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Nonlocal ψ-Hilfer Generalized Proportional Boundary Value Problems for Fractional Differential Equations and Inclusions

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Abstract: In this paper, we establish existence and uniqueness results for a new class of boundary value problems involving the ψ-Hilfer generalized proportional fractional derivative operator, supplemented with mixed nonlocal boundary conditions including multipoint, fractional integral multiorder and derivative multiorder operators. The given problem is first converted into an equivalent fixed point problem, which is then solved by means of the standard fixed point theorems. The Banach contraction mapping principle is used to establish the existence of a unique solution, while the Krasnosel'skiĭ and Schaefer fixed point theorems as well as the Leray–Schauder nonlinear alternative are applied for obtaining the existence results. We also discuss the multivalued analogue of the problem at hand. The existence results for convex- and nonconvex-valued multifunctions are respectively proved by means of the Leray–Schauder nonlinear alternative for multivalued maps and Covitz–Nadler's fixed point theorem for contractive multivalued maps. Numerical examples illustrating the obtained results are also presented.

Keywords: ψ-Hilfer generalized proportional fractional derivative; fractional differential equations and inclusions; nonlocal boundary conditions; existence; fixed point

MSC: 26A33; 34A08; 34A60; 34B15

Citation: Ntouyas, S.K.; Ahmad, B.; Tariboon, J. Nonlocal ψ-Hilfer Generalized Proportional Boundary Value Problems for Fractional Differential Equations and Inclusions. *Foundations* **2022**, *2*, 377–398. https://doi.org/10.3390/foundations2020026

Academic Editor: Ioannis K. Argyros and Lorentz Jäntschi

Received: 3 March 2022
Accepted: 18 April 2022
Published: 22 April 2022

1. Introduction

Fractional differential equations frequently appear in different research areas, such as engineering, physics, economics, chemistry, viscoelasticity, robotics, control theory, etc. We refer the reader to the monographs [1–9] for a systematic development of fractional calculus and fractional differential equations. In particular, boundary value problems for fractional differential equations constitute an important and interesting area of research in applied analysis. Recent development on the topic contains different types of fractional-order differential operators such as Caputo, Riemann–Liouville, Hadamard, Erdeyl–Kober, Katugampola, etc. In [10], Hilfer introduced a new derivative, which interpolates between the Riemann–Liouville and Caputo fractional derivatives. For some applications of the Hilfer fractional derivative, and some recent results on initial and boundary value problems, for instance, see [11–24] and the references cited therein. One can find some recent works on Hilfer–Hadamard fractional differential equations in [25–28].

In [29,30], Katugampola introduced a generalized fractional operator combining Riemann–Liouville and Hadamard fractional operators. These operators were modified by Jarad et al. [31] to include the Caputo and Caputo–Hadamard fractional derivatives. The authors in [32] introduced a new type of fractional derivative, the generalized proportional fractional derivative. The work of [32] was generalized in [33,34] by using the concept of the proportional derivative of a function with respect to another function. In [35], the Hilfer generalized proportional fractional derivative was proposed. For some recent results on Hilfer generalized proportional fractional differential equations, see [36,37].

Recently, in [38], the authors introduced the ψ-Hilfer generalized proportional fractional derivative of a function with respect to another function and discussed its properties. As an application, the existence and uniqueness of solutions for the following nonlocal problem of order in $(0,1)$ were established:

$$\begin{cases} D_{c+}^{\alpha,\beta,\sigma,\psi} w(\vartheta) = f(\vartheta, w(\vartheta)), \quad \vartheta \in [c,d],\ d > c \geq 0, \\ I_{c+}^{1-\gamma,\sigma,\psi} w(c) = \sum_{i=1}^{m} \mu_i w(\tau_i), \end{cases}$$

where $D_{c+}^{\alpha,\beta,\sigma,\psi}$ is the ψ-Hilfer generalized proportional fractional derivative, $I_{c+}^{1-\gamma,\sigma,\psi}$ is the ψ-Hilfer generalized proportional fractional integral, $0 < \alpha < 1$, $0 \leq \beta \leq 1$, $\sigma \in (0,1]$, $\gamma = \alpha + \beta(1-\alpha)$, $\tau_i \in (c,d)$, $\mu_i \in \mathbb{R}$ and $f : [c,d] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Motivated by the work presented in [38], in this paper, we introduce and study a nonlocal mixed boundary value problem for a ψ-Hilfer generalized proportional fractional differential equation of order in $(1,2]$ given by

$$\begin{cases} D_{c^+}^{\alpha,\beta,\sigma,\psi} w(\vartheta) = f(\vartheta, w(\vartheta)), \quad \vartheta \in [c,d],\ d > c \geq 0, \\ w(c) = 0, \\ w(d) = \sum_{j=1}^{m} \eta_j w(\xi_j) + \sum_{i=1}^{n} \zeta_i\, I_{c^+}^{\phi_i,\sigma,\psi} w(\theta_i) + \sum_{k=1}^{r} \lambda_k\, D_{c^+}^{\delta_k,\beta,\sigma,\psi} w(\mu_k), \end{cases} \tag{1}$$

where $D_{c+}^{\chi,\beta,\sigma,\psi}$ denotes the ψ-Hilfer generalized proportional fractional derivative operator of order $\chi \in \{\alpha, \delta_k\}$, $\alpha, \delta_k \in (1,2]$ and type $\beta \in [0,1]$, respectively, $\sigma \in (0,1]$, $\eta_j, \zeta_i, \lambda_k \in \mathbb{R}$ are given constants, $f : [c,d] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $I_{c+}^{\phi_i;\sigma,\psi}$ is the generalized proportional fractional integral operator of order $\phi_i > 0$ and $\xi_j, \theta_i, \mu_k \in (c,d)$, $j = 1,2,\ldots,m$, $i = 1,2,\ldots,n$, $k = 1,2,\ldots,r$, are given points. We also study the multivalued analogue of Problem (1).

Notice that Problem (1) covers a variety of nonlocal mixed boundary value problems, for example,

- Problem (1) reduces to a nonlocal mixed boundary value problem of Hilfer generalized proportional fractional differential equations of order in $(1,2]$ by setting $\psi(\vartheta) = \vartheta$ in it;
- Problem (1) corresponds to a nonlocal mixed boundary value problem of ψ-Hilfer fractional differential equations of order in $(1,2]$ by fixing $\sigma = 1$ in it, which is studied in [39].

Nonlocal conditions are considered to be more plausible than the classical conditions, as they can correctly describe certain features of physical problems. We emphasize that the mixed nonlocal boundary condition considered in Problem (1) is of more general form, as it includes multipoint, fractional integral multiorder and fractional derivative multiorder contributions.

The rest of the manuscript is arranged as follows. Section 2 contains some basic notions of fractional derivatives and integrals and some known results useful in the forthcoming analysis. In Section 3, we first prove an auxiliary result which plays a key role in transforming the given problem into a fixed point problem. Then, by applying Banach's contraction mapping principle, the existence of a unique solution for Problem (1) is shown. Three existence results for Problem (1) are proved by using the fixed point theorems due to Krasnosel'skiĭ and Schaefer and the nonlinear alternative of the Leray–Schauder type. The methods employed to establish the desired results are based on the well-known tools of fixed point theory. Such methods are effectively applied for solving a variety of problems appearing in applied and mathematical sciences such as equilibrium, optimization, economic and variational inequality problems. Moreover, these methods facilitate developing existence theory for initial and boundary value problems. We investigate the existence of solutions for the multivalued analogue of Problem (1) in Section 4. Precisely, the existence results for convex- and nonconvex-valued multifunctions involved in the inclusion problem given by (19) formulated in Section 4 are respectively established by applying the Leray–Schauder nonlinear alternative and Covitz–Nadler's fixed point theorem for multivalued maps. It is imperative to note that the multivalued (inclusion) problems are helpful in the investigation of dynamical systems, stochastic processes, queuing networks, climate control, etc. Examples are provided to illustrate the applicability of the results obtained in Sections 3 and 4.

2. Preliminaries

In this section, we recall some basic concepts related to our study.

Definition 1 ([33,34])**.** *Let $\sigma \in (0,1]$ and $\alpha \in \mathbb{R}^+$. Then the generalized proportional fractional integral of order α of $f \in L^1([c,d],\mathbb{R})$ with respect to ψ is given by*

$$(I_{c^+}^{\alpha,\sigma,\psi} f)(\vartheta) = \frac{1}{\sigma^\alpha \Gamma(\alpha)} \int_{c^+}^{\vartheta} e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(s))}(\psi(\vartheta)-\psi(s))^{\alpha-1}\psi'(s)f(s)ds, \ \vartheta > c. \quad (2)$$

Remark 1. *We notice that Definition 1 unifies several known definitions of fractional integrals for $\sigma = 1$, for example, it corresponds to*

- *The Riemann–Liouville fractional integral when $\psi(t) = t$;*
- *The Hadamard fractional integral when $\psi(t) = \log t$;*
- *The Katugampola fractional integral when $\psi(t) = \frac{t^\alpha}{\alpha}, \alpha > 0$.*

Definition 2 ([33,34])**.** *For $\sigma \in (0,1]$ and $\alpha \in \mathbb{R}^+$ and $\psi \in C([c,d],\mathbb{R})$ with $\psi'(\vartheta) > 0$, the generalized proportional fractional derivative of order α for $f \in C([c,d],\mathbb{R})$ with respect to ψ is given by*

$$(D_{c^+}^{\alpha,\sigma,\psi} f)(\vartheta) = \frac{D^{n,\sigma,\psi}}{\sigma^{n-\alpha}\Gamma(n-\alpha)} \int_{c^+}^{\vartheta} e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(s))}(\psi(\vartheta)-\psi(s))^{n-\alpha-1}\psi'(s)f(s)ds, \ \vartheta > c, \quad (3)$$

where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of the real number α.

Lemma 1 ([33,34])**.** *If $\alpha \geq 0, \beta > 0$ and $\sigma > 0$, then*

(i) $$(I_{c^+}^{\alpha,\sigma,\psi} e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(c))}(\psi(\vartheta)-\psi(c))^{\beta-1})(\vartheta) = \frac{\Gamma(\beta)}{\sigma^\alpha\Gamma(\beta+\alpha)} e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(c))}(\psi(\vartheta)-\psi(c))^{\beta+\alpha-1};$$

(ii) $$(D_{c^+}^{\alpha,\sigma,\psi} e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(c))}(\psi(\vartheta)-\psi(c))^{\beta-1})(\vartheta) = \frac{\sigma^\alpha\Gamma(\beta)}{\Gamma(\beta-\alpha)} e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(c))}(\psi(\vartheta)-\psi(c))^{\beta-\alpha-1}.$$

Lemma 2 ([33,34])**.** *Suppose that $\sigma \in (0,1]$, $\alpha > 0$ and $\beta > 0$. Then, if f is continuous and defined for $\vartheta \geq c$, we have*

$$I_{c+}^{\alpha,\sigma,\psi}(I_{c+}^{\beta,\sigma,\psi} f)(\vartheta) = I_{c+}^{\beta,\sigma,\psi}(I_{c+}^{\alpha,\sigma,\psi} f)(\vartheta) = (I_{c+}^{\alpha+\beta,\sigma,\psi} f)(\vartheta).$$

Lemma 3 ([33,34])**.** *Suppose that $\sigma \in (0,1]$, $0 \leq n < [\alpha] < 1$ with $n \in \mathbb{N}$ and $f \in L^1((c,d),\mathbb{R})$. Then*

$$D_{c+}^{n,\sigma,\psi}(I_{c+}^{\alpha,\sigma,\psi} f)(\vartheta) = (I_{c+}^{\alpha-n,\sigma,\psi} f)(\vartheta).$$

Definition 3 ([38]). *Let $f, \psi \in C^m([c,d],\mathbb{R})$ and ψ be positive and strictly increasing with $\psi'(\vartheta) \neq 0$, for all $\vartheta \in [c,d]$. The ψ-Hilfer generalized proportional fractional derivative of order α and type β for f with respect to another function ψ is defined by*

$$(D_{c+}^{\alpha,\beta,\sigma,\psi} f)(\vartheta) = (I_{c+}^{\beta(n-\alpha),\sigma,\psi}(D^{n,\sigma,\psi}) I_{c+}^{(1-\beta)(n-\alpha),\sigma,\psi} f)(\vartheta), \tag{4}$$

where $n-1 < \alpha < n$, $0 \leq \beta \leq 1$ with $n \in \mathbb{N}$ and $\sigma \in (0,1]$. In addition, $D^{\sigma,\psi} f(\vartheta) = (1-\sigma) f(\vartheta) + \sigma \frac{f'(\vartheta)}{\psi'(\vartheta)}$ and $I_{c+}^{(\cdot)}$ is the generalized proportional fractional integral operator defined in (2).

