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Generalized Fractional Integrals Involving Product of a Generalized Mittag–Leffler Function and Two *H*-Functions

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Abstract: The objective of this research is to obtain some fractional integral formulas concerning products of the generalized Mittag–Leffler function and two *H*-functions. The resulting integral formulas are described in terms of the *H*-function of several variables. Moreover, we give some illustrative examples for the efficiency of the general approach of our results.

Keywords: generalized fractional calculus operators; *H*-functions of several variables; Mittag–Leffler function

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1. Introduction and Preliminaries

The Fractional calculus operators having various special functions have been used for modeling systems in turbulence and fluid dynamics, stochastic dynamic systems, thermonuclear fusion, image processing, nonlinear biological system, and quantum mechanics. For example, Beleanu et al. [1] described a fractional sub-equation method to solve fractional differential equations. Purohit and Kalla [2] studied the solutions of generalized fractional partial differential equations in quantum mechanics. Mathai et al. [3–5] examine the various application of *H*- function by using fractional calculus for the various problem of physics.

Especially the diverse applications of fractional calculus motivated us to establish, here two image formula for the product of two *H*- functions and generalized Mittag–Leffler functions involving left and right sided fractional operator of Saigo–Meada [6], the resulting integral formulas are described in terms of the *H*-function of several variables. Similarly in next two image formula established with the help of the product of two *H*-functions and generalized multi-index Mittag–Leffler functions involving left and right sided fractional operator of Saigo–Meada [6], the resulting integral formulas are described in terms of the *H*-function of several variables. By virtue of the unified nature of generalized Mittag–Leffler functions involved in our results, a large number of new and known results are shown to follow as special cases of our main results but here we presented two of them.

First of all, we recall Mittag–Leffler function and its generalizations.

In 1903, Mittag–Leffler [7] defined a function in term of a power series:

$$E_\rho(y) = \sum_{l=0}^{\infty} \frac{y^n}{\Gamma(\rho l + 1)}, (\rho > 0, y \in \mathbb{C}). \tag{1}$$

Then in 1905, Wiman [8] has given a two-index generalization of this function as:

$$E_{\rho,\chi}(y) = \sum_{l=0}^{\infty} \frac{y^n}{\Gamma(\rho l + \chi)}, (\rho > 0, \chi > 0, y \in \mathbb{C}). \tag{2}$$

Further, Prabhakar [9] presented the generalizing series representation of (2) as:

$$E^{\varpi}_{\rho,\chi}(y) = \sum_{l=0}^{\infty} \frac{(\varpi)_l}{\Gamma(\rho l+\chi)} \frac{y^n}{l!}, (\Re(\rho) > 0, \Re(\chi) > 0, \rho, \chi, \varpi, y \in \mathbb{C}). \tag{3}$$

In addition, Kilbas et al. [10] defined an extension of (3) as:

$$E_{\varpi}[(\rho,\chi)_n; y] = E_{\varpi}[(\rho_1,\chi_1), \cdots, (\rho_n,\chi_n); y] = \sum_{l=0}^{\infty} \frac{(\varpi)_l}{\prod_{i=1}^{n} \Gamma(\rho_i l+\chi_i)} \frac{y^l}{l!}. \tag{4}$$

If $\chi_i \in \mathbb{R}$ $(\chi_i \neq 0)$, $\rho_i \in \mathbb{C}$ $(i = 1, \cdots, n)$ and $y \in \mathbb{C}$, then

(i) if $\sum_{i=0}^{n} \chi_i > 0$, then the above extended Wright function is an entire function of y.
(ii) if $\sum_{i=0}^{n} \chi_i > 0$ and either $|y| < \sum_{i=1}^{n} |\chi_i|^{\chi_i}$ or $y = \sum_{i=1}^{n} |\chi_i|^{\chi_i}, \sum_{i=1}^{n} \Re(\rho) > \Re(\varpi) + n/2$, then the series in $\sum_{l=0}^{\infty} \frac{(\varpi)_l}{\prod_{i=1}^{n} \Gamma(\rho_i l+\chi_i)} \frac{y^l}{l!}$ is absolutely convergent.

Then, after some time, generalization of multi-index Mittag–Leffler function was studied by Saxena and Nishimoto [11] in 2010, and defined in the following manner:

$$\begin{gathered} E_{\varpi,r}[(\rho,\chi)_n; y] = E^{\varpi,r}_{(\rho_i,\chi_i)_n}[y] = \sum_{l=0}^{\infty} \frac{(\varpi)_{rl}}{\prod_{i=1}^{n} \Gamma(\rho_i l+\chi_i)} \frac{y^l}{l!}, \\ (\Re(\rho_i) > 0, \rho_j, \chi_i, \varpi, r, y \in \mathbb{C}, i = 1, 2, \cdots, n; \Re\left(\sum_{i=1}^{m} \rho_i\right) > \max\{0, \Re(r) - 1\}). \end{gathered} \tag{5}$$

In the present paper, we use the generalized Mittag–Leffler function, defined by Saxena and Kalla [12] as:

$$\begin{aligned} &E^{\varpi_i}_{(\rho_i),\tau}(t^{\chi_1}(d-ct)^{-\delta_1}, \cdots, t^{\chi_m}(d-ct)^{-\delta_m}) = E^{(\varpi_1,\cdots,\varpi_m)}_{(\rho_1,\cdots,\rho_m),\tau}\left(t^{\chi_1}(d-ct)^{-\delta_1}, \cdots, t^{\chi_m}(d-ct)^{-\delta_m}\right) \\ &= \sum_{l_1,\cdots,l_m=0}^{\infty} \frac{(\varpi_1)_{l_1} \cdots (\varpi_m)_{l_m} (t^{\chi_1}(d-ct)^{-\delta_1})^{l_1} \cdots (t^{\chi_m}(d-ct)^{-\delta_m})^{l_m}}{\Gamma(\tau+\rho_1 l_1+\cdots+\rho_m l_m)(l_1)! \cdots (l_m)!}, \\ &(\varpi_i, \rho_i, \tau.y, t, c, d, \delta_i, \chi_i \in \mathbb{C}, \Re(\rho_i) > 0, i = 1, \cdots, m). \end{aligned} \tag{6}$$

Further, the generalized multi-index Mittag–Leffler function $E^{(\varpi,c);r}_{(\rho_j,\chi_j)_n}(t^{\chi}(d-ct)^{-\delta}; p)$, defined as [13]:

$$\begin{aligned} &E^{(\varpi,e);r}_{(\rho_j,\chi_j)_n}(t^{\tau}(d-ct)^{-\delta}; p) = \sum_{l=0}^{\infty} \frac{B_p(\varpi+rl, e-\varpi)}{B(\varpi, e-\varpi)} \frac{(e)_{rl}}{\prod_{i=1}^{n} \Gamma(\rho_i l+\chi_i)} \frac{(t^{\tau}(d-ct)^{-\delta})^l}{l!}, \\ &(\rho_i, \chi_i, \varpi, r, t, c, d, e, \delta \in \mathbb{C}, r \geq 0, \Re(e) > \Re(\varpi) > 0, \Re(\chi_i) > 0, (i = 1, 2, \cdots, n); \\ &\Re(\sum_{i=1}^{m} \rho_i) > \max\{0, \Re(r) - 1\}). \end{aligned} \tag{7}$$

Here, B_p is extended Beta function, define as $B_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{\frac{-p}{t(t-1)}}$, $(\Re(x) > 0, \Re(y) > 0, \Re(p) > 0.)$

B is classical Beta function, see [14].

