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A Note on a Coupled System of Hilfer Fractional Differential Inclusions

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Abstract: A coupled system of Hilfer fractional differential inclusions with nonlocal integral boundary conditions is considered. An existence result is established when the set-valued maps have non-convex values. We treat the case when the set-valued maps are Lipschitz in the state variables and we avoid the applications of fixed point theorems as usual. An illustration of the results is given by a suitable example.

Keywords: fractional derivative; differential inclusion; boundary condition

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1. Introduction

The fractional derivative introduced by Hilfer in [1] was recently used in the study of many boundary value problems concerned with fractional derivatives. This fractional derivative generalizes both Riemann–Liouville and Caputo derivatives; in fact, this derivative is an interpolation between Riemann–Liouville and Caputo derivatives. Several properties and applications of the Hilfer fractional derivative may be found in [2]. Additionally, we recall that the literature is full of explanations and motivations for considering systems defined by fractional order derivatives (e.g., [3–7] etc.).

Recently, many papers in the literature have been devoted to the study of fractional differential equations and inclusions defined by the Hilfer fractional derivative (e.g., [8–11] etc.). We point out that a complete survey on this field of study may be found in [8]. Taking into account this new trend in research, our intention is to contribute to the development of this topic by establishing new results for a particular class of problems.

The present note is devoted to the following boundary value problem

$$\begin{cases} D_H^{\alpha_1,\beta_1}x_1(t) \in F_1(t,x_1(t),x_2(t)), & a.e.\ t\in[a,b], \\ D_H^{\alpha_2,\beta_2}x_2(t) \in F_2(t,x_1(t),x_2(t)), & a.e.\ t\in[a,b], \end{cases} \quad (1)$$

$$\begin{cases} x_1(a)=0, & x_1(b)=\sum_{i=1}^m \theta_i I^{\varphi_i}x_2(\xi_i), \\ x_2(a)=0, & x_2(b)=\sum_{j=1}^n \zeta_j I^{\psi_j}x_1(z_j), \end{cases} \quad (2)$$

where $F_1(.,.,.):[a,b]\times\mathbf{R}^2\to\mathcal{P}(\mathbf{R})$, $F_2(.,.,.):[a,b]\times\mathbf{R}^2\to\mathcal{P}(\mathbf{R})$ are given set-valued maps, $D_H^{\alpha,\beta}$ is the Hilfer fractional derivative of order α and type β, I^{φ} is the Riemann–Liouville fractional integral of order $\varphi>0$, $\alpha_1,\alpha_2\in(1,2)$, $\beta_1,\beta_2\in[0,1]$, $\varphi_i,\psi_j>0$, $\xi_i,z_j\in[a,b]$, $a>0$, $\theta_i,\zeta_j\in\mathbf{R}$, $i=\overline{1,m}$, $j=\overline{1,n}$.

Our study is motivated by two recent papers [10,11]. In [10], problems (1) and (2) is studied in the single-valued case; namely, the right-hand side in (1) is given by single-valued maps. Existence and uniqueness results are provided by using well-known fixed point theorems: Banach, Leray–Schauder and Krasnoselskii. In [11], a "simple" (not coupled) set-valued problem as in (1) and (2) is studied, and existence results are also obtained by applying known set-valued fixed point results as Leray–Schauder and Nadler.

The purpose of our note is twofold. On one hand, we extend the study in [10] to the set-valued framework, and on the other hand, we generalize the study in [11] to the coupled case. The approach we present here avoids the applications of fixed point theorems and takes into account the case when the values of F_1 and F_2 are not convex; but these set-valued maps are assumed to be Lipschitz in the second and third variable. In this case, we establish an existence result for problems (1) and (2). Our result use Filippov's technique [12]; more exactly, the existence of solutions is obtained by starting from a pair of given "quasi"solutions. In addition, the result provides an estimate between the "quasi" solutions and the solutions obtained.

Even if the technique used here may be seen at other classes of coupled systems of fractional differential inclusions [13–15], to the best of our knowledge, the present paper is the first in literature which contains an existence result of Filippov type for coupled systems of Hilfer fractional differential inclusions.

2. Preliminaries

We set by I the interval $[a,b]$. We denote by $C(I,\mathbf{R})$ the Banach space of all continuous functions $x(.) : I \to \mathbf{R}$ endowed with the norm $|x(.)|_C = \sup_{t\in I}|x(t)|$ and by $L^1(I,\mathbf{R})$ the Banach space of all integrable functions $x(.) : I \to \mathbf{R}$ endowed with the norm $|x(.)|_1 = \int_a^b |x(t)|dt$.

