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Analysis of Sequential Caputo Fractional Differential Equations versus Non-Sequential Caputo Fractional Differential Equations with Applications

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Abstract: It is known that, from a modeling point of view, fractional dynamic equations are more suitable compared to integer derivative models. In fact, a fractional dynamic equation is referred to as an equation with memory. To demonstrate that the fractional dynamic model is better than the corresponding integer model, we need to compute the solutions of the fractional differential equations and compare them with an integer model relative to the data available. In this work, we will illustrate that the linear nq-order sequential Caputo fractional differential equations, which are sequential of order q where $q < 1$ with fractional initial conditions and/or boundary conditions, can be solved. The reason for choosing sequential fractional dynamic equations is that linear non-sequential Caputo fractional dynamic equations with constant coefficients cannot be solved in general. We used the Laplace transform method to solve sequential Caputo fractional initial value problems. We used fractional boundary conditions to compute Green's function for sequential boundary value problems. In addition, the solution of the sequential dynamic equations yields the solution of the corresponding integer-order differential equations as a special case as $q \to 1$.

Keywords: sequential Caputo fractional derivative; fractional initial and boundary value problems; Mittag–Leffler functions; Green's function

MSC: 34A08; 34A12

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1. Introduction

Dynamic equations with integer-order, ordinary, partial, and hybrid models with initial and/or boundary conditions have been used as mathematical models in various branches of science and engineering. The concept of fractional calculus was known over 300 years ago. However, the study of fractional dynamic equations gained importance due to its myriad applications in widespread fields of science and engineering. In the past four decades, the application of fractional dynamic equations has been felt in several scientific and engineering areas. See [1–30] for some applications. See [14–16,19,20,24,27,31–33] for monographs on analysis and applications of fractional dynamic equations. The paper [31] is entirely dedicated to the study of Mittag–Leffler functions. The Mittag–Leffler function plays a crucial role in the study of linear fractional differential equations. See [34–44] for some analysis and numerical work on Caputo fractional differential equations. Additionally, see [45–49] for some work on fractional boundary value problems and sequential fractional boundary value problems. In [20], the authors observed in their experiment that the use of half-order derivatives and integrals led to a formulation of certain electro-chemical problems that was more economical and useful than the classical approach in terms of Fick's law of diffusion. In short, we can establish that fractional-order differential equations represent a better model only when the solution of the fractional order is closer to the data compared to the integer model. To reach this conclusion or judgement, we should be able to solve the fractional differential equation whose solution reduces to the integer solution as the fractional order q tends to its nearest integer. For certain values of the fractional order, the solution of the fractional order has the least error with the data. In [1], the authors demonstrated that the modeling of some real phenomena by fractional differential equations of appropriate fractional order is better than the nearest integer-order model. If the fractional order is $q < 1$, then we can solve the linear initial value problem, which involves Mittag–Leffler functions. The two-parameter Mittag–Leffler function is the generalization of the exponential function, and it is, in fact, the exponential function when the two parameters are exactly 1. We can solve the linear Caputo fractional differential equation of the form

$$^{c}D^{q}_{0+}u = \lambda u + f(t), \tag{1}$$

when $(n-1) < q < n$ for any integer n. However, the initial conditions are those of the nth-order differential equations. The initial conditions will be of the form,

$$u^{(k)}(0) = b_k, \ for\ k = 0, 1, \ldots (n-1).$$

See [14,24] for more details. In order to obtain the solution, the authors start with the basis solution as

$$1, t, t^2, \ldots\ldots\ldots, t^{(n-1)}.$$

However, we cannot solve a general linear fractional differential equation of any order q when $(n-1) < q < n$, especially when we have lower-order fractional derivatives in the fractional differential equation. See, for example, Equation (4.2) from [24] presented below,

$$^{c}D^{Q}_{0+}u + b\,^{c}D^{q}_{0+}u = h(t). \tag{2}$$

It is to be noted that if the initial conditions are provided for the above equation, the initial conditions will be the same as those of the nearest integer derivative. For example, if $Q > q$ and $(n-1) < Q < n$, then the initial conditions will be the same as those of the nth-order differential equation. However, if Q and q are integers, theoretically we can solve the equation by assuming $u = e^{rt}$ to be the solution and find values of r to find the general solution. The main reason for this is that the integer-order derivative is sequential. In order to compute the solution of the linear Caputo fractional differential equation of order nq where $(n-1) < nq < n$ when it has terms involving kq-order fractional derivatives with $k = 0, 1, 2, \ldots (n-1)$, the Caputo fractional derivative must be sequential. In this case, we can use the Mittag–Leffler functions $E_{q,r}(\lambda t^q)$ with its parameter $q, r < 1$. In addition, as $q, r \to 1$, the integer solution can be obtained as a special case. In order to establish that the Caputo fractional differential equation represents a better model compared with the integer model, we need to compute the solution of the corresponding fractional model. In addition, the solution of the Caputo fractional model should tend to the solution of the integer model when the fractional order $q \to 1$. However, the standard method we adopt, like the use of the Wronskian to find two linearly independent solutions or the variations of the parameter method, cannot be used for sequential Caputo fractional differential equations. The only suitable method seems to be the Laplace transform method. For that purpose, the Laplace transform table, which includes the transforms of appropriate Mittag–Leffler functions or generalizations of Mittag–Leffler functions, has been included. See [4,20,41–43] for some sequential initial value problems.

