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Multiple Positive Solutions for Fractional Boundary Value Problems with Integral Boundary Conditions and Parameter Dependence

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Abstract: In this paper, we consider the existence of multiple positive solutions to boundary value problems of nonlinear fractional differential equation with integral boundary conditions and parameter dependence. To obtain our results, we used a functional-type cone expansion-compression fixed point theorem and the Leggett–Williams fixed point theorem. Examples are included to illustrate the main results.

Keywords: fractional differential equation; integral boundary conditions; positive solutions; Green's function; Leggett–Williams fixed point theorem

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Citation: Tabti, H.; Belmekki, M. Multiple Positive Solutions for Fractional Boundary Value Problems with Integral Boundary Conditions and Parameter Dependence. *Foundations* **2022**, *2*, 714–725. https://doi.org/10.3390/foundations2030049

Academic Editors: Sotiris K. Ntouyas and António Lopes

Received: 13 July 2022
Accepted: 26 August 2022
Published: 29 August 2022

1. Introduction

Over the last few years, fractional differential equations have attracted a great deal of attention for their numerous science and engineering applications: physics, electrical networks, polymer rheology, chemical technology, biology, control theory, and finance. For details, we refer the reader to the books [1–4]. Recently, some papers have dealt with the existence of positive solutions to different types of fractional differential equations by using nonlinear analysis (See [5–10] and the references therein).

We also noted that boundary value problems with integral boundary conditions for differential equations appear in applied mathematics and physics, chemical engineering, underground water flow, and thermo-elasticity. For details, the reader is referred to [11–18].

Motivated by these works, the aim of this paper is to study the existence of multiple solutions of the following nonlinear fractional differential equations with integral boundary conditions

$$D^{\delta}_{0+}u(t) + f(t, u(t)) = 0 \quad 0 < t < 1, \tag{1}$$

$$u(0) = 0 \quad u(1) = \lambda \int_0^1 h(r)u(r)dr, \tag{2}$$

where $1 < \delta \leq 2$ and $\lambda > 0$, D^{δ}_{0+} is the Riemann–Liouville fractional derivative, and f is a continuous function. We proved the existence of at least two positive solutions for the fractional boundary value problem (1) and (2) under suitable conditions on f. The main tools used were two well-known fixed point theorems on cones.

The text by Guo and Lakshmikantham [19] is an excellent resource for using fixed point theory in the study of solutions to boundary value problems.

First, we determined the corresponding Green's function and some of its properties; then the boundary value problem (1) and (2) was converted to an equivalent Fredholm integral equation of the second kind by using Green's function. In Section 3, by means of the properties of the function, the functional-type cone expansion–compression fixed-point theorem and the Leggett–Williams fixed-point theorem, we showed the existence of multiple positive solutions. Finally in Section 4, we give some illustrative examples to support the main results.

2. Preliminaries

In this section, for the convenience of the reader, we give some definitions and lemmas concerning the fractional calculus theory used in our proofs. For details, see [1,3,4].

Definition 1. *[3] The Riemann–Liouville fractional integral operator of the order $\delta > 0$ for a function $f : (0,+\infty) \longrightarrow \mathbb{R}$ is defined as*

$$I_{0^+}^{\delta} f(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} f(s)\, ds,$$

provided that the right side is pointwise defined on $(0,+\infty)$.

Definition 2. *[3] The Riemann–Liouville fractional derivative operator of the order $\delta > 0$ of a continuous function $f : (0,+\infty) \longrightarrow \mathbb{R}$ is given by*

$$D_{0^+}^{\delta} f(t) = \frac{1}{\Gamma(n-\delta)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\delta-1} f(s)\, ds,$$

where $n = [\delta] + 1$, and $[\delta]$ denotes the integral part of the number δ, provided the right-hand side is pointwisely defined on $(0,+\infty)$.

Lemma 1. *[3] Assume that $f \in \mathcal{C}(0,1) \cap \mathcal{L}(0,1)$ with a fractional derivative of the order $\delta > 0$ that belongs to $\mathcal{C}(0,1) \cap \mathcal{L}(0,1)$. Then*

$$I_{0^+}^{\delta} D_{0^+}^{\delta} f(t) = f(t) + c_1 t^{\delta-1} + c_2 t^{\delta-2} + \cdots + c_n t^{\delta-n},$$

where $c_i \in \mathbb{R}$, and $i = 1, 2, ..., n$ with $n - 1 < \delta \leq n$.