Further, we are recalling fractional integral operator of arbitrary order involving Appell function F_3 in the kernel defined and studied by Saigo and Maeda [6], as given below:

Let $\kappa > 0$, $y > 0$, $\Re(\kappa) > 0$, $\zeta, \zeta', \vartheta, \vartheta', \kappa \in \mathbb{C}$, then

$$(I^{\zeta,\zeta',\vartheta,\vartheta',\kappa}_{0,y} f)(y) = \frac{y^{-\zeta}}{\Gamma(\kappa)} \int_0^y (y-t)^{\kappa-1} t^{-\zeta'} F_3\left(\zeta, \zeta', \vartheta, \vartheta'; \kappa; 1-\frac{t}{y}, 1-\frac{y}{t}\right) f(t)dt, \tag{8}$$

$$(I^{\zeta,\zeta',\vartheta,\vartheta',\kappa}_{y,\infty} f)(y) = \frac{y^{-\zeta'}}{\Gamma(\kappa)} \int_y^{\infty} (t-y)^{\kappa-1} t^{-\zeta} F_3\left(\zeta, \zeta', \vartheta, \vartheta'; \kappa; 1-\frac{y}{t}, 1-\frac{t}{y}\right) f(t)dt, \tag{9}$$

where

$$F_3(\zeta,\zeta',\vartheta,\vartheta';\kappa;z,\omega)=\sum_{m,n=0}^{\infty}\frac{(\zeta)_m(\zeta')_n(\vartheta)_m(\vartheta')_n}{(\kappa)_{m+n}}\frac{z^m}{m!}\frac{\omega^n}{n!}(\max\{|z|,|\omega|\}<1). \tag{10}$$

The Fox's H-function is a generalized hypergeometric function, defined by means of the Mellin-Barnes type contour integral (see [15]),

$$H_{r,s}^{m,n}\left[z\middle|\begin{matrix}(a_i,\zeta_i)_{1,r}\\(b_j,\vartheta_i)_{1,s}\end{matrix}\right]=H_{r,s}^{m,n}\left[z\middle|\begin{matrix}(a_1,\zeta_1),\cdots,(a_r,\zeta_r)\\(b_1,\vartheta_1),\cdots,(b_s,\vartheta_s)\end{matrix}\right]=\frac{1}{2\pi i}\int_{\mathfrak{L}}\theta_s(k)z^{-k}\mathrm{d}k, \tag{11}$$

where, for convenience,

$$\theta_s(k)=\frac{\prod_{i=1}^{m}\Gamma(b_i-\vartheta_ik)\prod_{i=1}^{n}\Gamma(1-a_i-\zeta_ik)}{\prod_{i=n+1}^{r}\Gamma(a_i-\zeta_ik)\prod_{i=m+1}^{s}\Gamma(1-b_i-\vartheta_ik)}. \tag{12}$$

The H-function of several variables is defined as (see [15]):

$$\begin{aligned}H[y_1,\cdots,y_k]&=H_{r,s:r_1,s_1;\cdots;r_k,s_k}^{0,n:m_1,n_1;\cdots;m_k,n_k}\left[\begin{matrix}y_1\\\cdot\\y_k\end{matrix}\middle|\begin{matrix}\left(a_i;\zeta_i\cdots,\zeta_i^{(k)}\right)_{1,r}:(c_i',\kappa_i')_{1,r_1};\cdots;\left(c_i^{(k)},\kappa_i^{(k)}\right)_{1,r_k}\\\left(b_i;\vartheta_i',\cdots,\vartheta_i^{(k)}\right)_{1,s}:(d_i',\delta_i')_{1,s_1};\cdots;\left(d_i^{(k)},\delta_i^{(k)}\right)_{1,s_k}\end{matrix}\right]\\&=\frac{1}{(2\pi i)^k}\int_{L_1}\cdots\int_{L_k}\phi(\omega_1,\cdots,\omega_k)\theta_1(\omega_1)\cdots\theta_k(\omega_k)y_1^{\omega_1},\cdots,y_k^{\omega_k}d\omega_1\cdots d\omega_k,\end{aligned} \tag{13}$$

here

$$\phi(\omega_1,\cdots,\omega_k)=\frac{\prod_{i=1}^{n}\Gamma(1-a_i+\sum_{j=1}^{k}\zeta_i^{(j)}\omega_j)}{\prod_{i=n+1}^{r}\Gamma(a_i-\sum_{j=1}^{k}\zeta_i^{(j)}\omega_j)\prod_{i=1}^{s}\Gamma(1-b_i+\sum_{j=1}^{k}\vartheta_i^{(j)}\omega_j)}, \tag{14}$$

and

$$\theta_i(\omega_i)=\frac{\prod_{i=1}^{n_j}\Gamma(1-c_i^{(j)}+\kappa_i^{(j)}\omega_i)\prod_{i=1}^{m_j}\Gamma(d_i^{(j)}-\delta_i^{(j)}\omega_j)}{\prod_{i=n_j+1}^{r_j}\Gamma(c_i^{(j)}-\kappa_i^{(j)}\omega_j)\prod_{i=m_j+1}^{s_j}\Gamma(1-d_i^{(j)}+\delta_j^{(i)}\omega_j)}, \tag{15}$$

here for all $j\in\{1,\cdots,k\}$.

Here for convenience $\left(a_i;\zeta_i,\cdots,\zeta_i^{(k)}\right)_{1,r}$ abbreviate $\left(a_1;\zeta_1,\cdots,\zeta_1^{(k)}\right),\cdots,(a_r;\zeta_r,\cdots,\zeta_r^{(k)}\Big)$. While $\left(c_i^{(j)},\kappa_i^{(j)}\right)_{1,r_j}$ abbreviate the array of r_j pairs of parameters: $\left(c_1^{(j)},\kappa_1^{(j)}\right),\cdots,\left(c_{r_j}^{(j)},\kappa_{r_j}^{(j)}\right)$, $j\in\{1,\cdots,k\}$, and so on.

