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Investigation of a Nonlinear Coupled (k, ψ)–Hilfer Fractional Differential System with Coupled (k, ψ)–Riemann–Liouville Fractional Integral Boundary Conditions

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Abstract: This paper is concerned with the existence of solutions for a new boundary value problem of nonlinear coupled (k, ψ)–Hilfer fractional differential equations subject to coupled (k, ψ)–Riemann–Liouville fractional integral boundary conditions. We prove two existence results by applying the Leray–Schauder alternative, and Krasnosel'skiĭ's fixed-point theorem under different criteria, while the third result, concerning the uniqueness of solutions for the given problem, relies on the Banach's contraction mapping principle. Examples are included for illustrating the abstract results.

Keywords: (k, ψ)–Hilfer fractional derivative; nonlinear systems of equations; (k, ψ)–Riemann–Liouville fractional integral boundary conditions; existence of solutions; fixed point

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1. Introduction

Fractional-order boundary value problems involving classical, nonlocal, multi-point, and integral boundary conditions have attracted significant interest in the recent years, as such problems appear in diverse disciplines of engineering and applied sciences. For theoretical details of the topic, see the monographs [1–3]. Recently, some existence results for nonlocal integro-multipoint boundary value problems involving Hilfer-type fractional integro-differential equations and inclusions were presented in [4]. In [5], the authors analyzed a multipoint boundary value problem involving a Caputo–Hadamard fractional derivative operator, while a nonlinear Caputo–Hadamard fractional initial value problem was studied in [6].

Great interest has also been shown in studying the coupled systems of fractional-order differential equations, as such systems constitute the mathematical models of many physical phenomena occurring in bio-engineering [7], fractional dynamics [8], and financial economics [9], etc. Let us now dwell on some recent works on boundary value problems involving coupled fractional differential systems. In [10,11], the authors proved some interesting results for coupled systems of sequential fractional differential equations involving a ψ–Hilfer fractional derivative. Boundary value problems for Hilfer-type sequential fractional differential equations and inclusions with Riemann–Stieltjes integral multi-strip boundary conditions were studied in [12].

It is interesting to notice that there do exist different definitions of fractional derivative operators such as Riemann–Liouville, Caputo–Liouville, Hadamard, Erdélyi–Kober, ψ–Riemann–Liouville [1], Hilfer [13], k–Riemann–Liouville [14], ψ–Hilfer [15], (k, ψ)–Riemann–Liouville [14] and (k, ψ)–Hilfer [16], to name a few.

The topic of (k,ψ)–Hilfer-type fractional boundary value problems constitutes an interesting and useful area of research, as a (k,ψ)–Hilfer-type fractional derivative operator generalizes many well-known fractional derivative operators; for details, see [17] and the references cited therein. Recently, in [18], the authors studied the existence of solutions for a nonlocal nonlinear (k,ψ)–Hilfer-type fractional boundary value problem. For some more results on (k,ψ)–Hilfer fractional boundary value problems, see [19]. In [20], the authors studied a coupled system of nonlinear fractional differential equations involving the (k,φ)–Hilfer fractional derivative operators with nonlocal multipoint boundary conditions:

$$\begin{cases} {}^{k,H}D^{\overline{\alpha},\overline{\beta};\varphi}w(\theta) = f(\theta, w(\theta), z(\theta)), \quad \theta \in (c,d], \\ {}^{k,H}D^{\alpha_1,\beta_1;\varphi}z(\theta) = f_1(\theta, w(\theta), z(\theta)), \quad \theta \in (c,d], \\ w(c) = 0,\ w(d) = \sum\limits_{i=1}^{m} \lambda_i\, z(\xi_i),\ z(c) = 0,\ z(d) = \sum\limits_{j=1}^{k} \mu_j\, w(\eta_j), \end{cases}$$

where ${}^{k,H}D^{\overline{\alpha},\overline{\beta};\varphi}$, ${}^{k,H}D^{\alpha_1,\beta_1;\varphi}$ denote the (k,φ)–Hilfer fractional derivative operators of orders $\overline{\alpha}, \alpha_1$, $1 < \overline{\alpha}, \alpha_1 < 2$ and parameters $\overline{\beta}, \beta_1$, $0 \le \overline{\beta}, \beta_1 \le 1$, respectively; $f, f_1 : [c,d] \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $\lambda_i, \mu_j \in \mathbb{R}$; and $a < \xi_i, \eta_j < b, i = 1,2,\ldots,m, j = 1,2,\ldots,k$.

Exploring the literature on (k,ψ)–Hilfer fractional differential systems, it is found that the only work dealing with such a coupled system was discussed in [20]. In order to develop this new area of research, we consider a nonlinear coupled system of (k,ψ)–Hilfer fractional differential equations of different orders in $(1,2]$, complemented with coupled (k,ψ)–Riemann-Liouville fractional integral boundary conditions. In precise terms, we investigate the existence of solutions for the following problem:

$$\begin{cases} {}^{k,H}\mathcal{D}^{\hat{\alpha},\hat{\beta};\psi}\hat{r}(t) = f(t, \hat{r}(t), \hat{z}(t)), \quad t \in (a,b], \\ {}^{k,H}\mathcal{D}^{\hat{p},\hat{q};\psi}\hat{z}(t) = \tilde{f}(t, \hat{r}(t), \hat{z}(t)), \quad t \in (a,b], \\ \hat{r}(a) = 0, \quad \hat{r}(b) = \lambda\, \hat{z}(\xi) + \mu\, {}^{k}\mathcal{I}^{v,\psi}\hat{z}(\sigma), \\ \hat{z}(a) = 0, \quad \hat{z}(b) = \varsigma\, \hat{r}(u) + \theta\, {}^{k}\mathcal{I}^{w,\psi}\hat{r}(\tau), \end{cases} \tag{1}$$

where ${}^{k,H}\mathcal{D}^{\hat{\alpha},\hat{\beta};\psi}$, ${}^{k,H}\mathcal{D}^{\hat{p},\hat{q};\psi}$ denote the (k,ψ)–Hilfer fractional derivative operators of orders $\hat{\alpha}, \hat{p}$, $1 < \hat{\alpha}, \hat{p} < 2$ and parameters $\hat{\beta}, \hat{q}$, $0 \le \hat{\beta}, \hat{q} \le 1$, respectively, $f, \tilde{f} : [a,b] \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $\lambda, \mu, \varsigma, \theta \in \mathbb{R}$, ${}^{k}\mathcal{I}^{v,\psi}$, ${}^{k}\mathcal{I}^{w,\psi}$ are the (k,ψ)–Riemann–Liouville fractional integrals of order $v, w > 0$, and $a < \xi, \sigma, u, \tau < b$. The existence and uniqueness results for the nonlinear system (1) will be established with the aid of the Leray–Schauder alternative, Krasnosel'skiĭ's fixed-point theorem and Banach's contraction mapping principle. The tools of the fixed point theory provide a decent platform from which to develop the existence theory for initial and boundary value problems once such problems are convertible to fixed-point problems. In this context, the present strategy is preferred over the other existing methods such as Lie symmetry, quasilinearization, etc.

Here, we emphasize that the problem (1) formulated and investigated in the present work is novel and the existence and uniqueness results derived for this problem enrich the literature on nonlocal integral boundary value problems involving (k,ψ)–Hilfer fractional differential and (k,ψ)–Riemann–Liouville integral operators of different orders. For more details, see the Conclusion section.

The structure of this paper is as follows. In Section 2, we recall some definitions and prove an auxiliary result which plays a fundamental role in transforming the given nonlinear system into a fixed-point problem. The main results are proved in Section 3. In Section 4, we provide numerical examples for illustration of our theoretical results.

2. Preliminaries

Let us first recall the definitions of a (k,ψ)–Riemann–Liouville fractional integral and (k,ψ)–Hilfer fractional derivative operators.

Definition 1 ([21]). *Let $\psi : [a,b] \to \mathbb{R}$ be an increasing function with $\psi'(t) \neq 0$ for all $t \in [a,b]$. Then, the (k,ψ)–Riemann–Liouville fractional integral of order $\alpha > 0$ ($\alpha \in \mathbb{R}$) of a function $\mathfrak{h} \in L^1([a,b],\mathbb{R})$ is given by*

$$ {}^{k}\mathfrak{I}^{\alpha;\psi}_{a+}\mathfrak{h}(t) = \frac{1}{k\Gamma_k(\alpha)} \int_a^t \psi'(u)(\psi(t)-\psi(u))^{\frac{\alpha}{k}-1}\mathfrak{h}(u)du,\ k>0. \tag{2} $$

Definition 2 ([16]). *Let $\alpha, k \in \mathbb{R}^+ = (0,\infty)$, $\beta \in [0,1]$, $\psi \in C^n([a,b],\mathbb{R})$, $\psi'(t) \neq 0$, $t \in [a,b]$ and $\mathfrak{h} \in C^n([a,b],\mathbb{R})$. Then, the (k,ψ)–Hilfer fractional derivative of the function $\mathfrak{h}$ of order α and type β, is defined by*

$$ {}^{k,H}\mathcal{D}^{\alpha,\beta;\psi}\mathfrak{h}(t) = {}^{k}\mathfrak{I}^{\beta(nk-\alpha);\psi}_{a+}\left(\frac{k}{\psi'(t)}\frac{d}{dt}\right)^n {}^{k}\mathfrak{I}^{(1-\beta)(nk-\alpha);\psi}_{a+}\mathfrak{h}(t),\quad n = \left\lceil \frac{\alpha}{k} \right\rceil. \tag{3} $$

Now, we prove an auxiliary lemma that plays a key role in the study of the nonlinear system (1).

Lemma 1. *Assume that $a < b$, $k > 0$, $1 < \hat{\alpha}, \hat{p} \leq 2$, $\hat{\beta}, \hat{q} \in [0,1]$, $\vartheta_k = \hat{\alpha} + \hat{\beta}(2k-\hat{\alpha})$, $\eta_k = \hat{p} + \hat{q}(2k-\hat{p})$, $h, \tilde{h} \in C^2([a,b],\mathbb{R})$ and*

$$ \mathcal{A} := \mathcal{A}_1\mathcal{A}_4 - \mathcal{A}_2\mathcal{A}_3 \neq 0. \tag{4} $$

Then, the pair $(\hat{r},\hat{z})$ is a solution of the nonlocal (k,ψ)–Hilfer fractional system

$$ \begin{cases} {}^{k,H}\mathcal{D}^{\hat{\alpha},\hat{\beta};\psi}\hat{r}(t) = h(t), & t \in (a,b], \\ {}^{k,H}\mathcal{D}^{\hat{p},\hat{q};\psi}\hat{z}(t) = \tilde{h}(t), & t \in (a,b], \\ \hat{r}(a) = 0, \quad \hat{r}(b) = \lambda\,\hat{z}(\xi) + \mu\,{}^{k}\mathcal{I}^{v,\psi}\hat{z}(\sigma), \\ \hat{z}(a) = 0, \quad \hat{z}(b) = \varsigma\,\hat{r}(u) + \theta\,{}^{k}\mathcal{I}^{w,\psi}\hat{r}(\tau), \end{cases} \tag{5} $$

if and only if

$$ \begin{aligned} \hat{r}(t) \;=\; & {}^{k}\mathcal{I}^{\hat{\alpha},\psi}h(t) + \frac{(\psi(t)-\psi(a))^{\frac{\vartheta_k}{k}-1}}{\mathcal{A}\Gamma_k(\vartheta_k)}\Bigg[\mathcal{A}_4\bigg(\lambda\,{}^{k}\mathcal{I}^{\hat{p},\psi}\tilde{h}(\xi) + \mu\,{}^{k}\mathcal{I}^{\hat{p}+v,\psi}\tilde{h}(\sigma) - {}^{k}\mathcal{I}^{\hat{\alpha},\psi}h(b)\bigg) \\ & + \mathcal{A}_2\bigg(\varsigma\,{}^{k}\mathcal{I}^{\hat{\alpha},\psi}h(u) + \theta\,{}^{k}\mathcal{I}^{\hat{\alpha}+w,\psi}h(\tau) - {}^{k}\mathcal{I}^{\hat{p},\psi}\tilde{h}(b)\bigg)\Bigg], \end{aligned} \tag{6} $$

