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Existence and Stability Results for Fractional Hybrid q-Difference Equations with q-Integro-Initial Condition

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Abstract: This article is concerned with the study of a new class of hybrid fractional q-integro-difference equations involving Caputo type q-derivatives and Riemann-Liouville q-integrals of different orders with a nonlocal q-integro-initial condition. An existence result for the given problem is obtained by means of Krasnoselskii's fixed point theorem, whereas the uniqueness of its solutions is shown by applying the Banach contraction mapping principle. We also discuss the stability of solutions of the problem at hand and find that it depends on the nonlocal parameter in contrast to the initial position of the domain. To demonstrate the application of the obtained results, examples are constructed.

Keywords: q-integro-difference equation; initial value problem; existence of solutions; fixed point

MSC: 34A08; 39A12; 39A13

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1. Introduction

Fractional calculus is concerned with derivative and integral operators of arbitrary (non-integer) orders. This branch of mathematical analysis has received great attention during the last few decades, though its roots go back to the speculations between Leibniz (1697) and Euler (1730) about fractional-order derivatives. It is worthwhile to note that fractional order operators do have different forms (definitions) in contrast to the integer-order ones. The tools of fractional calculus have been extensively used in the mathematical modeling of many real-world phenomena. An interesting feature accounting for the popularity of this subject is the nonlocal nature of fractional-order operators. For theoretical and applications details of the topic, for instance, see the books [1–5] and the references therein.

Influenced by the overwhelming interest in the fractional calculus, many authors turned to enhancing the literature on fractional q-difference equations. One can find some interesting results on fractional q-difference equations in the articles [6–14]. For some recent works on systems of fractional q-difference equations with different kinds of boundary conditions, see [15,16] and the references cited therein. In a more recent work [17], the authors studied an initial value problem for fractional hybrid q-difference equations.

The objective of the present work is to introduce and study a fractional hybrid q-integro-difference equation complemented with a nonlocal q-integro-initial condition given by

$$^{c}D_q^{\alpha}[u(x) - f(x,u(x))] = ag(x,u(x)) + bI_q^{\delta}h(x,u(x)),\ 0 < q < 1,\ 0 \le x \le 1, \tag{1}$$

$$u(0) = u_0 \int_0^{\eta} \frac{(\eta - qs)^{(\gamma-1)}}{\Gamma_q(\gamma)} u(s)d_qs,\ \ \gamma > 0,\ 0 < \eta < 1,\ u_0 \in \mathbb{R}, \tag{2}$$

where $^{c}D_q^{\alpha}$ and I_q^{δ} denote the Caputo type fractional q-derivative of order $\alpha \in (0,1]$ and q-Riemann-Liouville integral with $0 < \delta < 1$, respectively, $a, b \in \mathbb{R}$, and $f, g, h : [0,1] \times \mathbb{R} \to$

$\mathbb{R}$ are given continuous functions. Without loss of generality, it is assumed that $f(0,0)=0$ (the case $f(0,0)\neq 0$ can be dealt with in a similar manner).

Hybrid fractional differential equations constitute a class of equations which contains the fractional derivative of an unknown function hybrid with the nonlinearity depending on it. Several dynamic systems involve such equations, for instance, infection transmission in population dynamic models [18]. The quadratic perturbations of nonlinear differential equations also give rise to hybrid equations. Our work is motivated by a recent study of fractional hybrid q-difference equations of the form: ${}^{c}D_q^{\gamma}[u(x)-f(x,u(x))]=g(x,u(x))$, $0<\gamma\leq 1$, $0<x<1$, $u(0)=u_0$ in [17]. The problem (1) and (2) proposed in this article is of more general nature as it deals with a hybrid fractional q-difference equation with mixed nonlinearities: $g(x,u(x))$ and $I_q^{\delta}h(x,u(x))$ subject to nonlocal q-integro-initial condition.

The rest of the article is arranged as follows. In Section 2, we recall some general concepts of fractional q-calculus and prove an auxiliary lemma for the linear variant of the problem (1) and (2). In Section 3, we establish existence and uniqueness results for the problem (1) and (2). Section 4 is concerned with the stability of solutions for the given problem. The article concludes with examples illustrating the main results.

