foundations

MDPI

Article

Treatment of a Coupled System for Quadratic Functional Integral Equation on the Real Half-Line via Measure of Noncompactness

Ahmed M. A. El-Sayed [1,*], Yasmin M. Y. Omar [2], Hind H. G. Hashem [1] and Shorouk M. Al-Issa [3,4]

1 Faculty of Science, Alexandria University, Alexandria 21544, Egypt; hindhghashem@gmail.com
2 Faculty of Science, Omar Al-Mukhtar University, Tripoli 22131, Libya; yasmin202020mohamed@gmail.com
3 Faculty of Arts and Sciences, Department of Mathematics, The International University of Beirut, Beirut 1107, Lebanon; shorouk.alissa@liu.edu.lb
4 Faculty of Arts and Sciences, Department of Mathematics, Lebanese International University, Saida 1600, Lebanon
* Correspondence: amasayed@alexu.edu.eg

Abstract: This article is devoted to the solvability and the asymptotic stability of a coupled system of a functional integral equation on the real half-axis. Our consideration is located in the space of bounded continuous functions on $\mathbb{R}_+$ $(BC(\mathbb{R}_+))$. The main tool applied in this work is the technique associated with measures of noncompactness in $BC(\mathbb{R}_+)$ by a given modulus of continuity. Next, we formulate and prove a sufficient condition for the solvability of that coupled system. We, additionally, provide an example and some particular cases to demonstrate the effectiveness and value of our results.

Keywords: space of functions continuous and bounded on the half-axis; measure of noncompactness; fixed-point theorem of Darbo type; coupled system of integral equations; asymptotic stability

check for updates

Citation: El-Sayed, A.M.A.; Omar, Y.M.Y.; Hashem, H.H.G.; Al-Issa, S.M. Treatment of a Coupled System for Quadratic Functional Integral Equation on the Real Half-Line via Measure of Noncompactness. *Foundations* **2023**, *3*, 37–48. https://doi.org/10.3390/foundations3010004

Academic Editors: Sotiris K. Ntouyas and Ioannis K. Argyros

Received: 10 October 2022
Revised: 18 November 2022
Accepted: 3 January 2023
Published: 6 January 2023

1. Preliminaries and Introduction

Measures of noncompactness are frequently employed in fixed-point theory, and they are especially useful in work on the concepts of differential equations, optimization theory, functional integral equations, and integral equations (see [1–5]).

Nonlinear integral equations are useful for describing many real-world phenomena and nonlinear analysis [4,5].

It is worthwhile mentioning that Darbo fixed-point theorem and the measures of noncompactness create a powerful and convenient technique which is very applicable in establishing theorems of existence for various types of operator equations (functional integral, integral, differential). For solvability on bounded domain, see [6–8].

Investigation on the real half-axis of the integral equations on different spaces of functions has received a great attention (see [9–14]).

In [11], measures of noncompactness in the space of functions which are defined, continuous and bounded on the real half-axis, and taking values in an arbitrary Banach space E, are constructed. One of the constructed measures of noncompactness is applied to prove the existence of solutions of an infinite system of quadratic integral equations in the space of functions defined, continuous and bounded on the real half-axis.

In addition, the solvability of an infinite system of integral equations of the Volterra–Hammerstein type in the space of functions defined, continuous and bounded on the real half-axis with values in the sequence space l_1 is discussed [13]. Moreover, this result is extended to a wider class of considered infinite systems [13].

Motivated by these results, in this article, we discuss a coupled system of a functional integral equation, abbreviated by CSFIE

$$
\begin{aligned}
x(t) &= f_1(t, y(t)) \cdot g_1\left(t, \int_0^t v_1(t, s, y(s))\, ds\right), \ t \geq 0 \\
y(t) &= f_2(t, x(t)) \cdot g_2\left(t, \int_0^t v_2(t, s, x(s))\, ds\right), \ t \geq 0,
\end{aligned} \tag{1}
$$

and establish the existence of the solution of that coupled system on $\mathbb{R}_+$ utilizing Darbo's fixed-point and the measure of noncompactness theorem. Furthermore, for the solution of (1), we study the asymptotic stability.

