{X\times X\times X} &\le r. \end{aligned}$$

Hence T is bounded. To show that T is relatively compact, it is need to prove that T is equi-continues operator. To achieve the goal, let $t_1, t_2 \in [0, b]$ and considering

$$\begin{aligned} |T_1(u,v,w)(t_2) - T_1(u,v,w)(t_1)| &= \int_0^{t_2} \frac{1}{\Gamma(\alpha)} (t_2-\theta)^{\alpha-1} f_1(\theta, u(\theta), v(\theta), w(\theta)) d\theta \\ &- \int_0^{t_1} (t_1-\theta)^{\alpha-1} f_1(\theta, u(\theta), v(\theta), w(\theta)) d\theta \\ &\le \frac{M_{f_1} + C_{f_1} r}{\Gamma(\alpha)} \\ &\times \left[\int_0^{t_2} \frac{1}{\Gamma(\alpha)} (t_2-\theta)^{\alpha-1} - \int_0^{t_1} \frac{1}{\Gamma(\alpha)} (t_1-\theta)^{\alpha-1} \right] \\ &\le \frac{M_{f_1} + C_{f_1} r}{\Gamma(\alpha+1)} (t_2^\alpha - t_1^\alpha). \end{aligned} \tag{34}$$

Since as $t_2 \to t_1$ in right side of (34) leads to zero, so we have $\|T_1(u,v,w)(t_2) - T_1(u,v,w)(t_1)\|_X \to 0$ as $t_2 \to t_1$. Hence T_1 is equi-continues operator. Similarly T_2, T_3 are also equi-continues. Hence $T = (T_1, T_2, T_3)$ is equi-continues. Thus T is completely continues operator and thanking to Arzelaá Theorem, T has at least one fixed point. Hence the considered system (2) has at least one solution. □

4.2. Numerical Procedure to Problem II

In this section, we establish and extend the numerical procedure of Section 3.2 for the system (3). Applying DTM to system (3), we have

$$\begin{aligned} U(k+\alpha\alpha_1) &= \frac{\Gamma(1+\frac{k}{\alpha_1})}{\Gamma(\alpha+1+\frac{k}{\alpha_1})} \Big[-U(k) \Big] \\ V(k+\alpha\alpha_1) &= \frac{\Gamma(1+\frac{k}{\alpha_1})}{\Gamma(\alpha+1+\frac{k}{\alpha_1})} \left[U(k) + \sum_{l=0}^{k} V(l)V(k-l) \right] \\ W(k+\alpha\alpha_1) &= \frac{\Gamma(1+\frac{k}{\alpha_1})}{\Gamma(\alpha+1+\frac{k}{\alpha_1})} \left[\sum_{l=0}^{k} V(l)V(k-l) \right], \end{aligned} \tag{35}$$

where $k = 0, 1, 2, ..., \alpha\alpha_1 - 1$, we are computing $U(k), V(k), W(k)$. We see that $U(0) = 0,\ V(0) = W(0) = 0$. We have

$$\begin{aligned}
U(1) &= \frac{-1}{\Gamma(\alpha+1)},\ V(1) = \frac{1}{\Gamma(\alpha+1)},\ W(1) = \frac{1}{\Gamma(\alpha+1)}[0] = 0 \\
U(2) &= \frac{\Gamma(1+\frac{1}{\alpha_1})}{\Gamma(\alpha+1+\frac{1}{\alpha_1})} \frac{-1}{\Gamma(\alpha+1)} \\
V(2) &= \frac{\Gamma(1+\frac{1}{\alpha_1})}{\Gamma(\alpha+1+\frac{1}{\alpha_1})} \frac{-1}{\Gamma(\alpha+1)} \\
W(2) &= \frac{\Gamma(1+\frac{1}{\alpha_1})}{\Gamma(\alpha+1+\frac{1}{\alpha_1})}[0] = 0 \\
U(3) &= \frac{\Gamma(1+\frac{2}{\alpha_1})}{\Gamma(\alpha+1+\frac{2}{\alpha_1})} \frac{\Gamma(1+\frac{1}{\alpha_1})}{\Gamma(\alpha+1+\frac{1}{\alpha_1})} \frac{1}{\Gamma(\alpha+1)} \\
V(3) &= \frac{\Gamma(1+\frac{2}{\alpha_1})}{\Gamma(\alpha+1+\frac{2}{\alpha_1})} \left[\frac{\Gamma(1+\frac{1}{\alpha_1})}{\Gamma(\alpha+1+\frac{1}{\alpha_1})} \frac{-1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)} \right] \\
W(3) &= \frac{\Gamma(1+\frac{2}{\alpha_1})}{\Gamma(\alpha+1+\frac{2}{\alpha_1})} \frac{1}{\Gamma(\alpha+1)}
\end{aligned} \tag{36}$$