2. Preliminaries

Let us first recall the general concepts of q-fractional calculus ([19,20]).

A q-real number denoted by $[a]_q$ is defined by

$$[a]_q=\frac{1-q^a}{1-q},\ \ a\ \in\mathbb{R},\, q\in\mathbb{R}^+\setminus\{1\}.$$

The q-shifted factorial (q-analogue of the Pochhammer symbol) is

$$(a;q)_0=1,\ \ (a;q)_k=\prod_{i=0}^{k-1}(1-aq^i),\ k\in\mathbb{N}\cup\{\infty\}.$$

We define the q-analogue of the exponent $(x-y)^k$ as

$$(x-y)^{(0)}=1,\ \ (x-y)^{(k)}=\prod_{j=0}^{k-1}(x-yq^j),\ k\in\mathbb{N},\ x,y\in\mathbb{R}.$$

In general, if ρ is a real number, then, $(x-y)^{(\rho)}=x^{\rho}\prod_{j=0}^{\infty}\frac{x-yq^j}{x-yq^{\rho+j}}$ and $x^{(\rho)}=x^{\rho}$ when $y=0$. If $\rho>0$ and $0\leq x\leq y\leq t$, then $(t-y)^{(\rho)}\leq(t-x)^{(\rho)}$. The q-Gamma function $\Gamma_q(\rho)$ is defined as

$$\Gamma_q(\rho)=\frac{(1-q)^{(\rho-1)}}{(1-q)^{\rho-1}},\qquad \rho\in\mathbb{R}\setminus\{0,-1,-2,\ldots\},$$

which satisfies the relation $\Gamma_q(\rho+1)=[\rho]_q\Gamma_q(\rho)$ [21].

Definition 1 ([19])**.** *Let f be a function defined on $[0,1]$. The Riemann-Liouville type fractional q-integral of order $\beta\geq 0$ is defined as $(I_q^0 f)(t)=f(t)$ and*

$$I_q^{\beta}f(t):=\int_0^t\frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}f(s)d_qs=t^{\beta}(1-q)^{\beta}\sum_{k=0}^{\infty}q^k\frac{(q^{\beta};q)_k}{(q;q)_k}f(tq^k),\ \ \beta>0,\ \ t\in[0,1].$$

Moreover, the semigroup property holds for Riemann-Liouville type fractional q-integrals, that is, $I_q^{\gamma} I_q^{\beta} f(t) = I_q^{\beta+\gamma} f(t)$, $\gamma, \beta \in \mathbb{R}^+$ (Proposition 4.3 [21]). Further, according to the Lemma 2.8 in [16],

$$I_q^{\beta}\Big((x-a)^{(\sigma)}\Big) = \frac{\Gamma_q(\sigma+1)}{\Gamma_q(\beta+\sigma+1)}(x-a)^{(\beta+\sigma)}, \quad 0 < a < x < b, \beta \in \mathbb{R}^+, \sigma \in (-1,\infty).$$

In particular, for $\sigma = 0, a = 0$, using q-integration by parts, we have

$$\begin{aligned}(I_q^{\beta}1)(x) &= \frac{1}{\Gamma_q(\beta)}\int_0^x (x-qt)^{(\beta-1)} d_q t = \frac{1}{\Gamma_q(\beta)}\int_0^x \frac{D_q((x-t)^{(\beta)})}{-[\beta]_q} d_q t \\ &= \frac{-1}{\Gamma_q(\beta+1)}\int_0^x D_q((x-t)^{(\beta)}) d_q t = \frac{1}{\Gamma_q(\beta+1)} x^{(\beta)}.\end{aligned}$$

Definition 2. *The q-derivative of a function f is defined as*

$$(D_q f)(t) = \frac{f(t) - f(qt)}{t - qt}, \; t \neq 0, \; (D_q f)(0) = \lim_{t\to 0}(D_q f)(t).$$

Furthermore,

$$D_q^0 f = f, \;\; D_q^n f = D_q(D_q^{n-1} f), \;\; n = 1, 2, 3, \ldots.$$

Definition 3 ([21])**.** *The Caputo fractional q-derivative of order $\beta > 0$ is defined by*

$${}^c D_q^{\beta} f(t) = I_q^{\lceil \beta \rceil - \beta} D_q^{\lceil \beta \rceil} f(t),$$

where $\lceil \beta \rceil$ is the smallest integer greater than or equal to β.