In [11], the authors derived Green's function in relation to problem (1) and (2). More precisely, the authors demonstrated the following lemma:

Lemma 2. *[11] We have*

$$D_{0^+}^{\delta} u(t) + y(t) = 0 \quad 0 < t < 1, \tag{3}$$

$$u(0) = 0 \quad u(1) = \lambda \int_0^1 h(r) u(r) dr, \tag{4}$$

where $1 < \delta \leq 2$. Assume that $1 - \lambda \int_0^1 h(r) r^{\delta-1} dr \neq 0$ and $y \in \mathcal{C}[0,1]$ then the boundary value problem (3) and (4) has the unique solution $u \in \mathcal{C}[0,1]$ as defined by the expression

$$u(t) = \int_0^1 G(t,s) y(s)\, ds,$$

where $G(t,s)$ is Green's function given by

$$G(t,s) = G_1(t,s) + G_2(t,s)$$

with

$$G_1(t,s) = \begin{cases} \frac{t^{\delta-1}(1-s)^{\delta-1} - (t-s)^{\delta-1}}{\Gamma(\delta)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\delta-1}(1-s)^{\delta-1}}{\Gamma(\delta)}, & 0 \leq t \leq s \leq 1, \end{cases} \tag{5}$$

and

$$G_2(t,s) = \frac{\lambda t^{\delta-1}}{1 - \lambda \int_0^1 h(r) r^{\delta-1} dr} \int_0^1 h(r) G_1(r,s)\, dr. \tag{6}$$

Lemma 3. *[6] The function $G_1(t,s)$ given in Lemma 2 has the following properties:*

(i) $G_1(t,s)$ *is a continuous function for all* $t,s \in [0,1]$.

(ii) $G_1(t,s) > 0$ *for all* $t,s \in (0,1)$.

(iii) $\max_{0\le t\le 1} G_1(t,s) = G_1(s,s) = \dfrac{s^{\delta-1}(1-s)^{\delta-1}}{\Gamma(\delta)}$ *for* $s \in (0,1)$.

(iv) There exists a positive function $\gamma \in \mathcal{C}(0,1)$ *such that*

$$\gamma(s)G_1(s,s) \le \min_{1/4\le t\le 3/4} G_1(t,s) \quad \textit{for} \quad s \in (0,1)$$

with

$$\gamma(s) = \begin{cases} \frac{[\frac{3}{4}(1-s)]^{\delta-1}-(\frac{3}{4}-s)^{\delta-1}}{s^{\delta-1}(1-s)^{\delta-1}}, & 0 < s \le r, \\ \frac{1}{(4s)^{\delta-1}}, & r \le s < 1, \end{cases}$$

where $1/4 < r < 3/4$ *is the unique solution of the equation*

$$\left[\frac{3}{4}(1-s)\right]^{\delta-1} - \left(\frac{3}{4}-s\right)^{\delta-1} = \frac{1}{4^{\delta-1}}(1-s)^{\delta-1}.$$

In the following lemma, we present two inequalities that will be used in the next section to prove the existence of solutions to the problem (1) and (2).

Lemma 4. *Assume that* $h \ge 0$ *on* $[0,1]$, *and* h *was introduced at the boundary conditions (2) as denoted by* $A = \int_0^1 h(r)r^{\delta-1}dr$, $B = \int_0^1 h(r)dr$ *and* $C = \int_{1/4}^{3/4} h(r)dr$. *Suppose that* $1-\lambda A > 0$ *with* $\lambda > 0$. *Then the Green's function* $G(t,s)$ *defined in Lemma 2 satisfies the following inequalities:*

$$\max_{0\le t\le 1} G(t,s) \le \left(1+\frac{\lambda B}{1-\lambda A}\right)G_1(s,s) \quad \textit{for all} \quad s \in (0,1) \tag{7}$$

$$k(s)G_1(s,s) \le \min_{1/4\le t\le 3/4} G(t,s) \quad \textit{for all} \quad s \in (0,1) \tag{8}$$

with $k(s) = \left[1+\dfrac{\lambda C}{4(1-\lambda A)}\right]\gamma(s)$ *for* $s \in (0,1)$.

Proof. First, from the expression of G and using Lemma 3 part (iii), we have

$$\begin{aligned} \max_{0\le t\le 1} G(t,s) &\le G_1(s,s) + \tfrac{\lambda}{1-\lambda A}\int_0^1 h(r)G_1(s,s)\,dr \\ &= \left(1+\tfrac{\lambda B}{1-\lambda A}\right)G_1(s,s), \qquad \forall s \in (0,1). \end{aligned}$$

Second, inequality (8) follows from Lemma 3 part (iv); in fact,

$$\begin{aligned} \min_{1/4\le t\le 3/4} G(t,s) &\ge \min_{1/4\le t\le 3/4} G_1(t,s) + \min_{1/4\le t\le 3/4} \frac{\lambda t^{\delta-1}}{1-\lambda A}\int_0^1 h(r)G_1(r,s)\,dr \\ &\ge \gamma(s)G_1(s,s) + \frac{\lambda}{4^{\delta-1}(1-\lambda A)}\int_0^1 h(r)G_1(r,s)\,dr \\ &\ge \gamma(s)G_1(s,s) + \frac{\lambda}{4(1-\lambda A)}\int_{1/4}^{3/4} h(r)\gamma(s)G_1(s,s)\,dr \\ &= \gamma(s)G_1(s,s) + \frac{\lambda C}{4(1-\lambda A)}\gamma(s)G_1(s,s) \\ &= k(s)G_1(s,s), \qquad \forall s \in (0,1). \end{aligned}$$

The proof is complete. □

Remark 1. *From the continuity and the non-negativeness of the function* γ *on* $(0,1)$ *(see [6]), then* $k \in \mathcal{C}\big((0,1),(0,+\infty)\big)$.