In this work, we will also look at some sequential boundary value problems. The need for this is mainly to take advantage of the parameter q of the Caputo fractional derivative to enhance the mathematical model. The majority of the work done on fractional boundary value problems is, in general, on non-sequential boundary value problems. In addition, most works have used integer boundary conditions. The integer derivative involved in boundary value problems is known to be sequential. In short, most work has used Green's functions of the integer order in the majority of the fractional boundary value problems.

As far as the boundary value problem is concerned, in literature, the linear operator has only the highest derivative term of order q, such that the value of q is $(n-1) < q < n$. For example, when $n = 1$, most literature has used the corresponding boundary condition as that of the second-order boundary value problem. Thus, it is relatively easy to compute Green's function, which is the same as the integer order. Green's function will not have any effect from the order of the fractional derivative. Related to the sequential boundary value problem, we have included the lower-order fractional derivative terms and have used the fractional derivative in the boundary condition as well. Thus, the Green's function that is computed will involve the sequential derivative of order q.

Recently, in [44], we published a result on sequential Caputo versus non-sequential Caputo fractional initial and boundary value problems. Please see the references in [44] for more work on the sequential boundary value problem. Before proceeding, a remark on the research article [50] is in order. Their conclusion on the coincidence of the left and the right boundary conditions hinges on Theorem 4 of their result. They claim that the Caputo left derivative computed at $x = a$ and the Caputo right derivative computed at $x = b$ are true only for constant functions. Their proof is based on ${}^cD^q_{a+}f(x)|_{x=a} = 0$ and ${}^cD^q_{b-}f(x)|_{x=b} = 0$ for every increasing function $f(x)$ on $[a, b]$. A simple counter-example is that $f(x) = (x-a)^q$ when computed from the left, is $\Gamma(q+1)$ for all $x \in (a, b)$. By continuity, the derivative is identically $\Gamma(q+1)$ for all $x \in [a, b]$. Similarly, when we compute $f(x) = (b-x)^q$ from the right (using the definition of the right derivative with the proper sign), it will yield $\Gamma(q+1)$ for all $x \in [a, b]$. It is to be noted that $f(x) = (x-a)^q$ is an increasing function in x, as they assumed in their Theorem 4. Our example is a direct generalization of the integer result. In short, when we say that the left derivative is equal to the right derivative, we mean that the left Caputo derivative of a function $f(x-a)$ can be transformed to the right Caputo derivative of the function $f(b-x)$. However, for symmetric functions such as $f((x-a),(b-x)) = f((b-x),(x-a))$, the Caputo left derivative will be exactly equal to the right Caputo derivative for any $x \in [a, b]$. In other cases, we need to replace $(x-a)$ and $(b-x)$ with $(b-x)$ and $(x-a)$, respectively, which is illustrated by our example of $f(x) = (x-a)^q$. One can construct many such examples.

In this work, we will recall some results we have developed relative to linear sequential Caputo fractional boundary value problems, which cannot be solved in general if they are non-sequential. The advantage of our results is that we can compute the numerical solution of the linear sequential Caputo fractional differential equations with fractional boundary conditions. This numerical solution tends to the corresponding solution of the integer boundary value problems. However, in the literature, fractional boundary conditions have been used only for Riemann–Liouville boundary value problems. As an example, see [51]. However, these authors obtained an estimate for Green's function instead of computing Green's function. In our work, we will recall the exact expression of Green's function that can be obtained for linear sequential Caputo fractional boundary value problems involving fractional boundary conditions.

In this work, our major contributions are: (1) we provide a methodology for solving the linear sequential Caputo fractional differential equations with fractional initial conditions, and (2) we provide a methodology for solving linear sequential Caputo fractional boundary value problems with fractional boundary conditions.

Our work here in solving linear sequential Caputo fractional differential equations with fractional initial conditions and fractional boundary conditions is the initial step in solving weakly non-linear fractional differential equations. With this initial step, and by developing appropriate comparison theorems, we can solve the weakly non-linear problem by the monotone method, generalized monotone method, quasilinearization method, and generalized quasilinearization method. These are open problems of importance in applications.

Some applications, such as ice-melting problems, in integer-order partial differential equations with initial and boundary conditions have been developed in [52,53]. It is an interesting open problem to study Caputo time fractional partial differential equations of the same model, and the model can be improved by choosing the value of q as a parameter using the available data.