Next, we recall the following results, which were established in Theorem 5.2 of [21]:

$$I_q^{\beta}\, {}^c D_q^{\beta} f(t) = f(t) - \sum_{k=0}^{\lceil \beta \rceil - 1} \frac{t^k}{\Gamma_q(k+1)} (D_q^k f)(0^+), \; \forall\, t \in (0, a], \; \beta > 0; \tag{3}$$

$${}^c D_q^{\beta}\, I_q^{\beta} f(t) = f(t), \quad \forall\, t \in (0, a], \; \beta > 0.$$

The following lemma plays a key role in transforming the problem (1) and (2) into a fixed point problem.

Lemma 1. *Let $y \in C([0,1], \mathbb{R})$. Then, the unique solution of the problem*

$$\begin{cases} {}^c D_q^{\alpha}[u(x) - f(x, u(x))] = y(x), \;\; 0 < x < 1, \\ u(0) = u_0 \displaystyle\int_0^{\eta} \frac{(\eta - qs)^{(\gamma-1)}}{\Gamma_q(\gamma)} u(s) d_q s, \;\; \gamma > 0, \end{cases} \tag{4}$$

is given by

$$\begin{aligned} u(x) \;&= f(x, u(x)) + \int_0^x \frac{(x-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s) d_q s \\ &+ u_0 \int_0^{\eta} \frac{(\eta - qs)^{(\gamma-1)}}{\Gamma_q(\gamma)} u(s) d_q s - f(0, u_0 I_q^{\gamma} u(\eta)). \end{aligned} \tag{5}$$

Proof. Applying the operator I_q^α on both sides of the fractional q-difference equation in (4), we obtain

$$u(x) = f(x, u(x)) + I_q^\alpha y(x) + c_1, \tag{6}$$

where $c_1 = u(0) - f(0, u(0))$ is a constant deduced from the expansion (3) and that $\alpha \in (0, 1]$. Inserting (6) in the nonlocal q-integro-initial condition in (4), we find that

$$c_1 = u_0 \int_0^\eta \frac{(\eta - qs)^{(\gamma-1)}}{\Gamma_q(\gamma)} u(s) d_q s - f(0, u_0 I_q^\gamma u(\eta)).$$

Substituting the value of c_1 in (6) yields the solution (5). By direct computation, one can obtain the converse of the lemma. This completes the proof. □

3. Main Results

By Lemma 1, we transform the problem (1) and (2) into a fixed point problem $x = \mathcal{G}x$, where the operator $\mathcal{G} : \mathcal{C} \to \mathcal{C}$ is defined by

$$\begin{aligned}(\mathcal{G}u)(x) &= f(x, u(x)) + a \int_0^x \frac{(x - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s, u(s)) d_q s \\ &+ b \int_0^x \frac{(x - qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha + \delta)} h(s, u(s)) d_q s \\ &+ u_0 \int_0^\eta \frac{(\eta - qs)^{(\gamma-1)}}{\Gamma_q(\gamma)} u(s) d_q s - f(0, u_0 I_q^\gamma u(\eta)).\end{aligned} \tag{7}$$

Here, $\mathcal{C} = C([0, 1], \mathbb{R})$ denotes the Banach space of all continuous functions from $[0, 1]$ into $\mathbb{R}$ endowed with the usual norm defined by $\|u\| = \sup\{|u(x)|, x \in [0, 1]\}$.

Notice that the problem (1) and (2) has solutions only if the operator $\mathcal{G}$ has fixed points.