Now, we use the following fixed-point theorems to prove the main results. First, we give the definition of a cone.

Definition 3. *[19]. Let E be a real Banach space. A nonempty closed convex set* $P \subset E$ *is called a cone if it satisfies the following two conditions:*

1. $u \in P$, $\lambda \geq 0$ *implies* $\lambda u \in P$;
2. $u \in P$, $-u \in P$ *implies* $u = 0$.

Every cone $P \subset E$ induced an ordering in E given by $u \leq v$ if and only if $v - u \in P$.

Lemma 5. *[19]. Let E be an ordered Banach space such that* $P \subset E$ *is a cone. Furthermore, suppose that* Ω_1, Ω_2, Ω_3 *are bounded open subsets of E with* $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, $\overline{\Omega_2} \subset \Omega_3$. *Finally, let* $T : P \cap (\overline{\Omega_3} \backslash \Omega_1) \longrightarrow P$ *be a completely continuous operator such that*

(A_1) $\|Tu\| \geq \|u\|, \quad \forall u \in P \cap \partial\Omega_1$;
(A_2) $\|Tu\| \leq \|u\|, \quad Tu \neq u, \forall u \in P \cap \partial\Omega_2$;
(A_3) $\|Tu\| \geq \|u\|, \quad \forall u \in P \cap \partial\Omega_3$.

T has at least two fixed points u^* *and* u^{**} *in* $P \cap (\overline{\Omega_3} \backslash \Omega_1)$; *moreover,* $u^* \in P \cap (\Omega_2 \backslash \Omega_1)$ *and* $u^{**} \in P \cap (\overline{\Omega_3} \backslash \overline{\Omega_2})$.

Definition 4. *We say that the map* ψ *is a non-negative continuous concave functional on a cone P of a real Banach space E provided that* $\psi : P \longrightarrow [0,+\infty)$ *is continuous and*

$$\psi\big(tu + (1-t)v\big) \geq t\psi(u) + (1-t)\psi(v),$$

for all $u, v \in P$ *and* $0 \leq t \leq 1$.

Let

$$P(\psi, b, d) = \{u \in P / \ b \leq \psi(u), \|u\| \leq d\}.$$

Theorem 1. *(Leggett–Williams) [20]. Let P be a cone in a real Banach space E,* $\overline{P_c} = \{u \in P / \ \|u\| \leq c\}$, ψ *be a non-negative continuous concave positive function on cone P such that* $\psi(u) \leq \|u\|$ *for all* $u \in \overline{P_c}$. *Suppose* $T : \overline{P_c} \longrightarrow \overline{P_c}$ *is completely continuous and there exist constants* $0 < a < b < d \leq c$ *such that*

(B_1) $\{u \in P(\psi, b, d) / \ \psi(u) > b\} \neq \emptyset$ *and* $\psi(Tu) > b$ *for* $u \in P(\psi, b, d)$,
(B_2) $\|Tu\| < a$ *for* $u \in \overline{P_a}$,
(B_3) $\psi(Tu) > b$ *for* $u \in P(\psi, b, c)$ *with* $\|Tu\| > d$.

Then T has at least three fixed points u_1, u_2 *and* u_3 *such that* $\|u_1\| < a$, $b < \psi(u_2)$, $\|u_3\| > a$ *with* $\psi(u_3) < b$.

3. Existence of Multiple Positive Solutions

This section is devoted to proving the existence of multiple positive solutions for problem (1) and (2). For this end, we introduce the following notations:

$$K = \frac{1}{\Gamma(\delta+1)} + \frac{\Gamma(\delta)}{\Gamma(2\delta)} \cdot \frac{\lambda h^*}{1-\lambda A}, \tag{9}$$

$$M = \left[\left(1 + \frac{\lambda B}{1-\lambda A}\right) \int_0^1 G_1(s,s)\, ds\right]^{-1}, \tag{10}$$

$$N = \left(\int_{1/4}^{3/4} k(s)G_1(s,s)\,ds\right)^{-1}. \tag{11}$$

Let E be the Banach space $(\mathcal{C}[0,1], \|\cdot\|)$ where $\|u\| = \max_{0\le t\le 1} |u(t)|$. Define the cone $P \subset E$ by

$$P = \{u \in E/\ u(t) \ge 0,\ t \in [0,1]\}$$

and the operator $T : E \longrightarrow E$ by

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))\,ds \tag{12}$$

with G defined in Lemma 2.

It is clear that the fixed points of the operator T are the solutions to the problem (1) and (2).