and

$$ \begin{aligned} \hat{z}(t) \;=\; & {}^{k}\mathcal{I}^{\hat{p},\psi}\tilde{h}(t) + \frac{(\psi(t)-\psi(a))^{\frac{\eta_k}{k}-1}}{\mathcal{A}\Gamma_k(\eta_k)}\Bigg[\mathcal{A}_1\bigg(\varsigma\,{}^{k}\mathcal{I}^{\hat{\alpha},\psi}h(u) + \theta\,{}^{k}\mathcal{I}^{\hat{\alpha}+w,\psi}h(\tau) - {}^{k}\mathcal{I}^{\hat{p},\psi}\tilde{h}(b)\bigg) \\ & + \mathcal{A}_3\bigg(\lambda\,{}^{k}\mathcal{I}^{\hat{p},\psi}\tilde{h}(\xi) + \mu\,{}^{k}\mathcal{I}^{\hat{p}+v,\psi}\tilde{h}(\sigma) - {}^{k}\mathcal{I}^{\hat{\alpha},\psi}h(b)\bigg)\Bigg], \end{aligned} \tag{7} $$

where

$$ \begin{aligned} \mathcal{A}_1 \;&=\; \frac{(\psi(b)-\psi(a))^{\frac{\vartheta_k}{k}-1}}{\Gamma_k(\vartheta_k)},\ \mathcal{A}_2 = \lambda\frac{(\psi(\xi)-\psi(a))^{\frac{\eta_k}{k}-1}}{\Gamma_k(\eta_k)} + \mu\frac{(\psi(\sigma)-\psi(a))^{\frac{\eta_k+v}{k}-1}}{\Gamma_k(\eta_k+v)}, \\ \mathcal{A}_3 \;&=\; \varsigma\frac{(\psi(u)-\psi(a))^{\frac{\vartheta_k}{k}-1}}{\Gamma_k(\vartheta_k)} + \theta\frac{(\psi(\tau)-\psi(a))^{\frac{\vartheta_k+w}{k}-1}}{\Gamma_k(\vartheta_k+w)},\ \mathcal{A}_4 = \frac{(\psi(b)-\psi(a))^{\frac{\eta_k}{k}-1}}{\Gamma_k(\eta_k)}. \end{aligned} \tag{8} $$

Proof. Let $(\hat{r},\hat{z})$ be a solution of the system (5). As argued in [18], operating ${}^{k}\mathcal{I}^{\hat{\alpha},\psi}$ and ${}^{k}\mathcal{I}^{\hat{\rho},\psi}$ on both sides of the first and second equations in (5), respectively, we obtain

$$\begin{aligned}\hat{r}(t) &= {}^{k}\mathcal{I}^{\hat{\alpha},\psi}h(t)+c_0\frac{(\psi(b)-\psi(a))^{\frac{\vartheta_k}{k}-1}}{\Gamma_k(\vartheta_k)}+c_1\frac{(\psi(t)-\psi(a))^{\frac{\vartheta_k}{k}-2}}{\Gamma_k(\vartheta_k-k)},\\ \hat{z}(t) &= {}^{k}\mathcal{I}^{\hat{\rho},\psi}\tilde{h}(t)+d_0\frac{(\psi(t)-\psi(a))^{\frac{\eta_k}{k}-1}}{\Gamma_k(\eta_k)}+d_1\frac{(\psi(t)-\psi(a))^{\frac{\eta_k}{k}-2}}{\Gamma_k(\eta_k-k)},\end{aligned} \tag{9}$$

where c_0, c_1, d_0 and d_1 are unknown arbitrary constants. Using the conditions $\hat{r}(a)=0$ and $\hat{z}(a)=0$ in (9), we obtain $c_1=0$ and $d_1=0$, since $\frac{\vartheta_k}{k}-2<0$, $\frac{\eta_k}{k}-2<0$. Therefore, the equations in (9) take the form:

$$\begin{aligned}\hat{r}(t) &= {}^{k}\mathcal{I}^{\hat{\alpha},\psi}h(t)+c_0\frac{(\psi(b)-\psi(a))^{\frac{\vartheta_k}{k}-1}}{\Gamma_k(\vartheta_k)},\\ \hat{z}(t) &= {}^{k}\mathcal{I}^{\hat{\rho},\psi}\tilde{h}(t)+d_0\frac{(\psi(t)-\psi(a))^{\frac{\eta_k}{k}-1}}{\Gamma_k(\eta_k)},\end{aligned} \tag{10}$$

which, on inserting in the boundary conditions: $\hat{r}(b)=\lambda\,\hat{z}(\xi)+\mu\,{}^{k}\mathcal{I}^{v,\psi}\hat{z}(\sigma)$ and $\hat{z}(b)=\varsigma\,\hat{r}(u)+\theta\,{}^{k}\mathcal{I}^{w,\psi}\hat{r}(\tau)$, yields

$$\begin{aligned}{}^{k}\mathcal{I}^{\hat{\alpha},\psi}h(b)+c_0\frac{(\psi(b)-\psi(a))^{\frac{\vartheta_k}{k}-1}}{\Gamma_k(\vartheta_k)} &= \lambda\,{}^{k}\mathcal{I}^{\hat{\rho},\psi}\tilde{h}(\xi)+\lambda\,d_0\frac{(\psi(\xi)-\psi(a))^{\frac{\eta_k}{k}-1}}{\Gamma_k(\eta_k)}\\ &\quad+\mu\,{}^{k}\mathcal{I}^{\hat{\rho}+v,\psi}\tilde{h}(\sigma)+\frac{d_0\,\mu(\psi(\sigma)-\psi(a))^{\frac{\eta_k+v}{k}-1}}{\Gamma_k(\eta_k+v)},\end{aligned} \tag{11}$$

and

$$\begin{aligned}{}^{k}\mathcal{I}^{\hat{\rho},\psi}\tilde{h}(b)+d_0\frac{(\psi(b)-\psi(a))^{\frac{\vartheta_k}{k}-1}}{\Gamma_k(\eta_k)} &= \varsigma\,{}^{k}\mathcal{I}^{\hat{\alpha},\psi}h(u)+\tau\,c_0\frac{(\psi(u)-\psi(a))^{\frac{\vartheta_k}{k}-1}}{\Gamma_k(\vartheta_k)}\\ &\quad+\theta\,{}^{k}\mathcal{I}^{\hat{\alpha}+w,\psi}h(\tau)+\theta\,c_0\frac{(\psi(\tau)-\psi(a))^{\frac{\vartheta_k+w}{k}-1}}{\Gamma_k(\vartheta_k+w)}.\end{aligned} \tag{12}$$

Solving the system (11) and (12) for c_0 and d_0 together with the notation in (4) and (8), we find that

$$\begin{aligned}c_0 &= \frac{1}{\mathcal{A}}\Big[\mathcal{A}_4\Big(\lambda\,{}^{k}\mathcal{I}^{\hat{\rho},\psi}\tilde{h}(\xi)+\mu\,{}^{k}\mathcal{I}^{\hat{\rho}+v,\psi}\tilde{h}(\sigma)-{}^{k}\mathcal{I}^{\hat{\alpha},\psi}h(b)\Big)\\ &\quad+\mathcal{A}_2\Big(\varsigma\,{}^{k}\mathcal{I}^{\hat{\alpha},\psi}h(u)+\theta\,{}^{k}\mathcal{I}^{\hat{\alpha}+w,\psi}h(\tau)-{}^{k}\mathcal{I}^{\hat{\rho},\psi}\tilde{h}(b)\Big)\Big],\\ d_0 &= \frac{1}{\mathcal{A}}\Big[\mathcal{A}_1\Big(\varsigma\,{}^{k}\mathcal{I}^{\hat{\alpha},\psi}h(u)+\theta\,{}^{k}\mathcal{I}^{\hat{\alpha}+w,\psi}h(\tau)-{}^{k}\mathcal{I}^{\hat{\rho},\psi}\tilde{h}(b)\Big)\\ &\quad+\mathcal{A}_3\Big(\lambda\,{}^{k}\mathcal{I}^{\hat{\rho},\psi}\tilde{h}(\xi)+\mu\,{}^{k}\mathcal{I}^{\hat{\rho}+v,\psi}\tilde{h}(\sigma)-{}^{k}\mathcal{I}^{\hat{\alpha},\psi}h(b)\Big)\Big].\end{aligned}$$

Replacing c_0 and d_0 by their above values in (10), we obtain (6) and (7). The converse is obtained by direct calculation and this completes the proof. □

In the following, we denote by $\mathbb{X}=C([a,b],\mathbb{R})$ the Banach space of all continuous real-valued functions on $[a,b]$ equipped with the norm $\|\hat{r}\|=\max\{|\hat{r}(t)|; t\in[a,b]\}$. The space $(\mathbb{X}\times\mathbb{X},\|(\hat{r},\hat{z})\|)$ is a Banach space endowed with norm $\|(\hat{r},\hat{z})\|=\|\hat{r}\|+\|\hat{z}\|$.

3. Existence and Uniqueness Results

In the following, for $t \in [a,b]$, we set $f_{\hat{r},\hat{z}}(t) = f(t,\hat{r}(t),\hat{z}(t))$ and $\tilde{f}_{\hat{r},\hat{z}}(t) = \tilde{f}(t,\hat{r}(t),\hat{z}(t))$. By Lemma 1, we define an operator $\mathcal{T} : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{X} \times \mathbb{X}$ associated with the nonlinear system (1) as

$$\mathcal{T}(\hat{r},\hat{z})(t) = \begin{pmatrix} \mathcal{T}_1(\hat{r},\hat{z})(t) \\ \mathcal{T}_2(\hat{r},\hat{z})(t) \end{pmatrix},$$

where

$$\begin{aligned} \mathcal{T}_1(\hat{r},\hat{z})(t) &= {}^k\mathcal{I}^{\hat{\alpha},\psi} f_{\hat{r},\hat{z}}(t) \\ &+ \frac{(\psi(t)-\psi(a))^{\frac{\vartheta_k}{k}-1}}{\mathcal{A}\Gamma_k(\vartheta_k)} \Big[\mathcal{A}_4 \Big(\lambda\, {}^k\mathcal{I}^{\hat{p},\psi} \tilde{f}_{\hat{r},\hat{z}}(\xi) + \mu\, {}^k\mathcal{I}^{\hat{p}+v,\psi} \tilde{f}_{\hat{r},\hat{z}}(\sigma) - {}^k\mathcal{I}^{\hat{\alpha},\psi} f_{\hat{r},\hat{z}}(b) \Big) \\ &+ \mathcal{A}_2 \Big(\varsigma\, {}^k\mathcal{I}^{\hat{\alpha},\psi} f_{\hat{r},\hat{z}}(u) + \theta\, {}^k\mathcal{I}^{\hat{\alpha}+w,\psi} f_{\hat{r},\hat{z}}(\tau) - {}^k\mathcal{I}^{\hat{p},\psi} \tilde{f}_{r,z}(b) \Big) \Big], \end{aligned} \tag{13}$$

and

$$\begin{aligned} \mathcal{T}_2(\hat{r},\hat{z})(t) &= {}^k\mathcal{I}^{\hat{p},\psi} \tilde{f}_{\hat{r},\hat{z}}(t) \\ &+ \frac{(\psi(t)-\psi(a))^{\frac{\eta_k}{k}-1}}{\mathcal{A}\Gamma_k(\eta_k)} \Big[\mathcal{A}_1 \Big(\varsigma\, {}^k\mathcal{I}^{\hat{\alpha},\psi} f_{\hat{r},\hat{z}}(u) + \theta\, {}^k\mathcal{I}^{\hat{\alpha}+w,\psi} h(\tau) - {}^k\mathcal{I}^{\hat{p},\psi} \tilde{f}_{\hat{r},\hat{z}}(b) \Big) \\ &+ \mathcal{A}_3 \Big(\lambda\, {}^k\mathcal{I}^{\hat{p},\psi} \tilde{f}_{\hat{r},\hat{z}}(\xi) + \mu\, {}^k\mathcal{I}^{\hat{p}+v,\psi} \tilde{f}_{\hat{r},\hat{z}}(\sigma) - {}^k\mathcal{I}^{\hat{\alpha},\psi} f_{\hat{r},\hat{z}}(b) \Big) \Big]. \end{aligned} \tag{14}$$

Observe that the fixed-point problem $\mathcal{T}(\hat{r},\hat{z}) = (\hat{r},\hat{z})$ is equivalent to the nonlinear system (1) and, hence, the existence of a fixed point for the operator $\mathcal{T}$ will imply the existence of a solution of the nonlinear system (1).