Remark 2. *From the above definition, we remark that*

$$(D_{c+}^{\alpha,\beta,\sigma,\psi} f)(\vartheta) = \begin{cases} (D^{n,\sigma,\psi}) I_{c+}^{(n-\alpha),\sigma,\psi} f)(\vartheta), & \beta = 0, \\ (I_{c+}^{(n-\alpha),\sigma,\psi})(D^{n,\sigma,\psi} f)(\vartheta), & \beta = 1. \end{cases}$$

This means that $D_{c+}^{\alpha,\beta,\sigma,\psi}$ interpolates between the Riemann–Liouville and Caputo generalized proportional fractional derivatives.

Remark 3 ([38]). *The ψ-Hilfer generalized proportional fractional derivative $D_{c+}^{\alpha,\beta,\sigma,\psi}$ is equivalent to*

$$(D^{\alpha,\beta,\sigma,\psi} f)(\vartheta) = I_{c+}^{\beta(n-\alpha),\sigma,\psi}(D^{n,\sigma,\psi})(I_{c+}^{(1-\beta)(n-\alpha),\sigma,\psi} f)(\vartheta) = (I^{\beta(n-\alpha),\sigma,\psi} D_{c+}^{\gamma,\sigma,\psi} f)(\vartheta),$$

where $\gamma = \alpha + \beta(n-\alpha)$.

Remark 4 ([38]). *It is assumed that the parameters α, β and γ (involved in the above definitions) satisfy the relations:*

$$\gamma = \alpha + \beta(n-\alpha), \;\; n-1 < \alpha, \;\; \gamma \leq n, \;\; 0 \leq \beta \leq 1,$$

and

$$\gamma \geq \alpha, \;\; \gamma > \beta, \;\; n-\gamma < n - \beta(n-\alpha).$$

Lemma 4 ([38]). *Let $n-1 < \alpha < n$, with $n \in \mathbb{N}$, $0 \leq \beta \leq 1$, $\sigma \in (0,1]$ and $\gamma = \alpha + \beta(n-\alpha)$. For $\eta \in \mathbb{R}$ such that $\eta > n$, we have*

$$D_{c+}^{\alpha,\beta,\sigma,\psi} e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(c))}(\psi(\vartheta)-\psi(c))^{\eta-1} = \frac{\sigma^\alpha \Gamma(\eta)}{\Gamma(\eta-\alpha)} e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(c))}(\psi(\vartheta)-\psi(c))^{\eta-\alpha-1}.$$

Lemma 5 ([38]). *Let $n-1 < \alpha < n$, $n \in \mathbb{N}$, $\sigma \in (0,1]$, $0 \leq \beta \leq 1$ and $\gamma = \alpha + \beta(n-\alpha)$ be such that $n-1 < \gamma < n$. If $f \in C([c,d],\mathbb{R})$ and $I_{c+}^{n-\gamma,\sigma,\psi} f \in C^n([c,d],\mathbb{R})$, then*

$$I_{c+}^{\alpha,\sigma,\psi} D_{c+}^{\alpha,\beta,\sigma,\psi} f(\vartheta) = f(\vartheta) - \sum_{k=1}^{n} \frac{e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(c))}(\psi(\vartheta)-\psi(c))^{\gamma-k}}{\sigma^{\gamma-k}\Gamma(\gamma-k+1)} (I_{c+}^{k-\gamma,\sigma,\psi} f)(c).$$

3. Main Results

In this section, we establish existence and uniqueness results for the ψ-Hilfer generalized proportional fractional boundary value problem given by (1). Let us first prove an auxiliary lemma concerning a linear variant of the boundary value problem in (1), which facilitates transforming the given nonlinear problem into an equivalent integral equation. In our case, $n = [\alpha] + 1 = 2$ and $\gamma = \alpha + (2-\alpha)\beta$.

Lemma 6. *Let $h \in C([c,d],\mathbb{R})$ and*

$$\Lambda := \frac{e^{\frac{\sigma-1}{\sigma}(\psi(d)-\psi(c))}(\psi(d)-\psi(c))^{\gamma-1}}{\sigma^{\gamma-1}\Gamma(\gamma)} - \sum_{j=1}^{m}\eta_j \frac{e^{\frac{\sigma-1}{\sigma}(\psi(\xi_j)-\psi(c))}(\psi(\xi_j)-\psi(c))^{\gamma-1}}{\sigma^{\gamma-1}\Gamma(\gamma)}$$
$$-\sum_{i=1}^{n}\zeta_i \frac{e^{\frac{\sigma-1}{\sigma}(\psi(\theta_i)-\psi(c))}(\psi(\theta_i)-\psi(c))^{\gamma+\phi_1-1}}{\sigma^{\gamma+\phi_1-1}\Gamma(\gamma+\phi_1)}$$
$$-\sum_{k=1}^{r}\lambda_k \frac{e^{\frac{\sigma-1}{\sigma}(\psi(\mu_k)-\psi(c))}(\psi(\mu_k)-\psi(c))^{\gamma-\delta_k-1}}{\sigma^{\gamma-\delta_k-1}\Gamma(\gamma-\delta_k)} \neq 0. \quad (5)$$

Then w is a solution of the following linear ψ-Hilfer generalized proportional fractional boundary value problem

$$\begin{cases} D_{c^+}^{\alpha,\beta,\sigma,\psi} w(\vartheta) = h(\vartheta), \quad \vartheta \in [c,d], \\ w(c) = 0, \\ w(d) = \sum_{j=1}^{m}\eta_j w(\xi_j) + \sum_{i=1}^{n}\zeta_i\, I_{c^+}^{\phi_i,\sigma,\psi} w(\theta_i) + \sum_{k=1}^{r}\lambda_k\, D_{c^+}^{\delta_k,\beta,\sigma,\psi} w(\mu_k), \end{cases} \quad (6)$$

if and only if

$$w(\vartheta) = I_{c^+}^{\alpha,\sigma,\psi} h(\vartheta) + \frac{e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(c))}(\psi(\vartheta)-\psi(c))^{\gamma-1}}{\Lambda\sigma^{\gamma-1}\Gamma(\gamma)}\Bigg\{\sum_{j=1}^{m}\eta_j I^{\alpha,\sigma,\psi}h(\xi_j)$$
$$+\sum_{i=1}^{n}\zeta_i I^{\alpha+\phi_i,\sigma,\psi}h(\theta_i) + \sum_{k=1}^{r}\lambda_k I^{\alpha-\delta_k,\sigma,\psi}h(\mu_k) - I^{\alpha,\sigma,\psi}h(\vartheta)\Bigg\}, \vartheta \in [c,d]. \quad (7)$$

Proof. From Lemma 5 with $n = 2$, we have

$$I_{c+}^{\alpha,\sigma,\psi} D_{c+}^{\alpha,\beta,\sigma,\psi} w(\vartheta) = w(\vartheta) - \frac{e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(c))}(\psi(\vartheta)-\psi(c))^{\gamma-1}}{\sigma^{\gamma-1}\Gamma(\gamma)}(I_{c+}^{1-\gamma,\sigma,\psi}w)(c)$$
$$-\frac{e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(c))}(\psi(\vartheta)-\psi(c))^{\gamma-2}}{\sigma^{\gamma-2}\Gamma(\gamma-1)}(I_{c+}^{2-\gamma,\sigma,\psi}w)(c),$$

which implies that

$$w(\vartheta) = I_{c+}^{\alpha,\sigma,\psi} h(\vartheta) + c_0 \frac{e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(c))}(\psi(\vartheta)-\psi(c))^{\gamma-1}}{\sigma^{\gamma-1}\Gamma(\gamma)}$$
$$+c_1 \frac{e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(c))}(\psi(\vartheta)-\psi(c))^{\gamma-2}}{\sigma^{\gamma-2}\Gamma(\gamma-1)}, \quad (8)$$

where $c_0 = (I_{c+}^{1-\gamma,\sigma,\psi}w)(c)$ and $c_1 = (I_{c+}^{2-\gamma,\sigma,\psi}w)(c)$. Using the first boundary condition ($w(c) = 0$) in (8) yields $c_1 = 0$, since $\gamma \in [\alpha, 2]$. In consequence, (8) takes the form:

$$w(\vartheta) = I_{c+}^{\alpha,\sigma,\psi} h(\vartheta) + c_0 \frac{e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(c))}(\psi(\vartheta)-\psi(c))^{\gamma-1}}{\sigma^{\gamma-1}\Gamma(\gamma)}. \quad (9)$$

From (9), we have

$$w(\xi_j) = I^{\alpha,\sigma,\psi} h(\xi_j) + c_0 \frac{e^{\frac{\sigma-1}{\sigma}(\psi(\xi_j)-\psi(c))}(\psi(\xi_j)-\psi(c))^{\gamma-1}}{\sigma^{\gamma-1}\Gamma(\gamma)},$$
$$I_{c^+}^{\phi_i,\sigma,\psi} w(\theta_i) = I^{\phi_i+\alpha,\sigma,\psi} h(\theta_i) + c_0 \frac{e^{\frac{\sigma-1}{\sigma}(\psi(\theta_i)-\psi(c))}(\psi(\theta_i)-\psi(c))^{\phi_i+\gamma-1}}{\sigma^{\phi_i+\gamma-1}\Gamma(\phi_i+\gamma)},$$

$$D_{c^+}^{\delta_k,\beta,\sigma,\psi}w(\mu_k) = I^{\alpha-\delta_k,\sigma,\psi}h(\mu_k)+c_0\frac{e^{\frac{\sigma-1}{\sigma}(\psi(\mu_k)-\psi(c))}(\psi(\mu_k)-\psi(c))^{\gamma-\delta_k-1}}{\sigma^{\gamma-\delta_k-1}\Gamma(\gamma-\delta_k)}.$$

Inserting the above values into the second boundary condition, $w(d)=\sum_{j=1}^{m}\eta_j w(\xi_j)+\sum_{i=1}^{n}\zeta_i\, I_{c^+}^{\phi_i,\sigma,\psi}w(\theta_i)+\sum_{k=1}^{r}\lambda_k\, D_{c^+}^{\delta_k,\beta,\sigma,\psi}w(\mu_k)$, and using the notation in (5), we obtain

$$c_0=\frac{1}{\Lambda}\left\{\sum_{j=1}^{m}I^{\alpha,\sigma,\psi}h(\xi_j)+\sum_{i=1}^{n}\zeta_i I^{\alpha+\phi_i,\sigma,\psi}h(\theta_i)+\sum_{k=1}^{r}\lambda_k I^{\alpha-\delta_k,\sigma,\psi}h(\mu_k)-I^{\alpha,\sigma,\psi}h(\vartheta)\right\}.$$

Substituting the value of c_0 in (9) yields Equation (7), as desired. The converse of the lemma can be obtained easily, by direct computation. The proof is finished. □