Our first result, based on the Banach contraction mapping principle, is concerned with existence of a unique solution to the problem at hand.

Theorem 1. *Let $f, g, h : [0, 1] \times \mathbb{R} \to \mathbb{R}$ satisfy the Lipschitz condition, that is, for all $u, v \in \mathbb{R}$ and $x \in [0, 1]$, there exist Lipschitz constants $L_i, i = 1, 2, 3$, such that (a) $|f(x, u) - f(x, v)| \le L_1|u - v|$; (b) $|g(x, u) - g(x, v)| \le L_2|u - v|$; and $|h(x, u) - h(x, v)| \le L_3|u - v|$. Then, the fractional hybrid q-difference Equation (1) supplemented with nonlocal q-integro-initial condition (2) has a unique continuous solution on $[0, 1]$, provided that*

$$\Delta := L_1 + \frac{|a|L_2}{\Gamma_q(\alpha + 1)} + \frac{|b|L_3}{\Gamma_q(\alpha + \delta + 1)} + \frac{|u_0|\eta^{(\gamma)}(1 + L_1)}{\Gamma_q(\gamma + 1)} < 1. \tag{8}$$

Proof. We verify the hypothesis of the Banach contraction mapping principle in two steps. For that, let us set $\sup_{x\in[0,1]} |f(x, 0)| = K_1 < \infty$, $\sup_{x\in[0,1]} |g(x, 0)| = K_2 < \infty$ and $\sup_{x\in[0,1]} |h(x, 0)| = K_3 < \infty$ and introduce a closed ball $B_\rho = \{u \in \mathcal{C} : \|u\| \le \rho\}$, where

$$\rho \ge \frac{K_1 + \frac{|a|K_2}{\Gamma_q(\alpha+1)} + \frac{|b|K_3}{\Gamma_q(\alpha+\delta+1)}}{1 - \Delta},$$

and Δ is defined in (8). In our first step, we show that $\mathcal{G}B_\rho \subset B_\rho$, where $\mathcal{G} : \mathcal{C} \to \mathcal{C}$ is defined by (7). For any $u \in B_\rho$, $x \in [0, 1]$, it follows from the given assumption that $f, g, h : [0, 1] \times \mathbb{R} \to \mathbb{R}$ satisfy the Lipschitz condition, we have

$$|f(x, u(x))| \le |f(x, u(x)) - f(x, 0)| + |f(x, 0)| \le L_1\,\rho + K_1,$$

$$|g(x, u(x))| \le |g(x, u(x)) - g(x, 0)| + |g(x, 0)| \le L_2\,\rho + K_2,$$

and

$$|h(x,u(x))| \leq |h(x,u(x)) - h(x,0)| + |h(x,0)| \leq L_3\,\rho + K_3.$$

Then, for $u \in B_\rho,\ x \in [0,1]$, we have

$$\begin{aligned}
\|\mathcal{G}u\| &\leq \sup_{x\in[0,1]} \Bigg\{ |f(x,u(x))| + |a| \int_0^x \frac{(x-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |g(s,u(s))| d_q s \\
&\quad + |b| \int_0^x \frac{(x-qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha+\delta)} |h(s,u(s))| d_q s \\
&\quad + |u_0| \int_0^\eta \frac{(\eta-qs)^{(\gamma-1)}}{\Gamma_q(\gamma)} |u(s)| d_q s + |f(0, u_0 I_q^\gamma u(\eta))| \Bigg\} \\
&\leq \sup_{x\in[0,1]} \Bigg\{ \Big(|f(x,u(x)) - f(x,0)| + |f(x,0)| \Big) \\
&\quad + |a| \int_0^x \frac{(x-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \Big(|g(s,u(s)) - g(s,0)| + |g(s,0)| \Big) d_q s \\
&\quad + |b| \int_0^x \frac{(x-qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha+\delta)} \Big(|h(s,u(s)) - h(s,0)| + |h(s,0)| \Big) d_q s \\
&\quad + |u_0| \int_0^\eta \frac{(\eta-qs)^{(\gamma-1)}}{\Gamma_q(\gamma)} |u(s)| d_q s + \Big(|f(0,u(0)) - f(0,0)| + |f(0,0)| \Big) \Bigg\} \\
&\leq \left(L_1 + \frac{|a|L_2}{\Gamma_q(\alpha+1)} + \frac{|b|L_3}{\Gamma_q(\alpha+\delta+1)} + \frac{|u_0|(1+L_1)\,\eta^{(\gamma)}}{\Gamma_q(\gamma+1)} \right)\rho \\
&\quad + K_1 + \frac{|a|K_2}{\Gamma_q(\alpha+1)} + \frac{|b|K_3}{\Gamma_q(\alpha+\delta+1)},
\end{aligned}$$