Suppose, as usual, that the parameters: a_i, $(i=1,\cdots,r)$; $c_i^{(j)}$, $(i=1,\cdots,r_j)$ b_i, $(i=1,\cdots,s)$; $d_i^{(j)}$, $(i=1,\cdots,s_j)$ here for all $j\in\{1,\cdots,k\}$, are complex number and the associated coefficients $\zeta_i^{(j)}$, $(i=1,\cdots,r)$; $\kappa_i^{(j)}$, $(i=1,\cdots,r_j)$, $\vartheta_i^{(j)}$, $(i=1,\cdots,s)$; $\delta_i^{(j)}$, $(i=1,\cdots,s_j)$, $j\in\{1,\cdots,k\}$, are positive real numbers such that:

$$\Lambda_j:=\sum_{i=1}^{r}\zeta_i^{(j)}-\sum_{i=1}^{s}\vartheta_i^{(j)}+\sum_{i=1}^{r_j}\kappa_i^{(j)}-\sum_{i=1}^{s_j}\delta_i^{(j)}\le 0, \tag{16}$$

$$\triangle_j:-\sum_{i=n+1}^{r}\zeta_i^{(j)}-\sum_{i=1}^{s}\vartheta_i^{(j)}+\sum_{i=1}^{n_j}\kappa_i^{(j)}-\sum_{i=n_j+1}^{r_k}\kappa_i^{(j)}+\sum_{i=1}^{m_j}\delta_i^{(j)}-\sum_{i=m_j+1}^{s_j}\delta_i^{(j)}>0, \tag{17}$$

where the integer n,r,s,m_j,n_j,r_k,s_k are constrained by the inequalities $0\le n\le r$, $s\ge 0$, $1\le m_j\le s_k$ and $0\le n_j\le r_j$, for all $j\in\{1,\cdots,k\}$ and the equality in (17) holds true for suitably restricted values of the complex variables $y_1,\cdots,y_k$.

The multiple Maline-Barnes contour integral representing the multivariate H function converge absolutely, under the condition (17) when: $|arg(y_j)| < \frac{1}{2}\triangle_j\pi$ for all $j \in \{1, \cdots, k\}$. The point $y_j = 0$ for $j \in \{1, \cdots, k\}$ and varius exceptional parameter values being tacitly excluded.

Remark 1. *The special case of* (13) *when r = 1 reduces to the Fox's H-function.*

To prove our main results we required the following Lemma [6]:

Lemma 1. *Let* $\zeta, \zeta', \vartheta, \vartheta', \kappa \in \mathbb{C}$*; if* $Re(\kappa) > 0$ *and* $Re(\rho) > max[0, Re(\zeta + \zeta' + \vartheta - \kappa), Re(\zeta' - \vartheta')]$*, then*

$$I_{0,y}^{\zeta,\zeta',\vartheta,\vartheta',\kappa} y^{\rho-1} = y^{\rho-\zeta-\zeta'+\kappa-1} \frac{\Gamma(\rho)\Gamma(\rho+\kappa-\zeta-\zeta'-\vartheta)\Gamma(\rho+\vartheta'-\zeta')}{\Gamma(\rho+\kappa-\zeta-\zeta')\Gamma(\rho+\kappa-\zeta'-\vartheta)\Gamma(\rho+\vartheta')}. \quad (18)$$

Lemma 2. *Let* $\zeta, \zeta', \vartheta, \vartheta', \kappa \in \mathbb{C}$*; if* $Re(\kappa) > 0$ *and* $Re(\rho) < 1 + min[\Re(-\vartheta), Re(\zeta + \zeta' + \vartheta - \kappa), Re(\zeta + \vartheta' - \kappa)]$*, then*

$$I_{y,\infty}^{\zeta,\zeta',\vartheta,\vartheta',\kappa} y^{\rho-1} = y^{\rho-\zeta-\zeta'+\kappa-1} \frac{\Gamma(1+\zeta+\zeta'-\kappa-\rho)\Gamma(1+\zeta+\vartheta'-\kappa-\rho)\Gamma(1-\vartheta-\rho)}{\Gamma(1-\rho)\Gamma(1+\zeta+\zeta'+\vartheta'-\kappa-\rho)\Gamma(1+\zeta-\vartheta-\rho)}. \quad (19)$$

2. Main Results

In the present section, we introduced fractional integrals involving the product of generalized Mittag–Leffler function and two Fox's H-functions. These Integrals are defined in form of generalized multivariate H-function.

Theorem 1. Let $\zeta, \zeta', \vartheta, \vartheta', \kappa, \mu, \eta, \delta_i, \varpi_i, \rho_i, \chi_i, v_1, v_2, z_1, z_2, c, d, y \in \mathbb{C}$ and $\tau, \alpha_1, \alpha_2 > 0$, $\Re(\rho_i) > 0$, $i = 1, \cdots, m$*. Then the following relation holds:*

$$\begin{aligned}
&\Bigg(I_{0,y}^{\zeta,\zeta',\vartheta,\vartheta',\kappa}\Bigg(t^{\mu-1}(d-ct)^{-\eta} E_{(\rho_i),\tau}^{\varpi_i}\Big(t^{\chi_1}(d-ct)^{-\delta_1}, \cdots, t^{\chi_m}(d-ct)^{-\delta_m}\Big) \\
&H_{r_1,s_1}^{m_1,n_1}\left[z_1 t^{\alpha_1}(d-ct)^{-v_1}\middle|\begin{matrix}(a_i,A_i)_{1,r_1}\\(b_i,B_i)_{1,s_1}\end{matrix}\right] H_{r_2,s_2}^{m_2,n_2}\left[z_2 t^{\alpha_2}(d-ct)^{-v_2}\middle|\begin{matrix}(c_i,C_i)_{1,r_2}\\(d_i,D_i)_{1,s_2}\end{matrix}\right]\Bigg)\Bigg)(y) \\
&= d^{-\eta} y^{\mu-\zeta-\zeta'+\kappa-1} \sum_{l_1,\cdots,l_m=0}^{\infty} \frac{(\varpi)_{l_1}\cdots(\varpi)_{l_m}}{\Gamma(\tau+\rho_1 l_1+\cdots+\rho_m l_m)} \left(\frac{c}{d}\right)^{\delta_l l_1} y^{\chi_1 l_1}\frac{1}{l_1!}\cdots\left(\frac{c}{d}\right)^{\delta_m l_m} y^{\chi_m l_m}\frac{1}{l_m!} \\
&H_{4,4:r_1,s_1;r_2,s_2;0,1}^{0,4:m_1,n_1;m_2,n_2;1,0}\left[\begin{matrix}z_1 y^{\alpha_1}\\ z_2 y^{\alpha_2}\\ -\frac{c}{d}y\end{matrix}\middle|\begin{matrix}E_1 : (a_i,A_i)_{1,r_1};(c_i,C_i)_{1,r_2};-\\ E_2 : (b_i,B_i)_{1,s_1};(d_i,D_i)_{1,s_2};0,1\end{matrix}\right],
\end{aligned} \quad (20)$$