For simplicity, we set

$$\begin{aligned} \mathfrak{Q}_1 &= \frac{(\psi(b)-\psi(a))^{\frac{\hat{\alpha}}{k}}}{\Gamma_k(\hat{\alpha}+k)} + \frac{(\psi(b)-\psi(a))^{\frac{\vartheta_k}{k}-1}}{|\mathcal{A}|\Gamma_k(\vartheta_k)} \Big[\mathcal{A}_4 \frac{(\psi(b)-\psi(a))^{\frac{\hat{\alpha}}{k}}}{\Gamma_k(\hat{\alpha}+k)} \\ &+ \mathcal{A}_2 \Big(|\varsigma| \frac{(\psi(u)-\psi(a))^{\frac{\hat{\alpha}}{k}}}{\Gamma_k(\hat{\alpha}+k)} + |\theta| \frac{(\psi(\tau)-\psi(a))^{\frac{\hat{\alpha}+w}{k}}}{\Gamma_k(\hat{\alpha}+w+k)} \Big) \Big], \\ \mathfrak{Q}_2 &= \frac{(\psi(b)-\psi(a))^{\frac{\vartheta_k}{k}-1}}{|\mathcal{A}|\Gamma_k(\vartheta_k)} \Big[\mathcal{A}_4 \Big(|\lambda| \frac{(\psi(\xi)-\psi(a))^{\frac{\hat{p}}{k}}}{\Gamma_k(\hat{p}+k)} + |\mu| \frac{(\psi(\sigma)-\psi(a))^{\frac{\hat{p}+v}{k}}}{\Gamma_k(\hat{p}+v+k)} \Big) \\ &+ \frac{(\psi(b)-\psi(a))^{\frac{\hat{p}}{k}}}{\Gamma_k(\hat{p}+k)} \Big], \\ \mathfrak{Q}_3 &= \frac{(\psi(t)-\psi(a))^{\frac{\eta_k}{k}-1}}{|\mathcal{A}|\Gamma_k(\eta_k)} \Big[\mathcal{A}_1 \Big(|\varsigma| \frac{(\psi(u)-\psi(a))^{\frac{\hat{\alpha}}{k}}}{\Gamma_k(\hat{\alpha}+k)} + |\theta| \frac{(\psi(\tau)-\psi(a))^{\frac{\hat{\alpha}+w}{k}}}{|\mathcal{A}|\Gamma_k(\hat{\alpha}+w+k)} \Big) \\ &+ \mathcal{A}_3 \frac{(\psi(b)-\psi(a))^{\frac{\hat{\alpha}}{k}}}{\Gamma_k(\hat{\alpha}+k)} \Big], \\ \mathfrak{Q}_4 &= \frac{(\psi(b)-\psi(a))^{\frac{\hat{p}}{k}}}{\Gamma_k(\hat{p}+k)} + \frac{(\psi(b)-\psi(a))^{\frac{\eta_k}{k}-1}}{|\mathcal{A}|\Gamma_k(\vartheta_k)} \Big[\mathcal{A}_1 \frac{(\psi(b)-\psi(a))^{\frac{\hat{p}}{k}}}{\Gamma_k(\hat{p}+k)} \\ &+ \mathcal{A}_3 \Big(|\lambda| \frac{(\psi(\xi)-\psi(a))^{\frac{\hat{p}}{k}}}{\Gamma_k(\hat{p}+k)} + |\mu| \frac{(\psi(\sigma)-\psi(a))^{\frac{\hat{p}+v}{k}}}{\Gamma_k(\hat{p}+v+k)} \Big) \Big], \end{aligned} \tag{15}$$

and

$$\mathfrak{Q}_1^* = \mathfrak{Q}_1 - \frac{(\psi(b)-\psi(a))^{\frac{\hat{\alpha}}{k}}}{\Gamma_k(\hat{\alpha}+k)}, \;\; \mathfrak{Q}_4^* = \mathfrak{Q}_4 - \frac{(\psi(b)-\psi(a))^{\frac{\hat{p}}{k}}}{\Gamma_k(\hat{p}+k)}. \tag{16}$$

3.1. Existence Results

We prove our first existence result by applying the Leray–Schauder alternative [22].

Theorem 1. *Suppose that $\mathcal{A} \neq 0$ ($\mathcal{A}$ is given by (4)) and $f, \tilde{f} : [a,b] \times \mathbb{R} \longrightarrow \mathbb{R}$ are two continuous functions such that, for all $t \in [a,b]$ and $\hat{r}_i, \hat{z}_i \in \mathbb{R}$, $i = 1,2$, we have*

$$|f(t, \hat{r}_1, \hat{z}_1)| \leq \hat{k}_0 + \hat{k}_1|\hat{r}_1| + \hat{k}_2|\hat{z}_1|,$$
$$|\tilde{f}(t, \hat{r}_2, \hat{z}_2)| \leq \hat{v}_0 + \hat{v}_1|\hat{r}_2| + \hat{v}_2|\hat{z}_2|,$$

where $\hat{k}_i, \hat{v}_i, i = 1,2$ are real positive constants with $\hat{k}_0, \hat{v}_0 > 0$. Moreover, it is assumed that

$$(\Omega_1 + \Omega_3)\hat{k}_1 + (\Omega_2 + \Omega_4)\hat{v}_1 < 1 \;\; and \;\; (\Omega_1 + \Omega_3)\hat{k}_2 + (\Omega_2 + \Omega_4)\hat{v}_2 < 1,$$

where $\Omega_i, i = 1,2,3,4$ are given in (15). Then, the nonlinear system (1) has at least one solution on $[a,b]$.

Proof. Observe that continuity of the functions f and $\tilde{f}$ ensures the continuity of the operator $\mathcal{T}$. Next, we establish that the operator $\mathcal{T}$ is uniformly bounded.

Let $\mathcal{M}$ be a bounded set of $\mathbb{X} \times \mathbb{X}$. Then, we can find positive constants $\mathcal{L}_1$ and $\mathcal{L}_2$ such that

$$|f(t, \hat{r}(t), \hat{z}(t)| \leq \mathcal{L}_1, \quad |\tilde{f}(t, \hat{r}(t), \hat{z}(t)| \leq \mathcal{L}_2, \;\; \forall (\hat{r}, \hat{z}) \in \mathcal{M}.$$

Thus, for all $(\hat{r}, \hat{z}) \in \mathcal{M}$, we have

$$\begin{aligned} |\mathcal{T}_1(\hat{r},\hat{z})(t)| \;\; \leq \;\; & \bigg\{ \frac{(\psi(b)-\psi(a))^{\frac{\hat{\alpha}}{k}}}{\Gamma_k(\hat{\alpha}+k)} + \frac{(\psi(b)-\psi(a))^{\frac{\vartheta_k}{k}-1}}{|\mathcal{A}|\Gamma_k(\vartheta_k)} \Big[\mathcal{A}_4 \frac{(\psi(b)-\psi(a))^{\frac{\hat{\alpha}}{k}}}{\Gamma_k(\hat{\alpha}+k)} \\ & + \mathcal{A}_2 \Big(|\varsigma| \frac{(\psi(u)-\psi(a))^{\frac{\hat{\alpha}}{k}}}{\Gamma_k(\hat{\alpha}+k)} + |\theta| \frac{(\psi(\tau)-\psi(a))^{\frac{\hat{\alpha}+w}{k}}}{\Gamma_k(\hat{\alpha}+w+k)} \Big) \bigg\} \mathcal{L}_1 \\ & + \frac{(\psi(b)-\psi(a))^{\frac{\vartheta_k}{k}-1}}{|\mathcal{A}|\Gamma_k(\vartheta_k)} \Big[\mathcal{A}_4 \Big(|\lambda| \frac{(\psi(\xi)-\psi(a))^{\frac{\hat{p}}{k}}}{\Gamma_k(\hat{p}+k)} + |\mu| \frac{(\psi(\sigma)-\psi(a))^{\frac{\hat{p}+v}{k}}}{\Gamma_k(\hat{p}+v+k)} \Big) \\ & + \mathcal{A}_2 \frac{(\psi(b)-\psi(a))^{\frac{\hat{p}}{k}}}{\Gamma_k(\hat{p}+k)} \Big] \mathcal{L}_2. \end{aligned}$$

Hence, we obtain

$$\|\mathcal{T}_1(\hat{r},\hat{z})\| \leq \Omega_1 \mathcal{L}_1 + \Omega_2 \mathcal{L}_2.$$

Analogously, we have

$$\|\mathcal{T}_2(\hat{r},\hat{z})\| \leq \Omega_3 \mathcal{L}_1 + \Omega_4 \mathcal{L}_2.$$

In view of the foregoing inequalities, it follows that

$$\|\mathcal{T}(\hat{r},\hat{z})\| \leq \|\mathcal{T}_1(\hat{r},\hat{z})\| + \|\mathcal{T}_2(\hat{r},\hat{z})\| \leq (\Omega_1+\Omega_3)\mathcal{L}_1 + (\Omega_2+\Omega_4)\mathcal{L}_2.$$

This shows that the operator $\mathcal{T}$ is uniformly bounded. To prove that the operator $\mathcal{T}$ is equicontinuous, we take $t_1, t_2 \in [a,b]$ with $t_1 < t_2$. Then, we have

$$
\begin{aligned}
&|\mathcal{T}_1(\hat{r}(t_2),\hat{z}(t_2)) - \mathcal{T}_1(\hat{r}(t_1),\hat{z}(t_1))| \\
\leq\ & \frac{1}{\Gamma_k(\hat{\alpha})}\left|\int_a^{t_1}\psi'(s)[(\psi(t_2)-\psi(s))^{\frac{\alpha}{k}-1} - (\psi(t_1)-\psi(s))^{\frac{\hat{\alpha}}{k}-1}]f(s,\hat{r}(s),\hat{z}(s))ds \right.\\
&\left. + \int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\frac{\hat{\alpha}}{k}-1}f(s,\hat{r}(s),\hat{z}(s))ds\right| \\
&+\frac{(\psi(t_2)-\psi(a))^{\frac{\theta_k}{k}-1} - (\psi(t_1)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\Lambda|\Gamma_k(\vartheta_k)}\Bigg[\mathcal{A}_4\Bigg(|\lambda|\ {}^k\mathcal{I}^{\hat{p},\psi}|\tilde{f}(\xi,\hat{r}(\xi),\hat{z}(\xi))| \\
&+|\mu|\ {}^k\mathcal{I}^{\hat{p}+v,\psi}|\tilde{f}(\sigma,\hat{r}(\sigma),\hat{z}(\sigma))| + {}^k\mathcal{I}^{\hat{p},\psi}|f(b,\hat{r}(b),\hat{z}(b))|\Bigg) \\
&+\mathcal{A}_2\Bigg(|\varsigma|\ {}^k\mathcal{I}^{\hat{\alpha},\psi}|f(u,\hat{r}(u),\hat{z}(u)) + |\theta|\ {}^k\mathcal{I}^{\hat{\alpha}+w,\psi}|f(\tau,\hat{r}(\tau),\hat{z}(\tau))| \\
&+{}^k\mathcal{I}^{\hat{p},\psi}|\tilde{f}(b,\hat{r}(b),\hat{z}(b))|\Bigg)\Bigg] \\
\leq\ & \frac{\mathcal{L}_1}{\Gamma_k(\hat{\alpha}+k)}[2(\psi(t_2)-\psi(t_1))^{\frac{\hat{\alpha}}{k}} + |(\psi(t_2)-\psi(c))^{\frac{\hat{\alpha}}{k}} - (\psi(t_1)-\psi(c))^{\frac{\hat{\alpha}}{k}}|] \\
&+\frac{(\psi(t_2)-\psi(a))^{\frac{\theta_k}{k}-1} - (\psi(t_1)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\Lambda|\Gamma_k(\vartheta_k)}\Bigg[\mathcal{A}_4\Bigg(|\lambda|\ \frac{(\psi(\xi)-\psi(a))^{\frac{\hat{p}}{k}}}{\Gamma_k(\hat{p}+k)}\mathcal{L}_2 \\
&+|\mu|\frac{(\psi(\sigma)-\psi(a))^{\frac{\hat{p}+v}{k}}}{\Gamma_k(\hat{p}+v+k)}\mathcal{L}_2 + \frac{(\psi(b)-\psi(a))^{\frac{\hat{\alpha}}{k}}}{\Gamma_k(\hat{\alpha}+k)}\mathcal{L}_1\Bigg) \\
&+\mathcal{A}_2\Bigg(|\varsigma|\ \frac{(\psi(u)-\psi(a))^{\frac{\hat{\alpha}}{k}}}{\Gamma_k(\hat{\alpha}+k)}\mathcal{L}_1 + |\theta|\ \frac{(\psi(\tau)-\psi(a))^{\frac{\hat{\alpha}+w}{k}}}{\Gamma_k(\hat{\alpha}+w+k)}\mathcal{L}_1 + \frac{(\psi(b)-\psi(a))^{\frac{\hat{p}}{k}}}{\Gamma_k(\hat{p}+k)}\mathcal{L}_2\Bigg)\Bigg].
\end{aligned}
$$

The right hand of the above inequality is independent of $(\hat{r},\hat{z})$ and tends to zero as $t_2 - t_1 \to 0$. Hence, $T_1(\hat{r},\hat{z})$ is equicontinuous. Similarly, it can be shown that $T_2(\hat{r},\hat{z})$ is equicontinuous. Accordingly, the operator $T(\hat{r},\hat{z})$ is completely continuous.