2. Preliminaries

In this section, we recall some basic definitions about the sequential initial and boundary value problems in our main results that are useful to discuss.

Definition 1. *Let $q > 0$ and $u(t) : (0, \infty) \longrightarrow \mathbb{R}$. Then, the Riemann–Liouville derivative of $u(t)$ of order q is given by*

$$D_{0+}^{q} u(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t \frac{u(s)}{(t-s)^{q-n+1}} ds,$$

where $n \in \mathbb{N}$ such that $(n-1) < q < n$.

Note that in the above definition, we can replace q by nq such that $(n-1) < nq < n$.

Definition 2. *The Riemann–Liouville fractional integral of order q is defined by*

$$D_{0+}^{-q} u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{(q-1)} u(s) ds, \tag{3}$$

where $0 < q < 1$.

Definition 3. *Let $nq > 0$, and $u(t) : (0, \infty) \longrightarrow \mathbb{R}$. Then, the Caputo derivative of $u(t)$ order nq is given by*

$${}^{c}D_{0+}^{nq} u(t) = \frac{1}{\Gamma(n-nq)} \int_0^t (t-s)^{-nq+n-1} u^{(n)}(s) ds,$$

where $n \in \mathbb{N}$ such that $(n-1) < nq < n$. In particular, if $q = 1$, then $nq = n$ is an integer and ${}^{c}D^{nq} u = u^{(n)}(t)$ and ${}^{c}D^{q} u = u'(x)$.

Note that the definition of the Caputo fractional integral of order q is same as that of the Riemann–Liouville fractional integral of order q for $0 < q < 1$.

The next definitions are useful for sequential boundary value problems.

Definition 4. *The Caputo (left-sided) fractional derivative of $u(x)$ of order q, when $(n-1) < q < n$, is given by*

$${}^{c}D_{a^+}^{q} u(x) = \frac{1}{\Gamma(n-q)} \int_a^x (x-s)^{n-q-1} u^{(n)}(s) ds, \ x > a, \tag{4}$$

and the (right-sided) fractional derivative is given by

$${}^{c}D_{b^-}^{q} u(x) = \frac{(-1)^n}{\Gamma(n-q)} \int_x^b (s-x)^{n-q-1} u^{(n)}(s) ds, \ x < b, \tag{5}$$

where $u^{(n)}(t) = \frac{d^n(u)}{dt^n}$.

In particular, if $q = n$ is an integer, then ${}^{c}D_{0+}^{q} u = u^{(n)}(x)$ and ${}^{c}D_{0+}^{q} u = u'(x)$ if $q = 1$. In this work, we choose the value of q such that it is replaced by nq and $(n-1) < nq < n$. In short, if $q = 1$, then we have the nth-order derivative. The next definition is that of the Mittag–Leffler function. It is the generalization of the exponential function and it plays the same role for fractional differential equations as the exponential function plays for the integer derivative dynamic equations, especially when $0 < q < 1$.

Definition 5. *The Mittag–Leffler function of two parameters q, r is given by*

$$E_{q,r}(\lambda(t-t_0)^q) = \sum_{k=0}^{\infty} \frac{(\lambda(t-t_0)^q)^k}{\Gamma(qk+r)},$$

where $q, r > 0$. Furthermore, for $t_0 = 0$ and $r = 1$, we obtain

$$E_{q,1}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk+1)},$$

where $q > 0$.

See [14,24,31] for more details on the Mittag–Leffler functions.

If $q = r = 1$, then the Mittag–Leffler (ML for short) function is the usual exponential function. In this work, we use the ML function when $0 < q \leq 1$. In addition, the Mittag–Leffler functions with $r = 1$, as well as $r = q$ and $0 < q < 1$, will be useful when sequential derivatives are involved. In the integer-order case, the solutions of linear equations with constant coefficients depend on the trigonometric functions of sine and cosine, which are defined in terms of the exponential function. In this work, we need fractional trigonometric sine and cosine functions that depend on the value of q as well. For that purpose, we define the following fractional trigonometric functions $\sin_{q,1}(\lambda t^q)$ and $\cos_{q,1}(\lambda t^q)$.

Definition 6. *The fractional trigonometric functions $\sin_{q,1}(\lambda t^q)$ and $\cos_{q,1}(\lambda t^q)$ are given by*

$$\sin_{q,1}(\lambda t^q) = \frac{1}{2i}[E_{q,1}(i\lambda t^q) - E_{q,1}(-i\lambda t^q)],$$

and

$$\cos_{q,1}(\lambda t^q) = \frac{1}{2}[E_{q,1}(i\lambda t^q) + E_{q,1}(-i\lambda t^q)],$$

respectively.

Using the above definition, similarly, we can define $\sin_{q,q}(\lambda t^q)$ and $\cos_{q,q}(\lambda t^q)$ using $E_{q,q}(\lambda t^q)$ in place of $E_{q,1}(\lambda t^q)$.