The Pompeiu–Hausdorff distance of the closed subsets $A, B \subset \mathbf{R}$ is defined by $d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}$, where $d^*(A,B) = \sup\{d(a,B); a \in A\}$ and $d(x,B) = \inf_{y\in B} d(x,y)$.

The fractional integral of order $\alpha > 0$ of a Lebesgue integrable function $f : (0,\infty) \to \mathbf{R}$ is defined by

$$I^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds,$$

provided the right-hand side is pointwise defined on $(0,\infty)$ and $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\mathrm{d}t$.

The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a Lebesgue integrable function $f : (0,\infty) \to \mathbf{R}$ is defined by

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{-\alpha+n-1} f(s)ds,$$

where $n = [\alpha] + 1$, provided the right-hand side is pointwise defined on $(0,\infty)$.

The Caputo fractional derivative of order $\alpha > 0$ of a function $f : [0,\infty) \to \mathbf{R}$ is defined by

$$D_C^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{-\alpha+n-1} f^{(n)}(s)\mathrm{d}s,$$

where $n = [\alpha] + 1$. It is assumed implicitly that f is n times differentiable whose n-th derivative is absolutely continuous.

The Hilfer fractional derivative of order $\alpha \in (n-1,n)$ *and type* $\beta \in [0,1]$ of a function $f : (0,\infty) \to \mathbf{R}$ is defined by

$$D_H^{\alpha,\beta} f(t) = I^{\beta(n-\alpha)} \frac{d^n}{dt^n} I^{(1-\beta)(n-\alpha)} f(t)$$

As it was already recalled, this derivative interpolates between Riemann–Liouville and Caputo derivatives. When $\beta = 0$ the Hilfer fractional derivative gives Riemann–Liouville fractional derivative $D_H^{\alpha,0} f(t) = \frac{d^n}{dt^n} I^{n-\alpha} f(t)$ and when $\beta = 1$ the Hilfer fractional derivative gives Caputo fractional derivative $D_H^{\alpha,1} f(t) = I^{n-\alpha} \frac{d^n}{dt^n} f(t)$.

The next technical result proved in [10] considers a linear version of problems (1) and (2), for which an integral representation of the solution is provided.

Lemma 1. *Let $f_1(.):[a,b]\to \mathbf{R}$, $f_2(.):[a,b]\to \mathbf{R}$ be continuous mappings and $\alpha_1,\alpha_2\in(1,2)$, $\beta_1,\beta_2\in[0,1)$, $\varphi_i,\psi_j>0$, $\gamma_1=\alpha_1+2\beta_1-\alpha_1\beta_1$, $\gamma_2=\alpha_2+2\beta_2-\alpha_2\beta_2$, $\xi_i,z_j\in[a,b]$, $a>0$, $\theta_i,\zeta_j\in\mathbf{R}$, $i=\overline{1,m}$, $j=\overline{1,n}$.*

Then, the solution of the system

$$\begin{cases} D_H^{\alpha_1,\beta_1}x_1=f_1(t) & t\in[a,b],\\ D_H^{\alpha_2,\beta_2}x_2=f_2(t) & t\in[a,b]\end{cases}$$

with boundary conditions (2) is given by

$$\begin{cases} x_1(t)=I^{\alpha_1}f_1(t)+\frac{(t-a)^{\gamma_1-1}}{C\Gamma(\gamma_1)}[\frac{(b-a)^{\gamma_2-1}}{\Gamma(\gamma_2)}(\sum_{i=1}^m\theta_iI^{\alpha_2+\varphi_i}f_2(\xi_i)-I^{\alpha_1}f_1(b))\\ +(\sum_{i=1}^m\frac{\theta_i(\xi_i-a)^{\gamma_2+\varphi_i-1}}{\Gamma(\gamma_2+\varphi_i)})(\sum_{j=1}^n\zeta_jI^{\alpha_1+\psi_j}f_1(z_j)-I^{\alpha_2}f_2(b))],\\ x_2(t)=I^{\alpha_2}f_2(t)+\frac{(t-a)^{\gamma_2-1}}{C\Gamma(\gamma_2)}[\frac{(b-a)^{\gamma_1-1}}{\Gamma(\gamma_1)}(\sum_{j=1}^n\zeta_jI^{\alpha_1+\psi_j}f_1(z_j)-I^{\alpha_2}f_2(b))\\ +(\sum_{j=1}^n\frac{\zeta_j(z_j-a)^{\gamma_1+\psi_j-1}}{\Gamma(\gamma_1+\psi_j)})(\sum_{i=1}^m\theta_iI^{\alpha_2+\varphi_i}f_2(\xi_i)-I^{\alpha_1}f_1(b))],\end{cases} \tag{3}$$

where $C=\frac{(b-a)^{\gamma_1+\gamma_2-2}}{\Gamma(\gamma_1)\Gamma(\gamma_2)}-(\sum_{i=1}^m\frac{\theta_i(\xi_i-a)^{\gamma_2+\varphi_i-1}}{\Gamma(\gamma_2+\varphi_i)})(\sum_{j=1}^n\frac{\zeta_j(z_j-a)^{\gamma_1+\psi_j-1}}{\Gamma(\gamma_1+\psi_j)})\neq 0$.