Finally, we show the boundedness of the set $\mathcal{D} = \{(\hat{r},\hat{z}) \in \mathbb{X}\times\mathbb{X} : (\hat{r},\hat{z}) = \nu\mathcal{T}(\hat{r},\hat{z}),\ 0 \leq \nu \leq 1\}$. Let $(\hat{r},\hat{z}) \in \mathcal{D}$. Then, $(\hat{r},\hat{z}) = \nu\mathcal{T}(\hat{r},\hat{z})$ and for all $t \in [a,b]$, we have

$$
\hat{r}(t) = \nu\mathcal{T}_1(\hat{r},\hat{z})(t), \quad z(t) = \nu\mathcal{T}_2(\hat{r},\hat{z})(t).
$$

Thus, we obtain

$$
\begin{aligned}
&|\hat{r}(t)| \\
\leq\ & \Bigg\{\frac{(\psi(b)-\psi(a))^{\frac{\hat{\alpha}}{k}}}{\Gamma_k(\hat{\alpha}+k)} + \frac{(\psi(b)-\psi(a))^{\frac{\vartheta_k}{k}-1}}{|\mathcal{A}|\Gamma_k(\vartheta_k)}\Big[\mathcal{A}_4\frac{(\psi(b)-\psi(a))^{\frac{\hat{\alpha}}{k}}}{\Gamma_k(\hat{\alpha}+k)} \\
&+\mathcal{A}_2\Big(|\varsigma|\ \frac{(\psi(u)-\psi(a))^{\frac{\hat{\alpha}}{k}}}{\Gamma_k(\hat{\alpha}+k)} + |\theta|\frac{(\psi(\tau)-\psi(a))^{\frac{\hat{\alpha}+w}{k}}}{\Gamma_k(\hat{\alpha}+w+k)}\Big)\Big]\Bigg\}(\hat{k}_0 + \hat{k}_1\|\hat{r}\| + \hat{k}_2\|\hat{z}\|) \\
&+\frac{(\psi(b)-\psi(a))^{\frac{\vartheta_k}{k}-1}}{|\mathcal{A}|\Gamma_k(\vartheta_k)}\Bigg\{\mathcal{A}_4\Big(|\lambda|\frac{(\psi(\xi)-\psi(a))^{\frac{\hat{p}}{k}}}{\Gamma_k(\hat{p}+k)} + |\mu|\ \frac{(\psi(\sigma)-\psi(a))^{\frac{\hat{p}+v}{k}}}{\Gamma_k(\hat{p}+v+k)}\Big) \\
&+\mathcal{A}_2\frac{(\psi(b)-\psi(a))^{\frac{\hat{p}}{k}}}{\Gamma_k(\hat{p}+k)}\Bigg\}(\hat{v}_0 + \hat{v}_1\|\hat{r}\| + \hat{v}_2\|\hat{z}\|),
\end{aligned}
$$

and

$$
\begin{aligned}
& |\hat{z}(t)| \\
\leq \; & \frac{(\psi(b)-\psi(a))^{\frac{\eta_k}{k}-1}}{|\mathcal{A}|\Gamma_k(\eta_k)}\Bigg[\mathcal{A}_1\Bigg(\frac{(|\varsigma|\;\psi(u)-\psi(a))^{\frac{\hat{\alpha}}{k}}}{\Gamma_k(\hat{\alpha}+k)}+|\theta|\;\frac{(\psi(\tau)-\psi(a))^{\frac{\hat{\alpha}+w}{k}}}{|\mathcal{A}|\Gamma_k(\hat{\alpha}+w+k)}\Bigg) \\
& +\mathcal{A}_3\frac{(\psi(b)-\psi(a))^{\frac{\hat{\alpha}}{k}}}{\Gamma_k(\hat{\alpha}+k)}\Bigg](\hat{k}_0+\hat{k}_1\|\hat{r}\|+\hat{k}_2\|\hat{z}\|)+\Bigg\{\frac{(\psi(b)-\psi(a))^{\frac{\hat{p}}{k}}}{\Gamma_k(\hat{p}+k)} \\
& +\frac{(\psi(t)-\psi(a))^{\frac{\eta_k}{k}-1}}{|\mathcal{A}|\Gamma_k(\vartheta_k)}\Bigg[\mathcal{A}_1\frac{(\psi(b)-\psi(a))^{\frac{\hat{p}}{k}}}{\Gamma_k(\hat{p}+k)}+\mathcal{A}_3\Bigg(|\lambda|\;\frac{(\psi(\xi)-\psi(a))^{\frac{\hat{p}}{k}}}{\Gamma_k(\hat{p}+k)} \\
& +|\mu|\frac{(\psi(\sigma)-\psi(a))^{\frac{\hat{p}+v}{k}}}{\Gamma_k(\hat{p}+v+k)}\Bigg)\Bigg]\Bigg\}(\hat{v}_0+\hat{v}_1\|\hat{r}\|+\hat{v}_2\|\hat{z}\|).
\end{aligned}
$$

Thus, we have

$$\|\hat{r}\| \leq \mathfrak{Q}_1(\hat{k}_0+\hat{k}_1\|\hat{r}\|+\hat{k}_2\|z\|)+\mathfrak{Q}_2((\hat{v}_0+\hat{v}_1\|\hat{r}\|+\hat{v}_2\|z\|)$$

and

$$\|\hat{z}\| \leq \mathfrak{Q}_3(\hat{k}_0+k_1\|\hat{r}\|+k_2\|z\|)+\mathfrak{Q}_4((\hat{v}_0+\hat{v}_1\|\hat{r}\|+\hat{v}_2\|z\|).$$

Hence, we obtain

$$
\begin{aligned}
\|\hat{r}\|+\|\hat{z}\| \quad \leq \quad & (\mathfrak{Q}_1+\mathfrak{Q}_3)\hat{k}_0+(\mathfrak{Q}_2+\mathfrak{Q}_4)\hat{v}_0+[(\mathfrak{Q}_1+\mathfrak{Q}_3)\hat{k}_1+(\mathfrak{Q}_2+\mathfrak{Q}_4)\hat{v}_1]\|\hat{r}\| \\
& +[(\mathfrak{Q}_1+\mathfrak{Q}_3)\hat{k}_2+(\mathfrak{Q}_2+\mathfrak{Q}_4)\hat{v}_2]\|\hat{z}\|,
\end{aligned}
$$

which implies that

$$\|(\hat{r},\hat{z}) \leq \frac{(\mathfrak{Q}_1+\mathfrak{Q}_3)\hat{k}_0+(\mathfrak{Q}_2+\mathfrak{Q}_4)\hat{v}_0}{\mathcal{M}_0},$$

where

$$\mathcal{M}_0=\min\{1-[(\mathfrak{Q}_1+\mathfrak{Q}_3)\hat{k}_1+(\mathfrak{Q}_2+\mathfrak{Q}_4)\hat{v}_1],1-[(\mathfrak{Q}_1+\mathfrak{Q}_3)\hat{k}_2+(\mathfrak{Q}_2+\mathfrak{Q}_4)\hat{v}_2]\}.$$

Consequently, using the Leray–Schauder alternative, it follows that the operator $\mathcal{T}$ has at least one fixed point, which is a solution of the nonlinear system (1) on $[a,b]$. □

In the following result, we apply Krasnosel'skiĭ's fixed-point theorem [23] to prove our second existence result.

Theorem 2. *Assume that:*

$(\mathcal{H}_1)$ *There exist positive real constants $\hat{m}_i, \hat{n}_i, i=1,2$ such that, $f,\tilde{f}:[a,b]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ are continuous and satisfy the following conditions for all $t\in[a,b]$ and $\hat{r}_i,\hat{z}_i\in\mathbb{R}$, $i=1,2$:*

$$|f(t,\hat{r}_1,\hat{z}_1)-f(t,\hat{r}_2,\hat{z}_2)| \leq \hat{m}_1|\hat{r}_1-\hat{r}_2|+\hat{m}_2|\hat{z}_1-\hat{z}_2|,$$

$$|\tilde{f}(t,\hat{r}_1,\hat{z}_1)-\tilde{f}(t,\hat{r}_2,\hat{z}_2)| \leq \hat{n}_1|\hat{r}_1-\hat{r}_2|+\hat{n}_2|\hat{z}_1-\hat{z}_2|.$$

$(\mathcal{H}_2)$ *There exist P and $Q\in C([a,b],\mathbb{R}_+)$ satisfying*

$$|f(t,\hat{r},\hat{z})| \leq P(t), \quad |\tilde{f}(t,\hat{r},\hat{z})| \leq Q(t), \;\; \textit{for each} \;\; (t,\hat{r},\hat{z})\in[a,b]\times\mathbb{R}\times\mathbb{R}.$$

Then, the nonlinear system (1) has at least one solution on $[a,b]$, provided that

$$[\mathfrak{Q}_1^*+\mathfrak{Q}_3](\hat{m}_1+\hat{m}_2)+[\mathfrak{Q}_2+\mathfrak{Q}_4^*](\hat{n}_1+\hat{n}_2)<1, \tag{17}$$

where $\mathfrak{Q}_i, i=1,2,3,4$ are given in (15).

Proof. First, we decompose the operator $\mathcal{T}$ into four operators $\mathcal{T}_{1,1}, \mathcal{T}_{1,2}, \mathcal{T}_{2,1}$ and $\mathcal{T}_{2,2}$ as

$$\begin{aligned}
\mathcal{T}_{1,1}(\hat{r},\hat{z})(t) &= {}^{k}\mathcal{I}^{\hat{\alpha},\psi} f(t,\hat{r}(t),\hat{z}(t)), \quad t\in[a,b],\\
\mathcal{T}_{1,2}(\hat{r},\hat{z})(t) &= \frac{(\psi(t)-\psi(a))^{\frac{\vartheta_k}{k}-1}}{\mathcal{A}\Gamma_k(\vartheta_k)}\Big[\mathcal{A}_4\Big(\lambda\, {}^{k}\mathcal{I}^{\hat{p},\psi}\tilde{f}(\xi,\hat{r}(\xi),\hat{z}(\xi))\\
&\quad +\mu\, {}^{k}\mathcal{I}^{\hat{p}+v,\psi}\tilde{f}(\sigma,\hat{r}(\sigma),\hat{z}(\sigma)) - {}^{k}\mathcal{I}^{\hat{\alpha},\psi} f(b,\hat{r}(b),\hat{z}(b))\Big)\\
&\quad +\mathcal{A}_2\Big(\varsigma\, {}^{k}\mathcal{I}^{\hat{\alpha},\psi} f(u,\hat{r}(u),\hat{z}(u))) + \theta\, {}^{k}\mathcal{I}^{\hat{\alpha}+w,\psi} h(\tau,\hat{r}(\tau),\hat{z}(\tau)\\
&\quad - {}^{k}\mathcal{I}^{\hat{p},\psi}\tilde{f}(b,\hat{r}(b),\hat{z}(b))\Big)\Big], \quad t\in[a,b],\\
\mathcal{T}_{2,1}(\hat{r},\hat{z})(t) &= {}^{k}\mathcal{I}^{\hat{p},\psi}\tilde{f}(t,\hat{r}(t),\hat{z}(t)), \quad t\in[a,b],\\
\mathcal{T}_{2,2}(\hat{r},\hat{z})(t) &= \frac{(\psi(t)-\psi(a))^{\frac{\eta_k}{k}-1}}{\mathcal{A}\Gamma_k(\eta_k)}\Big[\mathcal{A}_1\Big(\varsigma\, {}^{k}\mathcal{I}^{\hat{\alpha},\psi} f(u,\hat{r}(u),\hat{z}(u))\\
&\quad +\theta\, {}^{k}\mathcal{I}^{\hat{\alpha}+w,\psi} f(\tau,\hat{r}(\tau),\hat{z}(\tau)) - {}^{k}\mathcal{I}^{\hat{p},\psi}\tilde{f}(b,\hat{r}(b),\hat{z}(b))\Big)\\
&\quad +\mathcal{A}_3\Big(\lambda\, {}^{k}\mathcal{I}^{\hat{p},\psi}\tilde{f}(\xi,\hat{r}(\xi),\hat{z}(\xi)) + \mu\, {}^{k}\mathcal{I}^{\hat{p}+v,\psi}\tilde{f}(\sigma,\hat{r}(\sigma),\hat{z}(\sigma))\\
&\quad - {}^{k}\mathcal{I}^{\hat{\alpha},\psi} f(b,\hat{r}(b),\hat{z}(b))\Big)\Big], \quad t\in[a,b].
\end{aligned}$$