Another definition which we need is that of the fractional trigonometric function involving complex numbers of the form $\lambda + i\mu$. For that purpose, we will define the generalized fractional trigonometric functions $G\sin_{q,1}((\lambda + i\mu)t^q)$ and $G\cos_{q,1}((\lambda + i\mu)t^q)$ as follows:

Definition 7. *The generalized fractional trigonometric functions $G\sin_{q,1}((\lambda + i\mu)t^q)$ and $G\cos_{q,1}((\lambda + i\mu)t^q)$ are given by*

$$G\sin_{q,1}((\lambda+i\mu)t^q) = \frac{1}{2i}[E_{q,1}((\lambda+i\mu)t^q) - E_{q,1}((\lambda-i\mu)t^q)],$$

and

$$G\cos_{q,1}((\lambda+i\mu)t^q) = \frac{1}{2}[E_{q,1}((\lambda+i\mu)t^q)) + E_{q,1}((\lambda-i\mu)t^q)],$$

respectively.

One can also define $G\sin_{q,q}((\lambda+i\mu)t^q)$ and $G\cos_{q,q}((\lambda+i\mu)t^q)$ by replacing $E_{q,1}((\lambda+i\mu)t^q)$ and $E_{q,1}((\lambda-i\mu)t^q)$ by $E_{q,q}((\lambda+i\mu)t^q)$ and $E_{q,q}((\lambda-i\mu)t^q)$, respectively.

Remark 1. *If $q = 1$ in the above definition, then the functions $G\sin_{q,1}((\lambda + i\mu)t^q)$ and $G\sin_{q,q}((\lambda + i\mu)t^q)$ reduce to $e^{\lambda t}\sin(\mu t)$. Similarly, $G\cos_{q,1}((\lambda + i\mu)t^q)$ and $G\cos_{q,q}((\lambda + i\mu)t^q)$ also reduce to $e^{\lambda t}\cos(\mu t)$. Note that this simplification cannot be done when the Mittag–*

Leffler function is involved (with $q < 1$). The reason for that is the Mittag–Leffler function does not enjoy the properties of the exponential function. Furthermore , note that we will use the Mittag–Leffler functions and all fractional trigonometric functions with the independent variable t for initial value problems and x for boundary value problems.

Can we apply a method similar to the integer case for the linear homogeneous Caputo fractional differential equation of order nq when $(n-1) < nq < n$? In short, can we seek a solution of the form $E_{q,1}(rt^q)$, where the values of r would be a polynomial of degree n? For that purpose, we will consider linear Caputo fractional differential equations of order nq with constant coefficients of the form:

$$\sum_{k=0}^{n} a_k \, {}^cD_{0+}^{kq} u(t) = 0. \tag{6}$$

The answer is affirmative if the Caputo fractional derivative of order kq is sequential of order q for $k = 2, 3, 4 \ldots\ n$ in (6). We will provide the definition later. It is to be noted that the integer-order derivative is sequential, but the Caputo fractional derivative need not be sequential.

Recalling the definition of the Caputo fractional derivative of order nq when $(n-1) < nq < n$, we have

$${}^cD_{0+}^{2q}(u) = \frac{1}{\Gamma(2-2q)} \int_0^t \frac{u^{(2)}(s)}{(t-s)^{2q-1}} ds.$$

Let $u(t) = t^\omega$ in the above definition, where $2q$ is such that $1 < 2q < 2$.

Then,

$${}^cD_{0+}^{2q}(t^\omega) = \frac{\Gamma(\omega+1))t^{\omega-2q}}{\Gamma(1-2q+\omega)}.$$

In particular, if $\omega = q$, we obtain ${}^cD_{0+}^{2q}(t^q) = \frac{\Gamma(q+1)t^{-q}}{\Gamma(1-q)}$.

On the other hand, if we compute ${}^cD_{0+}^{q}(t^q) = \Gamma(1+q)$, then ${}^cD_{0+}^{q}({}^cD_{0+}^{q}(t^q)) = 0$.

In short

$${}^cD_{0+}^{2q}(E_{q,1}(\lambda t^q)) \neq (\lambda)^2 E_{q,1}(\lambda t^q),$$

if ${}^cD_{0+}^{2q}(u)$ is not sequential.

If ${}^cD_{0+}^{2q}(u)$ is sequential, then we have

$${}^cD_{0+}^{2q}(E_{q,1}(\lambda t^q)) = (\lambda)^2 E_{q,1}(\lambda t^q).$$

Next, we provide the definition for the sequential Caputo derivative.