Let $X=C([c,d],\mathbb{R})$ be the Banach space of all continuous functions from $[c,d]$ to $\mathbb{R}$ endowed with the norm $\|w\|:=\max_{\vartheta\in[c,d]}|w(\vartheta)|$.
In view of Lemma 6, we introduce an operator $\mathcal{F}:X\to X$ associated with Problem (1) as

$$\begin{aligned}\mathcal{F}(w)(\vartheta) &= I_{c^+}^{\alpha,\sigma,\psi}f(\vartheta,w(\vartheta))+\frac{e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(c))}(\psi(\vartheta)-\psi(c))^{\gamma-1}}{\Lambda\sigma^{\gamma-1}\Gamma(\gamma)}\left\{\sum_{j=1}^{m}\eta_j I^{\alpha,\sigma,\psi}f(\xi_j,w(\xi_j))\right.\\ &+\sum_{i=1}^{n}\zeta_i I^{\alpha+\phi_i,\sigma,\psi}f(\theta_i,w(\theta_i))+\sum_{k=1}^{r}\lambda_k I^{\alpha-\delta_k,\sigma,\psi}f(\mu_k,w(\mu_k))\\ &\left.-I^{\alpha,\sigma,\psi}f(d,w(d))\right\},\ \vartheta\in[c,d].\end{aligned}\tag{10}$$

In the sequel, we use the notation:

$$\begin{aligned}\Omega &= \frac{(\psi(d)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)}+\frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\sigma^{\gamma-1}\Gamma(\gamma)}\left\{\sum_{i=j}^{m}\frac{|\eta_j|(\psi(\xi_j)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)}\right.\\ &+\sum_{i=1}^{n}\frac{|\zeta_i|(\psi(\theta_i)-\psi(c))^{\alpha+\phi_i}}{\sigma^{\alpha}\Gamma(\alpha+\phi_i+1)}+\sum_{k=1}^{r}\frac{|\lambda_k|(\psi(\mu_k)-\psi(c))^{\alpha-\delta_k}}{\sigma^{\alpha}\Gamma(\alpha-\delta_k+1)}\\ &\left.+\frac{(\psi(d)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)}\right\}.\end{aligned}\tag{11}$$

3.1. Uniqueness Result

Here, the existence of a unique solution for the nonlinear ψ-Hilfer generalized proportional fractional boundary value problem in (1) is proved via Banach's contraction mapping principle [40].

Theorem 1. *Suppose that:*
(H_1) *There exists a constant $L>0$ such that, for all $\vartheta\in[c,d]$ and $u_i\in\mathbb{R}$, $i=1,2$,*

$$|f(\vartheta,u_1)-f(\vartheta,u_2)| \le L|u_1-u_2|.$$

If $L\Omega<1$, where Ω is defined by (11), then the nonlinear ψ-Hilfer generalized proportional fractional boundary value problem in (1) has a unique solution on $[c,d]$.

Proof. Let $N=\max_{\vartheta\in[c,d]}|f(\vartheta,0)|<\infty$. By assumption (H_1), we obtain

$$|f(\vartheta,w(\vartheta))|\le L|w(\vartheta)|+|f(\vartheta,0)|\le L\|w\|+N\le Lr+N.\tag{12}$$

We verify the hypothesis of Banach's contraction mapping principle in two steps.

Step I. Let $B_r = \{w \in X : \|w\| < r\}$ with $r \geq N\Omega/(1 - L\Omega)$. We show that $\mathcal{F}(B_r) \subset B_r$. For $w \in B_r$, by using the fact that $0 < e^{\frac{\sigma-1}{\sigma}(\psi(\cdot)-\psi(\cdot))} \leq 1$, we have

$$
\begin{aligned}
& |\mathcal{F}(w)(\vartheta)| \\
\leq\ & I_{c^+}^{\alpha,\sigma,\psi}|f(\vartheta, w(\vartheta))| + \frac{(\psi(d) - \psi(c))^{\gamma-1}}{|\Lambda|\sigma^{\gamma-1}\Gamma(\gamma)} \Bigg\{ \sum_{j=1}^{m} |\eta_j| I^{\alpha,\sigma,\psi}|f(\xi_j, w(\xi_j))| \\
& + \sum_{i=1}^{n} |\zeta_i| I^{\alpha+\phi_i,\sigma,\psi}|f(\theta_i, w(\theta_i))| + \sum_{k=1}^{r} |\lambda_k| I^{\alpha-\delta_k,\sigma,\psi}|f(\mu_k, w(\mu_k))| + I^{\alpha,\sigma,\psi}|f(d, w(d))| \Bigg\} \\
\leq\ & \frac{(\psi(d) - \psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)}(L\|w\| + N) + \frac{(\psi(d) - \psi(c))^{\gamma-1}}{|\Lambda|\sigma^{\gamma-1}\Gamma(\gamma)} \Bigg\{ \sum_{i=j}^{m} \frac{|\eta_j|(\psi(\xi_j) - \psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)} \\
& + \sum_{i=1}^{n} \frac{|\zeta_i|(\psi(\theta_i) - \psi(c))^{\alpha+\phi_i}}{\sigma^{\alpha}\Gamma(\alpha+\phi_i+1)} + \sum_{k=1}^{r} \frac{|\lambda_k|(\psi(\mu_k) - \psi(c))^{\alpha-\delta_k}}{\sigma^{\alpha}\Gamma(\alpha-\delta_k+1)} \\
& + \frac{(\psi(d) - \psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)} \Bigg\} (L\|w\| + N) \\
\leq\ & \Bigg[\frac{(\psi(d) - \psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)} + \frac{(\psi(d) - \psi(c))^{\gamma-1}}{|\Lambda|\sigma^{\gamma-1}\Gamma(\gamma)} \Bigg\{ \sum_{i=j}^{m} \frac{|\eta_j|(\psi(\xi_j) - \psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)} \\
& + \sum_{i=1}^{n} \frac{|\zeta_i|(\psi(\theta_i) - \psi(c))^{\alpha+\phi_i}}{\sigma^{\alpha}\Gamma(\alpha+\phi_i+1)} + \sum_{k=1}^{r} \frac{|\lambda_k|(\psi(\mu_k) - \psi(c))^{\alpha-\delta_k}}{\sigma^{\alpha}\Gamma(\alpha-\delta_k+1)} \\
& + \frac{(\psi(d) - \psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)} \Bigg\} \Bigg] (Lr + N) \\
=\ & \Omega(Lr + N) \leq r.
\end{aligned}
$$

Consequently,

$$
\|\mathcal{F}(w)\| = \max_{\vartheta \in [c,d]} |\mathcal{F}(w)(\vartheta)| \leq r,
$$

which means that $\mathcal{F}(B_r) \subset B_r$.

Step II. Now we will show that the operator $\mathcal{F}$ is a contraction. Let $w_1, w_2 \in X$. Then, for any $\vartheta \in [c, d]$, we have

$$
\begin{aligned}
& |\mathcal{F}(w_2)(\vartheta) - \mathcal{F}(w_1)(\vartheta)| \\
\leq\ & I_{c^+}^{\alpha,\sigma,\psi}|f(\vartheta, w_2(\vartheta)) - f(\vartheta, w_1(\vartheta))| + \frac{(\psi(d) - \psi(c))^{\gamma-1}}{|\Lambda|\sigma^{\gamma-1}\Gamma(\gamma)} \\
& \times \Bigg\{ \sum_{j=1}^{m} |\eta_j| I^{\alpha,\sigma,\psi}|f(\xi_j, w_2(\xi_j)) - f(\xi_j, w_1(\xi_j))| \\
& + \sum_{i=1}^{n} |\zeta_i| I^{\alpha+\phi_i,\sigma,\psi}|f(\theta_i, w_2(\theta_i)) - f(\theta_i, w_1(\theta_i))| \\
& + \sum_{k=1}^{r} |\lambda_k| I^{\alpha-\delta_k,\sigma,\psi}|f(\mu_k, w_2(\mu_k)) - f(\mu_k, w_1(\mu_k))| \\
& + I^{\alpha,\sigma,\psi}|f(d, w_2(d)) - f(d, w_1(d))| \Bigg\} \\
\leq\ & \Bigg[\frac{(\psi(d) - \psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)} + \frac{(\psi(d) - \psi(c))^{\gamma-1}}{|\Lambda|\sigma^{\gamma-1}\Gamma(\gamma)} \Bigg\{ \sum_{i=j}^{m} \frac{|\eta_j|(\psi(\xi_j) - \psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)} \\
& + \sum_{i=1}^{n} \frac{|\zeta_i|(\psi(\theta_i) - \psi(c))^{\alpha+\phi_i}}{\sigma^{\alpha}\Gamma(\alpha+\phi_i+1)} + \sum_{k=1}^{r} \frac{|\lambda_k|(\psi(\mu_k) - \psi(c))^{\alpha-\delta_k}}{\sigma^{\alpha}\Gamma(\alpha-\delta_k+1)}
\end{aligned}
$$

$$
\begin{aligned}
& \left. \left. + \frac{(\psi(d)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)} \right\} \right] L\|w_2-w_1\| \\
= \quad & L\Omega\|w_2-w_1\|.
\end{aligned}
$$

Hence,

$$
\|\mathcal{F}(w_2)-\mathcal{F}(w_1)\| = \max_{\vartheta\in[c,d]} |\mathcal{F}(w_2)(\vartheta)-\mathcal{F}(w_1)(\vartheta)| \leq L\Omega\|w_2-w_1\|,
$$

which shows that the operator $\mathcal{F}$ is a contraction, in view of the assumption $L\Omega < 1$. Therefore, the operator $\mathcal{F}$, by Banach's contraction mapping principle, has a unique fixed point, and thus, the ψ-Hilfer generalized proportional fractional boundary value problem in (1) has a unique solution on $[c,d]$. The proof is completed. □

3.2. Existence Results

In this subsection, we establish three existence results for the ψ-Hilfer generalized proportional fractional boundary value problem in (1). The first existence result is based on the fixed point theorem of Krasnosel'skiĭ [41].

Theorem 2. *Let $f : [c,d]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying (H_1). In addition, we assume that:*

(H_2) *There exists a continuous function $\phi\in C([c,d],\mathbb{R}^+)$ such that*

$$
|f(\vartheta,u)| \leq \phi(\vartheta), \;\; \text{for each} \;\; (\vartheta,u)\in[c,d]\times\mathbb{R}.
$$

Then the nonlinear ψ-Hilfer generalized proportional fractional boundary value problem in (1) has at least one solution on $[c,d]$, provided that

$$
\begin{aligned}
\Omega_1 := \; & \frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\sigma^{\gamma-1}\Gamma(\gamma)} \left\{ \sum_{i=j}^{m} \frac{|\eta_j|(\psi(\xi_j)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)} + \sum_{i=1}^{n} \frac{|\zeta_i|(\psi(\theta_i)-\psi(c))^{\alpha+\phi_i}}{\sigma^{\alpha}\Gamma(\alpha+\phi_i+1)} \right. \\
& \left. + \sum_{k=1}^{r} \frac{|\lambda_k|(\psi(\mu_k)-\psi(c))^{\alpha-\delta_k}}{\sigma^{\alpha}\Gamma(\alpha-\delta_k+1)} + \frac{(\psi(d)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)} \right\} L < 1. \qquad (13)
\end{aligned}
$$

Proof. Consider a closed ball

$$
B_\rho = \{w\in C([c,d],\mathbb{R}) : \|w\|\leq\rho\},
$$

where $\|\phi\| = \sup_{\vartheta\in[c,d]}|\phi(\vartheta)|$, $\rho\geq\Omega\|\phi\|$ and Ω is given by (11). We split the operator $\mathcal{F}$ defined by (10) on B_ρ to X as $\mathcal{F}=\mathcal{F}_1+\mathcal{F}_2$, where

$$
(\mathcal{F}_1 w)(\vartheta) \;\; = \;\; I_{c^+}^{\alpha,\sigma,\psi} f(\vartheta,w(\vartheta)), \;\; \vartheta\in[c,d],
$$

and

$$
\begin{aligned}
(\mathcal{F}_2 w)(\vartheta) \;\; = \;\; & \frac{e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(c))}(\psi(\vartheta)-\psi(c))^{\gamma-1}}{\Lambda\sigma^{\gamma-1}\Gamma(\gamma)} \left\{ \sum_{j=1}^{m} \eta_j I^{\alpha,\sigma,\psi} f(\xi_j,w(\xi_j)) \right. \\
& + \sum_{i=1}^{n} \zeta_i I^{\alpha+\phi_i,\sigma,\psi} f(\theta_i,w(\theta_i)) + \sum_{k=1}^{r} \lambda_k I^{\alpha-\delta_k,\sigma,\psi} f(\mu_k,w(\mu_k)) \\
& \left. - I^{\alpha,\sigma,\psi} f(d,w(d)) \right\}, \;\; \vartheta\in[c,d].
\end{aligned}
$$