where $E_1 = (1-\eta-\delta_1 l_1 - \cdots - \delta_m l_m; v_1, v_2, 1), (1-\mu-\chi_1 l_1 - \cdots - \chi_m l_m; \alpha_1, \alpha_2, 1), (1-\mu-\kappa+\zeta+\zeta'+\vartheta-\chi_1 l_1 - \cdots - \chi_m l_m; \alpha_1, \alpha_2, 1), (1-\mu+\zeta'-\vartheta'-\chi_1 l_1 - \cdots - \chi_m l_m; \alpha_1, \alpha_2, 1)$ *and* $E_2 = (1-\eta-\delta_1 l_1 - \cdots - \delta_m l_m; v_1, v_2, 0)$, $(1-\mu-\kappa+\zeta+\zeta'-\chi_1 l_1 - \cdots - \chi_m l_m; \alpha_1, \alpha_2, 1)$, $(1-\mu-\kappa+\zeta'+\vartheta-\chi_1 l_1 - \cdots - \chi_m l_m; \alpha_1, \alpha_2, 1), (1-\mu-\vartheta'-\chi_1 l_1 - \cdots - \chi_m l_m; \alpha_1, \alpha_2, 1)$.

and, satisfying the following condition

(i) $|argz_1| < \frac{1}{2}\triangle_1\pi$, $\triangle_1 > 0$*, where* $\triangle_1 = \sum_{i=1}^{m_1} B_i + \sum_{i=1}^{n_1} A_i - \sum_{i=m_1+1}^{s_1} B_i - \sum_{i=n_1+1}^{r_1} A_i$.

(ii) $|argz_2| < \frac{1}{2}\triangle_2\pi$, $\triangle_2 > 0$*, where* $\triangle_2 = \sum_{i=1}^{m_2} D_i + \sum_{i=1}^{n_2} C_i - \sum_{i=m_2+1}^{s_2} D_i - \sum_{i=n_2+1}^{r_2} C_i$.

(iii) $|\frac{a}{b}y| < 1$*, also we have* $Re(\mu) + \alpha_1 \min_{1\le i\le m_1} Re(\frac{b_i}{B_i}) + \alpha_2 \min_{1\le i\le m_2} Re(\frac{d_i}{D_i}) > \max[0, Re(\zeta + \zeta' + \vartheta - \kappa), Re(\zeta' -$

$\vartheta')]$,
$Re(\eta)+v_1\min_{1\le i\le m_1} Re(\frac{b_i}{B_i})+v_2\min_{1\le i\le m_2} Re(\frac{d_i}{D_i})>\max[0, Re(\zeta+\zeta'+\vartheta-\kappa), Re(\zeta'-\vartheta')]$.

Proof. Let $\jmath_1$ be the left-hand side of (20), then by apply Equation (6), we get

$$\begin{aligned}\jmath_1=&\left(I_{0,y}^{\zeta,\zeta',\vartheta,\vartheta',\kappa}\left(t^{\mu-1}(d-ct)^{-\eta}\sum_{l_1,\cdots,l_m=0}^{\infty}\frac{(\varpi_1)_{l_1}\cdots(\varpi_m)_{l_m}(t^{\chi_1}(d-ct)^{-\delta_1})^{l_1}\cdots(t^{\chi_m}(d-ct)^{-\delta_m})^{l_m}}{\Gamma(\tau+\rho_1 l_1+\cdots+\rho_m l_m)(l_1)!\cdots(l_m)!}\right.\right.\\&\left.\left.H_{r_1,s_1}^{m_1,n_1}\left[z_1t^{\alpha_1}(d-ct)^{-v_1}\middle|\begin{matrix}(a_i,A_i)_{1,r_1}\\(b_i,B_i)_{1,s_1}\end{matrix}\right]H_{r_2,s_2}^{m_2,n_2}\left[z_2t^{\alpha_2}(d-ct)^{-v_2}\middle|\begin{matrix}(c_i,C_i)_{1,r_2}\\(d_i,D_i)_{1,s_2}\end{matrix}\right]\right)\right)(y).\end{aligned}\tag{21}$$

Using (11) to replace H functions in its Mellin-Barnes contour integral, we get

$$\begin{aligned}\jmath_1=&\left(I_{0,y}^{\zeta,\zeta',\vartheta,\vartheta',\kappa}\left(t^{\mu-1}(d-ct)^{-\eta}\sum_{l_1,\cdots,l_m=0}^{\infty}\frac{(\varpi_1)_{l_1}\cdots(\varpi_m)_{l_m}(t^{\chi_1}(d-ct)^{-\delta_1})^{l_1}\cdots(t^{\chi_m}(d-ct)^{-\delta_m})^{l_m}}{\Gamma(\tau+\rho_1 l_1+\cdots+\rho_m l_m)(l_1)!\cdots(l_m)!}\right.\right.\\&\left.\left.\int_{L_1}\frac{1}{2\pi i}\phi_1(\zeta_1)(z_1(d-ct)^{-v_1})^{\zeta_1}d\zeta_1\int_{L_2}\frac{1}{2\pi i}\phi_2(\zeta_2)(z_2(d-ct)^{-v_2})^{\zeta_2}d\zeta_2\right)\right)(y).\end{aligned}\tag{22}$$

By using the generalized binomial theorem, expanding the term $(d-ct)^{-v_1}$, we get:

$$(d-ct)^{-v_1}=d^{-v_1}\sum_{k=0}^{\infty}\frac{(v_1)_k}{k!}(\frac{ct}{d})^k,|\frac{ct}{d}|<1,$$

similarly, we define the terms $(d-ct)^{-v_2}$ and $(d-ct)^{-\eta}$, and arranging the order of integral, we get

$$\begin{aligned}\jmath_1=&d^{-\eta}\sum_{l_1,\cdots,l_m=0}^{\infty}\frac{(\varpi_1)_{l_1}\cdots(\varpi_m)_{l_m}}{\Gamma(\tau+\rho_1 l_1+\cdots+\rho_m l_m)}\left(\frac{c}{d}\right)^{\delta_l l_1}\frac{1}{l_1!}\cdots\left(\frac{c}{d}\right)^{\delta_m l_m}\frac{1}{l_m!}\left(\frac{1}{2\pi i}\right)^3\int_{L_1}\phi_1(\zeta_1)(z_1)^{\zeta_1}d^{-v_1\zeta_1}d\zeta_1\\&\int_{L_2}\phi_2(\zeta_2)(z_2)^{\zeta_2}d^{-v_2\zeta_2}d\zeta_2\int_{L_3}\frac{\Gamma(\eta+\delta_1 l_1-\cdots-\delta_m l_m+v_1\zeta_1+v_2\zeta_2+\zeta_3)}{\Gamma(\eta+\delta_1 l_1-\cdots-\delta_m l_m+v_1\zeta_1+v_2\zeta_2)\Gamma(1+\zeta_3)}(-\frac{a}{b})^{\zeta_3}d\zeta_3\\&\left(I_{0,y}^{\zeta,\zeta',\vartheta,\vartheta',\kappa}t^{\chi_1 l_1+\cdots+\chi_m l_m+\mu+\alpha_1\zeta_1+\alpha_2\zeta_2+\zeta_3-1}\right)(y).\end{aligned}\tag{23}$$