Definition 1. *The mappings $x_1(.),x_2(.)\in C(I,\mathbf{R})$ are said to be solutions of problems (1) and (2) if there exists $f_1(.),f_2(.)\in L^1(I,\mathbf{R})$ such that $f_1(t)\in F_1(t,x_1(t),x_2(t))$ a.e. (I), $f_2(t)\in F_2(t,x_1(t),x_2(t))$ a.e. (I) and $x_1(.)$ and $x_2(.)$ are given by (3).*

In what follows, $\chi_A(\cdot)$ denotes the characteristic function of the set $A\subset\mathbf{R}$.

Remark 1. *Let us introduce the following notations*

$$\mathcal{K}_1(t,\tau)=\frac{(t-\tau)^{\alpha_1-1}\chi_{[a,t]}(\tau)}{\Gamma(\alpha_1)}-\frac{(t-a)^{\gamma_1-1}(b-a)^{\gamma_2-1}(b-\tau)^{\alpha_1-1}}{C\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\alpha_1)}+\\ (\sum_{i=1}^m\frac{\theta_i(\xi_i-a)^{\gamma_2+\varphi_i-1}}{\Gamma(\gamma_2+\varphi_i)})(\sum_{j=1}^n\frac{\zeta_j(z_j-a)^{\alpha_1+\psi_j-1}}{\Gamma(\alpha_1+\psi_j)}\chi_{[a,z_j]}(\tau)),$$

$$\mathcal{K}_2(t,\tau)=\frac{(t-a)^{\gamma_1-1}(b-a)^{\gamma_2-1}}{C\Gamma(\gamma_1)\Gamma(\gamma_2)}\sum_{i=1}^m\frac{\theta_i(\xi_i-\tau)^{\alpha_2+\varphi_i-1}\chi_{[a,\xi_i]}(\tau)}{\Gamma(\alpha_2+\varphi_i)}-\\ \sum_{i=1}^m\frac{\theta_i(\xi_i-a)^{\gamma_2+\varphi_i-1}(b-a)^{\alpha_2-1}}{\Gamma(\gamma_2+\varphi_i)\Gamma(\alpha_2)},$$

$$\mathcal{K}_3(t,\tau)=\frac{(t-a)^{\gamma_2-1}(b-a)^{\gamma_1-1}}{C\Gamma(\gamma_1)\Gamma(\gamma_2)}\sum_{j=1}^n\frac{\zeta_j(z_j-\tau)^{\alpha_1+\psi_j-1}\chi_{[a,z_j]}(\tau)}{\Gamma(\alpha_1+\psi_j)}-\\ \sum_{j=1}^n\frac{\zeta_j(z_j-a)^{\gamma_1+\psi_j-1}(b-a)^{\alpha_1-1}}{\Gamma(\gamma_1+\psi_j)\Gamma(\alpha_1)},$$

$$\mathcal{K}_4(t,\tau)=\frac{(t-\tau)^{\alpha_2-1}\chi_{[a,t]}(\tau)}{\Gamma(\alpha_2)}-\frac{(t-a)^{\gamma_2-1}(b-a)^{\gamma_1-1}(b-\tau)^{\alpha_2-1}}{C\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\alpha_2)}+\\ (\sum_{j=1}^n\frac{\zeta_j(z_j-a)^{\gamma_1+\psi_j-1}}{\Gamma(\gamma_1+\psi_j)})(\sum_{i=1}^m\frac{\theta_i(\xi_i-a)^{\alpha_2+\varphi_i-1}}{\Gamma(\alpha_2+\varphi_i)}\chi_{[a,\xi_i]}(\tau)).$$

Then, the solutions $(x_1(.),x_2(.))$ in Lemma 1 may be put as

$$x_1(t)=\int_a^b\mathcal{K}_1(t,\tau)f_1(\tau)d\tau+\int_a^b\mathcal{K}_2(t,\tau)f_2(\tau)d\tau,\quad t\in I\\ x_2(t)=\int_a^b\mathcal{K}_3(t,\tau)f_1(\tau)d\tau+\int_a^b\mathcal{K}_4(t,\tau)f_2(\tau)d\tau,\quad t\in I.$$