For any $w, y \in B_\rho$, we have

$$\begin{aligned}
& |(\mathcal{F}_1 w)(\vartheta) + (\mathcal{F}_2 y)(\vartheta)| \\
\leq\; & I_{c^+}^{\alpha,\sigma,\psi}|f(d,w(d))| + \frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\sigma^{\gamma-1}\Gamma(\gamma)}\Bigg\{\sum_{j=1}^{m}|\eta_j| I^{\alpha,\sigma,\psi}|f(\xi_j,y(\xi_j))| \\
& + \sum_{i=1}^{n}|\zeta_i| I^{\alpha+\phi_i,\sigma,\psi}|f(\theta_i,y(\theta_i))| + \sum_{k=1}^{r}|\lambda_k| I^{\alpha-\delta_k,\sigma,\psi}|f(\mu_k,y(\mu_k))| \\
& + I^{\alpha,\sigma,\psi}|f(d,y(d))|\Bigg\} \\
\leq\; & \Bigg[\frac{(\psi(d)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)} + \frac{(\psi(\vartheta)-\psi(c))^{\gamma-1}}{|\Lambda|\sigma^{\gamma-1}\Gamma(\gamma)}\Bigg\{\sum_{i=j}^{m}\frac{|\eta_j|(\psi(\xi_j)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)} \\
& + \sum_{i=1}^{n}\frac{|\zeta_i|(\psi(\theta_i)-\psi(c))^{\alpha+\phi_i}}{\sigma^{\alpha}\Gamma(\alpha+\phi_i+1)} + \sum_{k=1}^{r}\frac{|\lambda_k|(\psi(\mu_k)-\psi(c))^{\alpha-\delta_k}}{\sigma^{\alpha}\Gamma(\alpha-\delta_k+1)} \\
& + \frac{(\psi(d)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)}\Bigg\}\Bigg]\|\phi\| \\
=\; & \Omega\|\phi\| \leq \rho.
\end{aligned}$$

Hence $\|\mathcal{F}_1 w + \mathcal{F}_2 y\| \leq \rho$, which shows that $\mathcal{F}_1 w + \mathcal{F}_2 y \in B_\rho$. We can prove easily with the help of (13) that the operator $\mathcal{F}_2$ is a contraction mapping. Note that the operator $\mathcal{F}_1$ is continuous since f is continuous. In addition, since

$$\|\mathcal{F}_1 w\| \leq \frac{(\psi(d)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)}\|\phi\|,$$

$\mathcal{F}_1$ is uniformly bounded on B_ρ.

Finally, we have to prove that the operator $\mathcal{F}_1$ is completely continuous. For this, let $\vartheta_1, \vartheta_2 \in [c,d]$, $\vartheta_1 < \vartheta_2$. Then we have

$$\begin{aligned}
& |\mathcal{F}_1 w(\vartheta_2) - \mathcal{F}_1 w(\vartheta_1)| \\
\leq\; & \frac{1}{\sigma^{\alpha}\Gamma(\alpha)}\Bigg|\int_{c^+}^{\vartheta_2} e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta_2)-\psi(c))}(\psi(\vartheta_2)-\psi(c))^{\alpha-1}\psi'(s)f(s,w(s))ds \\
& - \int_{c^+}^{\vartheta_1} e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta_1)-\psi(c))}(\psi(\vartheta_1)-\psi(c))^{\alpha-1}\psi'(s)f(s,w(s))ds\Bigg| \\
\leq\; & \frac{1}{\sigma^{\alpha}\Gamma(\alpha)}\Bigg|\int_{c^+}^{\vartheta_1}\Big[(\psi(\vartheta_2)-\psi(c))^{\alpha-1} - (\psi(\vartheta_1)-\psi(c))^{\alpha-1}\Big]\psi'(s)f(s,w(s))ds\Bigg| \\
& + \frac{1}{\sigma^{\alpha}\Gamma(\alpha)}\Bigg|\int_{\vartheta_1}^{\vartheta_2}(\psi(\vartheta_2)-\psi(c))^{\alpha-1}\psi'(s)f(s,w(s))ds\Bigg| \\
\leq\; & \frac{\|\phi\|}{\sigma^{\alpha}\Gamma(\alpha+1)}\Big[2(\psi(\vartheta_2-\psi(\vartheta_1))^{\alpha} + |(\psi(\vartheta_2)-\psi(c))^{\alpha} - (\psi(\vartheta_1)-\psi(c))^{\alpha}|\Big],
\end{aligned}$$

which tends to zero, independently of $w \in B_\rho$, as $\vartheta_1 \to \vartheta_2$. Thus, $\mathcal{F}_1$ is equicontinuous, and by the Arzelá–Ascoli Theorem, it is compact on B_ρ. Thus, the hypotheses of Krasnosel'skiĭ's fixed point theorem [41] are satisfied, and hence, its conclusion implies that there exists at least one solution for the nonlinear ψ-Hilfer generalized proportional fractional boundary value problem in (1) on $[c,d]$, which completes the proof. □

Our next existence result relies on Schaefer's fixed point theorem [42].

Theorem 3. *Let $f : [c,d] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumption:*
(H_3) *There exists a real constant $M > 0$ such that for all $\vartheta \in [c,d], u \in \mathbb{R}$,*

$$|f(\vartheta,u)| \ \leq M.$$

Then there exists at least one solution for the nonlinear ψ-Hilfer generalized proportional fractional boundary value problem in (1) on $[c,d]$.

Proof. We will give the proof in two steps.
Step I. The operator $\mathcal{F} : X \to X$ is completely continuous.
To prove the continuity of $\mathcal{F}$, let $\{w_n\}$ be a sequence such that $w_n \to w$ in X. Then, for each $\vartheta \in [c,d]$, we have

$$\begin{aligned} |\mathcal{F}(w_n)(\vartheta) - \mathcal{F}(w)(\vartheta)| \ \leq \ & I_{c^+}^{\alpha,\sigma,\psi}|f(\vartheta, x_n(\vartheta)) - f(\vartheta, w(\vartheta))| + \frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\sigma^{\gamma-1}\Gamma(\gamma)} \\ & \times\Bigg\{ \sum_{j=1}^{m} |\eta_j| I^{\alpha,\sigma,\psi}|f(\xi_j, w_n(\xi_j)) - f(\xi_j, w(\xi_j))| \\ & + \sum_{i=1}^{n} |\zeta_i| I^{\alpha+\phi_i,\sigma,\psi}|f(\theta_i, w_n(\theta_i)) - f(\theta_i, w(\theta_i))| \\ & + \sum_{k=1}^{r} |\lambda_k| I^{\alpha-\delta_k,\sigma,\psi}|f(\mu_k, w_n(\mu_k)) - f(\mu_k, w(\mu_k))| \\ & + I^{\alpha,\sigma,\psi}|f(d, w_n(d)) - f(d, w(d))| \Bigg\}. \end{aligned}$$

Since $|f(s,w_n(s)) - f(s,w(s))| \to 0$ *as* $w_n \to w$ due to continuity of f, therefore

$$\|\mathcal{F}(w_n) - \mathcal{F}(w)\| \to 0 \ as \ w_n \to w,$$

which proves that $\mathcal{F}$ is continuous.

In the next step, we show that the operator $\mathcal{F}$ maps bounded sets into bounded sets in X. For $R > 0$, let $B_R = \{w \in X : \|w\| \leq R\}$. Then, for $\vartheta \in [c,d]$, we have

$$\begin{aligned} |\mathcal{F}(w)(\vartheta)| \ \leq \ & I_{c^+}^{\alpha,\sigma,\psi}|f(\vartheta, w(\vartheta))| + \frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\sigma^{\gamma-1}\Gamma(\gamma)} \Bigg\{ \sum_{j=1}^{m} |\eta_j| I^{\alpha,\sigma,\psi}|f(\xi_j, w(\xi_j))| \\ & + \sum_{i=1}^{n} |\zeta_i| I^{\alpha+\phi_i,\sigma,\psi}|f(\theta_i, w(\theta_i))| + \sum_{k=1}^{r} |\lambda_k| I^{\alpha-\delta_k,\sigma,\psi}|f(\mu_k, w(\mu_k))| \\ & + I^{\alpha,\sigma,\psi}|f(d, w(d))| \Bigg\} \\ \leq \ & \Bigg[\frac{(\psi(d)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)} + \frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\sigma^{\gamma-1}\Gamma(\gamma)} \Bigg\{ \sum_{i=j}^{m} \frac{|\eta_j|(\psi(\xi_j)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)} \\ & + \sum_{i=1}^{n} \frac{|\zeta_i|(\psi(\theta_i)-\psi(c))^{\alpha+\phi_i}}{\sigma^{\alpha}\Gamma(\alpha+\phi_i+1)} + \sum_{k=1}^{r} \frac{|\lambda_k|(\psi(\mu_k)-\psi(c))^{\alpha-\delta_k}}{\sigma^{\alpha}\Gamma(\alpha-\delta_k+1)} \\ & + \frac{(\psi(d)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)} \Bigg\} \Bigg] M, \end{aligned}$$

which, after taking the norm for $\vartheta \in [c,d]$, leads to $\|\mathcal{F}(w)\| \leq \Omega M$.

Finally, we show that bounded sets are mapped into equicontinuous sets by $\mathcal{F}$. For $\vartheta_1, \vartheta_2 \in [c,d]$, $\vartheta_1 < \vartheta_2$ and $u \in B_R$, we obtain

$$\begin{aligned}
&|\mathcal{F}(w)(\vartheta_2) - \mathcal{F}(w)(\vartheta_1)| \\
\leq\ & \frac{1}{\sigma^{\alpha}\Gamma(\alpha)}\left|\int_{c^+}^{\vartheta_1}\left[(\psi(\vartheta_2)-\psi(c))^{\alpha-1}-(\psi(\vartheta_1)-\psi(c))^{\alpha-1}\right]\psi'(s)f(s,w(s))ds\right| \\
&+\frac{1}{\sigma^{\alpha}\Gamma(\alpha)}\left|\int_{\vartheta_1}^{\vartheta_2}(\psi(\vartheta_2)-\psi(c))^{\alpha-1}\psi'(s)f(s,w(s))ds\right| \\
&+\frac{(\psi(\vartheta_2)-\psi(c))^{\gamma-1}-(\psi(\vartheta_1)-\psi(c))^{\gamma-1}}{|\Lambda|\sigma^{\gamma-1}\Gamma(\gamma)}\Bigg\{\sum_{j=1}^{m}|\eta_j|I^{\alpha,\sigma,\psi}|f(\xi_j,y(\xi_j))| \\
&+\sum_{i=1}^{n}|\zeta_i|I^{\alpha+\phi_i,\sigma,\psi}|f(\theta_i,y(\theta_i))|+\sum_{k=1}^{r}|\lambda_k|I^{\alpha-\delta_k,\sigma,\psi}|f(\mu_k,y(\mu_k))| \\
&+I^{\alpha,\sigma,\psi}|f(d,y(d))|\Bigg\} \\
\leq\ & \frac{M}{\sigma^{\alpha}\Gamma(\alpha+1)}\left[2(\psi(\vartheta_2-\psi(\vartheta_1))^{\alpha}+|(\psi(\vartheta_2)-\psi(c))^{\alpha}-(\psi(\vartheta_1)-\psi(c))^{\alpha}|\right] \\
&+\frac{(\psi(\vartheta_2)-\psi(c))^{\gamma-1}-(\psi(\vartheta_1)-\psi(c))^{\gamma-1}}{|\Lambda|\sigma^{\gamma-1}\Gamma(\gamma)}\Bigg\{\sum_{i=j}^{m}\frac{|\eta_j|(\psi(\xi_j)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)} \\
&+\sum_{i=1}^{n}\frac{|\zeta_i|(\psi(\theta_i)-\psi(c))^{\alpha+\phi_i}}{\sigma^{\alpha}\Gamma(\alpha+\phi_i+1)}+\sum_{k=1}^{r}\frac{|\lambda_k|(\psi(\mu_k)-\psi(c))^{\alpha-\delta_k}}{\sigma^{\alpha}\Gamma(\alpha-\delta_k+1)} \\
&+\frac{(\psi(d)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)}\Bigg\}M,
\end{aligned}$$

which tends to zero, independently of $w \in B_R$, as $\vartheta_1 \to \vartheta_2$. Thus, by the Arzelá–Ascoli theorem, the operator $\mathcal{F} : X \to X$ is completely continuous.