Lemma 6. *Assume that $f \in \mathcal{C}\big([0,1]\times[0,+\infty),[0,+\infty)\big)$. The operator $T : P \longrightarrow P$ defined by (12) is completely continuous.*

Proof. By Lemma 4 $G(t,s) \ge 0$,so $Tu(t) \ge 0$ for all $u \in P$. The operator $T : P \longrightarrow P$ is continuous in view of continuity of the function $G(t,s)$ and $f(s,u(s))$. Let $\mathcal{M} \subset P$ be bounded, which is to say there exists a positive $R > 0$ such that $\mathcal{M} = \{u \in P/\ \|u\| \le R\}$. Let

$$L = \max_{0\le t\le 1, 0\le u\le R} |f(t,u(t))| + 1.$$

From inequality (7) and for all $u \in \mathcal{M}$, we have

$$\begin{aligned} |Tu(t)| &= \left|\int_0^1 G(t,s)f(s,u(s))\,ds\right| \\ &\le \frac{L}{\Gamma(\delta)}\left(1+\frac{\lambda B}{1-\lambda A}\right)\int_0^1 s^{\delta-1}(1-s)^{\delta-1}\,ds \\ &= L\left(1+\frac{\lambda B}{1-\lambda A}\right)\frac{\Gamma(\delta)}{\Gamma(2\delta)}. \end{aligned}$$

Hence, $T(\mathcal{M})$ is bounded. On the other hand given $\epsilon > 0$, set

$$\eta = \frac{1}{2}\left(\frac{\epsilon}{LK}\right)^{1/(\delta-1)}.$$

Now, we show that whenever $t_1, t_2 \in [0,1]$ and $0 < t_2 - t_1 < \eta$, then $\big|Tu(t_2) - Tu(t_1)\big| < \epsilon$. In fact,

$$\begin{aligned} \big|Tu(t_2) - Tu(t_1)\big| &= \left|\int_0^1 \Big[G(t_2,s) - G(t_1,s)\Big]f(s,u(s))\,ds\right| \\ &\le \int_0^1 \Big|G(t_2,s) - G(t_1,s)\Big|f(s,u(s))\,ds \\ &< L\int_0^1 \Big|G(t_2,s) - G(t_1,s)\Big|\,ds. \end{aligned}$$

Then, we have

$$\begin{aligned} \int_0^1 \Big|G(t_2,s) - G(t_1,s)\Big|\,ds &\le \int_0^1 \Big|G_1(t_2,s) - G_1(t_1,s)\Big|\,ds \\ &\quad + \int_0^1 \Big|G_2(t_2,s) - G_2(t_1,s)\Big|\,ds. \end{aligned}$$

From the definition of $G_1(t,s)$, we obtained

$$\begin{aligned}
\int_0^1 \Big|G_1(t_2,s)-G_1(t_1,s)\Big|\,ds &= \int_0^{t_1} \Big|G_1(t_2,s)-G_1(t_1,s)\Big|\,ds \\
&\quad + \int_{t_1}^{t_2} \Big|G_1(t_2,s)-G_1(t_1,s)\Big|\,ds + \int_{t_2}^{1} \Big|G_1(t_2,s)-G_1(t_1,s)\Big|\,ds \\
&= \frac{1}{\Gamma(\delta)}\int_0^{t_1}\Big|t_2^{\delta-1}(1-s)^{\delta-1}-(t_2-s)^{\delta-1}-t_1^{\delta-1}(1-s)^{\delta-1}+(t_1-s)^{\delta-1}\Big|\,ds \\
&\quad +\frac{1}{\Gamma(\delta)}\int_{t_1}^{t_2}\Big|t_2^{\delta-1}(1-s)^{\delta-1}-(t_2-s)^{\delta-1}-t_1^{\delta-1}(1-s)^{\delta-1}\Big|\,ds \\
&\quad +\frac{1}{\Gamma(\delta)}\int_{t_2}^{1}\Big|t_2^{\delta-1}(1-s)^{\delta-1}-t_1^{\delta-1}(1-s)^{\delta-1}\Big|\,ds \\
&< \frac{1}{\Gamma(\delta)}\bigg[\int_0^{t_1}(t_2^{\delta-1}-t_1^{\delta-1})(1-s)^{\delta-1}\,ds+\int_{t_1}^{t_2}(t_2^{\delta-1}-t_1^{\delta-1})(1-s)^{\delta-1}\,ds \\
&\quad +\int_{t_2}^{1}(t_2^{\delta-1}-t_1^{\delta-1})(1-s)^{\delta-1}\,ds\bigg] \\
&\leq \frac{1}{\Gamma(\delta+1)}(t_2^{\delta-1}-t_1^{\delta-1}).
\end{aligned}$$