Moreover, we have the following estimates:

$$|\mathcal{K}_1(t,\tau)| \leq \frac{(b-a)^{\alpha_1-1}}{\Gamma(\alpha_1)} + \frac{(b-a)^{\gamma_1+\gamma_2+\alpha_1-3}}{|C|\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\alpha_1)} + \sum_{i=1}^{m} \frac{|\theta_i|(\xi_i-a)^{\gamma_2+\varphi_i-1}}{\Gamma(\gamma_2+\varphi_i)} \cdot \sum_{j=1}^{n} \frac{|\zeta_j|(z_j-a)^{\alpha_1+\psi_j-1}}{\Gamma(\alpha_1+\psi_j)} =: k_1 \quad \forall\, t,\tau \in I,$$

$$|\mathcal{K}_2(t,\tau)| \leq \frac{(b-a)^{\gamma_1+\gamma_2-2}}{|C|\Gamma(\gamma_1)\Gamma(\gamma_2)} \sum_{i=1}^{m} \frac{|\theta_i|(\xi_i-a)^{\alpha_2+\varphi_i-1}}{\Gamma(\alpha_2+\varphi_i)} + \sum_{i=1}^{m} \frac{|\theta_i|(\xi_i-a)^{\gamma_2+\varphi_i-1}(b-a)^{\alpha_2-1}}{\Gamma(\gamma_2+\varphi_i)\Gamma(\alpha_2)} =: k_2 \qquad \forall\, t,\tau \in I,$$

$$|\mathcal{K}_3(t,\tau)| \leq \frac{(b-a)^{\gamma_2+\gamma_1-2}}{|C|\Gamma(\gamma_1)\Gamma(\gamma_2)} \sum_{j=1}^{n} \frac{|\zeta_j|(z_j-a)^{\alpha_1+\psi_j-1}}{\Gamma(\alpha_1+\psi_j)} + \sum_{j=1}^{n} \frac{|\zeta_j|(z_j-a)^{\gamma_1+\psi_j-1}(b-a)^{\alpha_1-1}}{\Gamma(\gamma_1+\psi_j)\Gamma(\alpha_1)} =: k_3 \qquad \forall\, t,\tau \in I,$$

$$|\mathcal{K}_4(t,\tau)| \leq \frac{(b-a)^{\alpha_2-1}}{\Gamma(\alpha_2)} - \frac{(b-a)^{\gamma_1+\gamma_2+\alpha_2-3}}{|C|\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\alpha_2)} + \sum_{j=1}^{n} \frac{|\zeta_j|(z_j-a)^{\gamma_1+\psi_j-1}}{\Gamma(\gamma_1+\psi_j)} \cdot \sum_{i=1}^{m} \frac{|\theta_i|(\xi_i-a)^{\alpha_2+\varphi_i-1}}{\Gamma(\alpha_2+\varphi_i)} =: k_4 \quad \forall\, t,\tau \in I.$$

Finally, in the proof of our main result we need the following selection result for set-valued maps (e.g., [16]). It is, in fact, a variant of the well known selection theorem due to Kuratowski and Ryll-Nardzewski which, briefly, states that a measurable set-valued map with nonempty closed values admits a measurable selection.

Lemma 2. *Let Z be a separable Banach space, B its closed unit ball, $A : I \to \mathcal{P}(Z)$ is a set-valued map whose values are nonempty closed and $b : I \to Z, c : I \to \mathbf{R}_+$ are two measurable functions. If*

$$A(t) \cap (b(t) + c(t)B) \neq \emptyset \quad a.e.\ (I),$$

then the set-valued map $t \to A(t) \cap (b(t) + c(t)B)$ admits a measurable selection.

3. Main Result

Our results are proved under the following hypotheses.