Now using the Equation (18), we get

$$\begin{aligned}\jmath_1=&d^{-\eta}\sum_{l_1,\cdots,l_m=0}^{\infty}\frac{(\varpi_1)_{l_1}\cdots(\varpi_m)_{l_m}}{\Gamma(\chi+\rho_1 l_1+\cdots+\rho_m l_m)}\left(\frac{c}{d}\right)^{\delta_l l_1}\frac{1}{l_1!}\cdots\left(\frac{c}{d}\right)^{\delta_m l_m}\frac{1}{l_m!}\\&\left(\frac{1}{2\pi i}\right)^3\int_{L_1}\phi_1(\zeta_1)(z_1)^{\zeta_1}d^{-v_1\zeta_1}d\zeta_1\int_{L_2}\phi_2(\zeta_2)(z_2)^{\zeta_2}d^{-v_2\zeta_2}d\zeta_2\\&\int_{L_3}\frac{\Gamma(\eta+\delta_1 l_1+\cdots+\delta_m l_m+v_1\zeta_1+v_2\zeta_2+\zeta_3)}{\Gamma(\eta+\delta_1 l_1+\cdots+\delta_m l_m+v_1\zeta_1+v_2\zeta_2)\Gamma(1+\zeta_3)}(-\frac{c}{d})^{\zeta_3}d\zeta_3\\&\frac{\Gamma(\mu+\chi_1 l_1+\cdots+\chi_m l_m+\alpha_1\zeta_1+\alpha_2\zeta_2)}{\Gamma(\mu+\chi_1 l_1+\cdots+\chi_m l_m+\alpha_1\zeta_1+\alpha_2\zeta_2+\zeta_3+\kappa-\zeta-\zeta')}\\&\frac{\Gamma(\mu+\chi_1 l_1+\cdots+\chi_m l_m+\delta_1\zeta_1+\alpha_2\zeta_2+\zeta_3+\kappa-\zeta-\zeta'-\vartheta)}{\Gamma(\mu+\chi_1 l_1+\cdots+\chi_m l_m+\alpha_1\zeta_1+\alpha_2\zeta_2+\zeta_3-\kappa-\zeta'-\vartheta)}\\&\frac{\Gamma(\mu+\chi_1 l_1+\cdots+\chi_m l_m+\delta_1\zeta_1+\alpha_2\zeta_2+\zeta_3+\vartheta'-\zeta')}{\Gamma(\mu+\chi_1 l_1+\cdots+\chi_m l_m+\alpha_1\zeta_1+\alpha_2\zeta_2+\zeta_3+\vartheta')}y^{(\mu+\chi_1 l_1+\cdots+\chi_m l_m\alpha_1\zeta_1+\alpha_2\zeta_2+\zeta_3-\zeta-\zeta'+\kappa-1)}.\end{aligned}\tag{24}$$

Hence the above equation can be written in form of R.H.S. of the Equation (20) by using the Equation (13) and we get our desired result. □

Theorem 2. Let $\zeta,\zeta',\vartheta,\vartheta',\kappa,\mu,\eta,\delta, v_1,v_2,z_1,z_2,c,d,\varpi_i,\chi_i,\rho_i,\delta_i \in \mathbb{C}$ and $\tau,\alpha_1,\alpha_2>0$, $\Re(\rho_i)>0, i=1,\cdots,m$. *Then the relation holds:*

$$\Big(I^{\zeta,\zeta',\vartheta,\vartheta',\kappa}_{y,\infty}\Big(t^{\mu-1}(d-ct)^{-\eta}E^{\varpi_i}_{(\rho_i),\tau}\Big(t^{\chi_1}(d-ct)^{-\delta_1},\cdots,t^{\chi_m}(d-ct)^{-\delta_m}\Big)$$
$$H^{m_1,n_1}_{r_1,s_1}\left[z_1t^{\alpha_1}(d-ct)^{-v_1}\middle|\begin{matrix}(a_i,A_i)_{1,r_1}\\(b_i,B_i)_{1,s_1}\end{matrix}\right]H^{m_2,n_2}_{r_2,s_2}\left[z_2t^{\alpha_2}(d-ct)^{-v_2}\middle|\begin{matrix}(c_i,C_i)_{1,r_2}\\(d_i,D_i)_{1,s_2}\end{matrix}\right]\Big)\Big)(y)$$
$$=d^{-\eta}y^{\mu-\zeta-\zeta'+\kappa-1}\sum_{l_1,\cdots,l_m=0}^{\infty}\frac{(\varpi)_{l_1}\cdots(\varpi)_{l_m}}{\Gamma(\tau+\rho_1l_1+\cdots+\rho_ml_m)}\Big(\frac{c}{d}\Big)^{\delta_ll_1}y^{\chi_il}\frac{1}{l_1!}\cdots\Big(\frac{c}{d}\Big)^{\delta_ml_m}y^{\chi_ml_m}\frac{1}{l_m!} \tag{25}$$
$$H^{0,4:m_1,n_1;m_2,n_2;1,0}_{4,4:r_1,s_1;r_2,s_2;0,1}\left[\begin{matrix}z_1y^{\alpha_1}\\z_2y^{\alpha_2}\\-\frac{c}{d}y\end{matrix}\middle|\begin{matrix}F_1:(a_i,A_i)_{1,r_1};(c_i,C_i)_{1,r_2};-\\F_2:(b_i,B_i)_{1,s_1};(d_i,D_i)_{1,s_2};0,1\end{matrix}\right],$$

where $F_1=(1-\eta-\delta_1l_1-\cdots-\delta_ml_m;v_1,v_2,1)$, $(1-\mu-\chi_1l_1-\cdots-\chi_ml_m;\alpha_1,\alpha_2,1)$, $(1+\zeta+\zeta'+\vartheta'-\kappa-\mu-\chi_1l_1-\cdots-\chi_ml_m;\alpha_1,\alpha_2,1)$, $(1+\zeta-\vartheta-\mu-\chi_1l_1-\cdots-\chi_ml_m;\alpha_1,\alpha_2,1)$, *and* $F_2=(1-\eta-\delta_1l_1-\cdots-\delta_ml_m;v_1,v_2,0)$, $(1+\zeta+\zeta'-\kappa-\mu-\chi_1l_1-\cdots-\chi_ml_m;\alpha_1,\alpha_2,1)$, $(1+\zeta+\vartheta'\kappa-\chi_1l_1-\cdots-\chi_ml_m;\alpha_1,\alpha_2,1)$, $(1-\vartheta-\mu-\chi_1l_1-\cdots-\chi_ml_m;\alpha_1,\alpha_2,1)$ *and the condition (i)–(iii) in* (20) *are also satisfied.*