Now, denote by $H(s)=\int_0^1 h(r)G_1(r,s)\,dr$ and $h^*=\max_{0\leq t\leq 1} h(t)$, then from the expression $G_2(t,s)$ represented by (6), bearing in mind that

$$\begin{aligned}
\int_0^1 H(s)\,ds &\leq h^*\int_0^1\int_0^1 G_1(r,s)\,dr\,ds \\
&\leq \frac{h^*}{\Gamma(\delta)}\int_0^1 s^{\delta-1}(1-s)^{\delta-1}\,ds = h^*\frac{\Gamma(\delta)}{\Gamma(2\delta)},
\end{aligned}$$

we obtained

$$\begin{aligned}
\int_0^1 \Big|G_2(t_2,s)-G_2(t_1,s)\Big|\,ds &= \frac{\lambda(t_2^{\delta-1}-t_1^{\delta-1})}{1-\lambda A}\int_0^1 H(s)\,ds \\
&\leq \frac{\Gamma(\delta)}{\Gamma(2\delta)}\cdot\frac{\lambda h^*}{1-\lambda A}(t_2^{\delta-1}-t_1^{\delta-1}).
\end{aligned}$$

Then, we deduced that

$$\begin{aligned}
\big|Tu(t_2)-Tu(t_1)\big| &< L\bigg(\frac{1}{\Gamma(\delta+1)}+\frac{\Gamma(\delta)}{\Gamma(2\delta)}\cdot\frac{\lambda h^*}{1-\lambda A}\bigg)(t_2^{\delta-1}-t_1^{\delta-1}) \\
&= LK(t_2^{\delta-1}-t_1^{\delta-1}).
\end{aligned}$$

To estimate $t_2^{\delta-1}-t_1^{\delta-1}$, we used a method applied in [6].
Case 01. $\eta\leq t_1<t_2<1$

$$\begin{aligned}
\big|Tu(t_2)-Tu(t_1)\big| &< LK(t_2^{\delta-1}-t_1^{\delta-1})\leq LK\frac{\delta-1}{\eta^{2-\delta}}(t_2-t_1) \\
&\leq LK\eta^{\delta-1}<\epsilon.
\end{aligned}$$

Case 02. $0\leq t_1<\eta,\, t_2<2\eta$

$$\big|Tu(t_2)-Tu(t_1)\big| < LK(t_2^{\delta-1}-t_1^{\delta-1})\leq LKt_2^{\delta-1}\leq LK(2\eta)^{\delta-1}\leq\epsilon.$$

Thus, the set $T(\mathcal{M})$ is equicontinuous in E. As a consequence of the Arzelà–Ascoli theorem, we concluded that $T: P \longrightarrow P$ is completely continuous. $\square$

In our first result, we proved the existence of at least two positives solutions for the problem (1) and (2).

Theorem 2. *Assume that $f \in \mathcal{C}\big([0,1] \times [0,+\infty),[0,+\infty)\big)$. There exist three positive constants $0 < \sigma_1 < \sigma_2 < \sigma_3$ such that*

(C_1) $f(t,u) \geq N\sigma_1$, for $(t,u) \in [0,1] \times [0,\sigma_1]$
(C_2) $f(t,u) \leq M\sigma_2$, for $(t,u) \in [0,1] \times [0,\sigma_2]$
(C_3) $f(t,u) \geq N\sigma_3$, for $(t,u) \in [0,1] \times [0,\sigma_3]$.

Then the problem (1) and (2) has at least two positive solutions $u^*, u^{**} \in P$ with

$$\sigma_1 \leq \|u^*\| < \sigma_2 \qquad \text{and} \qquad \sigma_2 < \|u^{**}\| \leq \sigma_3.$$

Proof. We know by Lemma 6 that $T: P \longrightarrow P$ is completely continuous. Now, we divided the proof into three steps.

Step 01. Let $\Omega_1 = \{u \in P / \ \|u\| < \sigma_1\}$. For any $u \in P \cap \partial\Omega_1$, we have $\|u\| = \sigma_1$ and $0 \leq u(t) \leq \sigma_1$ for all $t \in [0,1]$. It follows from condition (C_1) and Lemma 4 inequality (8) that for $t \in [0,1]$

$$\begin{aligned}
\|Tu\| &= \max_{0\leq t\leq 1}\left|\int_0^1 G(t,s)f(s,u(s))\,ds\right| \\
&\geq \max_{1/4\leq t\leq 3/4}\left|\int_0^1 G(t,s)f(s,u(s))\,ds\right| \\
&\geq N\sigma_1\left[\int_{1/4}^{3/4} k(s)G_1(s,s)\,ds\right] = \sigma_1 = \|u\|
\end{aligned}$$

which implies that $\|Tu\| \geq \|u\| \ \forall u \in P \cap \partial\Omega_1$.