Hypothesis 1. *(i) $F_1 : I \times \mathbf{R}^2 \to \mathcal{P}(\mathbf{R})$ and $F_2 : I \times \mathbf{R}^2 \to \mathcal{P}(\mathbf{R})$ have nonempty closed values and the set-valued maps $F_1(.,y_1,y_2)$, $F_2(.,y_1,y_2)$ are measurable for any $y_1, y_2 \in \mathbf{R}$.*

(ii) There exist $l_1(.), l_2(.) \in L^1(I,(0,\infty))$ such that, for almost all $t \in I$, $F_1(t,.,.)$ is $l_1(t)$-Lipschitz and $F_2(t,.,.)$ is $l_2(t)$-Lipschitz; i.e.,

$$d_H(F_1(t,y_1,z_1), F_1(t,y_2,z_2)) \leq l_1(t)(|y_1-y_2| + |z_1-z_2|) \quad \forall\, y_1,y_2,z_1,z_2 \in \mathbf{R}.$$

$$d_H(F_2(t,y_1,z_1), F_2(t,y_2,z_2)) \leq l_2(t)(|y_1-y_2| + |z_1-z_2|) \quad \forall\, y_1,y_2,z_1,z_2 \in \mathbf{R}.$$

In what follows $l(t) = k_1 l_1(t) + k_2 l_2(t) + k_3 l_1(t) + k_4 l_2(t)$, $t \in I$.

Theorem 1. *Assume that $C \neq 0$, Hypothesis 1 is satisfied and $|l(.)|_1 < 1$. $(y_1(.), y_2(.)) \in C(I,\mathbf{R})^2$ are considered such that there exist $q_1(.), q_2(.) \in L^1(I,\mathbf{R})$ that verify $d(D_H^{\alpha_1,\beta_1} y_1(t), F_1(t,y_1(t),y_2(t))) \leq q_1(t)$ a.e. $t \in I$, $d(D_H^{\alpha_2,\beta_2} y_2(t), F_2(t,y_1(t),y_2(t))) \leq q_2(t)$ a.e. $t \in I$, $y_1(a) = y_2(a) = 0$, $y_1(b) = \sum_{i=1}^{m} \theta_i I^{\varphi_i} y_2(\xi_i)$ and $y_2(b) = \sum_{j=1}^{n} \zeta_j I^{\psi_j} y_1(z_j)$.*

Then, there exists $(x_1(.), x_2(.)) \in C(I,\mathbf{R})^2$ a solution of problem (1) and (2) satisfying for all $t \in I$

$$|x_1(t) - y_1(t)| + |x_2(t) - y_2(t)| \leq \frac{(k_1+k_3)|q_1(.)|_1 + (k_2+k_4)|q_2(.)|_1}{1 - |l(.)|_1}. \tag{4}$$

Proof. From the assumptions of the theorem

$$F_1(t, y_1(t), y_2(t)) \cap \{D_H^{\alpha_1,\beta_1} y_1(t) + q_1(t)[-1,1]\} \neq \emptyset \quad a.e.\ (I),$$
$$F_2(t, y_1(t), y_2(t)) \cap \{D_H^{\alpha_2,\beta_2} y_2(t) + q_2(t)[-1,1]\} \neq \emptyset \quad a.e.\ (I).$$

By Lemma 2, there exist mesurable selections $f_1^1(t) \in F_1(t, y_1(t), y_2(t))$, $f_2^1(t) \in F_2(t, y_1(t), y_2(t))$ *a.e.* (I) such that

$$|f_1^1(t) - D_H^{\alpha_1,\beta_1} y_1(t)| \leq q_1(t), \quad |f_2^1(t) - D_H^{\alpha_2,\beta_2} y_2(t)| \leq q_2(t) \quad a.e.\ (I).$$

Define

$$x_1^1(t) = \int_a^b \mathcal{K}_1(t,\tau) f_1^1(\tau) d\tau + \int_a^b \mathcal{K}_2(t,\tau) f_2^1(\tau) d\tau, \quad t \in I$$
$$x_2^1(t) = \int_a^b \mathcal{K}_3(t,\tau) f_1^1(\tau) d\tau + \int_a^b \mathcal{K}_4(t,\tau) f_2^1(\tau) d\tau, \quad t \in I.$$

We have the estimates

$$|x_1^1(t) - y_1(t)| \leq k_1 |q_1(.)|_1 + k_2 |q_2(.)|_1 \quad \forall t \in I,$$
$$|x_2^1(t) - y_2(t)| \leq k_3 |q_1(.)|_1 + k_4 |q_2(.)|_1 \quad \forall t \in I,$$

and so,

$$|x_1^1(t) - y_1(t)| + |x_2^1(t) - y_2(t)| \leq (k_1 + k_3)|q_1(.)|_1 + (k_2 + k_4)|q_2(.)|_1 =: k.$$