Definition 8. *The Caputo fractional derivative of $u(t)$ of order nq for $(n-1) < nq < n$ is said to be the sequential Caputo fractional derivative of order q if the relation*

$$({}^cD_{0+}^{nq})u(t) = {}^cD_{0+}^{q}({}^cD_{0+}^{(n-1)q})u(t), \tag{7}$$

holds for $n = 2, 3, \ldots$

In short, we can rewrite (7) as follows:

$$({}^cD_{0+}^{nq})u(t) = {}^cD_{0+}^{q}({}^cD_{0+}^{q}({}^cD_{0+}^{q} \ldots {}^cD_{0+}^{q})u),$$

n times. Although we can find the general solution of the homogeneous sequential Caputo fractional differential Equation (6) using $E_{q,1}(rt^q)$, we also require the initial conditions in terms of the Caputo fractional derivative of the lower order evaluated at $t = 0$. Basically, the initial conditions will be of the form:

$$({}^{c}D_{0+}^{kq})u(t)|_{t=0} = b_k \ (b_k \in R;\ k = 0, 1, \ldots\ldots, (n-1)). \tag{8}$$

Once we have the initial conditions of the above form, then we can also use the Laplace transform method to solve the linear homogeneous sequential Caputo fractional differential Equation (6) with initial conditions. The Laplace transform method is also useful to find the solution of the linear non-homogeneous sequential Caputo fractional differential equation with fractional initial conditions, since the variation-of-parameter method will not be useful.

The next result is related to taking the Laplace transform of the Caputo sequential derivative of order nq, which is sequential of order q. This will be very useful in solving the linear Caputo sequential fractional differential equation of order nq, which is sequential of order q with constant coefficients.

From now on, we will use the notation ${}^{sc}D_{0+}^{nq}u(t)$ for a Caputo derivative of order nq, which is sequential of order q. The next known result is related to the sequential Caputo fractional derivative of order nq.

Theorem 1. *The Laplace transform of a sequential Caputo fractional derivative of $u(t)$ of order nq on $[0, \infty)$, when nq is such that $n-1 < nq < n$ is given by*

$$\begin{aligned}\mathcal{L}({}^{sc}D_{0+}^{nq}u(t)) = s^{nq}U(s) - s^{(nq-1)}u(0) - s^{(n-1)q-1}({}^{sc}D_{0+}^{q}u(t)|_{t=0}) \\ - s^{(n-2)q-1}({}^{sc}D_{0+}^{2q}u(t)|_{t=0}) \ldots - s^{q-1}({}^{sc}D_{0+}^{(n-1)q}u(t)|_{t=0}),\end{aligned} \tag{9}$$

where $U(s) = \mathcal{L}(u(t))$.

For proof, see [42].

Next, we provide Laplace transform tables that will be useful in solving the nq-order linear non-homogeneous sequential Caputo fractional differential equation (which is sequential of order q with $0 < q < 1$) with Caputo fractional initial conditions. In fact, for all practical and computational purposes, we choose the value of q such that $0.5 \leq q < 1$, when $n = 2$. This way, we use the value of q such that $q < 1$ and provide a solution with the least error with the available data. See [1] as an example.

Now, we will recall the Laplace transform table for some basic fractional functions that will be needed in our work. See [22,23] for more detailed Laplace transform tables.

Note that one can define the (right) sequential Caputo fractional derivative of order nq in terms of the (right) sequential Caputo fractional derivative of order q.

Definition 9. *The Caputo fractional derivative of order nq, for $(n-1) < nq < n$, which is (left at $x = a$) the sequential Caputo fractional derivative of order q, can be written as*

$$({}^{c}D_{a^+}^{nq})u(x) = \ ({}^{c}D_{a^+}^{q}({}^{c}D_{a^+}^{q}(\ldots\ldots n\ times)u(x). \tag{10}$$

Definition 10. *The Caputo fractional derivative of order nq for $(n-1) < nq < n$ (right at $x = b$) is a sequential Caputo fractional derivative of order q if*

$$({}^{c}D_{b^-}^{nq})u(x) = \ ({}^{c}D_{b^-}^{q}({}^{c}D_{b^-}^{q}(\ldots\ldots n\ times)u(x). \tag{11}$$