Step 02. Let $\Omega_2 = \{u \in P / \ \|u\| < \sigma_2\}$. For any $u \in P \cap \partial\Omega_2$ we have $0 \leq u(t) \leq \sigma_2$ for all $t \in [0,1]$. It follows from condition (C_2) and Lemma 4 inequality (7) that for $t \in [0,1]$

$$\begin{aligned}
\|Tu\| &= \max_{0\leq t\leq 1}\left|\int_0^1 G(t,s)f(s,u(s))\,ds\right| \\
&\leq \left(1+\frac{\lambda B}{1-\lambda A}\right)\int_0^1 G_1(s,s)f(s,u(s))\,ds \\
&\leq M\sigma_2\left[\left(1+\frac{\lambda B}{1-\lambda A}\right)\int_0^1 G_1(s,s)\,ds\right] = \sigma_2 = \|u\|
\end{aligned}$$

so $\|Tu\| \leq \|u\| \ \forall u \in P \cap \partial\Omega_2$.

Step 03. Let $\Omega_3 = \{u \in P / \ \|u\| < \sigma_3\}$. For any $u \in P \cap \partial\Omega_3$ we have $\|u\| = \sigma_3$, then $0 \leq u(t) \leq \sigma_3$ for all $t \in [0,1]$. Then by condition (C_3) we have

$$\begin{aligned}
\|Tu\| &= \max_{0\leq t\leq 1}\left|\int_0^1 G(t,s)f(s,u(s))\,ds\right| \\
&\geq \max_{1/4\leq t\leq 3/4}\left|\int_0^1 G(t,s)f(s,u(s))\,ds\right| \\
&\geq N\sigma_3\left[\int_{1/4}^{3/4} k(s)G_1(s,s)\,ds\right] = \sigma_3 = \|u\|
\end{aligned}$$

which implies that $\|Tu\| \geq \|u\| \; \forall u \in P \cap \partial\Omega_3$. By Lemma 5, T has at least two fixed points (u^* and u^{**}) in $P \cap (\overline{\Omega_3} \backslash \Omega_1)$; therefore the problem (1) and (2) has at least two positive solutions, u^* and $u^{**}, \in P$ such that

$$\sigma_1 \leq \|u^*\| < \sigma_2 \qquad \text{and} \qquad \sigma_2 < \|u^{**}\| \leq \sigma_3.$$

□

In the next result, we show the existence of at least three positive solutions of the boundary value problem (1) and (2).

Let the non-negative continuous concave positive functional ψ on the cone P be defined by

$$\psi(u) = \min_{1/4 \leq t \leq 3/4} |u(t)|.$$

It is easy to verify that, $\forall u \in P \quad \psi(u) \leq \|u\|$.

Theorem 3. *Assume that $f \in \mathcal{C}\big([0,1] \times [0,+\infty), [0,+\infty)\big)$ and there exist constants $0 < a < b < c$ such that*

(H_1) $f(t,u) < Ma$, for $(t,u) \in [0,1] \times [0,a]$
(H_2) $f(t,u) \geq Nb$, for $(t,u) \in [1/4,3/4] \times [b,c]$
(H_3) $f(t,u) \leq Mc$, for $(t,u) \in [0,1] \times [0,c]$.

Then the problem (1) and (2) has at least three positive solutions—u_1, u_2 and u_3—with

$$\max_{0 \leq t \leq 1} |u_1(t)| < a, \qquad b < \min_{1/4 \leq t \leq 3/4} |u_2(t)| < \max_{0 \leq t \leq 1} |u_2(t)| \leq c,$$

$$a < \max_{0 \leq t \leq 1} |u_3(t)| \leq c, \qquad \min_{1/4 \leq t \leq 3/4} |u_3(t)| < b.$$

Proof. We show that all the conditions of Theorem 1 were satisfied. Let $u \in \overline{P_c}$ then $\|u\| \leq c$ and by (H_3) with Equation (10), we have

$$\begin{aligned} \|Tu\| &= \max_{0 \leq t \leq 1} \left| \int_0^1 G(t,s) f(s,u(s))\, ds \right| \\ &\leq \left(1 + \frac{\lambda B}{1 - \lambda A}\right) \int_0^1 G_1(s,s) f(s,u(s))\, ds \\ &\leq Mc\left[\left(1 + \frac{\lambda B}{1 - \lambda A}\right) \int_0^1 G_1(s,s)\, ds\right] = c. \end{aligned}$$

Hence $T : \overline{P_c} \longrightarrow \overline{P_c}$, and by Lemma 6 is completely continuous.