In the next part of the proof, we construct, by induction, the sequences $x_n^1(.), x_n^2(.) \in C(I, \mathbf{R})$ and $f_n^1(.)$, $f_n^2(.) \in L^1(I, \mathbf{R})$, $n \geq 1$ with the following properties

$$x_1^n(t) = \int_a^b \mathcal{K}_1(t,\tau) f_1^n(\tau) d\tau + \int_a^b \mathcal{K}_2(t,\tau) f_2^n(\tau) d\tau, \quad t \in I$$
$$x_2^n(t) = \int_a^b \mathcal{K}_3(t,\tau) f_1^n(\tau) d\tau + \int_a^b \mathcal{K}_4(t,\tau) f_2^n(\tau) d\tau, \quad t \in I. \tag{5}$$

$$f_1^n(t) \in F_1(t, x_1^{n-1}(t), x_2^{n-1}(t)), \quad f_2^n(t) \in F_2(t, x_1^{n-1}(t), x_2^{n-1}(t)) \quad a.e.\ (I), \tag{6}$$

$$|f_1^{n+1}(t) - f_1^n(t)| \leq l_1(t)(|x_1^n(t) - x_1^{n-1}(t)| + |x_2^n(t) - x_2^{n-1}(t)|) \quad a.e.\ (I),$$
$$|f_2^{n+1}(t) - f_2^n(t)| \leq l_2(t)(|x_1^n(t) - x_1^{n-1}(t)| + |x_2^n(t) - x_2^{n-1}(t)|) \quad a.e.\ (I). \tag{7}$$

We point out that from (5) to (7) it follows

$$|x_1^{n+1}(t) - x_1^n(t)| + |x_2^{n+1}(t) - x_2^n(t)| \leq k(|l(.)|_1)^n \quad a.e.\ (I) \quad \forall n \in \mathbf{N}. \tag{8}$$

The case $n = 0$ is already proved. Now, we assume (8) valid for $n - 1$. For almost all $t \in I$,

$$|x_1^{n+1}(t) - x_1^n(t)| \leq \int_a^b |\mathcal{K}_1(t,\tau)|.|f_1^{n+1}(\tau) - f_1^n(\tau)| d\tau + \int_a^b |\mathcal{K}_2(t,\tau)|.|f_2^{n+1}(\tau)$$
$$-f_2^n(\tau)| d\tau \leq k_1 \int_a^b |f_1^{n+1}(\tau) - f_1^n(\tau)| d\tau + k_2 \int_a^b |f_2^{n+1}(\tau) - f_2^n(\tau)| d\tau \leq$$
$$k_1 \int_a^b l_1(\tau)(|x_1^n(\tau) - x_1^{n-1}(\tau)| + |x_2^n(\tau) - x_2^{n-1}(\tau)|) d\tau + k_2 \int_a^b l_2(\tau)(|x_1^n(\tau) -$$
$$x_1^{n-1}(\tau)| + |x_2^n(\tau) - x_2^{n-1}(\tau)|) d\tau \leq k(|l(.)|_1)^{n-1}(k_1 \int_a^b l_1(\tau) d\tau + k_2 \int_a^b l_2(\tau) d\tau).$$

In a similar way, we obtain for almost all $t \in I$,

$$|x_2^{n+1}(t) - x_2^n(t)| \leq k(|l(.)|_1)^{n-1}(k_3 \int_a^b l_1(\tau) d\tau + k_4 \int_a^b l_2(\tau) d\tau).$$

Therefore, (8) is true for n.

Inequality (8) shows that the sequences $\{x_1^n(.)\}, \{x_2^n(.)\}$ are Cauchy in the space $C(I, \mathbf{R})$. Let $x_1(.) \in C(I, \mathbf{R})$ and $x_2(.) \in C(I, \mathbf{R})$ be their limits in $C(I, \mathbf{R})$. Additionally, from (7) we deduce that, for almost all $t \in I$, the sequences $\{f_1^n(t)\}$, $\{f_2^n(t)\}$ are Cauchy in $\mathbf{R}$. We consider $f_1(.)$, $f_2(.)$ their pointwise limit.

At the same time, inequality (8) and Hypothesis 1 give

$$|x_1^n(t)-y_1(t)|+|x_2^n(t)-y_2(t)|\leq |x_1^1(t)-y_1(t)|+|x_2^1(t)-y_2(t)|+ \sum_{i=1}^{n-1}(|x_1^{i+1}(t)-x_1^i(t)|+|x_2^{i+1}(t)-x_2^i(t)|)\leq k+\sum_{i=1}^{n}k(|l(.)|_1)^i\leq \frac{k}{1-|l(.)|_1}. \quad (9)$$