and so on. Therefore the general series solution is given by

$$\begin{aligned}
u(t) &= U(0) + U(1)t^{\frac{1}{\alpha}} + U(2)t^{\frac{2}{\alpha}} + U(3)t^{\frac{3}{\alpha}} + U(4)t^{\frac{4}{\alpha}} + \dots \\
v(t) &= V(0) + V(1)t^{\frac{1}{\alpha}} + V(2)t^{\frac{2}{\alpha}} + V(3)t^{\frac{3}{\alpha}} + V(4)t^{\frac{4}{\alpha}} + \dots \\
w(t) &= W(0) + W(1)t^{\frac{1}{\alpha}} + W(2)t^{\frac{2}{\alpha}} + W(3)t^{\frac{3}{\alpha}} + W(4)t^{\frac{4}{\alpha}} + \dots.
\end{aligned} \tag{37}$$

Here we provide plots at different fractional order upto fourth terms of the solutions (37) in Figures 4–6 respectively.

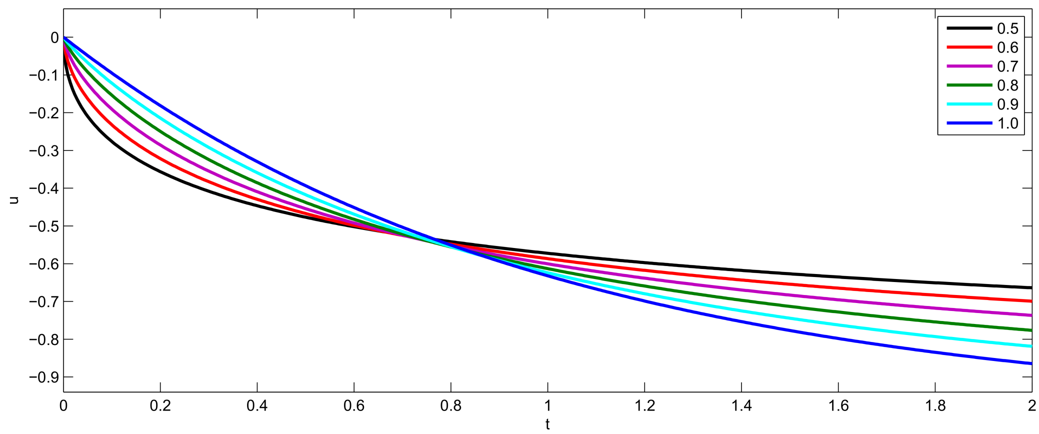

Figure 4. Plots of approximate solutions for $u(t)$ of the system (3) against various fractional order.