Choosing $u(t) = (b+c)/2$ for $t \in [0,1]$. It is clear that

$$\frac{b+c}{2} \in P(\psi,b,c) \qquad \text{and} \qquad \psi\left(\frac{b+c}{2}\right) > b;$$

therefore, $\{u \in P(\psi,b,c) / \; \psi(u) > b\} \neq \emptyset$. Let $u \in P(\psi,b,c)$ then $b \leq u(t) \leq c$ for $t \in [1/4,3/4]$. From assumption (H_2) and Equation (11), we obtain

$$\begin{aligned} \psi(Tu) &= \min_{1/4 \leq t \leq 3/4} \int_0^1 G(t,s) f(s,u(s))\, ds \\ &\geq \int_0^1 k(s) G_1(s,s) f(s,u(s))\, ds \\ &> Nb \int_{1/4}^{3/4} k(s) G_1(s,s)\, ds. \end{aligned}$$

Consequently, we have $\quad \psi(Tu) > b \qquad \forall u \in P(\psi,b,c).$

Hence, condition (B_1) from Theorem 1 holds. Now, we show that the condition (B_2) of Theorem 1 is satisfied. If $u \in \overline{P_a}$ then $\|u\| \leq a$. Assumption (H_1) implies that $f(t,u) < Ma$ for $t \in [0,1]$. Thus

$$\begin{aligned} \|Tu\| &= \max_{0\leq t\leq 1}\left|\int_0^1 G(t,s)f(s,u(s))\,ds\right| \leq \left(1+\frac{\lambda B}{1-\lambda A}\right)\int_0^1 G_1(s,s)f(s,u(s))\,ds \\ &< Ma\left(1+\frac{\lambda B}{1-\lambda A}\right)\int_0^1 G_1(s,s)\,ds = a. \end{aligned}$$

This implies that condition (B_2) from Theorem 1 is satisfied.

Finally, we prove that the condition (B_3) of Theorem 1 w satisfied. If $u \in P(\psi, b, c)$ then $b \leq u(t) \leq c$ for $1/4 \leq t \leq 3/4$. From assumption (H_2), we have

$$\begin{aligned} \psi(Tu) &= \min_{1/4\leq t\leq 3/4}\int_0^1 G(t,s)f(s,u(s))\,ds \\ &\geq \int_0^1 k(s)G_1(s,s)f(s,u(s))\,ds \\ &> Nb\left(\int_{1/4}^{3/4} k(s)G_1(s,s)\,ds\right). \end{aligned}$$

Thus, the condition (B_3) from Theorem 1 is also satisfied. By Theorem 1, problem (1) and (2) has at least three positives solutions u_1, u_2 and u_3 with

$$\max_{0\leq t\leq 1}|u_1(t)| < a, \qquad b < \min_{1/4\leq t\leq 3/4}|u_2(t)| < \max_{0\leq t\leq 1}|u_2(t)| \leq c,$$

$$a < \max_{0\leq t\leq 1}|u_3(t)| \leq c, \qquad \min_{1/4\leq t\leq 3/4}|u_3(t)| < b.$$

□

4. Examples

In this section, we provide the following examples to demonstrate the consistency of the main theorems.

Example 1. *Consider the following problem*

$$\begin{cases} D_{0^+}^{3/2}u(t) + f(t,u(t)) = 0, & 0 < t < 1 \\ u(0) = 0 \quad u(1) = 2\int_0^1 su(s)ds, \end{cases} \tag{13}$$

with

$$f(t,u) = \frac{u^3}{16} + \frac{e^{t/2}}{8}; \quad t \in [0,1].$$

In system (13), we see that $\delta = \frac{3}{2}$ *and* $\lambda = 2$ *and* $h(t) = t$. *Then* $A = \int_0^1 h(r)r^{1/2}\,dr = \int_0^1 r^{3/2}\,dr = \frac{2}{5}$ *with condition*

$$1 - \lambda A = 1 - 4/5 > 0 \quad \textit{holds.}$$

$B = \int_0^1 r\,dr = \frac{1}{2}$ *and* $C = \int_{1/4}^{3/4} r\,dr = \frac{1}{4}$. *By simple calculation, we obtain*

$$M \approx 0.3761 \qquad \textit{and} \qquad N \approx 8.4092.$$

Choosing $\sigma_1 = \frac{1}{80}$, $\sigma_2 = 2$ *and* $\sigma_3 = 14$. *We get*

$(C_1)'$ $f(t,u) = \dfrac{u^3}{16} + \dfrac{e^{t/2}}{8} > 0.125 \geq N\sigma_1 \approx 0.1051$ *for* $t \in [0,1]$ *and* $\|u\| = \dfrac{1}{80}$

$(C_2)'$ $f(t,u) = \dfrac{u^3}{16} + \dfrac{e^{t/2}}{8} \leq 0.7061 \leq M\sigma_2 \approx 0.7522$ *for* $t \in [0,1]$ *and* $\|u\| = 2$

$(C_3)'$ $f(t,u) = \dfrac{u^3}{16} + \dfrac{e^{t/2}}{8} \geq 171.6 \geq N\sigma_3 \approx 117.7288$ *for* $t \in [0,1]$ *and* $\|u\| = 14$.