Laplace transform Table			
S.N	$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}(f(t))$	
1.	$E_{q,1}(\pm\lambda t^q)$	$\frac{s^{q-1}}{s^q \mp \lambda}$	$s^q > \lambda, q > -1$
2.	$t^{q-1}E_{q,q}(\pm\lambda t^q)$	$\frac{1}{s^q \mp \lambda}$	$s^q > \lambda, q > -1$
3.	$\frac{t^q}{q}E_{q,q}(\pm\lambda t^q)$	$\frac{s^{q-1}}{(s^q \mp \lambda)^2}$	$s^q > \lambda, q > -1$
4.	$sin_{q,1}(\lambda t^q)$	$\frac{\lambda s^{q-1}}{s^{2q}+\lambda^2}$	$s > 0$
5.	$cos_{q,1}(\lambda t^q)$	$\frac{s^{2q-1}}{s^{2q}+\lambda^2}$	$s > 0$
6.	$t^{q-1}sin_{q,q}(\lambda t^q)$	$\frac{\lambda}{s^{2q}+\lambda^2}$	$s > 0$
7.	$t^{q-1}cos_{q,q}(\lambda t^q)$	$\frac{s^q}{s^{2q}+\lambda^2}$	$s > 0$
8.	$E_{q,1}(\lambda t^q) + \frac{\lambda t^q}{q}E_{q,q}(\lambda t^q)$	$\frac{s^{2q-1}}{(s^q-\lambda)^2}$	
9.	$t^{q-1}\sum_{k=0}^{\infty}\frac{(k+1)\lambda^k t^{qk}}{\Gamma(qk+q)}$	$\frac{s^q}{(s^q-\lambda)^2}$	
10.	$t^{2q-1}\sum_{k=0}^{\infty}\frac{(k+1)\lambda^k t^{qk}}{\Gamma(qk+2q)}$	$\frac{1}{(s^q-\lambda)^2}$	
11.	$\sum_{k=0}^{\infty}\frac{k(k+1)}{2}\frac{(\lambda t^q)^{k-1}}{\Gamma(q(k-1)+1)}$	$\frac{s^{3q-1}}{(s^q-\lambda)^3}$	
12.	$Gcos_{q,1}\{(\lambda+i\mu)t^q\}$	$\frac{s^{q-1}(s^q-\lambda)}{(s^q-\lambda)^2+\mu^2}$	
13.	$Gsin_{q,1}\{(\lambda+i\mu)t^q\}$	$\frac{\mu s^{q-1}}{(s^q-\lambda)^2+\mu^2}$	
15.	$t^{q-1}Gcos_{q,q}\{(\lambda+i\mu)t^q\}$	$\frac{s^q-\lambda}{(s^q-\lambda)^2+\mu^2}$	
16.	$t^{q-1}Gsin_{q,q}\{(\lambda+i\mu)t^q\}$	$\frac{\mu}{(s^q-\lambda)^2+\mu^2}$	

Note that in this work, we consider the left sequential Caputo fractional derivative only for initial value problems. However, we need both the left and right sequential Caputo fractional derivatives for boundary value problems. We use notation t for the initial value problem and x for boundary value problems.

3. Main Results

3.1. Solution of Linear Sequential Caputo Fractional Differential Equations with Fractional Initial Conditions

In this section, we will provide a method to solve a linear non-homogeneous sequential Caputo fractional differential equation of order nq, which is sequential of order q. In addition, the initial conditions will involve the sequential Caputo fractional derivatives of $u(t)$ of order kq for $k = 1, 2, \ldots (n-1)$ at $t = 0$.

For that purpose, consider the linear non-homogeneous sequential fractional differential equations of order nq with initial conditions involving Caputo fractional derivatives of order kq with $k = 0, 1, \ldots, (n-1)$ of the form:

$$\sum_{k=0}^{n} a_k \, ({}^{sc}D_{0+}^{kq})u(t) = f(t), \tag{12}$$

with initial conditions of the form,

$$({}^{sc}D_{0+}^{kq})u(t)|_{t=0} = b_k \; (b_k \in R; \; k = 0, 1, \ldots\ldots, (n-1)). \tag{13}$$

Note that we need to use the Laplace transform method only to solve (12) and (13). The main reason for this is that the product rule for the Caputo derivative is not available, but it is needed to apply the method of undetermined coefficients and for the variation-of-parameter method. One needs to use Equation (1) in order to apply the Laplace transform method. Here, we only recall the results resulting for $n = 2$ and $f(t) = 0$.

$${}^{sc}D_{0+}^{2q}u + b\,{}^{sc}D_{0+}^{q}u + cu = 0, \; t \in (0, \infty) \tag{14}$$

with initial conditions as,

$$u(0) = A, \ {}^{sc}D^{q}_{0+}u(t)|_{t=0} = B. \tag{15}$$

We assume that $0.5 < q < 1$. Now, applying Laplace transform on (14) and (15) using Theorem 1, we obtain

$$s^{2q}U(s) - s^{(2q-1)}u(0) - s^{(q-1)}({}^{sc}D^{q}_{0+}u(t)|_{t=0}) + b(s^{q}U(s) - s^{(q-1)}u(0)) + cU(s) = 0, \tag{16}$$

where $U(s) = \mathcal{L}(u(t))$.

Our aim here is to show that when $q = 1$, our results yield the results of the second-order linear homogeneous differential equation with initial conditions as a special case.

Now, solving for $U(s)$ from Equation (16) and substituting the initial conditions from Equation (15), we obtain

$$U(s) = \frac{As^{(2q-1)} + (B + bA)s^{(q-1)}}{s^{2q} + bs^{q} + c}. \tag{17}$$

For convenience, we will denote $As^{(2q-1)} + (B + bA)s^{(q-1)} = s^{q-1}G(s)$.

Now, if we can take the inverse Laplace transform on both sides of Equation (17), we obtain the solution of the sequential Caputo initial value problem (14) and (15) of order $2q$.