and

$$|f_1^n(t)-D_H^{\alpha_1,\beta_1}y_1(t)|+|f_2^n(t)-D_H^{\alpha_2,\beta_2}y_2(t)|\leq |f_1^1(t)-D_H^{\alpha_1,\beta_1}y_1(t)|+ |f_2^1(t)-D_H^{\alpha_2,\beta_2}y_2(t)|+\sum_{i=1}^{n-1}(|f_1^{i+1}(t)-f_1^i(t)|+|f_2^{i+1}(t)-f_2^i(t)|)\leq |f_1^1(t)-D_H^{\alpha_1,\beta_1}y_1(t)|+|f_2^1(t)-D_H^{\alpha_2,\beta_2}y_2(t)|+\sum_{i=1}^{n-1}(l_1(t)+l_2(t))(|x_1^i(t) -x_1^{i-1}(t)|+|x_2^i(t)-x_2^{i-1}(t)|)\leq q_1(t)+q_2(t)+(l_1(t)+l_2(t))\frac{k}{1-|l(.)|_1}$$

for almost all $t\in I$.

This means that the sequences $f_1^n(.)$, $f_2^n(.)$ are integrably bounded and, therefore, their limits $f_1(.),f_2(.)$ belong to $L^1(I,\mathbf{R})$.

The next step of the proof contains the construction in (5)–(7). By induction, we suppose that for $M\geq 1$, $x_1^m(.),x_2^m(.)\in C(I,\mathbf{R})$ and $f_1^m(.),f_2^m(.)\in L^1(I,\mathbf{R})$, $m=1,2,...M$ with (5) and (7) for $m=1,2,...M$ and (6) for $m=1,2,...M-1$ are constructed.

Using again Hypothesis 1

$$F_1(t,x_1^M(t),x_2^M(t))\cap\{f_1^M(t)+(l_1(t)|x_1^M(t)-x_1^{M-1}(t)|+l_1(t)|x_2^M(t)- x_2^{M-1}(t)|)[-1,1]\}\neq\emptyset,$$
$$F_2(t,x_1^M(t),x_2^M(t))\cap\{f_2^M(t)+(l_2(t)|x_1^M(t)-x_1^{M-1}(t)|+l_2(t)|x_2^M(t)- x_2^{M-1}(t)|)[-1,1]\}\neq\emptyset$$

for almost all $t\in I$. By Lemma 2, we obtain the existence of measurable selections $f_1^{M+1}(.)$ of $F_1(.,x_M^1(.),x_M^2(.))$ and $f_2^{M+1}(.)$ of $F_2(.,x_M^1(.),x_M^2(.))$ such that

$$|f_1^{M+1}(t)-f_1^M(t)|\leq l_1(t)(|x_1^M(t)-x_1^{M-1}(t)|+|x_2^M(t)-x_2^{M-1}(t)|)\quad a.e.\ (I),$$
$$|f_2^{M+1}(t)-f_2^M(t)|\leq l_2(t)(|x_1^M(t)-x_1^{M-1}(t)|+|x_2^M(t)-x_2^{M-1}(t)|)\quad a.e.\ (I).$$

We define $x_1^{M+1}(.),x_2^{M+1}(.)$ as in (5) with $n=M+1$.

Finally, it remains to take $n\to\infty$ in (5) and (9) in order to finish the proof. □

Theorem 1 is the first in the literature that contains an existence result of Filippov type for coupled systems of Hilfer fractional differential inclusions. Due to the presence in its statement of the "quasi" solutions, the formulation of Theorem 1 seems to look complicated. However, for a particular choice of the "quasi" solutions (namely, $y_1(.)=y_2(.)=0$), one may obtain a statement similar to a result that can be derived by using the set-valued contration principle. This may be seen in the following consequence of Theorem 1.

Corollary 1. *Assume that* $C\neq 0$, *Hypothesis 1 is satisfied,* $|l(.)|_1<1$, $d(0,F_1(t,0,0))\leq l_1(t)$ *a.e.* $t\in I$ *and* $d(0,F_2(t,0,0))\leq l_2(t)$ *a.e.* $t\in I$.

Then, there exists $(x_1(.),x_2(.))\in C(I,\mathbf{R})^2$ *a solution of problem (1) and (2) satisfying for all* $t\in I$

$$|x_1(t)|+|x_2(t)|\leq\frac{(k_1+k_3)|l_1(.)|_1+(k_2+k_4)|l_2(.)|_1}{1-|l(.)|_1}.$$

Proof. We apply Theorem 1 with $y_1(.)=y_2(.)=0$, $q_1(.)=l_1(.)$ and $q_2(.)=l_2(.)$. □

Remark 2. *If in (1),* F_1 *and* F_2 *are single-valued maps, Corollary 1 provides a generalization to the set-valued framework of Theorem 1 in [10].*