With the use of Theorem 2, problem (13) has at least two positive solutions, u^* *and* u^{**} *such that*

$$\frac{1}{80} \leq \|u^*\| < 2 \qquad \text{and} \qquad 2 < \|u^{**}\| \leq 14.$$

Example 2. *Now, we consider the same boundary value problem*

$$\begin{cases} D_{0^+}^{3/2} u(t) + f(t,u(t)) = 0, & 0 < t < 1 \\ u(0) = 0 \quad u(1) = 2\displaystyle\int_0^1 su(s)ds, \end{cases} \tag{14}$$

where

$$f(t,u) = \begin{cases} \dfrac{t}{25} + 9u^4, & u < 1, \\ \\ \dfrac{89}{10} + \dfrac{t}{25} + \dfrac{u}{10}, & u \geq 1. \end{cases}$$

We have

$$M \approx 0.3761 \qquad \text{and} \qquad N \approx 8.4092.$$

Choosing $a = \dfrac{1}{5}$, $b = 1$ *and* $c = 35$ *there hold*

$(H_1)'$ $f(t,u) = \dfrac{t}{25} + 9u^4 \leq 0.0544 < Ma \approx 0.0752$ *for* $(t,u) \in [0,1] \times [0,1/5]$

$(H_2)'$ $f(t,u) = \dfrac{89}{10} + \dfrac{t}{25} + \dfrac{u}{10} \geq 9.01 > Nb \approx 8.4092$ *for* $(t,u) \in [1/4,3/4] \times [1,35]$

$(H_3)'$ $f(t,u) = \dfrac{89}{10} + \dfrac{t}{25} + \dfrac{u}{10} \leq 12.44 < Mc \approx 13.1635$ *for* $(t,u) \in [0,1] \times [0,35]$.

With the use of Theorem 3, problem (14) has at least three positive solutions, u_1, u_2 *and* u_3 *with*

$$\max_{0\leq t\leq 1} |u_1(t)| < \frac{1}{5}, \qquad 1 < \min_{1/4\leq t\leq 3/4} |u_2(t)| < \max_{0\leq t\leq 1} |u_2(t)| \leq 35,$$

$$\frac{1}{5} < \max_{0\leq t\leq 1} |u_3(t)| \leq 35, \qquad \min_{1/4\leq t\leq 3/4} |u_3(t)| < 1.$$

Example 3. *Let us consider the following problem*

$$\begin{cases} D_{0^+}^{3/2} u(t) + f(t,u(t)) = 0, & 0 < t < 1 \\ u(0) = 0 \quad u(1) = \displaystyle\int_0^1 \sqrt{s}u(s)ds, \end{cases} \tag{15}$$

with

$$f(t,u) = \begin{cases} \dfrac{1}{20}t^2 + 12u^3, & u < 1, \\ \\ \dfrac{35}{3} + \dfrac{1}{20}t^2 + \dfrac{u}{3}, & u \geq 1. \end{cases}$$

In system (15), we see that $\delta = \frac{3}{2}$ *and* $\lambda = 1$ *and* $h(t) = \sqrt{t}$. *Then* $A = \int_0^1 h(r) r^{1/2}\, dr = \int_0^1 r\, dr = \frac{1}{2}$ *on the condition that*

$$1 - \lambda A = 1 - 1/2 > 0 \quad \textit{holds.}$$

$B = \int_0^1 \sqrt{r}\, dr = \frac{2}{3}$ *and* $C = \int_{1/4}^{3/4} \sqrt{r}\, dr = \frac{3\sqrt{3}-1}{12}$. *By simple calculation, we obtain*

$$M \approx 0.9671 \qquad \textit{and} \qquad N \approx 11.6313.$$

Choosing $a = \frac{1}{10}$, $b = 1$ *and* $c = 20$ *there satisfy*

$(H_1)''f(t,u) = \frac{1}{20}t^2 + 12u^3 \leq 0.062 < Ma \approx 0.0967$ *for* $(t,u) \in [0,1] \times [0,1/10]$

$(H_2)''f(t,u) = \frac{35}{3} + \frac{1}{20}t^2 + \frac{u}{3} \geq 12.003 > Nb \approx 11.6313$ *for* $(t,u) \in [1/4,3/4] \times [1,20]$

$(H_3)''f(t,u) = \frac{35}{3} + \frac{1}{20}t^2 + \frac{u}{3} \leq 18.384 < Mc \approx 19.342$ *for* $(t,u) \in [0,1] \times [0,20]$.

Then all conditions of Theorem 3 hold. Thus, the problem (15) has at least three positive solutions—u_1, u_2 *and* u_3*—with*

$$\max_{0 \leq t \leq 1} |u_1(t)| < \frac{1}{10}, \qquad 1 < \min_{1/4 \leq t \leq 3/4} |u_2(t)| < \max_{0 \leq t \leq 1} |u_2(t)| \leq 20,$$

$$\frac{1}{10} < \max_{0 \leq t \leq 1} |u_3(t)| \leq 20, \qquad \min_{1/4 \leq t \leq 3/4} |u_3(t)| < 1.$$

Author Contributions: All authors participated in obtaining the results and writing the paper. All authors have read and agreed with the final manuscript.

Funding: This research received no external funding.

Acknowledgments: The authors express their sincere thanks to the referees and editors for their comments and suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

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