We can compute the solution of (14) and (15) based on the roots of the quadratic equation $s^{2q} + bs^{q} + c = 0$ in terms of s^{q}.

1. Let $b \neq 0, \ b^2 - 4c > 0$. In this case, the quadratic equation $s^{2q} + bs^{q} + c = 0$ will have real and distinct roots, say λ_1 and λ_2. That is, we can factor $s^{2q} + bs^{q} + c = (s^{q} - \lambda_1)(s^{q} - \lambda_2)$. In this case, using the partial fraction method, we can write (17) as

$$\frac{s^{q-1}G(s)}{s^{2q} + bs^{q} + c} = \frac{c_1 s^{q-1}}{(s^{q} - \lambda_1)} + \frac{c_2 s^{q-1}}{(s^{q} - \lambda_2)}.$$

   Here, the constants c_i for $i = 1, 2$ depend on A, B, b, c, λ_1, and λ_2. Using the above relation in Equation (17) and taking the inverse Laplace transform, we can write the solutions of (14) and (15) as

$$u(t) = c_1 E_{q,1}(\lambda_1 t^{q}) + c_2 E_{q,1}(\lambda_2 t^{q}).$$

   Note that in the above case, if $b = 0$ and $c < 0$, we will have two real and distinct roots. The solution can be obtained on the same lines as above.
2. $b \neq 0, \ b^2 - 4c = 0$. In this case, the quadratic equation $s^{2q} + bs^{q} + c = 0$ will have real and coincident roots, say λ. Then, we can factor as $s^{2q} + bs^{q} + c = (s^{q} - \lambda)^2$. In this case, by algebraic manipulation, we can write (17) as

$$\frac{s^{q-1}G(s)}{s^{2q} + bs^{q} + c} = \frac{As^{q-1}}{(s^{q} - \lambda)} + \frac{(\lambda A + B + bA)s^{q-1}}{(s^{q} - \lambda)^2}.$$

   Using the Laplace transform table, the above relation, and Relation (17), we can write the solutions of (14) and (15) as

$$u(t) = AE_{q,1}(\lambda t^{q}) + \frac{(\lambda A + B + bA)t^{q}}{q}E_{q,q}(\lambda t^{q}).$$

3. If $b \neq 0, then b^2 - 4c < 0$. In this case, the quadratic equation $s^{2q} + bs^{q} + c = 0$ will have complex roots of the form $\lambda \pm i\mu$. We can write $s^{2q} + bs^{q} + c = (s^{q} - \lambda)^2 + \mu^2$. Then by algebraic manipulation, we can write (17) as

$$\frac{s^{q-1}G(s)}{s^{2q} + bs^{q} + c} = \frac{g_1 s^{q-1}(s^{q} - \lambda)}{(s^{q} - \lambda)^2 + \mu^2} + \frac{\mu g_2}{(s^{q} - \lambda)^2 + \mu^2}$$

and
$$\frac{1}{s^{2q}+bs^q+c}=\frac{1}{\mu}\frac{\mu}{(s^q-\lambda)^2+\mu^2},$$
where g_1 and g_2 are constants that can be determined.
Now, using the table and the above relation, we can take the inverse Laplace transform of Relation (17) to obtain the solutions of (14) and (15) as
$$u(t)=g_1Gcos_{q,1}\{(\lambda+i\mu)t^q\}+g_2Gsin_{q,1}\{(\lambda+i\mu)t^q\}.$$
Further, if $\lambda=0$, then the solution of (14) and (15) will be
$$u(t)=g_1cos_{q,1}(\mu t^q)+g_2sin_{q,1}(\mu t^q).$$
Furthermore, note that if $q=1$, one can easily observe that the integer results can be obtained as a special case, except when the factors are complex numbers with the real part not equal to zero. Even in that situation, the results of the integer case can be obtained using the exponential rules.
Note that we can also use the above technique to solve non-homogeneous sequential Caputo fractional differential equations when $f(t)\neq 0$. In this case, Mittag–Leffler functions $E_{q,q}(\lambda t^q)$ will be needed in the convolution integral.

Remark 2. *It is to be noted that the linear sequential Caputo fractional differential equations with fractional initial conditions can be reduced to a system of Caputo fractional differential equations with initial conditions. However, the converse is, in general, not true, as in the integer case. See [22,23] for recent work on two and three systems of non-homogeneous linear Caputo fractional differential equations with initial conditions. It is to be noted that the standard method of finding the fundamental matrix cannot be used to solve Caputo fractional differential systems with initial conditions. In [23], a numerical method to solve nonlinear Caputo fractional system of a SIR model is developed. A more detailed Laplace transform is also provided in [23].*

3.2. Linear Sequential Caputo Fractional Boundary Value Problems with Fractional Boundary Conditions

In [47,48,51,54–60], the authors studied fractional boundary value problems. For the non-homogeneous boundary value problem, they constructed Green's function. It is to be noted that the boundary conditions for non-sequential differential equations will be the same as those of the nearest integer boundary value problem. In addition, even solving the homogeneous Caputo fractional differential equations with the corresponding boundary conditions is not easy and sometimes not possible. However, if we consider the sequential boundary value problem, we can adopt the same technique that we use to solve the corresponding integer problem. Observe that the boundary conditions will also involve the left and right fractional derivatives. In [40,41,44,61,62], sequential initial value problems and sequential boundary value problems were studied.