Step II. We show that the set $\mathcal{E} = \{w \in X \mid w = \nu\mathcal{F}(w), 0 \leq \nu \leq 1\}$ is bounded. Let $w \in \mathcal{E}$. Then $w = \nu\mathcal{F}(w)$. For any $\vartheta \in [c,d]$, we have $w(\vartheta) = \nu\mathcal{F}(w)(\vartheta)$. As in Step I, it follows with the aid of the hypothesis (H_3) that

$$\begin{aligned}
|w(\vartheta)| \leq\ & \Bigg[\frac{(\psi(d)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)}+\frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\sigma^{\gamma-1}\Gamma(\gamma)}\Bigg\{\sum_{i=j}^{m}\frac{|\eta_j|(\psi(\xi_j)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)} \\
&+\sum_{i=1}^{n}\frac{|\zeta_i|(\psi(\theta_i)-\psi(c))^{\alpha+\phi_i}}{\sigma^{\alpha}\Gamma(\alpha+\phi_i+1)}+\sum_{k=1}^{r}\frac{|\lambda_k|(\psi(\mu_k)-\psi(c))^{\alpha-\delta_k}}{\sigma^{\alpha}\Gamma(\alpha-\delta_k+1)} \\
&+\frac{(\psi(d)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)}\Bigg\}\Bigg]M.
\end{aligned}$$

Hence,

$$\|w\| \leq \Omega M,$$

which means that the set $\mathcal{E}$ is bounded. Thus, the operator $\mathcal{F}$ has at least one fixed point by Schaefer's fixed point theorem [42], which is a solution for the nonlinear ψ-Hilfer generalized proportional fractional boundary value problem in (1) on $[c,d]$. This completes the proof. □

Our last existence result is based on the Leray–Schauder nonlinear alternative [43].

Theorem 4. *Let $f \in C([c,d] \times \mathbb{R}, \mathbb{R})$. In addition, the following assumptions hold:*

(H_4) *There exist* $p \in C([c,d], \mathbb{R}^+)$ *and a continuous nondecreasing function* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $|f(\vartheta, u)| \leq p(\vartheta)\psi(\|u\|)$ *for each* $(\vartheta, u) \in [c,d] \times \mathbb{R}$;

(H_5) *There exists a constant* $K > 0$ *such that*

$$\frac{K}{\Omega\|p\|\psi(K)} > 1,$$

where Ω *is defined by (11).*

Then the nonlinear ψ*-Hilfer generalized proportional fractional boundary value problem in (1) has at least one solution on* $[c,d]$.

Proof. Here, we prove only that there exists an open set $U \subseteq C([c,d], \mathbb{R})$ with $w \neq \mu\mathcal{F}(w)$ for $\mu \in (0,1)$ and $w \in \partial U$, since the operator $\mathcal{F}$ is already shown to be completely continuous in Theorem 3.

Let $w \in C([c,d], \mathbb{R})$ be such that $w = \mu\mathcal{F}(w)$ for some $0 < \mu < 1$. Then, for each $\vartheta \in [c,d]$, we have

$$\begin{aligned}
|w(\vartheta)| \quad &\leq \quad I_{c^+}^{\alpha,\sigma,\psi}|f(\vartheta, w(\vartheta))| + \frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\sigma^{\gamma-1}\Gamma(\gamma)}\Bigg\{\sum_{j=1}^{m}|\eta_j|I^{\alpha,\sigma,\psi}|f(\xi_j, w(\xi_j))| \\
&\quad + \sum_{i=1}^{n}|\zeta_i|I^{\alpha+\phi_i,\sigma,\psi}|f(\theta_i, w(\theta_i))| + \sum_{k=1}^{r}|\lambda_k|I^{\alpha-\delta_k,\sigma,\psi}|f(\mu_k, w(\mu_k))| \\
&\quad + I^{\alpha,\sigma,\psi}|f(d, w(d))|\Bigg\} \\
&\leq \Bigg[\frac{(\psi(d)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)} + \frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\sigma^{\gamma-1}\Gamma(\gamma)}\Bigg\{\sum_{i=j}^{m}\frac{|\eta_j|(\psi(\xi_j)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)} \\
&\quad + \sum_{i=1}^{n}\frac{|\zeta_i|(\psi(\theta_i)-\psi(c))^{\alpha+\phi_i}}{\sigma^{\alpha}\Gamma(\alpha+\phi_i+1)} + \sum_{k=1}^{r}\frac{|\lambda_k|(\psi(\mu_k)-\psi(c))^{\alpha-\delta_k}}{\sigma^{\alpha}\Gamma(\alpha-\delta_k+1)} \\
&\quad + \frac{(\psi(d)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)}\Bigg\}\Bigg]\|p\|\psi(\|w\|).
\end{aligned}$$

Hence, we obtain $\|w\| \leq \Omega\|p\|\psi(\|w\|)$, which implies that

$$\frac{\|w\|}{\Omega\|p\|\psi(\|w\|)} \leq 1.$$

In view of (H_5), there is no solution w such that $\|w\| \neq K$. Let us set

$$\varphi = \{w \in C([c,d], \mathbb{R}) : \|w\| < K\}.$$

Observe that the operator $\mathcal{F} : \overline{\varphi} \to C([c,d], \mathbb{R})$ is continuous and completely continuous. Notice that we cannot find any $w \in \partial\varphi$ satisfying $w = \mu\mathcal{F}(w)$ for some $\mu \in (0,1)$ in view of the definition of φ. In consequence, we deduce that there exists a fixed point $w \in \overline{\varphi}$ for the operator $\mathcal{F}$ by the application of the Leray–Schauder nonlinear alternative [43], which is indeed a solution of Problem (1). This finishes the proof. □

3.3. Illustrative Examples for the Single-Valued Case

Consider the following ψ-Hilfer generalized proportional fractional boundary value problem:

$$\begin{cases} D_{\frac{1}{9}}^{\frac{3}{2},\frac{1}{2},\frac{3}{4},(\vartheta^2+1)} w(\vartheta) = f(\vartheta, w(\vartheta)), \quad \vartheta \in [1/9, 13/9], \\ w\left(\frac{1}{9}\right) = 0, \\ w\left(\frac{13}{9}\right) = \frac{1}{22} w\left(\frac{1}{3}\right) + \frac{3}{44} w\left(\frac{7}{9}\right) + \frac{7}{66} w\left(\frac{11}{9}\right) + \frac{9}{101} I_{\frac{1}{9}}^{\frac{2}{5},\frac{3}{4},(\vartheta^2+1)} w\left(\frac{5}{9}\right) \\ \qquad + \frac{11}{123} I_{\frac{1}{9}}^{\frac{7}{5},\frac{3}{4},(\vartheta^2+1)} w\left(\frac{10}{9}\right) + \frac{13}{135} D_{\frac{1}{9}}^{\frac{1}{4},\frac{1}{2},\frac{3}{4},(\vartheta^2+1)} w\left(\frac{2}{9}\right) \\ \qquad + \frac{15}{147} D_{\frac{1}{9}}^{\frac{1}{2},\frac{1}{2},\frac{3}{4},(\vartheta^2+1)} w\left(\frac{8}{9}\right) + \frac{17}{159} D_{\frac{1}{9}}^{\frac{3}{4},\frac{1}{2},\frac{3}{4},(\vartheta^2+1)} w\left(\frac{4}{3}\right). \end{cases} \tag{14}$$

Here $\alpha = 3/2$, $\beta = 1/2$, $\sigma = 3/4$, $c = 1/9$, $d = 13/9$, $m = 3$, $\eta_1 = 1/22$, $\eta_2 = 3/44$, $\eta_3 = 7/66$, $\xi_1 = 1/3$, $\xi_2 = 7/9$, $\xi_3 = 11/9$, $n = 2$, $\zeta_1 = 9/101$, $\zeta_2 = 11/123$, $\phi_1 = 2/5$, $\phi_2 = 7/5$, $\theta_1 = 5/9$, $\theta_2 = 10/9$, $r = 3$, $\lambda_1 = 13/135$, $\lambda_2 = 15/147$, $\lambda_3 = 17/159$, $\delta_1 = 1/4$, $\delta_2 = 1/2$, $\delta_3 = 3/4$, $\mu_1 = 2/9$, $\mu_2 = 8/9$, $\mu_3 = 4/3$ and a function $\psi(\vartheta) = \vartheta^2 + 1$. Next, we can find that $\gamma = 7/4$, $\Lambda \approx 0.7194876167$, $\Omega \approx 16.98683713$ and $\Omega_1 \approx 13.52737724$.

(i) Let the nonlinear Lipschitzian function $f : [1/9, 13/9] \times \mathbb{R} \to \mathbb{R}$ be presented in the form

$$f(\vartheta, w) = \frac{1}{2(9\vartheta + 4)^2 + 1}\left(\frac{3|w| + 2w^2}{1 + |w|}\right) + \frac{1}{3}\vartheta^2 + \frac{1}{4}. \tag{15}$$

It is easy to see that $|f(\vartheta, u_1) - f(\vartheta, u_2)| \leq (3/(2(9\vartheta + 4)^2 + 1))|u_1 - u_2|$ for all $\vartheta \in [1/9, 13/9]$ and $u_1, u_2 \in \mathbb{R}$, which implies that $L = 1/17$. Moreover, $L\Omega \approx 0.9992257135 < 1$. From Theorem 1, we conclude that Problem (14) with f defined in (15) has a unique solution on $[1/9, 13/9]$.

(ii) Let the function $f : [1/9, 13/9] \times \mathbb{R} \to \mathbb{R}$ be given by

$$f(\vartheta, w) = \frac{1}{9\vartheta + 13}\left(\frac{|w|}{1 + |w|}\right) + \frac{1}{2}e^{-\vartheta^2} + \frac{1}{3}. \tag{16}$$

Clearly, (H_1) holds true, as $|f(\vartheta, u_1) - f(\vartheta, u_2)| \leq (1/14)|u_1 - u_2|$ for all $\vartheta \in [1/9, 13/9]$, $u_1, u_2 \in \mathbb{R}$ with $L = 1/14$. Observe that the unique solution to Problem (14) is not possible since $L\Omega \approx 1.213345509 > 1$. However, we have

$$|f(\vartheta, w)| \leq \frac{1}{9\vartheta + 13} + \frac{1}{2}e^{-\vartheta^2} + \frac{1}{3} := \phi(\vartheta),$$

and $L\Omega_1 \approx 0.9662412314 < 1$. Therefore, applying the result in Theorem 2, we deduce that Problem (14) with f defined in (16) has at least one solution on $[1/9, 13/9]$.

(iii) Consider a nonlinear function $f : [1/9, 13/9] \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(\vartheta, w) = \sin^8 w + \frac{w^{184}}{1 + w^{184}} + \vartheta e^{-w^4} + \frac{1}{2}. \tag{17}$$

Since $|f(\vartheta, w)| \leq 71/18$ for all $\vartheta \in [1/9, 13/9]$, $w \in \mathbb{R}$, Theorem 3 implies that Problem (14) with f defined in (17) has at least one solution on $[1/9, 13/9]$.