which implies that $\|\mathcal{G}u\| \leq \rho$. Because $u \in B_\rho$ is arbitrary, $\mathcal{G}B_\rho \subset B_\rho$. For any $x \in [0,1]$ and any $u, v \in \mathbb{R}$, we obtain

$$\begin{aligned}
\|\mathcal{G}u - \mathcal{G}v\| &\leq \sup_{x\in[0,1]} \Bigg\{ |f(x,u(x)) - f(x,v(x))| \\
&\quad + |a| \int_0^x \frac{(x-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |g(s,u(s)) - g(s,v(s))| d_q s \\
&\quad + |b| \int_0^x \frac{(x-qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha+\delta)} |h(s,u(s)) - h(s,v(s))| d_q s \\
&\quad + |u_0| \int_0^\eta \frac{(\eta-qs)^{(\gamma-1)}}{\Gamma_q(\gamma)} |u(s) - v(s)| d_q s + |f(0, u_0 I_q^\gamma u(\eta)) - f(0, u_0 I_q^\gamma v(\eta))| \Bigg\} \\
&\leq \left(L_1 + \frac{|a|L_2}{\Gamma_q(\alpha+1)} + \frac{|b|L_3}{\Gamma_q(\alpha+\delta+1)} + \frac{|u_0|\eta^{(\gamma)}(1+L_1)}{\Gamma_q(\gamma+1)} \right) \|u - v\| \\
&= \Delta \|u - v\|.
\end{aligned}$$

As $\Delta < 1$, therefore, $\mathcal{G}$ is a contraction. Hence, we deduce by the conclusion of the Banach contraction mapping principle that the operator $\mathcal{G}$ has a unique fixed point, which is the unique continuous solution of the problem (1) and (2). The proof is completed. □

The following existence result is based on Krasnoselskii's fixed point theorem.

Lemma 2 (Krasnoselskii [22]). *Let Y be a closed, convex, bounded and nonempty subset of a Banach space X. Let Θ_1, Θ_2 be the operators such that (i) $\Theta_1 x + \Theta_2 y \in Y$ whenever $x, y \in Y$; (ii)*

Θ_2 is compact and continuous; (iii) Θ_1 is a contraction mapping. Then, there exists $\theta \in Y$ such that $\theta = \Theta_1\theta + \Theta_2\theta$.

Theorem 2. *Assume that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies the Lipschitz condition, that is, for all $u, v \in \mathbb{R}$ and $x \in [0,1]$, there exists a Lipschitz constant L_1 such that $|f(x,u) - f(x,v)| \le L_1|u - v|$ with $\sup_{x\in[0,1]} |f(x,0)| = K_1 < \infty$. Moreover, there exist $\sigma_i \in C([0,1], \mathbb{R}^+)$, $i = 1,2$ with $\|\sigma_i\| = \sup_{x\in[0,1]} |\sigma_i(x)|$ such that $|g(x,u)| \le \sigma_1(x)$, $|h(x,u)| \le \sigma_2(x)$, $\forall(x,u) \in [0,1] \times \mathbb{R}$. Then, the problem (1) and (2) has at least one continuous solution on $[0,1]$, provided that*

$$L_1 + \frac{|u_0|(1 + L_1)\eta^{(\gamma)}}{\Gamma_q(\gamma + 1)} < 1. \tag{9}$$