Remark 3. *In the research article [50], the authors claimed that the only function whose left derivative of a function at $x=a^+$ equals the right derivative of the function at $x=b^-$ is true for constant functions. Their claim is based on the assumption that ${}^cD^q_{a+}f(x)|_{x=a}=0$ and ${}^cD^q_{b-}f(x)|_{x=b}=0$ for every increasing function $f(x)$. Certainly, it is not true when $f(x)=(x-a)^q$, which is an increasing function of $f(x)$.*

In fact, ${}^cD^q_{a+}(x-a)^q=\Gamma(q+1)$ on $[a.b)$.

In a sequential derivative, we need ${}^cD^q_{a+}f(x)$ to be differentiable on $[a,b)$, which implies that ${}^cD^q_{a+}f(x)$ is continuous on $[a,b]$. From this, we obtain
$${}^cD^q_{a+}(x-a)^q|_{x=a}={}^cD^q_{b-}(b-x)^q|_{x=b}=\Gamma(q+1).$$

See [58] for the mathematical model, which requires symmetric fractional derivatives.

Remark 4. *Note that for symmetric functions, the left derivative at any point $x = x_1$ will be exactly equal to the right derivative at $x = x_1$. For example, if $f(x) = (x-a)^{nq}(b-x)^{nq}$ for any integer n and q such that $0 < q \leq 1$, it is easy to show that the left derivative is exactly equal to the right derivative at any point $x = x_1$. For functions which are not symmetric, we can show that they are the same when we replace $(x-a)$ by $(b-x)$ and $(b-x)$ by $(x-a)$.*

Next, we recall a result from [61,62]. See [62] for numerical results on linear sequential Caputo boundary value problems.

Consider the sequential Caputo fractional boundary value problem with mixed boundary conditions of the form:

$$\begin{aligned} -{}^cD^{2q}u + u &= f(x), \\ {}^cD^{q}u(0) &= 0, \\ {}^cD^{q}u(1) &= 0. \end{aligned} \tag{18}$$

The solution of (18) is given by

$$u(x) = \int_0^1 G(x,s)f(s)ds, \tag{19}$$

where $G(x,s)$ is Green's function given by,

$$G(x,s) = \begin{cases} \left[sin_q(s) + \frac{cos_q(s)}{tan_q(1)}\right]cos_q(x), & x < s \\ \left[sin_q(x) + \frac{cos_q(x)}{tan_q(1)}\right]cos_q(s), & x > s. \end{cases} \tag{20}$$

Hence,

$$u(x) = \int_0^1 G(x,s)f(s)ds, \tag{21}$$

which means that,

$$\begin{aligned} u(x) = {} & \int_0^x \left[sin_q(x) + \frac{cos_q(x)}{tan_q(1)}\right]\left[cos_q(s)\right]f(s)ds \\ & + \int_x^1 \left[sin_q(s) + \frac{cos_q(s)}{tan_q(1)}\right]\left[cos_q(x)\right]f(s)ds. \end{aligned} \tag{22}$$

This is just to illustrate that the above representation of the solution is possible when we assume that the fractional derivative ${}^cD^{2q}_{0+}u$ is sequential. Note that the majority of the Caputo fractional boundary value problems available in the literature use the linear operator ${}^cD^{2q}_{0+}u$, which is not sequential. Consequently, the boundary conditions will not involve left and right fractional derivatives. In addition, Green's function will be the same as that of the linear operator, being the second derivative.

4. Conclusions

In this work, we have demonstrated the advantage of studying linear sequential Caputo fractional differential equations with fractional initial value problems, and linear sequential Caputo fractional boundary value problems with fractional boundary conditions. In particular, differential equations with Caputo fractional derivatives of order nq is sequential of order q where $q < 1$ are very useful. The advantage is that the solution of the sequential Caputo fractional differential equation yields the solution of the corresponding integer differential equation as a special case. In addition, it is possible to find an appropriate value of q that matches the data available for the specific models. This is the first step in order to solve the nonlinear sequential Caputo fractional initial and boundary value problems with fractional initial and boundary conditions. In our future work, we plan to develop comparison theorems for relative sequential dynamic equations, which will be useful to solve nonlinear sequential Caputo dynamic equations with initial and boundary conditions. The advantage of our work is that the value of q can be used as a parameter to enhance the model driven by data. Solving linear sequential Caputo fractional dynamic equations is immensely useful in solving weakly non-linear sequential Caputo fractional differential equations by any of the iterative methods.

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