Proof. Let J_2 be the left-hand side of(25), using Equation (6) and (11), we get:

$$J_2=\Big(I^{\zeta,\zeta',\vartheta,\vartheta',\kappa}_{y,\infty}\Big(t^{\mu-1}(d-ct)^{-\eta}\sum_{l_1,\cdots,l_m=0}^{\infty}\frac{(\varpi_1)_{l_1}\cdots(\varpi_m)_{l_m}(t^{\chi_1}(d-ct)^{-\delta_1})^{l_1}\cdots(t^{\chi_m}(d-ct)^{-\delta_m})^{l_m}}{\Gamma(\tau+\rho_1l_1+\cdots+\rho_ml_m)(l_1)!\cdots(l_m)!}$$
$$\int_{L_1}\frac{1}{2\pi i}\phi_1(\zeta_1)(z_1(d-ct)^{-v_1})^{\zeta_1}d\zeta_1\int_{L_2}\frac{1}{2\pi i}\phi_2(\zeta_2)(z_2(d-ct)^{-v_2})^{\zeta_2}d\zeta_2\Big)\Big)(y). \tag{26}$$

By using the generalized binomial theorem, expanding the term $(d-ct)^{-v_1}$, we get:

$$(d-ct)^{-v_1}=d^{-v_1}\sum_{k=0}^{\infty}\frac{(v_1)_k}{k!}(\frac{ct}{d})^k,|\frac{ct}{d}|<1,$$

similarly, we define the terms $(d-ct)^{-v_2}$ and $(d-ct)^{-\eta}$, and arranging the order of integral, we get:

$$J_2=d^{-\eta}\sum_{l_1,\cdots,l_m=0}^{\infty}\frac{(\varpi_1)_{l_1}\cdots(\varpi_m)_{l_m}}{\Gamma(\tau+\rho_1l_1+\cdots+\rho_ml_m)}\Big(\frac{c}{d}\Big)^{\delta_ll_1}\frac{1}{l_1!}\cdots\Big(\frac{c}{d}\Big)^{\delta_ml_m}\frac{1}{l_m!}\Big(\frac{1}{2\pi i}\Big)^3\int_{L_1}\phi_1(\zeta_1)(z_1)^{\zeta_1}d^{-v_1\zeta_1}d\zeta_1$$
$$\int_{L_2}\phi_2(\zeta_2)(z_2)^{\zeta_2}d^{-v_2\zeta_2}d\zeta_2\int_{L_3}\frac{\Gamma(\eta+\delta_1l_1+\cdots+\delta_ml_m+v_1\zeta_1+v_2\zeta_2+\zeta_3)}{\Gamma(\eta+\delta_1l_1+\cdots+\delta_ml_m+v_1\zeta_1+v_2\zeta_2)\Gamma(1+\zeta_3)}(-\frac{c}{d})^{\zeta_3}d\zeta_3 \tag{27}$$
$$\Big(I^{\zeta,\zeta',\vartheta,\vartheta',\kappa}_{y,\infty}t^{\chi_1l_1+\cdots+\chi_ml_m+\mu+\alpha_1\zeta_1+\alpha_2\zeta_2+\zeta_3-1}\Big)(y).$$

Next by using the Equation (19), we get:

$$J_2 = d^{-\eta} \sum_{l_1,\cdots,l_m=0}^{\infty} \frac{(\varpi_1)_{l_1}\cdots(\varpi_m)_{l_m}}{\Gamma(\tau+\rho_1 l_1+\cdots+\rho_m l_m)} \left(\frac{c}{d}\right)^{\delta_l l_1} \frac{1}{l_1!} \cdots \left(\frac{c}{d}\right)^{\delta_m l_m} \frac{1}{l_m!}$$
$$\left(\frac{1}{2\pi i}\right)^3 \int_{L_1} \phi_1(\zeta_1)(z_1)^{\zeta_1} d^{-v_1\zeta_1} d\zeta_1 \int_{L_2} \phi_2(\zeta_2)(z_2)^{\zeta_2} d^{-v_2\zeta_2} d\zeta_2$$
$$\int_{L_3} \frac{\Gamma(\eta+\delta_1 l_1+\cdots+\delta_m l_m+v_1\zeta_1+v_2\zeta_2+\zeta_3)}{\Gamma(\eta+\delta_1 l_1+\cdots+\delta_m l_m+v_1\zeta_1+v_2\zeta_2)\Gamma(1+\zeta_3)} (-\frac{c}{d})^{\zeta_3} d\zeta_3$$
$$\frac{\Gamma(1+\zeta+\zeta'-\kappa-\mu-\chi_1 l_1-\cdots-\chi_m l_m-\alpha_1\zeta_1-\alpha_2\zeta_2-\zeta_3)}{\Gamma(1-\mu-\chi_1 l_1-\cdots-\chi_m l_m-\alpha_1\zeta_1-\alpha_2\zeta_2-\zeta_3)} \tag{28}$$
$$\frac{\Gamma(1+\zeta+\vartheta'-\kappa-\mu-\chi_1 l_1-\cdots-\chi_m l_m-\alpha_1\zeta_1-\alpha_2\zeta_2-\zeta_3)}{\Gamma(1+\zeta+\zeta'+\vartheta'-\kappa-\mu-\chi_1 l_1-\cdots-\chi_m l_m-\alpha_1\zeta_1-\alpha_2\zeta_2-\zeta_3)}$$
$$\frac{\Gamma(1-\vartheta-\mu-\chi_1 l_1-\cdots-\chi_m l_m-\alpha_1\zeta_1-\alpha_2\zeta_2-\zeta_3)}{\Gamma(1+\zeta-\vartheta-\mu-\chi_1 l_1-\cdots-\chi_m l_m-\alpha_1\zeta_1-\alpha_2\zeta_2-\zeta_3)} y^{(\mu+\chi_1 l_1+\cdots+\chi_m l_m+\alpha_1\zeta_1+\alpha_2\zeta_2+\zeta_3-\zeta-zeta'+\kappa-1)}.$$

Hence, the above equation can be written in form of R.H.S of the Equation (25) by using Equation (13). Hence, we get our desired result. □