(iv) Let us consider $f : [1/9, 13/9] \times \mathbb{R} \to \mathbb{R}$ as

$$f(\vartheta, w) = \frac{1}{9\vartheta + 4}\left(\frac{w^{198}}{6(1 + w^{196})} + \frac{1}{8}\right), \tag{18}$$

which is bounded since

$$|f(\vartheta, w)| \leq \frac{1}{9\vartheta + 4}\left(\frac{1}{6}w^2 + \frac{1}{8}\right).$$

Choosing $p(\vartheta) = 1/(9\vartheta + 4)$ and $\psi(K) = (1/6)K^2 + (1/8)$, we have that $\|p\| = 1/5$ and there exists a constant K such that $K \in (0.7105439089, 1.055529420)$ satisfying the inequality in (H_5). Hence, by Theorem 4, Problem (14) with f defined in (18) has at least one solution on $[1/9, 13/9]$.

4. Multivalued Case

In this section, we study the multivalued case of the nonlinear ψ-Hilfer generalized proportional fractional boundary value problem in (1), given by

$$\begin{cases} D_{c^+}^{\alpha,\beta,\sigma,\psi} w(\vartheta) \in F(\vartheta, w(\vartheta)), \quad \vartheta \in [c,d], \\ w(c) = 0, \\ w(d) = \sum_{j=1}^{m} \eta_j w(\xi_j) + \sum_{i=1}^{n} \zeta_i \, I_{c^+}^{\phi_i,\sigma,\psi} w(\theta_i) + \sum_{k=1}^{r} \lambda_k \, D_{c^+}^{\delta_k,\beta,\sigma,\psi} w(\mu_k), \end{cases} \tag{19}$$

where $F : [c,d] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map ($\mathcal{P}(\mathbb{R})$ denotes the family of all nonempty subsets of $\mathbb{R}$) and the other symbols are the same as defined in Problem (1).

Let $(X, \|\cdot\|)$ be a normed space. In the following, we denote the classes of all closed, bounded, compact and compact and convex sets in X, by $\mathcal{P}_{cl}$, $\mathcal{P}_b$, $\mathcal{P}_{cp}$ and $\mathcal{P}_{cp,c}$, respectively.

The set of selections of F for each $u \in C([c,d], \mathbb{R})$ is defined as

$$S_{F,\omega} := \{z \in L^1([c,d], \mathbb{R}) : z(\vartheta) \in F(\vartheta, \omega(\vartheta)) \text{ for a.e. } \vartheta \in [c,d]\}.$$

For details on multivalued analysis, see [44–46].

4.1. Existence Results for Problem (19)

Definition 4. *A function $w \in C([c,d], \mathbb{R})$ is called a solution of the ψ-Hilfer generalized proportional inclusion fractional boundary value problem in (19) if there exists a function $v \in L^1([c,d], \mathbb{R})$ with $v(\vartheta) \in F(\vartheta, w)$ almost everywhere (a.e.) on $[c,d]$ such that*

$$\begin{aligned} w(\vartheta) &= I_{c^+}^{\alpha,\sigma,\psi} v(\vartheta) + \frac{e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(c))}(\psi(\vartheta)-\psi(c))^{\gamma-1}}{\Lambda \sigma^{\gamma-1}\Gamma(\gamma)} \Bigg\{ \sum_{j=1}^{m} \eta_j I^{\alpha,\sigma,\psi} v(\xi_j) \\ &\quad + \sum_{i=1}^{n} \zeta_i I^{\alpha+\phi_i,\sigma,\psi} v(\theta_i) + \sum_{k=1}^{r} \lambda_k I^{\alpha-\delta_k,\sigma,\psi} v(\mu_k) - I^{\alpha,\sigma,\psi} v(\vartheta) \Bigg\}. \end{aligned}$$

4.1.1. Case 1: Convex-Valued Multifunctions

Here we consider the case when the multifunction F has convex values and we prove an existence result for the ψ-Hilfer generalized proportional inclusion fractional boundary value problem in (19) by using the nonlinear alternative for Kakutani maps [43] and a closed graph operator theorem [47], under the assumption that F is L^1-Carathéodory.

Theorem 5. *Assume that:*

(A_1) *The multifunction $F : [c,d] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory;*

(A_2) *There exist a nondecreasing function $\chi \in C([c,d], \mathbb{R}^+)$ and a continuous function $q : [c,d] \to \mathbb{R}^+$ such that*
$\|F(\vartheta, \omega)\|_{\mathcal{P}} := \sup\{|z| : z \in F(\vartheta, \omega)\} \le q(\vartheta)\chi(\|\omega\|)$ for each $(\vartheta, \omega) \in [c,d] \times \mathbb{R}$;

(A_3) *There exists a positive number M such that*

$$\frac{M}{\chi(M)\|q\|\Omega} > 1,$$

where Ω is given by (11).

Then the ψ-Hilfer generalized proportional inclusion fractional boundary value problem (19) has at least one solution on $[c,d]$.

Proof. We introduce a multivalued operator $N : C([c,d],\mathbb{R}) \to \mathcal{P}(C([c,d],\mathbb{R}))$ as

$$N(w) = \left\{ \begin{array}{l} h \in C([c,d],\mathbb{R}) : \\ h(\vartheta) = \left\{ \begin{array}{l} I_{c^+}^{\alpha,\sigma,\psi} v(\vartheta) + \dfrac{e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(c))}(\psi(\vartheta)-\psi(c))^{\gamma-1}}{\Lambda \sigma^{\gamma-1}\Gamma(\gamma)} \\ \times \left\{ \displaystyle\sum_{j=1}^{m} \eta_j I^{\alpha,\sigma,\psi} v(\xi_j) + \sum_{i=1}^{n} \zeta_i I^{\alpha+\phi_i,\sigma,\psi} v(\theta_i) \right. \\ \left. + \displaystyle\sum_{k=1}^{r} \lambda_k I^{\alpha-\delta_k,\sigma,\psi} v(\mu_k) - I^{\alpha,\sigma,\psi} v(\vartheta) \right\}, \quad v \in S_{F,w}. \end{array} \right. \end{array} \right\} \tag{20}$$

We will prove that the operator N satisfies the hypotheses of the Leray–Schauder nonlinear alternative for Kakutani maps [43] in several steps.

Step 1. *N is bounded on bounded sets of $C([c,d],\mathbb{R})$.*

Let $B_r = \{w \in C([c,d],\mathbb{R}) : \|w\| \le r\}$, $r > 0$, be a bounded set in $C([c,d],\mathbb{R})$. For each $h \in N(w)$ and $w \in B_r$, there exists $v \in S_{F,w}$ such that

$$\begin{aligned} h(\vartheta) &= I_{c^+}^{\alpha,\sigma,\psi} v(\vartheta) + \frac{e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(c))}(\psi(\vartheta)-\psi(c))^{\gamma-1}}{\Lambda \sigma^{\gamma-1}\Gamma(\gamma)} \left\{ \sum_{j=1}^{m} \eta_j I^{\alpha,\sigma,\psi} v(\xi_j) \right. \\ &\quad \left. + \sum_{i=1}^{n} \zeta_i I^{\alpha+\phi_i,\sigma,\psi} v(\theta_i) + \sum_{k=1}^{r} \lambda_k I^{\alpha-\delta_k,\sigma,\psi} v(\mu_k) - I^{\alpha,\sigma,\psi} v(\vartheta) \right\}. \end{aligned}$$

For $\vartheta \in [c,d]$, using the assumption (A_2), we obtain

$$\begin{aligned} |h(\vartheta)| &\le I_{c^+}^{\alpha,\sigma,\psi} |v(\vartheta)| + \frac{(\psi(\vartheta)-\psi(c))^{\gamma-1}}{|\Lambda| \sigma^{\gamma-1}\Gamma(\gamma)} \left\{ \sum_{j=1}^{m} \eta_j I^{\alpha,\sigma,\psi} |v(\xi_j)| \right. \\ &\quad \left. + \sum_{i=1}^{n} \zeta_i I^{\alpha+\phi_i,\sigma,\psi} |v(\theta_i)| + \sum_{k=1}^{r} \lambda_k I^{\alpha-\delta_k,\sigma,\psi} |v(\mu_k)| + I^{\alpha,\sigma,\psi} |v(\vartheta)| \right\} \\ &\le \left[\frac{(\psi(d)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)} + \frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda| \sigma^{\gamma-1}\Gamma(\gamma)} \left\{ \sum_{i=j}^{m} \frac{|\eta_j|(\psi(\xi_j)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)} \right. \right. \\ &\quad + \sum_{i=1}^{n} \frac{|\zeta_i|(\psi(\theta_i)-\psi(c))^{\alpha+\phi_i}}{\sigma^{\alpha}\Gamma(\alpha+\phi_i+1)} + \sum_{k=1}^{r} \frac{|\lambda_k|(\psi(\mu_k)-\psi(c))^{\alpha-\delta_k}}{\sigma^{\alpha}\Gamma(\alpha-\delta_k+1)} \\ &\quad \left. \left. + \frac{(\psi(d)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)} \right\} \right] \|q\| \chi(\|w\|), \end{aligned}$$

and consequently

$$\|h\| \le \|q\| \chi(r) \Omega.$$

Step 2. *Bounded sets are mapped by N into equicontinuous sets of $C([c,d],\mathbb{R})$.*

Let $w \in B_r$ and $h \in N(w)$. Then there exists $v \in S_{F,w}$ such that

$$\begin{aligned} h(\vartheta) &= I_{c^+}^{\alpha,\sigma,\psi} v(\vartheta) + \frac{e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(c))}(\psi(\vartheta)-\psi(c))^{\gamma-1}}{\Lambda \sigma^{\gamma-1}\Gamma(\gamma)} \left\{ \sum_{j=1}^{m} \eta_j I^{\alpha,\sigma,\psi} v(\xi_j) \right. \\ &\quad \left. + \sum_{i=1}^{n} \zeta_i I^{\alpha+\phi_i,\sigma,\psi} v(\theta_i) + \sum_{k=1}^{r} \lambda_k I^{\alpha-\delta_k,\sigma,\psi} v(\mu_k) - I^{\alpha,\sigma,\psi} v(\vartheta) \right\}. \end{aligned}$$

Let $\vartheta_1,\ \vartheta_2 \in [c,d],\ \vartheta_1 < \vartheta_2$. Then

$$\begin{aligned}
& |h(\vartheta_2) - h(\vartheta_1)| \\
\leq\ & \frac{1}{\sigma^\alpha \Gamma(\alpha)} \left| \int_{c^+}^{\vartheta_1} \left[(\psi(\vartheta_2) - \psi(c))^{\alpha-1} - (\psi(\vartheta_1) - \psi(c))^{\alpha-1} \right] \psi'(s) v(s) ds \right| \\
& + \frac{1}{\sigma^\alpha \Gamma(\alpha)} \left| \int_{\vartheta_1}^{\vartheta_2} (\psi(\vartheta_2) - \psi(c))^{\alpha-1} \psi'(s) v(s) ds \right| \\
& + \frac{(\psi(\vartheta_2) - \psi(c))^{\gamma-1} - (\psi(\vartheta_1) - \psi(c))^{\gamma-1}}{|\Lambda| \sigma^{\gamma-1} \Gamma(\gamma)} \Bigg\{ \sum_{j=1}^m |\eta_j| I^{\alpha,\sigma,\psi} |v(\xi_j)| \\
& + \sum_{i=1}^n |\zeta_i| I^{\alpha+\phi_i,\sigma,\psi} |v(\theta_i)| + \sum_{k=1}^r |\lambda_k| I^{\alpha-\delta_k,\sigma,\psi} |v(\mu_k)| + I^{\alpha,\sigma,\psi} |v(d)| \Bigg\} \\
\leq\ & \frac{M}{\sigma^\alpha \Gamma(\alpha+1)} \Big[2(\psi(\vartheta_2 - \psi(\vartheta_1))^\alpha + |(\psi(\vartheta_2) - \psi(c))^\alpha - (\psi(\vartheta_1) - \psi(c))^\alpha| \Big] \\
& + \frac{(\psi(\vartheta_2) - \psi(c))^{\gamma-1} - (\psi(\vartheta_1) - \psi(c))^{\gamma-1}}{|\Lambda| \sigma^{\gamma-1} \Gamma(\gamma)} \Bigg\{ \sum_{i=j}^m \frac{|\eta_j| (\psi(\xi_j) - \psi(c))^\alpha}{\sigma^\alpha \Gamma(\alpha+1)} \\
& + \sum_{i=1}^n \frac{|\zeta_i| (\psi(\theta_i) - \psi(c))^{\alpha+\phi_i}}{\sigma^\alpha \Gamma(\alpha+\phi_i+1)} + \sum_{k=1}^r \frac{|\lambda_k| (\psi(\mu_k) - \psi(c))^{\alpha-\delta_k}}{\sigma^\alpha \Gamma(\alpha-\delta_k+1)} \\
& + \frac{(\psi(d) - \psi(c))^\alpha}{\sigma^\alpha \Gamma(\alpha+1)} \Bigg\} \|q\| \chi(r) \to 0,
\end{aligned}$$

as $\vartheta_1 \to \vartheta_2$ independently of $w \in B_r$. Hence, $N : C([c,d],\mathbb{R}) \to \mathcal{P}(C([c,d],\mathbb{R}))$, by the Arzelá-Ascoli theorem, is completely continuous.