Notice that $\mathcal{T}_1 = \mathcal{T}_{1,1} + \mathcal{T}_{1,2}$ and $\mathcal{T}_2 = \mathcal{T}_{2,1} + \mathcal{T}_{2,2}$. Assume that $\mathcal{B}_{\hat{\rho}} = \{(\hat{r},\hat{z}) \in \mathbb{X}\times\mathbb{X} : \|(\hat{r},\hat{z})\| \le \hat{\rho}\}$ is a ball, in which $\hat{\rho} \ge (\mathfrak{Q}_1+\mathfrak{Q}_3)\|P\| + (\mathfrak{Q}_2+\mathfrak{Q}_4)\|Q\|$. Similarly, as in Theorem 1, we have

$$|\mathcal{T}_{1,1}(\hat{r}_1,\hat{r}_2)(t) + \mathcal{T}_{1,2}(\hat{z}_1,\hat{z}_2)(t)| \le \mathfrak{Q}_1\|P\| + \mathfrak{Q}_2\|Q\|,$$

and

$$|\mathcal{T}_{1,1}(\hat{r}_1,\hat{r}_2)(t) + \mathcal{T}_{2,2}(\hat{z}_1,\hat{z}_2)(t)| \le \mathfrak{Q}_3\|P\| + \mathfrak{Q}_4\|Q\|.$$

Consequently, we obtain

$$\|\mathcal{T}_1(\hat{r}_1,\hat{r}_2) + \mathcal{T}_2(\hat{z}_1,\hat{z}_2)\| \le (\mathfrak{Q}_1+\mathfrak{Q}_3)\|P\| + (\mathfrak{Q}_2+\mathfrak{Q}_4)\|Q\| < \hat{\rho}.$$

Thus, $\mathcal{T}_1(\hat{r}_1,\hat{r}_2) + \mathcal{T}_2(\hat{z}_1,\hat{z}_2) \in \mathcal{B}_{\hat{\rho}}$. Next, it will be shown that the operator $(T_{1,2}, T_{2,2})$ is a contraction mapping. For $(\hat{r}_1,\hat{z}_1),(\hat{r}_2,\hat{z}_2) \in \mathcal{B}_{\hat{\rho}}$, we have

$$\begin{aligned}
&|\mathcal{T}_{1,2}(\hat{r}_1,\hat{r}_2)(t) - \mathcal{T}_{1,2}(\hat{z}_1,\hat{z}_2)(t)|\\
\le\;& \frac{(\psi(b)-\psi(a))^{\frac{\vartheta_k}{k}-1}}{|\mathcal{A}|\Gamma_k(\vartheta_k)}\Big[\mathcal{A}_4\frac{(\psi(b)-\psi(a))^{\frac{\hat{\alpha}}{k}}}{\Gamma_k(\hat{\alpha}+k)}\\
&+\mathcal{A}_2\Big(|\varsigma|\,\frac{(\psi(u)-\psi(a))^{\frac{\hat{\alpha}}{k}}}{\Gamma_k(\hat{\alpha}+k)} + |\theta|\frac{(\psi(\tau)-\psi(a))^{\frac{\hat{\alpha}+w}{k}}}{\Gamma_k(\hat{\alpha}+w+k)}\Big)\Big](\hat{m}_1\|\hat{r}_1-\hat{z}_1\| + \hat{m}_2\|\hat{r}_2-\hat{z}_2\|)\\
&+\frac{(\psi(b)-\psi(a))^{\frac{\vartheta_k}{k}-1}}{|\mathcal{A}|\Gamma_k(\vartheta_k)}\Big[\mathcal{A}_4\Big(|\lambda|\,\frac{(\psi(\xi)-\psi(a))^{\frac{\hat{p}}{k}}}{\Gamma_k(\hat{p}+k)} + |\mu|\,\frac{(\psi(\sigma)-\psi(a))^{\frac{\hat{p}+v}{k}}}{\Gamma_k(\hat{p}+v+k)}\Big)\\
&+\mathcal{A}_2\frac{(\psi(b)-\psi(a))^{\frac{\hat{p}}{k}}}{\Gamma_k(\hat{p}+k)}\Big](\hat{n}_1\|\hat{r}_1-\hat{z}_1\| + \hat{n}_2\|\hat{r}_2-\hat{z}_2\|)\\
=\;& \mathfrak{Q}_1^*(\hat{m}_1\|\hat{r}_1-\hat{z}_1\| + \hat{m}_2\|\hat{r}_2-\hat{z}_2\|) + \mathfrak{Q}_2(\hat{n}_1\|\hat{r}_1-\hat{z}_1\| + \hat{n}_2\|\hat{r}_2-\hat{z}_2\|)\\
=\;& [\mathfrak{Q}_1^*\hat{m}_1 + \mathfrak{Q}_2\hat{n}_1]\|\hat{r}_1-\hat{z}_1\| + [\mathfrak{Q}_1^*\hat{m}_2 + \mathfrak{Q}_2\hat{n}_2]\|\hat{r}_2-\hat{z}_2\|. \qquad (18)
\end{aligned}$$

In a similar fashion, one can find that

$$\begin{aligned} & |\mathcal{T}_{2,2}(\hat{r}_1,\hat{r}_2)(t) - \mathcal{T}_{2,2}(\hat{z}_1,\hat{z}_2)(t)| \\ \le\ & [\mathfrak{Q}_3\hat{m}_1 + \mathfrak{Q}_4^*\hat{n}_1]\|\hat{r}_1 - \hat{z}_1\| + [\mathfrak{Q}_3\hat{m}_2 + \mathfrak{Q}_4^*\hat{n}_2]\|\hat{r}_2 - \hat{z}_2\|. \end{aligned} \tag{19}$$

From (18) and (19), we obtain

$$\begin{aligned} & \|(\mathcal{T}_{1,2},\mathcal{T}_{2,2})(\hat{r}_1,\hat{z}_1) - (\mathcal{T}_{1,2},\mathcal{T}_{2,2})(\hat{z}_1,\hat{z}_2)\| \\ \le\ & \{[\mathfrak{Q}_1^* + \mathfrak{Q}_3](\hat{m}_1+\hat{m}_2) + [\mathfrak{Q}_2 + \mathfrak{Q}_4^*](\hat{n}_1+\hat{n}_2)\}(\|\hat{r}_1 - \hat{z}_1\| + \|\hat{r}_2 - \hat{z}_2\|). \end{aligned} \tag{20}$$

which shows that the operator $(\mathcal{T}_{1,2},\mathcal{T}_{2,1})$ is a contraction owing to the condition (17). On the other hand, in view of the continuity property of f and $\tilde{f}$, we conclude that the operator $(\mathcal{T}_{1,1},\mathcal{T}_{2,1})$ is continuous. In addition, we have

$$\|\mathcal{T}_{1,1}(\hat{r},\hat{z})\| \le \frac{\psi(b)-\psi(a))^{\frac{\hat{\alpha}}{k}}}{\Gamma_k(\hat{\alpha}+k)}\|P\| \text{ and } \|\mathcal{T}_{2,1}(r,z)\| \le \frac{\psi(b)-\psi(a))^{\frac{\hat{\rho}}{k}}}{\Gamma_k(\hat{\rho}+k)}\|Q\|.$$

Consequently, we obtain

$$\|(\mathcal{T}_{1,1},\mathcal{T}_{2,1})(\hat{r},\hat{z})\| \le \frac{(\psi(b)-\psi(a))^{\frac{\hat{\alpha}}{k}}}{\Gamma_k(\hat{\alpha}+k)}\|P\| + \frac{(\psi(b)-\psi(a))^{\frac{\hat{\rho}}{k}}}{\Gamma_k(\hat{\rho}+k)}\|Q\|.$$

Hence, $(\mathcal{T}_{1,1},\mathcal{T}_{2,1})B_\rho$ is uniformly bounded. In the next step, it will be shown that the set $(\mathcal{T}_{1,1},\mathcal{T}_{2,1})\mathcal{B}_\rho$ is equicontinuous. For $t_1,t_2 \in [a,b]$ with $t_1 < t_2$ and for all $(\hat{r},\hat{z}) \in \mathcal{B}_{\hat{\rho}}$, we obtain

$$\begin{aligned} & |\mathcal{T}_{1,1}(\hat{r},\hat{z})(t_2) - \mathcal{T}_{1,1}(\hat{r},\hat{z})(t_1) \\ \le\ & \frac{1}{\Gamma_k(\hat{\alpha})}\bigg|\int_a^{t_1}\psi'(s)[(\psi(t_2)-\psi(s))^{\frac{\hat{\alpha}}{k}-1} - (\psi(t_1)-\psi(s))^{\frac{\alpha}{k}-1}]f(s,\hat{r}(s),\hat{z}(s))ds \\ & + \int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\frac{\hat{\alpha}}{k}-1}f(s,\hat{r}(s),\hat{z}(s))ds\bigg| \\ \le\ & \frac{\|P\|}{\Gamma_k(\hat{\alpha}+k)}[2(\psi(t_2)-\psi(t_1))^{\frac{\hat{\alpha}}{k}} + |(\psi(t_2)-\psi(a))^{\frac{\hat{\alpha}}{k}} - (\psi(t_1)-\psi(a))^{\frac{\hat{\alpha}}{k}}|]. \end{aligned}$$

The right hand side of the above inequality is independent of $(\hat{r},\hat{z}) \in \mathcal{B}_{\hat{\rho}}$ and tends to zero as $t_1 \longrightarrow t_2$. Analogously, we have that $|(\mathcal{T}_{2,1}(\hat{r},\hat{z})(t_2) - \mathcal{T}_{2,1}(\hat{r},\hat{z})(t_1)| \to 0$ as $t_1 \longrightarrow t_2$, independent of $(\hat{r},\hat{z}) \in \mathcal{B}_{\hat{\rho}}$.

Thus, $|(\mathcal{T}_{1,1},\mathcal{T}_{2,1})(\hat{r},\hat{z})(t_2) - (\mathcal{T}_{1,1},\mathcal{T}_{2,1})(\hat{r},\hat{z})(t_1)| \to 0$ as $t_1 \longrightarrow t_2$. Therefore, $(\mathcal{T}_{1,1},\mathcal{T}_{2,1})$ is equicontinuous. Now, applying the Arzelá–Ascoli theorem, we deduce that the operator $(\mathcal{T}_{1,1},\mathcal{T}_{2,1})$ is compact on $\mathcal{B}_{\hat{\rho}}$. Thus, the hypotheses of Krasnosel'skiĭ's fixed-point theorem are satisfied, and, hence, there exists at least one solution of the nonlinear system (1) on $[a,b]$. □

3.2. Existence of a Unique Solution

In this subsection, we present a result concerning the uniqueness of solutions for the nonlinear system (1), which is derived with the aid of the Banach fixed-point theorem.

Theorem 3. *Suppose that $\mathcal{A} \neq 0$ ($\mathcal{A}$ is given by (4)) and $(\mathcal{H}_1)$ holds. Moreover, it is assumed that*

$$(\mathfrak{Q}_1+\mathfrak{Q}_3)(\hat{m}_1+\hat{m}_2) + (\mathfrak{Q}_2+\mathfrak{Q}_4)(\hat{n}_1+\hat{n}_2) < 1, \tag{21}$$

where $\mathfrak{Q}_i, i = 1,2,3,4$ are defined in (15). Then, the nonlinear system (1) has a unique solution on $[a,b]$.