Proof. Consider the set $B_r = \{u \in \mathcal{C} : \|u\| \le r\}$, where r is given by

$$r \ge \left(K_1 + \frac{|a|\|\sigma_1\|}{\Gamma_q(\alpha+1)} + \frac{|b|\|\sigma_2\|}{\Gamma_q(\alpha+\delta+1)}\right)\left[1 - \left(L_1 + \frac{|u_0|(1+L_1)\eta^{(\gamma)}}{\Gamma_q(\gamma+1)}\right)\right]^{-1}.$$

Define operators $\mathcal{G}_1$ and $\mathcal{G}_2$ on B_r to $\mathbb{R}$ as

$$(\mathcal{G}_1u)(x) = f(x, u(x)) - f(0, u_0 I_q^{\gamma} u(\eta)) + u_0 \int_0^{\eta} \frac{(\eta - qs)^{(\gamma-1)}}{\Gamma_q(\gamma)} u(s) d_q s,$$

and

$$(\mathcal{G}_2u)(x) = a\int_0^x \frac{(x-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s,u(s))d_qs + b\int_0^x \frac{(x-qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha+\delta)} h(s,u(s))d_qs.$$

For $u, v \in B_r$, we find that

$$\|\mathcal{G}_1u + \mathcal{G}_2v\| \le K_1 + \frac{|a|\|\sigma_1\|}{\Gamma_q(\alpha+1)} + \frac{|b|\|\sigma_2\|}{\Gamma_q(\alpha+\delta+1)} + \left(L_1 + \frac{|u_0|(1+L_1)\eta^{(\gamma)}}{\Gamma_q(\gamma+1)}\right) r \le r.$$

Thus, $\mathcal{G}_1u + \mathcal{G}_2v \in B_r$. By the continuity of g and h, it follows that the operator $\mathcal{G}_2$ is continuous. In addition, $\mathcal{G}_2$ is uniformly bounded on B_r as

$$\|\mathcal{G}_2x\| \le \frac{|a|\|\sigma_1\|}{\Gamma_q(\alpha+1)} + \frac{|b|\|\sigma_2\|}{\Gamma_q(\alpha+\delta+1)}.$$

Now, we prove the compactness of the operator $\mathcal{G}_2$. We define

$$\sup_{(x,u)\in[0,1]\times B_r} |g(x,u)| = g_1, \qquad \sup_{(x,u)\in[0,1]\times B_r} |h(x,u)| = h_1.$$

Consequently, we have

$$\begin{aligned} & \|(\mathcal{G}_2u)(x_2) - (\mathcal{G}_2u)(x_1)\| \\ \le\ & |a|g_1\left|\int_0^{x_2} \frac{(x_2-qs)^{(\alpha-1)} - (x_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_qs + \int_{x_1}^{x_2} \frac{(x_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_qs\right| \\ +\ & |b|h_1\left|\int_0^{x_2} \frac{(x_2-qs)^{(\alpha+\delta-1)} - (x_1-qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha+\delta)} d_qs + \int_{x_1}^{x_2} \frac{(x_1-qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha+\delta)} d_qs\right|, \end{aligned}$$

which is independent of u and tends to zero as $x_2 \to x_1$. Thus, $\mathcal{G}_2$ is relatively compact on B_r. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{G}_2$ is compact on B_r. Now, we shall show that $\mathcal{G}_1$ is a contraction. For $u, v \in B_r$, we have

$$\begin{aligned}\|\mathcal{G}_1u-\mathcal{G}_1v\| &\leq \sup_{x\in[0,1]}\Big\{|f(x,u(x))-f(x,v(x))|\Big\}\\ &+ \Big|f\Big(0,u_0\int_0^\eta \frac{(\eta-qs)^{(\gamma-1)}}{\Gamma_q(\gamma)}u(s)d_qs\Big)-f\Big(0,u_0\int_0^\eta \frac{(\eta-qs)^{(\gamma-1)}}{\Gamma_q(\gamma)}v(s)d_qs\Big)\Big|\\ &+ \Big|u_0\Big(\int_0^\eta \frac{(\eta-qs)^{(\gamma-1)}}{\Gamma_q(\gamma)}u(s)d_qs-\int_0^\eta \frac{(\eta-qs)^{(\gamma-1)}}{\Gamma_q(\gamma)}v(s)d_qs\Big)\Big|\\ &\leq \Big(L_1+\frac{|u_0|(1+L_1)\eta^{(\gamma)}}{\Gamma_q(\gamma+1)}\Big)\|u-v\|,\end{aligned}$$