Step 3. *For each $w \in C([c,d],\mathbb{R})$, $N(w)$ is convex.*

It is obvious, since $S_{F,w}$ is convex by the assumption that F has convex values.

Step 4. *The graph of N is closed.*

Let $w_n \to w_*$, $h_n \in N(w_n)$ and $h_n \to h_*$. Then we show that $h_* \in N(w_*)$. Observe that $h_n \in N(w_n)$ implies that there exists $v_n \in S_{F,w_n}$ such that, for each $\vartheta \in [c,d]$, we have

$$\begin{aligned}
h_n(\vartheta) \quad = \quad & I_{c^+}^{\alpha,\sigma,\psi} v_n(\vartheta) + \frac{e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(c))} (\psi(\vartheta) - \psi(c))^{\gamma-1}}{\Lambda \sigma^{\gamma-1} \Gamma(\gamma)} \Bigg\{ \sum_{j=1}^m \eta_j I^{\alpha,\sigma,\psi} v_n(\xi_j) \\
& + \sum_{i=1}^n \zeta_i I^{\alpha+\phi_i,\sigma,\psi} v_n(\theta_i) + \sum_{k=1}^r \lambda_k I^{\alpha-\delta_k,\sigma,\psi} v_n(\mu_k) - I^{\alpha,\sigma,\psi} v_n(\vartheta) \Bigg\}.
\end{aligned}$$

For each $\vartheta \in [c,d]$, we must have $v_* \in S_{F,w_*}$ such that

$$\begin{aligned}
h_*(\vartheta) \quad = \quad & I_{c^+}^{\alpha,\sigma,\psi} v_*(\vartheta) + \frac{e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(c))} (\psi(\vartheta) - \psi(c))^{\gamma-1}}{\Lambda \sigma^{\gamma-1} \Gamma(\gamma)} \Bigg\{ \sum_{j=1}^m \eta_j I^{\alpha,\sigma,\psi} v_*(\xi_j) \\
& + \sum_{i=1}^n \zeta_i I^{\alpha+\phi_i,\sigma,\psi} v_*(\theta_i) + \sum_{k=1}^r \lambda_k I^{\alpha-\delta_k,\sigma,\psi} v_*(\mu_k) - I^{\alpha,\sigma,\psi} v_*(\vartheta) \Bigg\}.
\end{aligned}$$

We introduce a continuous linear operator $\Phi : L^1([c,d],\mathbb{R}) \to C([c,d],\mathbb{R})$ as

$$v \to \Phi(v)(\vartheta) \quad = \quad I_{c^+}^{\alpha,\sigma,\psi} v(\vartheta) + \frac{e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(c))} (\psi(\vartheta) - \psi(c))^{\gamma-1}}{\Lambda \sigma^{\gamma-1} \Gamma(\gamma)} \Bigg\{ \sum_{j=1}^m \eta_j I^{\alpha,\sigma,\psi} v(\xi_j)$$

$$+\sum_{i=1}^{n}\zeta_i I^{\alpha+\phi_i,\sigma,\psi}v(\theta_i)+\sum_{k=1}^{r}\lambda_k I^{\alpha-\delta_k,\sigma,\psi}v(\mu_k)-I^{\alpha,\sigma,\psi}v(\vartheta)\Bigg\}.$$

Clearly, $\|h_n - h_*\| \to 0$ as $n \to \infty$, and consequently, by the closed graph operator theorem [47], $\Phi \circ S_{F,w}$ is a closed graph operator. In addition, we have $h_n \in \Phi(S_{F,w_n})$ and

$$\begin{aligned} h_*(\vartheta) &= I_{c^+}^{\alpha,\sigma,\psi}v_*(\vartheta)+\frac{e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(c))}(\psi(\vartheta)-\psi(c))^{\gamma-1}}{\Lambda\sigma^{\gamma-1}\Gamma(\gamma)}\Bigg\{\sum_{j=1}^{m}\eta_j I^{\alpha,\sigma,\psi}v_*(\xi_j) \\ &\quad+\sum_{i=1}^{n}\zeta_i I^{\alpha+\phi_i,\sigma,\psi}v_*(\theta_i)+\sum_{k=1}^{r}\lambda_k I^{\alpha-\delta_k,\sigma,\psi}v_*(\mu_k)-I^{\alpha,\sigma,\psi}v_*(\vartheta)\Bigg\}, \end{aligned}$$

for some $v_* \in S_{F,w_*}$. Thus, N has a closed graph, which implies that the operator N is upper semicontinuous, because, by [44], Proposition 1.2, a completely continuous operator is upper semicontinuous if it has a closed graph.

Step 5. *There exists an open set $U \subseteq C([c,d],\mathbb{R})$, such that, for any $\kappa \in (0,1)$ and all $w \in \partial U$, $w \notin \kappa N(w)$.*

Let $w \in \kappa N(w)$, $\kappa \in (0,1)$. Then there exists $v \in L^1([c,d],\mathbb{R})$ with $v \in S_{F,w}$ such that, for $\vartheta \in [c,d]$, we have

$$\begin{aligned} w(\vartheta) &= \kappa I_{c^+}^{\alpha,\sigma,\psi}v(\vartheta)+\kappa\frac{e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(c))}(\psi(\vartheta)-\psi(c))^{\gamma-1}}{\Lambda\sigma^{\gamma-1}\Gamma(\gamma)}\Bigg\{\sum_{j=1}^{m}\eta_j I^{\alpha,\sigma,\psi}v(\xi_j) \\ &\quad+\sum_{i=1}^{n}\zeta_i I^{\alpha+\phi_i,\sigma,\psi}v(\theta_i)+\sum_{k=1}^{r}\lambda_k I^{\alpha-\delta_k,\sigma,\psi}v(\mu_k)-I^{\alpha,\sigma,\psi}v(\vartheta)\Bigg\}. \end{aligned}$$

Following the computation as in Step 1, for each $\vartheta \in [c,d]$, we have

$$\begin{aligned} |w(\vartheta)| &\le \Bigg[\frac{(\psi(d)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)}+\frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\sigma^{\gamma-1}\Gamma(\gamma)}\Bigg\{\sum_{i=j}^{m}\frac{|\eta_j|(\psi(\xi_j)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)} \\ &\quad\times+\sum_{i=1}^{n}\frac{|\zeta_i|(\psi(\theta_i)-\psi(c))^{\alpha+\phi_i}}{\sigma^{\alpha}\Gamma(\alpha+\phi_i+1)}+\sum_{k=1}^{r}\frac{|\lambda_k|(\psi(\mu_k)-\psi(c))^{\alpha-\delta_k}}{\sigma^{\alpha}\Gamma(\alpha-\delta_k+1)} \\ &\quad+\frac{(\psi(d)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)}\Bigg\}\Bigg]\|q\|\chi(\|w\|) \\ &= \|q\|\chi(\|w\|)\Omega. \end{aligned}$$

In consequence, we obtain

$$\frac{\|w\|}{\chi(\|w\|)\|q\|\Omega}\le 1.$$

By the assumption (A_3), there exists a positive number M satisfying $\|w\| \neq M$. Let us define a set

$$\Theta=\{w\in C([c,d],\mathbb{R}):\|w\|<M\}.$$

Obviously, $N:\overline{\Theta}\to\mathcal{P}(C([c,d],\mathbb{R}))$ is a compact, conve- valued and upper semicontinuous multivalued map. By the definition of Θ, there does not exist any $w \in \partial\Theta$ for some $\kappa \in (0,1)$ satisfying $w \in \kappa N(w)$. Therefore, the conclusion of the Leray–Schauder nonlinear alternative for Kakutani maps [43] applies, and hence, the operator N has a fixed point $w \in \overline{\Theta}$. Thus, Problem (19) has at least one solution on $[c,d]$, which concludes the proof. □

4.1.2. Case 2: Nonconvex-Valued Multifunctions

Here, we prove the existence of a solution for the ψ-Hilfer generalized proportional fractional inclusion boundary value problem in (19) with a nonconvex-valued multivalued map via the fixed point theorem for contractive multivalued maps from Covitz and Nadler [48].

Definition 5 ([49]). *Let (X, d) be a metric space induced from the normed space $(X, \|\cdot\|)$ and $H_{\bar{d}} : \mathcal{P}(w) \times \mathcal{P}(w) \to \mathbb{R} \cup \{\infty\}$ be defined by*

$$H_{\bar{d}}(A, B) = \max\{\sup_{c\in A} \bar{d}(c, d), \sup_{d\in B} \bar{d}(c, d)\},$$

where $\bar{d}(c, d) = \inf_{c\in A} \bar{d}(c, d)$ and $\bar{d}(c, d) = \inf_{d\in B} \bar{d}(c, d)$.

Theorem 6. *Assume that:*

(B_1) *$F : [c, d] \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is such that $F(\cdot, w) : [c, d] \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $x \in \mathbb{R}$;*

(B_2) *$H_{\bar{d}}(F(\vartheta, w), F(\vartheta, \bar{w})) \leq \varrho(\vartheta)|w - \bar{w}|$ for almost all $\vartheta \in [c, d]$ and $w, \bar{w} \in \mathbb{R}$ with $\varrho \in C([c, d], \mathbb{R}^+)$ and $\bar{d}(0, F(\vartheta, 0)) \leq \varrho(\vartheta)$ for almost all $\vartheta \in [c, d]$.*

Then the ψ-Hilfer generalized proportional inclusion fractional boundary value problem in (19) has at least one solution on $[c, d]$ if

$$\Omega\|\varrho\| < 1,$$

where Ω is given by (11).

Proof. We will prove that the operator $N : C([c, d], \mathbb{R}) \to \mathcal{P}(C([c, d], \mathbb{R}))$, defined by (20), satisfies the hypotheses of Covitz–Nadler's fixed point theorem for multivalued maps [48].

Step I. N is nonempty and closed for every $v \in S_{F,w}$.

The set-valued map $F(\cdot, w(\cdot))$ is measurable by the measurable selection theorem ([50], Theorem III.6), and hence, it admits a measurable selection $v : [c, d] \to \mathbb{R}$. By the assumption (B_2), we obtain $|v(\vartheta)| \leq \varrho(\vartheta)(1 + |w(\vartheta)|)$, that is, $v \in L^1([c, d], \mathbb{R})$, and hence, F is integrably bounded. Therefore, we deduce that $S_{F,w} \neq \emptyset$.