Proof. Letting $\sup_{t\in[a,b]}|f(t,0,0)| = \mathfrak{N} < \infty$ and $\sup_{t\in[a,b]}|\hat{f}(t,0,0)| = \mathfrak{N}_1$, we fix

$$r \geq \frac{(\mathfrak{Q}_1+\mathfrak{Q}_3)\mathfrak{N} + (\mathfrak{Q}_2+\mathfrak{Q}_4)\mathfrak{N}_1}{1-[(\mathfrak{Q}_1+\mathfrak{Q}_3)(\hat{m}_1+\hat{m}_2)+(\mathfrak{Q}_2+\mathfrak{Q}_4)(\hat{n}_1+\hat{n}_2)]},$$

and show that $T\mathcal{B}_r \subseteq \mathcal{B}_r$, where $\mathcal{B}_r = \{(\hat{r},\hat{z}) \in \mathbb{X}\times\mathbb{X} : \|(\hat{r},\hat{z})\| \leq r\}$. For $(\hat{r},\hat{z}) \in \mathcal{B}_r$, we have

$$\begin{aligned}
&|\mathcal{T}_1(\hat{r},\hat{z})(t)| \\
\leq\;& {}^k\mathcal{I}^{\hat{\alpha},\psi}[|f(t,\hat{r}(t),\hat{z}(t)) - f(t,0,0)| + |f(t,0,0)|] \\
&+\frac{(\psi(b)-\psi(a))^{\frac{\vartheta_k}{k}-1}}{|\mathcal{A}|\Gamma_k(\vartheta_k)}\Bigg[\mathcal{A}_4\Bigg({}^k\mathcal{I}^{\hat{p},\psi}[|\hat{f}(\xi,\hat{r}(\xi),\hat{z}(\xi)) - \tilde{f}(\xi,0,0)| + |\tilde{f}(\xi,0,0)|] \\
&+|\mu|\,{}^k\mathcal{I}^{\hat{p}+v,\psi}[|f(\sigma,\hat{r}(\sigma),\hat{z}(\sigma)) - f(\sigma,0,0)| + |f(\sigma,0,0)|] \\
&+{}^k\mathcal{I}^{\hat{\alpha},\psi}[|f(b,\hat{r}(b),\hat{z}(b)) - f(b,0,0)| + |f(b,0,0)|]\Bigg) \\
&+\mathcal{A}_2\Bigg(|\varsigma|\,{}^k\mathcal{I}^{\hat{\alpha},\psi}[|f(u,\hat{r}(u),\hat{z}(u) - f(u,0,0)| + |f(u,0,0)|] \\
&+|\theta|\,{}^k\mathcal{I}^{\hat{\alpha},\psi}|f(\tau,\hat{r}(\tau),\hat{z}(\tau)) - f(\tau,0,0)| + |f(\tau,0,0)|] \\
&+{}^k\mathcal{I}^{\hat{p},\psi}[[\tilde{f}(b,\hat{r}(b),\hat{z}(b)) - \tilde{f}(b,0,0)| + |\tilde{f}(b,0,0)|]\Bigg)\Bigg] \\
\leq\;& {}^k\mathcal{I}^{\hat{\alpha},\psi}[\hat{m}_1\|\hat{r}\| + \hat{m}_2\|\hat{z}\| + \mathfrak{N}](b) \\
&+\frac{(\psi(b)-\psi(a))^{\frac{\vartheta_k}{k}-1}}{|\mathcal{A}|\Gamma_k(\vartheta_k)}\Bigg[\mathcal{A}_4\Bigg(|\lambda|\,{}^k\mathcal{I}^{\hat{p},\psi}[\hat{n}_1\|\hat{r}\| + \hat{n}_2\|\hat{z}\| + \mathfrak{N}_1](\xi) \\
&+|\mu|\,{}^k\mathcal{I}^{\hat{p}+v,\psi}[\hat{n}_1\|\hat{r}\| + \hat{n}_2\|\hat{z}\| + \mathfrak{N}_1](\sigma) + {}^k\mathcal{I}^{\hat{p},\psi}[\hat{m}_1\|\hat{r}\| + \hat{m}_2\|\hat{z}\| + \mathfrak{N}](b) \\
&+\mathcal{A}_2\Bigg(|\varsigma|\,{}^k\mathcal{I}^{\hat{\alpha},\psi}[\hat{m}_1\|\hat{r}\| + \hat{m}_2\|\hat{z}\| + \mathfrak{N}](u) + |\theta|{}^k\mathcal{I}^{\alpha\hat{+}w,\psi}[\hat{m}_1\|\hat{r}\| + \hat{m}_2\|\hat{z}\| + \mathfrak{N}](\tau) \\
&+{}^k\mathcal{I}^{\hat{p},\psi}[\hat{n}_1\|\hat{r}\| + \hat{n}_2\|\hat{z}\| + \mathfrak{N}_1](b)\Bigg)\Bigg] \\
\leq\;& \frac{(\psi(b)-\psi(a))^{\frac{\hat{\alpha}}{k}}}{\Gamma_k(\hat{\alpha}+k)}[\hat{m}_1\|\hat{r}\| + \hat{m}_2\|\hat{z}\| + \mathfrak{N}] \\
&+\frac{(\psi(b)-\psi(a))^{\frac{\vartheta_k}{k}-1}}{|\mathcal{A}|\Gamma_k(\vartheta_k)}\Bigg[\mathcal{A}_4\Bigg(|\lambda|\,\frac{(\psi(\xi)-\psi(a))^{\frac{\hat{p}}{k}}}{\Gamma_k(\hat{p}+k)}[\hat{n}_1\|\hat{r}\| + \hat{n}_2\|\hat{z}\| + \mathfrak{N}_1] \\
&+|\mu|\,\frac{(\psi(\sigma)-\psi(a))^{\frac{\hat{p}+v}{k}}}{\Gamma_k(\hat{p}+v+k)}[\hat{n}_1\|\hat{r}\| + \hat{n}_2\|\hat{z}\| + \mathfrak{N}_1] \\
&+\frac{(\psi(b)-\psi(a))^{\frac{\hat{\alpha}}{k}}}{\Gamma_k(\hat{\alpha}+k)}[\hat{m}_1\|\hat{r}\| + \hat{m}_2\|\hat{z}\| + \mathfrak{N}]\Bigg) \\
&+\mathcal{A}_2\Bigg(|\varsigma|\,\frac{(\psi(u)-\psi(a))^{\frac{\hat{\alpha}}{k}}}{\Gamma_k(\hat{\alpha}+k)}[\hat{m}_1\|\hat{r}\| + \hat{m}_2\|\hat{z}\| + \mathfrak{N}] \\
&+|\theta|\,\frac{(\psi(\tau)-\psi(a))^{\frac{\hat{\alpha}+w}{k}}}{\Gamma_k(\hat{\alpha}+w+k)}[\hat{m}_1\|\hat{r}\| + \hat{m}_2\|\hat{z}\| + \mathfrak{N}] \\
&+\mathcal{A}_2\frac{(\psi(b)-\psi(a))^{\frac{\hat{p}}{k}}}{\Gamma_k(\hat{p}+k)}[\hat{n}_1\|\hat{r}\| + \hat{n}_2\|\hat{z}\| + \mathfrak{N}_1]\Bigg)\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= \Bigg\{\frac{(\psi(b)-\psi(a))^{\frac{\hat\alpha}{k}}}{\Gamma_k(\hat\alpha+k)} + \frac{(\psi(b)-\psi(a))^{\frac{\vartheta_k}{k}-1}}{|\mathcal{A}|\Gamma_k(\vartheta_k)}\Big[\mathcal{A}_4\frac{(\psi(b)-\psi(a))^{\frac{\hat\alpha}{k}}}{\Gamma_k(\hat\alpha+k)} \\
&\quad +\mathcal{A}_2\Big(|\varsigma|\,\frac{(\psi(u)-\psi(a))^{\frac{\hat\alpha}{k}}}{\Gamma_k(\hat\alpha+k)} + |\theta|\frac{(\psi(\tau)-\psi(a))^{\frac{\hat\alpha+w}{k}}}{\Gamma_k(\hat\alpha+w+k)}\Big)\Big]\Bigg\}[\hat m_1\|\hat r\|+\hat m_2\|\hat z\|+\mathfrak{N}] \\
&\quad +\Bigg\{\mathcal{A}_4\Big(|\lambda|\,\frac{(\psi(\xi)-\psi(a))^{\frac{\hat p}{k}}}{\Gamma_k(\hat p+k)} + |\mu|\,\frac{(\psi(\sigma)-\psi(a))^{\frac{\hat p+v}{k}}}{\Gamma_k(\hat p+v+k)}\Big) \\
&\quad +\mathcal{A}_2\frac{(\psi(b)-\psi(a))^{\frac{\hat p}{k}}}{\Gamma_k(\hat p+k)}\Bigg\}[\hat n_1\|\hat r\|+\mathcal{A}\hat n_2\|\hat z\|+\mathfrak{N}_1] \\
&= \mathfrak{Q}_1[\hat m_1\|\hat r\|+\hat m_2\|\hat z\|+\mathfrak{N}]+\mathfrak{Q}_2[\hat n_1\|\hat r\|+\hat n_2\|\hat z\|+\mathfrak{N}_1] \\
&= (\mathfrak{Q}_1\hat m_1+\mathfrak{Q}_2\hat n_1)\|\hat r\|+(\mathfrak{Q}_1\hat m_2+\mathfrak{Q}_2\hat n_2)\|\hat z\|+\mathfrak{Q}_1\mathfrak{N}+\mathfrak{Q}_2\mathfrak{N}_1 \\
&\le \mathfrak{Q}_1\hat m_1+\mathfrak{Q}_2\hat n_1+\mathfrak{Q}_1\hat m_2+\mathfrak{Q}_2\hat n_2)r+\mathfrak{Q}_1\mathfrak{N}+\mathfrak{Q}_2\mathfrak{N}_1.
\end{aligned}
$$

Analogously, we obtain

$$
\begin{aligned}
&|\mathcal{T}_2(\hat r,\hat z)(t)| \\
&\le \Bigg\{\frac{(\psi(b)-\psi(a))^{\frac{\eta_k}{k}-1}}{|\mathcal{A}|\Gamma_k(\eta_k)}\Big[\mathcal{A}_1\Big(|\varsigma|\,\frac{(\psi(u)-\psi(a))^{\frac{\hat\alpha}{k}}}{\Gamma_k(\hat\alpha+k)}+|\theta|\,\frac{(\psi(\tau)-\psi(a))^{\frac{\hat\alpha+w}{k}}}{|\mathcal{A}|\Gamma_k(\hat\alpha+w+k)}\Big) \\
&\quad +\mathcal{A}_3\frac{(\psi(b)-\psi(a))^{\frac{\hat\alpha}{k}}}{\Gamma_k(\hat\alpha+k)}\Bigg\}[\hat m_1\|\hat r\|+\hat m_2\|\hat z\|+\mathfrak{N}]+\Bigg\{\frac{(\psi(b)-\psi(a))^{\frac{\hat p}{k}}}{\Gamma_k(\hat p+k)} \\
&\quad +\frac{(\psi(b)-\psi(a))^{\frac{\eta_k}{k}-1}}{|\mathcal{A}|\Gamma_k(\vartheta_k)}\Big[\mathcal{A}_1\frac{(\psi(b)-\psi(a))^{\frac{\hat p}{k}}}{\Gamma_k(\hat p+k)}+\mathcal{A}_3\Big(|\lambda|\,\frac{(\psi(\xi)-\psi(a))^{\frac{\hat p}{k}}}{\Gamma_k(\hat p+k)} \\
&\quad +|\mu|\frac{(\psi(\sigma)-\psi(a))^{\frac{\hat p+v}{k}}}{\Gamma_k(\hat p+v+k)}\Big)\Big]\Bigg\}[\hat n_1\|\hat r\|+\hat n_2\|\hat z\|+\mathfrak{N}_1] \\
&= \mathfrak{Q}_3[\hat m_1\|\hat r\|+\hat m_2\|\hat z\|+\mathfrak{N}]+\mathfrak{Q}_4[\hat n_1\|\hat r\|+\hat n_2\|\hat z\|+\mathfrak{N}_1] \\
&= (\mathfrak{Q}_3\hat m_1+\mathfrak{Q}_4\hat n_1)\|\hat r\|+(\mathfrak{Q}_3\hat m_2+\mathfrak{Q}_4\hat n_2)\|\hat z\|+\mathfrak{Q}_3\mathfrak{N}+\mathfrak{Q}_4\mathfrak{N}_1 \\
&\le \mathfrak{Q}_3\hat m_1+\mathfrak{Q}_4\hat n_1+\mathfrak{Q}_3\hat m_2+\mathfrak{Q}_4\hat n_2)r+\mathfrak{Q}_3\mathfrak{N}+\mathfrak{Q}_4\mathfrak{N}_1.
\end{aligned}
$$