which, in view of the condition (9), implies that $\mathcal{G}_1$ is a contraction. Because the hypothesis of Lemma 3.1 is satisfied, its conclusion applies and hence the problem (1)–(2) has at least one continuous solution on $[0,1]$. □

4. Stability Result

In this section, we present the stability criteria for the solutions of the problem (1) and (2).

Theorem 3. *Suppose that the assumptions of Theorem 1 are satisfied. Then, the solution of the problem (1) and (2) is stable with respect to the nonlocal values, that is,*

$$|u_1(x)-u_2(x)| \leq \frac{\widehat{u}(1+L_1)}{1-(L_1+M_1+M_2)}|u_1(\eta)-u_2(\eta)|,\ 0\leq x\leq 1,$$

where

$$\widehat{u}=\frac{|u_0|\eta^{(\gamma)}}{\Gamma_q(\gamma+1)},\ M_1=\frac{|a|L_2}{\Gamma_q(\alpha+1)},\ M_2=\frac{|b|L_3}{\Gamma_q(\alpha+\delta+1)}, \tag{10}$$

and u_1,u_2 satisfy the problem (1) and (2) with nonlocal values $u_1(\eta)$ and $u_2(\eta)$, respectively.

Proof. Let u_1 and u_2 satisfy (1) and (2) and solve the fixed point equations $u_j=\mathcal{G}u_j$, for $j=1,2$, where $\mathcal{G}:\mathcal{C}\to\mathcal{C}$ is defined by (7). Then, by the given assumptions, we obtain

$$\begin{aligned}|u_1(x)-u_2(x)| &\leq |f(x,u_1(x))-f(x,u_2(x))|\\ &\quad+|a|\int_0^x \frac{(x-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|g(s,u_1(s))-g(s,u_2(s))|d_qs\\ &\quad+|b|\int_0^x \frac{(x-qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha+\delta)}|h(s,u_1(s))-h(s,u_2(s))|d_qs\\ &\quad+|u_1(0)-u_2(0)|+|f(0,u_1(0))-f(0,u_2(0))|\\ &\leq L_1|u_1(x)-u_2(x)|+M_1|u_1(x)-u_2(x)|\\ &\quad+M_2|u_1(x)-u_2(x)|+(1+L_1)|u_1(0)-u_2(0)|\\ &\leq (L_1+M_1+M_2)|u_1(x)-u_2(x)|+(1+L_1)|u_1(0)-u_2(0)|,\end{aligned}$$

that is

$$(1-L_1-M_1-M_2)|u_1(x)-u_2(x)| \leq (1+L_1)|u_1(0)-u_2(0)|. \tag{11}$$

On the other hand, we have

$$\begin{aligned} |u_1(0)-u_2(0)| &\leq |u_0|\int_0^{\eta}\frac{(\eta-qs)^{(\gamma-1)}}{\Gamma_q(\gamma)}|u_1(s)-u_2(s)|d_qs \\ &\leq |u_0|\Big(\int_0^{\eta}\frac{(\eta-qs)^{(\gamma-1)}}{\Gamma_q(\gamma)}d_qs\Big)|u_1(\eta)-u_2(\eta)| = \widehat{u}\,|u_1(\eta)-u_2(\eta)|, \end{aligned} \tag{12}$$

where $\widehat{u}$ is given in (10). Combining (11) and (12), we obtain

$$|u_1(x)-u_2(x)| \leq \frac{\widehat{u}(1+L_1)}{1-(L_1+M_1+M_2)}|u_1(\eta)-u_2(\eta)|.$$

This completes the proof. □

5. Examples

Here, we present examples to illustrate the obtained results.