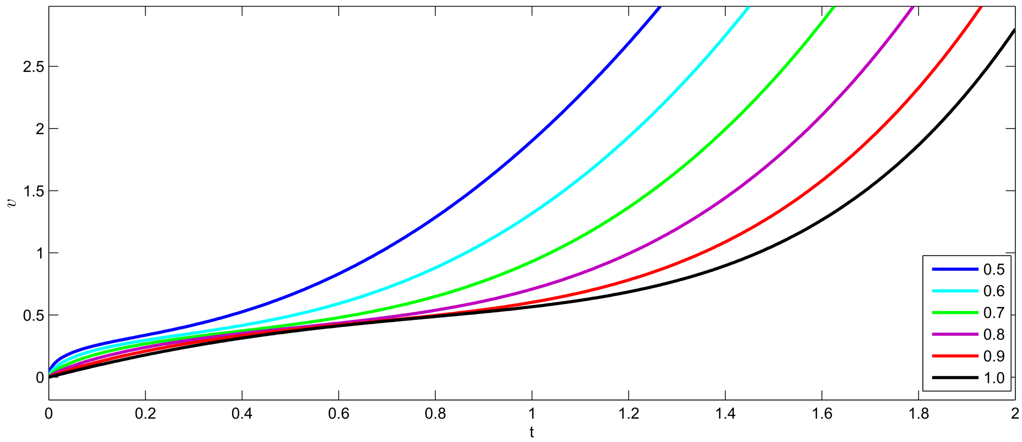

Figure 5. Plots of approximate solutions for $v(t)$ of the system (3) against various fractional order.

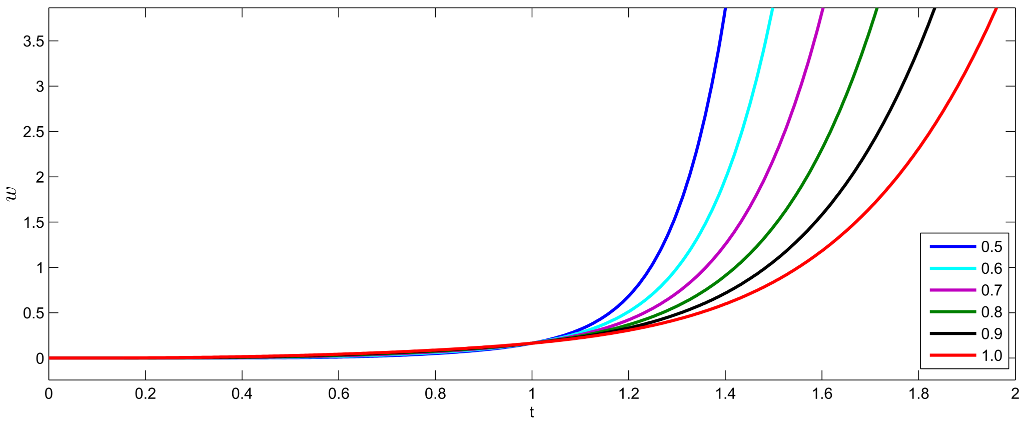

Figure 6. Plots of approximate solutions for $w(t)$ of the system (3) against various fractional order.

Further if we take $\alpha = 1$ in (37), then the system (3) becomes classical ODEs for which we have

$$\begin{aligned}
U(1) &= -1,\ V(1) = 1,\ W(1) = 0 \\
U(2) &= \frac{-1}{2},\ V(2) = \frac{1}{2},\ W(2) = 0 \\
U(3) &= \frac{-1}{6}, V(3) = \frac{-1}{6},\ W(3) = \frac{1}{3} \\
U(4) &= \frac{1}{24},\ V(4) = \frac{5}{24},\ W(4) = \frac{-1}{4}, \\
U(5) &= \frac{-1}{120},\ V(5) = \frac{1}{40},\ W(5) = \frac{-1}{60},
\end{aligned} \tag{38}$$

and so on, we have

$$\begin{aligned}
u(t) &= -t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24} - \frac{t^5}{120} + ... \\
v(t) &= t - \frac{t^2}{2} - \frac{t^3}{6} + \frac{5t^4}{24} + \frac{t^5}{40} + ... \\
w(t) &= \frac{t^3}{3} - \frac{t^4}{4} - \frac{t^5}{60} + \frac{7t^6}{72} + ...
\end{aligned} \tag{39}$$

Note: Thus the solution as received in (39). Such like results have computed in [32].

5. Concluding Remarks

In this article, we have developed analytical as well as qualitative theory for some families of FODEs by using DTM and some fixed point theorems due to Banach and Schauder. The obtained results have been demonstrated by several examples.

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