Accordingly, we have

$$
\begin{aligned}
\|\mathcal{T}(\hat r,\hat z)\| &= \|\mathcal{T}_1(\hat r,\hat z)\|+\|\mathcal{T}_2(\hat r,\hat z)\| \\
&\le [(\mathfrak{Q}_1+\mathfrak{Q}_3))(\hat m_1+\hat m_2)+(\mathfrak{Q}_2+\mathfrak{Q}_4)(\hat n_1+\hat n_2)]r \\
&\quad +(\mathfrak{Q}_1+\mathfrak{Q}_3)\mathfrak{N}+(\mathfrak{Q}_2+\mathfrak{Q}_4)\mathfrak{N}_1\le r,
\end{aligned}
$$

which implies that $\mathcal{T}(\mathcal{B}_r)\subseteq\mathcal{B}_r$. On the other hand, for $(\hat r_2,\hat z_2),(\hat r_1,\hat z_1)\in\mathbb{X}\times\mathbb{X}$ and $t\in[a,b]$, we have

$$
\begin{aligned}
&|T_1(r_2,z_2)(t)-T_1(r_1,z_1)(t)| \\
&\le {}^k\mathcal{I}^{\hat\alpha,\psi}|f(t,\hat r_2(t),\hat z_2(t))-f(t,\hat r_1(t),\hat z_1(t))| \\
&\quad +\frac{(\psi(b)-\psi(a))^{\frac{\vartheta_k}{k}-1}}{|\mathcal{A}|\Gamma_k(\vartheta_k)}\Big[\mathcal{A}_4\Big(|\lambda|\,{}^k\mathcal{I}^{\hat p,\psi}|\tilde f(\xi,\hat r_2(\xi),\hat z_2(\xi))-\tilde f(\xi,\hat r_1(\xi),\hat z_1(\xi))| \\
&\quad +|\mu|\,{}^k\mathcal{I}^{\hat p+v,\psi}|\tilde f(\sigma,\hat r_2(\sigma),\hat z_2(\sigma))-\tilde f(\sigma,\hat r_1(\sigma),\hat z_1(\sigma))| \\
&\quad +{}^k\mathcal{I}^{\hat p,\psi}|f(b,\hat r_2(b),\hat z_2(b))-f(b,\hat r_1(b),\hat z_1(b))|\Big) \\
&\quad +\mathcal{A}_2\Big(|\varsigma|\,{}^k\mathcal{I}^{\hat\alpha,\psi}|f(u,\hat r_2(u),\hat z_2(u))-f(u,\hat r_1(u),\hat z_1(u))|
\end{aligned}
$$

$$
\begin{aligned}
&+|\theta|\,{}^{k}\mathcal{I}^{\hat{\alpha}+w,\psi}|f(\tau,\hat{r}_2(\tau),\hat{z}_2(\tau))-f(\tau,\hat{r}_1(\tau),\hat{z}_1(\tau))| \\
&+{}^{k}\mathcal{I}^{\hat{p},\psi}|\tilde{f}(b,\hat{r}_2(b),\hat{z}_2(b))-\tilde{f}(b,\hat{r}_1(b),\hat{z}_1(b))|\Big)\Big] \\
\leq\ & \Bigg\{\frac{(\psi(b)-\psi(a))^{\frac{\hat{\alpha}}{k}}}{\Gamma_k(\hat{\alpha}+k)}+\frac{(\psi(b)-\psi(a))^{\frac{\vartheta_k}{k}-1}}{|\mathcal{A}|\Gamma_k(\vartheta_k)}\Bigg[\mathcal{A}_4\frac{(\psi(b)-\psi(a))^{\frac{\hat{\alpha}}{k}}}{\Gamma_k(\hat{\alpha}+k)} \\
&+\mathcal{A}_2\Bigg(|\varsigma|\,\frac{(\psi(u)-\psi(a))^{\frac{\hat{\alpha}}{k}}}{\Gamma_k(\hat{\alpha}+k)}+|\theta|\frac{(\psi(\tau)-\psi(a))^{\frac{\alpha+w}{k}}}{\Gamma_k(\hat{\alpha}+w+k)}\Bigg)\Bigg]\Bigg\}(\hat{m}_1\|\hat{r}_2-\hat{r}_1\|+\hat{m}_2\|\hat{z}_2-\hat{z}_1\|) \\
&+\frac{(\psi(b)-\psi(a))^{\frac{\vartheta_k}{k}-1}}{|\mathcal{A}|\Gamma_k(\vartheta_k)}\Bigg[\Bigg\{\mathcal{A}_4\Bigg(|\lambda|\,\frac{(\psi(\xi)-\psi(a))^{\frac{\hat{p}}{k}}}{\Gamma_k(\hat{p}+k)}+|\mu|\,\frac{(\psi(\sigma)-\psi(a))^{\frac{\hat{p}+v}{k}}}{\Gamma_k(\hat{p}+v+k)}\Bigg) \\
&+\mathcal{A}_2\frac{(\psi(b)-\psi(a))^{\frac{\hat{p}}{k}}}{\Gamma_k(\hat{p}+k)}\Bigg\}\Bigg](\hat{n}_1\|\hat{r}_2-\hat{r}_1\|+\hat{n}_2\|\hat{z}_2-\hat{z}_1\|) \\
=\ & (\mathfrak{Q}_1\hat{m}_1+\mathfrak{Q}_2\hat{n}_1)(\|\hat{r}_2-\hat{r}_1\|)+(\mathfrak{Q}_1\hat{m}_2+\mathfrak{Q}_2\hat{n}_2)(\|\hat{z}_2-\hat{z}_1\|).
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
&\|\mathcal{T}_1(\hat{r}_2,\hat{z}_2)(t)-\mathcal{T}_1(\hat{r}_1,\hat{z}_1)\| \\
&\leq((\mathfrak{Q}_1\hat{m}_1+\mathfrak{Q}_2\hat{n}_1+\mathfrak{Q}_1\hat{m}_2+\mathfrak{Q}_2\hat{n}_2)[\|\hat{r}_2-\hat{r}_1\|+\|\hat{z}_2-\hat{z}_1\|]. \qquad (22)
\end{aligned}
$$

Similarly, one can obtain

$$
\begin{aligned}
&\|\mathcal{T}_2(\hat{r}_2,\hat{z}_2)(t)-\mathcal{T}_2(\hat{r}_1,\hat{z}_1)\| \\
&\leq((\mathfrak{Q}_3\hat{m}_1+\mathfrak{Q}_4\hat{n}_1+\mathfrak{Q}_3\hat{m}_2+\mathfrak{Q}_4\hat{n}_2)[\|\hat{r}_2-\hat{r}_1\|+\|\hat{z}_2-\hat{z}_1\|]. \qquad (23)
\end{aligned}
$$

From (22) and (23), we deduce that

$$
\begin{aligned}
&\|\mathcal{T}(\hat{r}_2,\hat{z}_2)(t)-\mathcal{T}(\hat{r}_1,\hat{z}_1)\| \\
&\leq[(\mathfrak{Q}_1+\mathfrak{Q}_3)(\hat{m}_1+\hat{m}_2)+(\mathfrak{Q}_2+\mathfrak{Q}_4)(\hat{n}_1+\hat{n}_2)](\|\hat{r}_2-\hat{r}_1\|+\|\hat{z}_2-\hat{z}_1\|), \qquad (24)
\end{aligned}
$$

which, by the condition (21), implies that $\mathcal{T}$ is a contraction. Thus, we deduce by the conclusion of Banach's contraction mapping principle that $\mathcal{T}$ has a unique fixed point which corresponds to a unique solution of the nonlinear system (1). □

4. Examples

Consider the following six-point boundary value problems for a (k,ϕ)–Hilfer fractional system after fixing the coefficients in problem (1):

$$
\begin{cases}
{}^{\frac{7}{6},H}\mathfrak{D}^{\frac{4}{3},\frac{3}{4};2+\log t}\hat{r}(t)=f(t,\hat{r}(t),\hat{z}(t)), & \frac{1}{4}<t<\frac{11}{4}, \\
{}^{\frac{7}{6},H}\mathcal{D}^{\frac{5}{3},\frac{1}{4};2+\log t}\hat{z}(t)=\tilde{f}(t,\hat{r}(t),\hat{z}(t)), & \frac{1}{4}<t<\frac{11}{4}, \\
\hat{r}\left(\frac{1}{4}\right)=0, \quad \hat{r}\left(\frac{11}{4}\right)=\frac{1}{\pi^2}\hat{z}\left(\frac{3}{4}\right)+\frac{7}{39}\,{}^{\frac{7}{6}}\mathcal{I}^{\frac{5}{3};2+\log t}\hat{z}\left(\frac{7}{4}\right), \\
\hat{z}\left(\frac{1}{4}\right)=0, \quad \hat{z}\left(\frac{11}{4}\right)=\frac{9}{37}\hat{r}\left(\frac{5}{4}\right)+\frac{1}{e^2}\,{}^{\frac{7}{6}}\mathcal{I}^{\frac{3}{5};2+\log t}\hat{r}\left(\frac{9}{4}\right).
\end{cases} \qquad (25)
$$

Here $k=7/6$, $\hat{\alpha}=4/3$, $\hat{p}=5/3$, $\hat{\beta}=3/4$, $\hat{q}=1/4$, $v=5/3$, $w=3/5$, $\psi(t)=2+\log t$, $\lambda=1/\pi^2$, $\mu=7/39$, $\varsigma=9/37$, $\theta=1/e^2$, $a=1/4$, $b=11/4$, $\xi=3/4$, $\sigma=7/4$, $u=5/4$, $\tau=9/4$. Using the given data, we find that $\vartheta_k=25/12$, $\eta_k=11/6$, $\Gamma_k(\vartheta_k)\approx 1.047114902$, $\Gamma_k(\eta_k)\approx 0.9726277286$, $\Gamma_k(\vartheta_k+w)\approx 1.425589394$, $\Gamma_k(\eta_k+v)\approx 2.722222222$, $\Gamma_k(\hat{\alpha}+k)\approx 1.275021091$, $\Gamma_k(\hat{p}+k)\approx 1.577652779$, $\Gamma_k(\hat{\alpha}+w+k)\approx 1.928101260$,

$\Gamma_k(\hat{p}+v+k) \approx 7.812151797$, $\mathcal{A}_1 \approx 1.898644621$, $\mathcal{A}_2 \approx 0.3595879078$, $\mathcal{A}_3 \approx 0.6017756924$, $\mathcal{A}_4 \approx 1.694724814$ ($\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$, and $\mathcal{A}_4$ are given in (8)), $\mathcal{A} \approx 3.001288890$ ($\mathcal{A}$ is given by (4)), $\mathfrak{Q}_1 \approx 4.549161327$, $\mathfrak{Q}_2 \approx 0.7467564627$, $\mathfrak{Q}_3 \approx 1.353804071$, $\mathfrak{Q}_4 \approx 4.484708590$ ($\mathfrak{Q}_i, i = 1,2,3,4$ are defined in (15)), $\mathfrak{Q}_1^* \approx 2.418206215$, $\mathfrak{Q}_4^* \approx 2.273635184$ ($\mathfrak{Q}_1^*$, $\mathfrak{Q}_4^*$ are given in (16)).

(a) Illustration of Theorem 1.