Now we show that $N(w) \in \mathcal{P}_{cl}(C([c, d], \mathbb{R}))$ for each $w \in C([c, d], \mathbb{R})$. For that, let $\{u_n\}_{n\geq 0} \in N(w)$ with $u_n \to u$ $(n \to \infty)$ in $C([c, d], \mathbb{R})$. Then $u \in C([c, d], \mathbb{R})$ and we can find $v_n \in S_{F,w_n}$ such that, for each $\vartheta \in [c, d]$,

$$\begin{aligned} u_n(\vartheta) \quad &= \quad I_{c^+}^{\alpha,\sigma,\psi} v_n(\vartheta) + \frac{e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(c))}(\psi(\vartheta) - \psi(c))^{\gamma-1}}{\Lambda\sigma^{\gamma-1}\Gamma(\gamma)} \Bigg\{ \sum_{j=1}^{m} \eta_j I^{\alpha,\sigma,\psi} v_n(\xi_j) \\ &+ \sum_{i=1}^{n} \zeta_i I^{\alpha+\phi_i,\sigma,\psi} v_n(\theta_i) + \sum_{k=1}^{r} \lambda_k I^{\alpha-\delta_k,\sigma,\psi} v_n(\mu_k) - I^{\alpha,\sigma,\psi} v_n(\vartheta) \Bigg\}. \end{aligned}$$

Then we can obtain a subsequence (if necessary) v_n converging to v in $L^1([c, d], \mathbb{R})$ as F has compact values. Thus, $v \in S_{F,w}$ and, for each $\vartheta \in [c, d]$, we have

$$\begin{aligned} h_n(\vartheta) \to v(\vartheta) \quad &= \quad I_{c^+}^{\alpha,\sigma,\psi} v(\vartheta) + \frac{e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(c))}(\psi(\vartheta) - \psi(c))^{\gamma-1}}{\Lambda\sigma^{\gamma-1}\Gamma(\gamma)} \Bigg\{ \sum_{j=1}^{m} \eta_j I^{\alpha,\sigma,\psi} v(\xi_j) \\ &+ \sum_{i=1}^{n} \zeta_i I^{\alpha+\phi_i,\sigma,\psi} v(\theta_i) + \sum_{k=1}^{r} \lambda_k I^{\alpha-\delta_k,\sigma,\psi} v(\mu_k) - I^{\alpha,\sigma,\psi} v(\vartheta) \Bigg\}. \end{aligned}$$

Thus, $u \in N(w)$.

Step II. Here, we establish that there exists $0 < \overline{m_0} < 1$ ($\overline{m_0} = \Omega\|\varrho\|$) such that

$$H_{\bar{d}}(N(w), N(\bar{w})) \leq \overline{m_0}\|w - \bar{w}\| \text{ for each } w, \bar{w} \in C([c,d], \mathbb{R}).$$

Let $w, \bar{w} \in C([c,d], \mathbb{R})$ and $h_1 \in N(w)$. Then there exists $v_1(\vartheta) \in F(\vartheta, w(\vartheta))$ such that, for each $\vartheta \in [c,d]$,

$$\begin{aligned} h_1(\vartheta) &= I_{c^+}^{\alpha,\sigma,\psi} v_1(\vartheta) + \frac{e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(c))}(\psi(\vartheta)-\psi(c))^{\gamma-1}}{\Lambda\sigma^{\gamma-1}\Gamma(\gamma)} \Bigg\{ \sum_{j=1}^{m} \eta_j I^{\alpha,\sigma,\psi} v_1(\xi_j) \\ &\quad + \sum_{i=1}^{n} \zeta_i I^{\alpha+\phi_i,\sigma,\psi} v_1(\theta_i) + \sum_{k=1}^{r} \lambda_k I^{\alpha-\delta_k,\sigma,\psi} v_1(\mu_k) - I^{\alpha,\sigma,\psi} v_1(\vartheta) \Bigg\}. \end{aligned}$$

By (B_2), we have

$$H_{\bar{d}}(F(\vartheta, w), F(\vartheta, \bar{w})) \leq \varrho(\vartheta)|w(\vartheta) - \bar{w}(\vartheta)|.$$

Thus, there exists $z \in F(\vartheta, \bar{w}(\vartheta))$ such that

$$|v_1(\vartheta) - z| \leq \varrho(\vartheta)|w(\vartheta) - \bar{w}(\vartheta)|, \ \ \vartheta \in [c,d].$$

Let us define $\mathcal{V} : [c,d] \to \mathcal{P}(\mathbb{R})$ by

$$\mathcal{V}(\vartheta) = \{z \in \mathbb{R} : |v_1(\vartheta) - z| \leq \varrho(\vartheta)|w(\vartheta) - \bar{w}(\vartheta)|\}.$$

Then there exists a function $v_2(\vartheta)$ which is a measurable selection of $\mathcal{V}$, since the multivalued operator $\mathcal{V}(\vartheta) \cap F(\vartheta, \bar{w}(\vartheta))$ is measurable by Proposition III.4 in [50]. Hence, $v_2(\vartheta) \in F(\vartheta, \bar{x}(\vartheta))$, and for each $\vartheta \in [c,d]$, we have $|v_1(\vartheta) - v_2(\vartheta)| \leq \varrho(\vartheta)|w(\vartheta) - \bar{w}(\vartheta)|$. Thus, for each $\vartheta \in [c,d]$, we have

$$\begin{aligned} h_2(\vartheta) &= I_{c^+}^{\alpha,\sigma,\psi} v_2(\vartheta) + \frac{e^{\frac{\sigma-1}{\sigma}(\psi(\vartheta)-\psi(c))}(\psi(\vartheta)-\psi(c))^{\gamma-1}}{\Lambda\sigma^{\gamma-1}\Gamma(\gamma)} \Bigg\{ \sum_{j=1}^{m} \eta_j I^{\alpha,\sigma,\psi} v_2(\xi_j) \\ &\quad + \sum_{i=1}^{n} \zeta_i I^{\alpha+\phi_i,\sigma,\psi} v_2(\theta_i) + \sum_{k=1}^{r} \lambda_k I^{\alpha-\delta_k,\sigma,\psi} v_2(\mu_k) - I^{\alpha,\sigma,\psi} v_2(\vartheta) \Bigg\}. \end{aligned}$$

In consequence, we obtain

$$\begin{aligned} &|h_1(\vartheta) - h_2(\vartheta)| \\ \leq\ & I_{c^+}^{\alpha,\sigma,\psi}|v_2(\vartheta) - v_1(\vartheta)| + \frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\sigma^{\gamma-1}\Gamma(\gamma)} \Bigg\{ \sum_{j=1}^{m} \eta_j I^{\alpha,\sigma,\psi}|v_2(\xi_j - v_1(\xi_j)| \\ & + \sum_{i=1}^{n} \zeta_i I^{\alpha+\phi_i,\sigma,\psi}|v_2(\theta_i) - v_1(\theta_i)| + \sum_{k=1}^{r} \lambda_k I^{\alpha-\delta_k,\sigma,\psi}|v_2(\mu_k) - v_1(\mu_k)| \\ & + I^{\alpha,\sigma,\psi}|v_2(\vartheta) - v_1(\vartheta)| \Bigg\} \\ \leq\ & \Bigg[\frac{(\psi(d)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)} + \frac{(\psi(d)-\psi(c))^{\gamma-1}}{|\Lambda|\sigma^{\gamma-1}\Gamma(\gamma)} \Bigg\{ \sum_{i=j}^{m} \frac{|\eta_j|(\psi(\xi_j)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)} \\ & + \sum_{i=1}^{n} \frac{|\zeta_i|(\psi(\theta_i)-\psi(c))^{\alpha+\phi_i}}{\sigma^{\alpha}\Gamma(\alpha+\phi_i+1)} + \sum_{k=1}^{r} \frac{|\lambda_k|(\psi(\mu_k)-\psi(c))^{\alpha-\delta_k}}{\sigma^{\alpha}\Gamma(\alpha-\delta_k+1)} \\ & + \frac{(\psi(d)-\psi(c))^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)} \Bigg\} \Bigg] \|\varrho\| \|w - \bar{w}\|, \end{aligned}$$

which leads to

$$\|h_1 - h_2\| \le \Omega\|\varrho\|\|w - \bar{w}\|.$$

On switching the roles of w and $\bar{w}$, we obtain

$$H_{\bar{d}}(N(w), N(\bar{w})) \le \Omega\|\varrho\|\|w - \bar{w}\|,$$

which shows that N is a contraction. Consequently, by Covitz–Nadler's fixed point theorem [48], the operator N has a fixed point w which corresponds to a solution of the ψ-Hilfer generalized proportional inclusion fractional boundary value problem (19). The proof is complete. □

4.2. Illustrative Examples for the Multivalued Case

Let us consider the ψ-Hilfer generalized proportional inclusion:

$$D_{\frac{1}{9}}^{\frac{3}{2},\frac{1}{2},\frac{3}{4},(\vartheta^2+1)} w(\vartheta) \in F(\vartheta, w(\vartheta)), \quad \vartheta \in \left[\frac{1}{9}, \frac{13}{9}\right], \tag{21}$$

with the boundary conditions given by (14). Here, all the parameters are the same, as considered in Problem (14).

(a) Consider $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ defined by

$$F(\vartheta, w) = \left[\frac{1}{9\vartheta+5}\left(\frac{|w|^{177}}{13(1+|w|^{175})} + \frac{1}{15}\right), \frac{1}{9\vartheta+3}\left(\frac{|w|^{177}}{10(1+|w|^{175})} + \frac{1}{12}\right)\right], \tag{22}$$

and note that F is an L^1-Carathéodory, as

$$\|F(\vartheta, w)\|_{\mathcal{P}} \le \frac{1}{9\vartheta+3}\left(\frac{1}{10}w^2 + \frac{1}{12}\right).$$

Letting $q(\vartheta) = 1/(9\vartheta+3)$ and $\chi(M) = (1/10)M^2 + (1/12)$, we find that $\|q\| = 1/4$ and there exists a constant

$$M \in (0.4338127490, 1.920951689),$$

satisfying the inequality in (A_3). Hence, by Theorem 5, the ψ-Hilfer generalized proportional inclusion in (21) with the boundary conditions in (14), and F defined in (22), has at least one solution on $[1/9, 13/9]$.

(b) Let $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be defined by

$$F(\vartheta, w) = \left[0, \frac{1}{2(81\vartheta+8)}\left(\frac{w^2 + 2|w|}{1+|w|} + \frac{1}{2}\right)\right]. \tag{23}$$

It is easy to check that F is measurable and satisfies $H_{\bar{d}}(F(\vartheta, w), F(\vartheta, \overline{w})) \le (1/(81\vartheta+8))|w - \overline{w}|$. Setting $\varrho(\vartheta) = 1/(81\vartheta+8)$, we obtain $\|\varrho\| = 1/17$, $d(0, F(\vartheta, 0)) \le (1/4)\varrho(\vartheta) \le \varrho(\vartheta)$ and $\Omega\|\varrho\| \approx 0.9992257135 < 1$. Thus, all the assumptions of Theorem 6 are satisfied, and hence, there exists at least one solution for the inclusion in (21) with the boundary conditions in (14) and F defined in (23) on $[1/9, 13/9]$.

5. Conclusions

In this paper, we have presented the criteria ensuring the existence and uniqueness of solutions for a ψ-Hilfer generalized proportional fractional differential equation complemented with mixed nonlocal boundary conditions. The boundary conditions considered in the present study are more general, as they include multipoint, multiorder fractional integral and fractional derivative operators. After converting the given nonlinear problem into a fixed point problem, we applied the standard fixed point theorems to derive the desired results. We have also presented two existence results for the ψ-Hilfer generalized proportional fractional differential inclusion with mixed nonlocal boundary conditions for the cases when the multivalued map (involved in the inclusion problem) takes convex as well as nonconvex values. All the results presented in this paper are well-illustrated by numerical examples. Our results not only enrich the literature on the mixed nonlocal boundary value problems involving ψ-Hilfer generalized proportional fractional differential equations and inclusions but also yield several new results as special cases by fixing the parameters involved in the problems appropriately. For example, our results correspond to the ones with the boundary conditions containing multiorder fractional integrals as well as derivative operators when we fix all $\eta_j = 0, j = 1, \ldots, m$. For future study, we plan to consider ψ-Hilfer generalized proportional fractional differential and integro-differential equations and inclusions with other kinds of boundary conditions.

Author Contributions: Conceptualization, S.K.N., B.A. and J.T.; methodology, S.K.N., B.A. and J.T.; formal analysis, S.K.N., B.A. and J.T.; writing—original draft preparation, S.K.N., B.A. and J.T. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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