Example 1. *Let us consider the problem*

$$\begin{cases} D_H^{\frac{3}{2},\frac{1}{2}} x_1(t) \in [-\frac{1}{300}\frac{|x_1(t)|}{1+|x_1(t)|}, 0] \cup [0, \frac{1}{300}\frac{|x_2(t)|}{1+|x_2(t)|}] \quad a.e.\ [\frac{1}{3}, \frac{10}{3}], \\ D_H^{\frac{5}{4},\frac{2}{3}} x_2(t) \in [-\frac{1}{300}\frac{|\sin(x_1(t))|}{1+|\sin(x_1(t))|}, 0] \cup [0, \frac{1}{300}\frac{|\cos(x_2(t))|}{1+|\cos(x_2(t))|}] \quad a.e.\ [\frac{1}{3}, \frac{10}{3}], \end{cases} \tag{10}$$

with nonlocal integral boundary conditions as in [10]

$$\begin{cases} x_1(\frac{1}{3}) = 0, \quad x_1(\frac{10}{3}) = \frac{1}{2} I^{\frac{1}{3}} x_2(\frac{2}{3}) + \frac{2}{3} I^{\frac{1}{2}} x_2(1) + \frac{3}{4} I^{\frac{3}{5}} x_2(\frac{4}{3}) + \frac{4}{5} I^{\frac{2}{3}} x_2(\frac{5}{3}) \\ x_2(\frac{1}{3}) = 0, \quad x_2(\frac{10}{3}) = \frac{3}{2} I^{\frac{3}{2}} x_1(2) + \frac{4}{3} I^{\frac{5}{3}} x_1(\frac{7}{3}) + \frac{5}{4} I^{\frac{7}{4}} x_1(\frac{8}{3}) + \frac{6}{5} I^{\frac{9}{5}} x_1(3). \end{cases} \tag{11}$$

Thus, $F_1(t,(x_1,x_2)) = [-\frac{1}{300}\frac{|x_1|}{1+|x_1|}, 0] \cup [0, \frac{1}{300}\frac{|x_2|}{1+|x_2|}]$, $F_2(t,(x_1,x_2)) = [-\frac{1}{300}\frac{|\sin x_1|}{1+|\sin x_1|}, 0] \cup [0, \frac{1}{300}\frac{|\cos x_2|}{1+|\cos x_2|}]$, $\alpha_1 = \frac{3}{2}$, $\beta_1 = \frac{1}{2}$, $\gamma_1 = \frac{7}{4}$, $\alpha_2 = \frac{5}{4}$, $\beta_2 = \frac{2}{3}$, $\gamma_2 = \frac{41}{21}$, $a = \frac{1}{3}$, $b = \frac{10}{3}$, $m = n = 4$, $\theta_i = \frac{i}{i+1}$, $\zeta_i = \frac{2+i}{i+1}$, $\varphi_i = \frac{i}{i+2}$, $\psi_i = \frac{2i+1}{i+1}$, $\xi_i = \frac{i+1}{3}$, $z_i = \frac{i+5}{3}$, $i = 1,2,3,4$ *and* $C = 6,371398411$.

For all $t \in [\frac{1}{3}, \frac{10}{3}]$ *and all* $x_1, x_2, y_1, y_2 \in \mathbf{R}$ *we have*

$$\sup\{|z|;\ z \in F_i(t,(x_1,x_2))\} \le \frac{1}{300}, \qquad i = 1,2,$$

$$d_H(F_1(t,(x_1,x_2)), F_2(t,(y_1,y_2))) \le \frac{1}{300}|x_1 - y_1| + \frac{1}{300}|x_2 - y_2| \qquad i = 1,2.$$

By standard computations (e.g., [10]) $k_1 \approx 12.6$, $k_2 \approx 3.26$, $k_3 \approx 48.73$, $k_4 \approx 24.1$ *and* $(k_1 + k_3)3\frac{1}{300} + (k_2 + k_4)3\frac{1}{300} \approx 0.89 < 1$. *So, we may apply Corollary 1 and obtain the existence of a solution for problems (10) and (11).*

4. Conclusions

In the present paper, we extended the research in [10] to multivalued problems and the research in [11] to the situation of coupled Hilfer fractional differential inclusions. We established an existence result for problems (1) and (2) when the set-valued maps are Lipschitz in the state variables without any assumptions concerning the convexity of the values of the set-valued maps. Our approach uses a technique due to Filippov ([12]) instead of an usual application of set-valued fixed point theorems. An illustration of our result is provided by a numerical example.

It is worth mentioning that Theorem 1 may be a basic tool in the study of optimal control problems defined by such kinds of coupled systems of Hilfer fractional differential inclusions.

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