Theorem 3. *let $\zeta,\zeta',\vartheta,\vartheta',\kappa,\mu,\eta,\delta,v_1,v_2,z_1,z_2,c,d,e,\varpi,\chi_i,\rho_i \in \mathbb{C}$ and $\chi,\alpha_1,\alpha_2 > 0$, $\Re(\rho_i) > 0$. Then the following result holds:*

$$\left(I_{0,y}^{\zeta,\zeta',\vartheta,\vartheta',\kappa}\left(t^{\mu-1}(d-ct)^{-\eta}E_{(\rho_i,\chi_i)_n}^{(\varpi,e):r}(t^{\chi}(d-ct)^{-\delta};p)H_{r_1,s_1}^{m_1,n_1}\left[z_1 t^{\alpha_1}(d-ct)^{-v_1}\middle|\begin{matrix}(a_i,A_i)_{1,r_1}\\(b_i,B_i)_{1,s_1}\end{matrix}\right]\right.\right.$$
$$\left.\left.H_{r_2,s_2}^{m_2,n_2}\left[z_2 t^{\alpha_2}(d-ct)^{-v_2}\middle|\begin{matrix}(c_i,C_i)_{1,r_2}\\(d_i,D_i)_{1,s_2}\end{matrix}\right]\right)\right)(y) = d^{-\eta}y^{\mu-\zeta-\zeta'+\kappa-1}$$
$$\frac{e^{-2p}\sum_{a,b=0}^{\infty}L_a L_b \beta(\varpi+rl+a+1,e-\varpi+b+1)}{\beta(\varpi,e-\varpi)} \sum_{l=0}^{\infty}\frac{(e)_{rl}}{\prod_{i=1}^{n}\Gamma(\chi_i+\rho_i l)}\left(\frac{c}{d}\right)^{\delta l} y^{\chi l}\frac{1}{l!} \tag{29}$$
$$H_{4,4:r_1,s_1;r_2,s_2;0,1}^{0,4:m_1,n_1;m_2,n_2;1,0}\left[\begin{matrix}z_1 y^{\alpha_1}\\z_2 y^{\alpha_2}\\-\frac{c}{d}y\end{matrix}\middle|\begin{matrix}g_1:(a_i,A_i)_{1,r_1};(c_i,C_i)_{1,r_2};-\\g_2:\left(b_i,B_i'\right)_{1,s_1};(d_i,D_i)_{1,s_2};0,1\end{matrix}\right],$$

where $g_1 = (1-\eta-\delta l;v_1,v_2,1)$, $(1-\mu-\chi l;\alpha_1,\alpha_2,1)$, $(1-\mu-\kappa+\zeta+\zeta'+\vartheta-\chi l;\alpha_1,\alpha_2,1)$, $(1-\mu+\zeta'-\vartheta'-\chi l;\alpha_1,\alpha_2,1)$ and $g_2 = (1-\eta-\delta l;v_1,v_2,0)$, $(1-\mu-\kappa+\zeta+\zeta'-\chi l;\alpha_1,\alpha_2,1)$, $(1-\mu-\kappa+\zeta'+\vartheta-\chi l;\alpha_1,\alpha_2,1)$, $(1-\mu-\vartheta'-\chi l;\alpha_1,\alpha_2,1)$. and the condition (i)–(iii) in Theorem 1, are also satisfied.

Proof. The same argument as in the proof of Theorem 1 will establish the result in Theorem 3 by using the (7). So its proof details are omitted. □

Theorem 4. *let* $\zeta,\zeta',\vartheta,\vartheta',\kappa,\mu,\eta,\delta,\ v_1,v_2,z_1,z_2,c,d,e,\varpi,\chi_i,\rho_i,\delta,y \in \mathbb{C}$ *and* $\chi,\alpha_1,\alpha_2 > 0$, $\Re(\rho_i) > 0$. *Then the following result holds:*

$$\left(I_{y,\infty}^{\zeta,\zeta',\vartheta,\vartheta',\kappa}\left(t^{\mu-1}(d-ct)^{-\eta}E_{(\rho_i,\chi_i)_n}^{(\varpi,e):r}(t^{\chi}(d-ct)^{-\delta};p)H_{r_1,s_1}^{m_1,n_1}\left[z_1t^{\alpha_1}(d-ct)^{-v_1}\middle|\begin{matrix}(a_i,A_i)_{1,r_1}\\(b_i,B_i)_{1,s_1}\end{matrix}\right]\right.\right.$$
$$\left.\left.H_{r_2,s_2}^{m_2,n_2}\left[z_2t^{\alpha_2}(d-ct)^{-v_2}\middle|\begin{matrix}(c_i,C_i)_{1,r_2}\\(d_i,D_i)_{1,s_2}\end{matrix}\right]\right)\right)(y) = d^{-\eta}y^{\mu-\zeta-\zeta'+\kappa-1}$$
$$\frac{e^{-2p}\sum_{a,b,l=0}^{\infty}L_aL_b\beta(\varpi+rl+a+1,e-\varpi+b+1)}{\beta(\varpi,e-\varpi)}\sum_{l=0}^{\infty}\frac{(e)_{rl}}{\prod_{i=1}^{n}\Gamma(\chi_i+\rho_il)}\frac{1}{l!}\left(\frac{c}{d}\right)^{\delta l}y^{\chi l} \tag{30}$$
$$H_{4,4:r_1,s_1;r_2,s_2;0,1}^{0,4:m_1,n_1;m_2,n_2;1,0}\left[\begin{matrix}z_1y^{\alpha_1}\\z_2y^{\alpha_2}\\-\frac{c}{d}x\end{matrix}\middle|\begin{matrix}h_1:(a_i,A_i)_{1,r_1};(c_i,C_i)_{1,r_2};-\\h_2:\left(b_i,B_i'\right)_{1,s_1};(d_i,D_i)_{1,s_2};0,1\end{matrix}\right],$$

where $h_1 = (1-\eta-\delta l;v_1,v_2,1),(1-\mu-\chi l;\alpha_1,\alpha_2,1),(1+\zeta+\zeta'+\vartheta'-\kappa-\mu-\chi l;\alpha_1,\alpha_2,1)$, $(1+\zeta-\vartheta-\mu-\chi l;\alpha_1,\alpha_2,1)$ $h_2 = (1-\eta-\delta l;v_1,v_2,0),(1+\zeta+\zeta'-\kappa-\mu-\chi l;\alpha_1,\alpha_2,1)$, $(1+\zeta+\vartheta'\kappa-\chi l;\alpha_1,\alpha_2,1),(1-\vartheta-\mu-\chi l;\alpha_1,\alpha_2,1)$ *and the condition (i)–(iii) in Theorem 1 are also satisfied.*

Proof. The same argument as in the proof of Theorem 2 will establish the result in Theorem 4 by using the (7). So, its proof details are omitted. □

3. Special Cases

Firstly, here, it is important to remark the fact that the Saigo-Maeda fractional integral and fractional derivative operators involved in Theorems 1–4 are unified ones in nature. Secondly, the product of Fox's H-functions occurring in Theorems 1–4 can be suitably specialized to give a large number of useful functions, for example, the Bessel functions, Wright hypergeometric functions, and so on. Here, among a remarkably large number of possible special examples of the results in Theorems 1–4, we consider only the following two examples.