The present paper creates an essential extension of the investigations of the integral equation via the technique associated with measures of noncompactness on the real half line. However, we start by applying the technique associated with measures of noncompactness on a coupled system of integral equations in $BC(\mathbb{R}_+)$.

The following notations will be needed in our work. Assume that $BC(\mathbb{R}_+)$ is the class of all continuous and bounded functions in $\mathbb{R}_+$. The norm of $f \in BC(\mathbb{R}_+)$ is defined by

$$
\|f\| = \sup_{t \in \mathbb{R}_+} |f(t)|. \tag{2}
$$

$x \in X \subseteq BC(\mathbb{R}_+)$ and $\epsilon \geq 0$ are indicated by $w^T(x, \epsilon)$; $T > 0$ is the modulus of continuity of the function x on the interval $[0, T]$, i.e.,

$$
w^T(x, \epsilon) = \sup[|x(t) - x(s)| : t, s \in [0, T], \ |t - s| \leq \epsilon],
$$

$$
w^T(X, \epsilon) = \sup[w^T(x, \epsilon) : x \in X]
$$

and

$$
w_0^T(X) = \lim_{\epsilon \to 0} w^T(X, \epsilon), \qquad w_0(X) = \lim_{T \to \infty} w_0^T(X).
$$

In addition,

$$
\mathrm{diam} X(t) = sup\{|x(t) - y(t)|, \ x, y \in X\}, \qquad \alpha(X) \ = \ \lim_{t \to \infty} \sup \mathrm{diam} X(t)
$$

and the measure of noncompactness on $BC(\mathbb{R}_+)$ is given by [4]

$$
\mu(X) = w_0(X) \ + \ \alpha(X) \ = \ w_0(X) \ + \ \lim_{t \to \infty} \sup \mathrm{diam} X(t).
$$

Next, we state the Darbo fixed-point theorem [15].

Theorem 1. *Assume that $F : Q \to Q$ is a continuous operator, and Q is a nonempty closed bounded convex subset of the space E with $\mu(FX) \leq k\mu(X)$ for any nonempty subset X of Q, where the constant $k \in [0, 1)$. Then, F has a fixed point in the set Q.*

Now, let $E \ = \ BC(\mathbb{R}_+) \times BC(\mathbb{R}_+), \ X, Y \subset BC(\mathbb{R}_+)$ and

$$
U \ = \ \{u \in U : u \ = \ (x, y), \ x \in X, \ y \in Y\} \ = \ X \times Y.
$$

Define the following modulus of continuity:

$$
\omega(x, \epsilon) \ = \sup \{ \ | \ x(t) \ - \ x(s) \ | : \ \ t, s \ \in \ J = [0, T], \ \ | \ t \ - \ s \ | \ \leq \ \epsilon \ \},
$$

$$
\omega(y, \epsilon) \ = \sup \{ \ | \ y(t) \ - \ y(s) \ | : \ \ t, s \ \in \ J = [0, T], \ \ | \ t \ - \ s \ | \ \leq \ \epsilon \ \}
$$

and

$$
\omega(u, \epsilon) \ = \max \{\omega(x, \epsilon), \omega(y, \epsilon)\}.
$$

Then,

$$\omega(U,\epsilon) = \sup\{\,\omega(u,\epsilon) : u \in U\,\}, \quad \omega_0(U) = \lim_{\epsilon \to 0} \omega(U,\epsilon).$$

In addition,

$$\omega(U) = \omega(X \times Y) \le \max\{\omega(X,\epsilon), \omega(Y,\epsilon)\},$$

$$diam(U) = diam(X \times Y) \le \max\{diam(X), diam(Y)\},$$

$$\lim_{t \to \infty} \sup diam(U) \le \max\{\lim_{t\to\infty} \sup diam(X), \lim_{t\to\infty} \sup diam(Y)\}.$$

and

$$\mu(U) = \omega_0(U) + \alpha(U) = \omega_0(U) + \lim_{t\to\infty} \sup diam\, U(t).$$

2. Main Result

Consider the coupled system of functional integral Equation (1) with the following assumptions:

(i) $f_i : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $i = 1,2$ are continuous and $\sup_{t\in\mathbb{R}_+} |f_i(t,0)| = f_i^*$.