Let the functions $f, \tilde{f} : [1/4, 11/4] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$f(t,\hat{r},\hat{z}) = \frac{1+\cos^2(\hat{r}\hat{z})}{2\pi t} + \frac{e^{-\hat{z}^2}\sin\hat{r}}{4t+11} + \frac{|\hat{z}|^{15}e^{-(4t-1)^2}}{13(1+\hat{z}^{14})}, \tag{26}$$

$$\tilde{f}(t,\hat{r},\hat{z}) = \frac{4t(\hat{r}\hat{z})^2}{1+(\hat{r}\hat{z})^2} + \frac{\hat{r}^{12}}{11(1+|\hat{r}|^{11})} + \frac{\hat{z}\sin^8(\hat{r})}{4t+9}. \tag{27}$$

It is easy to find that $|f(t,\hat{r},\hat{z})| \le (4/\pi) + (1/12)|\hat{r}| + (1/13)|\hat{z}|$ and $|\tilde{f}(t,\hat{r},\hat{z})| \le 11 + (1/11)|\hat{r}| + (1/10)|\hat{z}|$. Setting $\hat{k}_0 = 4/\pi$, $\hat{k}_1 = 1/12$, $\hat{k}_2 = 1/13$, $\hat{v}_0 = 11$, $\hat{v}_1 = 1/11$, $\hat{v}_2 = 1/10$, we obtain $(\mathfrak{Q}_1+\mathfrak{Q}_3)\hat{k}_1 + (\mathfrak{Q}_2+\mathfrak{Q}_4)\hat{v}_1 \approx 0.9675015153 < 1$ and $(\mathfrak{Q}_1+\mathfrak{Q}_3)\hat{k}_2 + (\mathfrak{Q}_2+\mathfrak{Q}_4)\hat{v}_2 \approx 0.9772207667 < 1$. Therefore, we deduce by Theorem 1 that problem (25) with functions $f, \tilde{f}$ given by (26) and (27), respectively, has at least one solution on $[1/4, 11/4]$.

(b) Illustration of Theorem 2.

Define two nonlinear Lipschitzian functions $f, \tilde{f} : [1/4, 11/4] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(t,\hat{r},\hat{z}) = \frac{1}{3}\sin^2\pi t + \frac{|\hat{r}|e^{-(4t-1)}}{16(1+|\hat{r}|)} + \frac{1}{14}\tan^{-1}\hat{z}, \tag{28}$$

$$\tilde{f}(t,\hat{r},\hat{z}) = \frac{1}{5}\cos^2\pi t + \frac{1}{15}\sin\hat{r} + \frac{|\hat{z}|}{52t(1+|\hat{z}|)}. \tag{29}$$

Obviously, we have

$$|f(t,\hat{r},\hat{z})| \le \frac{1}{3}\sin^2\pi t + \frac{1}{16}e^{-(4t-1)} + \frac{\pi}{28}, \quad |\tilde{f}(t,\hat{r},\hat{z})| \le \frac{1}{5}\cos^2\pi t + \frac{1}{52t} + \frac{1}{15},$$

and

$$|f(t,\hat{r}_1,\hat{z}_1) - f(t,\hat{r}_2,\hat{z}_2)| \le \frac{1}{16}|\hat{r}_1 - \hat{r}_2| + \frac{1}{14}|\hat{z}_1 - \hat{z}_2|,$$

$$|\tilde{f}(t,\hat{r}_1,\hat{z}_1) - \tilde{f}(t,\hat{r}_2,\hat{z}_2)| \le \frac{1}{15}|\hat{r}_1 - \hat{r}_2| + \frac{1}{13}|\hat{z}_1 - \hat{z}_2|.$$

Setting $\hat{m}_1 = 1/16$, $\hat{m}_2 = 1/14$, $\hat{n}_1 = 1/15$, $\hat{n}_2 = 1/13$, we find that $[\mathfrak{Q}_1^* + \mathfrak{Q}_3](\hat{m}_1 + \hat{m}_2) + [\mathfrak{Q}_2 + \mathfrak{Q}_4^*](\hat{n}_1 + \hat{n}_2) \approx 0.9388772111 < 1$. Therefore, all the assumptions of Theorem 2 are fulfilled and, hence, the (k,ϕ)–Hilfer fractional system (25) with functions $f, \tilde{f}$ given by (28) and (29), respectively, has at least one solution on $[1/4, 11/4]$.

(c) Illustration of Theorem 3.

Let two nonlinear Lipschitzian unbounded functions $f, \tilde{f} : [1/4, 11/4] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be given by

$$f(t,\hat{r},\hat{z}) = \frac{e^{-(4t-1)^2}}{8(3t+5)}\left(\frac{\hat{r}^2+2|\hat{r}|}{1+|\hat{r}|}\right) + \frac{\sin|\hat{z}|}{4(4t+5)} + \frac{1}{3}, \tag{30}$$

$$\tilde{f}(t,\hat{r},\hat{z}) = \sin^2 t\,\frac{\tan^{-1}(\hat{r})}{4(2t+5)} + \frac{\cos^2(\pi t)}{28(2t+1)}\left(\frac{\hat{z}^2+2|\hat{z}|}{1+|\hat{z}|}\right) + \frac{1}{4}. \tag{31}$$

Clearly

$$|f(t,\hat{r}_1,\hat{z}_1) - f(t,\hat{r}_2,\hat{z}_2)| \le \frac{1}{23}|\hat{r}_1 - \hat{r}_2| + \frac{1}{24}|\hat{z}_1 - \hat{z}_2|,$$

and

$$|\tilde{f}(t,\hat{r}_1,\hat{z}_1) - \tilde{f}(t,\hat{r}_2,\hat{z}_2)| \leq \frac{1}{22}|\hat{r}_1 - \hat{r}_2| + \frac{1}{21}|\hat{z}_1 - \hat{z}_2|.$$

Setting $\hat{m}_1 = 1/23$, $\hat{m}_2 = 1/24$, $\hat{n}_1 = 1/22$ and $\hat{n}_2 = 1/21$, we have that $(\mathfrak{Q}_1 + \mathfrak{Q}_3)(\hat{m}_1 + \hat{m}_2) + (\mathfrak{Q}_2 + \mathfrak{Q}_4)(\hat{n}_1 + \hat{n}_2) \approx 0.9895188106 < 1$. Thus, the conclusion of Theorem 3 applies and, hence, the problem (25) with functions f, $\tilde{f}$ given by (30) and (31), respectively, has a unique solution on the interval $[1/4, 11/4]$.

5. Conclusions

In this paper, we presented sufficient criteria for the existence and uniqueness of solutions for a coupled system of nonlinear (k,ψ)–Hilfer fractional differential equations complemented with coupled (k,ψ)–Riemann–Liouville fractional integral boundary conditions. We proved the desired results for the given problem by applying the Leray–Schauder alternative, Krasnosel'skiǐ's fixed-point theorem and Banach's contraction mapping principle. As a special case, we obtained the new results for a coupled system of nonlinear (k,ψ)–Hilfer fractional differential equations equipped with
(i) four-point nonlocal coupled boundary conditions:

$$\hat{r}(a) = 0, \hat{r}(b) = \lambda\hat{z}(\xi), \hat{z}(a) = 0, \hat{z}(b) = \varsigma\hat{r}(u),$$

by taking $\mu = 0$ and $\theta = 0$ in the results of the paper; and
(ii) purely coupled (k,ψ)–Riemann–Liouville fractional integral boundary conditions:

$$\hat{r}(a) = 0, \hat{r}(b) = \mu\, {}^k\mathcal{I}^{v,\psi}\hat{z}(\sigma), \hat{z}(a) = 0, \hat{z}(b) = \theta\, {}^k\mathcal{I}^{w,\psi}\hat{r}(\tau),$$

by fixing $\lambda = 0, \varsigma = 0$ in the present results.

Comparing the present results with the related work on the topic, it is worthwhile to mention that the authors studied the (k,ψ)–Hilfer fractional differential equations and inclusions equipped with the different boundary conditions in the articles [17–19], while a system of (k,φ)–Hilfer fractional differential equations with nonlocal boundary conditions was investigated in [20]. Thus, the work established in this paper is not only new in the given configuration but also produces some new results as special cases. Therefore, our work is a significant contribution to the existing literature on the boundary value problems of (k,ψ)–Hilfer fractional differential systems. In future, we plan to investigate coupled systems of nonlinear sequential (k,ψ)–Hilfer fractional differential equations and inclusions equipped with different kinds of boundary conditions.

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References

1. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of the Fractional Differential Equations*; North-Holland Mathematics Studies; Elsevier: Amsterdam, The Netherlands, 2006; Volume 204.
2. Ahmad, B.; Ntouyas, S.K. *Nonlocal Nonlinear Fractional-Order Boundary Value Problems*; World Scientific: Singapore, 2021.
3. Zhou, Y. *Basic Theory of Fractional Differential Equations*; World Scientific: Singapore, 2014.

4. Nuchpong, C.; Ntouyas, S.K.; Tariboon, J. Boundary value problems of Hilfer-type fractional integro-differential equations and inclusions with nonlocal integro-multipoint boundary conditions. *Open Math.* **2020**, *18*, 1879–1894. [CrossRef]
5. Subramanian, M.; Gopal, T.N. Analysis of boundary value problem with multi-point conditions involving Caputo-Hadamard fractional derivative. *Proyecciones* **2020**, *39*, 155–1575. [CrossRef]
6. Belbali, H.; Benbachir, M.; Etemad, S.; Park, C.; Rezapour, S. Existence theory and generalized Mittag-Leffler stability for a nonlinear Caputo-Hadamard FIVP via the Lyapunov method. *AIMS Math.* **2022**, *7*, 14419–14433. [CrossRef]
7. Magin, R.L. *Fractional Calculus in Bioengineering*; Begell House Publishers: Danbury, CT, USA, 2006.
8. Zaslavsky, G.M. *Hamiltonian Chaos and Fractional Dynamics*; Oxford University Press: Oxford, UK, 2005.
9. Fallahgoul, H.A.; Focardi, S.M.; Fabozzi, F.J. *Fractional Calculus and Fractional Processes with Applications to Financial Economics. Theory and Application*; Elsevier/Academic Press: London, UK, 2017.
10. Almalahi, M.A.; Abdo, M.S.; Panchal, S.K. Existence and Ulam-Hyers stability results of a coupled system of ψ–Hilfer sequential fractional differential equations. *Results Appl. Math.* **2021**, *10*, 100142. [CrossRef]
11. Wongcharoen, A.; Ntouyas, S.K.; Wongsantisuk, P.; Tariboon, J. Existence results for a nonlocal coupled system of sequential fractional differential equations involving ψ–Hilfer fractional derivatives. *Adv. Math. Phys.* **2021**, *2021*, 5554619. [CrossRef]
12. Nuchpong, C.; Ntouyas, S.K.; Samadi, A.; Tariboon, J. Boundary value problems for Hilfer type sequential fractional differential equations and inclusions involving Riemann-Stieltjes integral multi-strip boundary conditions. *Adv. Differ. Equ.* **2021**, *2021*, 268. [CrossRef]
13. Hilfer, R. *Applications of Fractional Calculus in Physics*; World Scientific: Singapore, 2000.
14. Dorrego, G.A. An alternative definition for the k–Riemann-Liouville fractional derivative. *Appl. Math. Sci.* **2015**, *9*, 481–491. [CrossRef]
15. Vanterler da C. Sousa, J.; Capelas de Oliveira, E. On the ψ–Hilfer fractional derivative. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *60*, 72–91. [CrossRef]
16. Kucche, K.D.; Mali, A.D. On the nonlinear (k,ψ)–Hilfer fractional differential equations. *Chaos Solitons Fractals* **2021**, *152*, 111335. [CrossRef]
17. Ntouyas, S.K.; Ahmad, B.; Tariboon, J.; Alhodaly, M.S. Nonlocal integro-multi-point (k,ψ)–Hilfer type fractional boundary value problems. *Mathematics* **2022**, *10*, 2357. [CrossRef]
18. Ntouyas, S.K.; Ahmad, B.; Tariboon, J. Nonlocal boundary value problems for (k,ψ)–Hilfer fractional differential equations and inclusions. *Foundations* **2022**, *2*, 681–696. [CrossRef]
19. Ntouyas, S.K.; Ahmad, B.; Nuchpong, C.; Tariboon, J. On (k,ψ)–Hilfer fractional differential equations and inclusions with mixed (k,ψ)–derivative and integral boundary conditions. *Axioms* **2022**, *11*, 403. [CrossRef]
20. Samadi, A.; Ntouyas, S.K.; Tariboon, J. Nonlocal coupled system for (k,φ)–Hilfer fractional differential equations. *Fractal Fract.* **2022**, *6*, 234. [CrossRef]
21. Kwun, Y.C.; Farid, G.; Nazeer, W.; Ullah, S.; Kang, S.M. Generalized Riemann-Liouville k-fractional integrals associated with Ostrowski type inequalities and error bounds of Hadamard inequalities. *IEEE Access* **2018**, **6**, 64946–64953. [CrossRef]
22. Granas, A.; Dugundji, J. *Fixed Point Theory*; Springer: New York, NY, USA, 2003.
23. Krasnosel'skiĭ, M.A. Two remarks on the method of successive approximations. *Uspekhi Mat. Nauk* **1955**, *10*, 123–127.