Example 1. *Consider a fractional hybrid q-difference equation of fractional order with a nonlocal q-integro-initial condition given by*

$$\begin{cases} {}^cD_q^{\alpha}[u(x)-f(x,u(x))] = ag(x,u(x)) + bI_q^{\delta}h(x,u(x)), \ \ 0<x<1, \\ u(0) = u_0\displaystyle\int_0^{\eta}\frac{(\eta-qs)^{(\gamma-1)}}{\Gamma_q(\gamma)}u(s)d_qs, \end{cases} \tag{13}$$

where $q=\gamma=1/2, \delta=1/5, \eta=\alpha=1/4, a=b=1/3,$ *and* $u_0=0.8$.

I. Illustration of Theorem 1

Let us take

$$f(x,u)=\frac{7|\cos x||u|}{100(3+|u|)},\ g(x,u)=2x+\frac{3}{100}\sin u,\ h(x,u)=\frac{0.5|u|}{|u|+1}. \tag{14}$$

From (14), we obtain $L_1=0.07, L_2=0.03,$ *and* $L_3=0.5$ *as*

$$|f(x,u)-f(x,v)| \leq 0.07|u-v|, \ \textit{for each } x\in[0,1],$$

$$|g(x,u)-g(x,v)| \leq 0.03|u-v|, \ \textit{for each } x\in[0,1],$$

and

$$|h(x,u)-h(x,v)| \leq 0.5|u-v|, \ \textit{for each } x\in[0,1].$$

Moreover, $\Delta \simeq 0.726649 < 1$. *Clearly the assumptions of Theorem 1 hold. So, by the conclusion of Theorem 1, the problem (13) with the values of* $f, g,$ *and* h *given by (14) has a unique continuous solution on* $[0,1]$.

II. Illustration of Theorem 2

We take

$$f(x,u)=\frac{1}{8}\Big(\frac{x^2|u|}{1+|u|}+1\Big),\ g(x,u)=\frac{1}{2}\Big(|\sin u|x+\frac{1}{4}\Big),\ h(x,u)=\frac{2e^{-5x}\sin u}{x^2+10}+\frac{1}{5}. \tag{15}$$

Observe that $L_1=1/8$ *as* $|f(x,u)-f(x,v)|\leq\frac{1}{8}|u-v|$, *for each* $x\in[0,1]$, *and*

$$|g(x,u)| \leq \frac{1}{2}(x+\frac{1}{4}) = \sigma_1(x), \ \textit{for each } x\in[0,1],$$

$$|h(x,u)| \leq \frac{2e^{-5x}}{x^2+10} + \frac{1}{5} = \sigma_2(x), \ \ \textit{for each } x \in [0,1].$$

Further, we have

$$L_1 + \frac{|u_0|(1+L_1)\eta^{(\gamma)}}{\Gamma_q(\gamma+1)} \simeq 0.613665 < 1,$$

that is, the condition (9) is satisfied. Thus, all the conditions of Theorem 2 are satisfied and hence its conclusion applies to the problem (13) with the values of f, g, and h given by (15).

6. Conclusions

We have presented some new existence and uniqueness results for a nonlocal q-integro-initial value problem involving hybrid fractional q-integro-difference equations with mixed nonlinearities. Our study is based on the standard tools of the fixed point theory. The results presented in this article are more general and yield some new results as special cases corresponding to specific values of parameters α, δ, a , and b in the given problem. For instance, our results correspond to the ones with q-integral type nonlinearity if we take $a = 0$. In case $b = 0$, we obtain the results for the problem (1) and (2) with the non-integral type nonlinearity. Furthermore, it is worthwhile to notice that the stability of solutions of the problem at hand depends on the nonlocal parameter in contrast to the initial position of the domain. We believe that our results are new and indeed enrich the literature on hybrid fractional q-difference equations. In the future, we plan to study the higher order version of the hybrid fractional q-difference Equation (1) with different types of nonlocal and q-integral boundary conditions.

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