If we consider only single variable generalized Mittag–Leffler function $E_{(\rho),\tau}^{\varpi}$ by putting $i = 1$ in (6) and Multivariate H-functions and then apply the Theorems 1 and 2, then we get our following special cases as described below:

Corollary 1. *Let* $\zeta,\zeta',\vartheta,\vartheta',\kappa,\mu,\eta,\delta,\ v_1,v_2,z_1,z_2,c,d,\varpi,\chi,\rho,\delta,y \in \mathbb{C}$ *and* $\tau,\alpha_1,\alpha_2 > 0$, $\Re(\rho) > 0$. *Then the following result holds:*

$$\left(I_{0,y}^{\zeta,\zeta',\vartheta,\vartheta',\kappa}\left(t^{\mu-1}(d-ct)^{-\eta}E_{(\rho),\tau}^{\varpi}\left(t^{\chi}(d-ct)^{-\delta}\right)H_{r_1,s_1}^{m_1,n_1}\left[z_1t^{\alpha_1}(d-ct)^{-v_1}\middle|\begin{matrix}(a_i,A_i)_{1,r_1}\\(b_i,B_i)_{1,s_1}\end{matrix}\right]\right.\right.$$
$$\left.\left.H_{r_1,s_2}^{m_2,n_2}\left[z_2t^{\alpha_2}(d-ct)^{-v_2}\middle|\begin{matrix}(c_i,C_i)_{1,r_2}\\(d_i,D_i)_{1,s_2}\end{matrix}\right]\right)\right)(y) = d^{-\eta}y^{\mu-\zeta-\zeta'+\kappa-1}\sum_{l=0}^{\infty}\frac{(\varpi)_l}{\Gamma(\tau+\rho l)}\left(\frac{c}{d}\right)^{\delta l}y^{\chi l}\frac{1}{l!}$$
$$H_{4,4:r_1,s_1;r_2,s_2;0,1}^{0,4:m_1,n_1;m_2,n_2;1,0}\left[\begin{matrix}z_1y^{\alpha_1}\\z_2y^{\alpha_2}\\-\frac{c}{d}y\end{matrix}\middle|\begin{matrix}E_1:(a_i,A_i)_{1,r_1};(c_i,C_i)_{1,r_2};-\\E_2:\left(b_i,B_i'\right)_{1,s_1};(d_i,D_i)_{1,s_2};0,1\end{matrix}\right],$$

where $E_1' = (1-\eta-\delta l;v_1,v_2,1),(1-\mu-\chi l;\alpha_1,\alpha_2,1),(1-\mu-\kappa+\zeta+\zeta'+\vartheta-\chi l;\alpha_1,\alpha_2,1)$, $(1-\mu+\zeta'-\vartheta'-\chi l;\alpha_1,\alpha_2,1)$ *and* $E_2' = (1-\eta-\delta l;v_1,v_2,0),(1-\mu-\kappa+\zeta+\zeta'-\chi l_1;\alpha_1,\alpha_2,1),(1-\mu-\kappa+\zeta'+\vartheta-\chi l;\alpha_1,\alpha_2,1),(1-\mu-\vartheta'-\chi l;\alpha_1,\alpha_2,1)$.

Proof. In the similar manner as the proof of Theorem 1, we can easily get our desired result. □

Corollary 2. *let* $\zeta,\zeta',\vartheta,\vartheta',\kappa,\mu,\eta,\delta,\ v_1,v_2,z_1,z_2,c,d,\varpi,\chi,\rho,\delta,y\in\mathbb{C}$ *and* $\tau,\alpha_1,\alpha_2>0$, $Re(\rho)>0$. *Then the following result holds:*

$$\left(I_{y,\infty}^{\zeta,\zeta',\vartheta,\vartheta',\kappa}\left(t^{\mu-1}(d-ct)^{-\eta}E_{(\rho),\tau}^{\varpi}\left(t^{\chi}(d-ct)^{-\delta}\right)H_{r_1,s_1}^{m_1,n_1}\left[z_1t^{\alpha_1}(d-ct)^{-v_1}\middle|\begin{matrix}(a_i,A_i)_{1,r_1}\\(b_i,B_i)_{1,s_1}\end{matrix}\right]\right.\right.$$
$$\left.\left.H_{r_2,s_2}^{m_2,n_2}\left[z_2t^{\alpha_2}(d-ct)^{-v_2}\middle|\begin{matrix}(c_i,C_i)_{1,r_2}\\(d_i,D_i)_{1,s_2}\end{matrix}\right]\right)\right)(y)=d^{-\eta}y^{\mu-\zeta-\zeta'+\kappa-1}\sum_{l=0}^{\infty}\frac{(\varpi)_l}{\Gamma(\chi+\rho l)}\left(\frac{c}{d}\right)^{\delta l}y^{\chi l}\frac{1}{l!}$$
$$H_{4,4:r_1,s_1;r_2,s_2;0,1}^{0,4:m_1,n_1;m_2,n_2;1,0}\left[\begin{matrix}z_1y^{\alpha_1}\\z_2y^{\alpha_2}\\-\frac{c}{d}x\end{matrix}\middle|\begin{matrix}F_1:(a_i,A_i)_{1,r_1};(c_i,C_i)_{1,r_2};-\\F_2:\left(b_i,B_i'\right)_{1,s_1};(d_i,D_i)_{1,s_2};0,1\end{matrix}\right],$$

$F_1=(1-\eta-\delta l;v_1,v_2,1),(1-\mu-\chi l;\alpha_1,\alpha_2,1),(1+\zeta+\zeta'+\vartheta'-\kappa-\mu-\chi l;\alpha_1,\alpha_2,1),(1+\zeta-\vartheta-\mu-\chi l;\alpha_1,\alpha_2,1)$, $F_2=(1-\eta-\delta l;v_1,v_2,0),(1+\zeta+\zeta'-\kappa-\mu-\chi l;\alpha_1,\alpha_2,1),(1+\zeta+\vartheta'\kappa-\chi l;\alpha_1,\alpha_2,1),(1-\vartheta-\mu-\chi l;\alpha_1,\alpha_2,1)$ *and the condition (i)–(iii) in Theorem 1, are also satisfied.*

Proof. In the similar manner as the proof of Theorem (2), we can easily get our desired result. □

4. Conclusions

The H- functions associated with fractional calculus have been recognized to play a fundamental role in the probability theory as well as in their applications, including non-Gaussian stochastic processes and phenomena of nonstandard (i.e., anomalous) relaxation and diffusion. Another hand the H function and the generalized Mittag–Leffler function reduce to hypergeometric function and polynomials, so it becomes more important from the application viewpoint. Therefore, fractional calculus formulae involving hypergeometric functions and polynomials play an important role in the theory of special function and mathematical physics [16–18]. Finally, we conclude that our results presented in this paper are new and important from an application point of view. In the future, we are also trying to find some basic applications and examples of those results presented here to different research regions.

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