(ii) There exists a continuous function $m : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f_i(t,x) - f_i(t,y)| \le m_i(t)|x-y|, \qquad \forall\, x,\, y \in \mathbb{R},\ t \in \mathbb{R}_+,\ i = 1,2,$$

and $m_i = \sup_{t\in\mathbb{R}_+} m_i(t)$.

(iii) $g_i : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $i = 1,2$ are continuous and satisfy the Lipschitz condition

$$|g_i(t,x) - g_i(s,y)| \le l_i(|t-s| + |x-y|),\ l_i > 0, \qquad \forall (t,x),\ (s,y) \in \mathbb{R}_+ \times \mathbb{R}$$

$g_i^* = \sup_{t\in\mathbb{R}_+} |g_i(t,0)|$.

(iv) $v_i : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $i = 1,2$ are continuous and there exists a continuous function $k_i(t,s) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, such that $|v_i(t,s,x)| \le k_i(t,s)$, $\forall t,s \in \mathbb{R}_+$ and

$$\sup_{t\in\mathbb{R}_+} \int_0^t k_i(t,s)ds = k_i, \ \textit{and} \ \lim_{t\to\infty} \int_0^t k_i(t,s)ds = 0.$$

(v) Without loss of generality, we can write $m = \max\{m_1, m_2\}$, $g^* = \max\{g_1^*, g_2^*\}$, $k = \max\{k_1,k_2\}$, and $l = \max\{l_1,l_2\}$. Now there exists a positive constant C, such that $C = mg^* + mlk < 1$.

Remark 1. *From condition (ii) set $y = 0$, then*

$$\begin{aligned} |f_i(t,x(s))| - |f_i(t,0)| &\le |f_i(t,x(s)) - f_i(t,0)| \\ &\le m_i(t)|x|, \\ |f_i(t,x(s))| &\le m_i(t)|x| + |f_i(t,0)| \\ &\le m_i(t)|x| + f_i^*. \end{aligned}$$

Similarly, we have

$$|g_i(t,x(s))| \le l_i|x| + g_i^*.$$

Theorem 2. *Assume that conditions (i)–(v) hold; then, the coupled system (1) has at least one solution $(x,y) \in U \subset BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$.*

Proof. Suppose

$$B_r = \{u = (x, y) \in U : \ \|u\| \ = \ max\{||x||, \ ||y||\} \leq r\}.$$

Define the operator A by

$$A(x, y)(t) = (A_1 y(t), A_2 x(t))$$

where

$$A_1 y(t) = f_1(t, y(t)) \cdot g_1\left(t, \int_0^t v_1(t, s, y(s))\, ds\right)$$

$$A_2 x(t) = f_2(t, x(t)) \cdot g_2\left(t, \int_0^t v_2(t, s, x(s))\, ds\right).$$

Let $u = (x, y) \in U$; from our assumptions, we can deduce that the function Au is continues on U, and then we have

$$\begin{aligned}
|A_1 y(t)| &= \left| f_1(t, y(t)).g_1\left(t, \int_0^t v_1(t, s, y(s))\, ds\right)\right| \\
&= \left| f_1(t, y(t))|.|g_1\left(t, \int_0^t v_1(t, s, y(s))\, ds\right)\right| \\
&\leq (m(t)|y(t)| + f_1^*)(g_1^* + l_1\left|\int_0^t v_1(t, s, y(s))\, ds\right|) \\
&\leq |y(t)| m_1(t)(g_1^* + l_1 \int_0^t k_1(t, s) ds) \\
&+ (g^* + l_1 \int_0^t k_1(t, s) ds) f_1^* \\
&\leq \|y\|(m_1 g_1^* + m_1 l_1 k_1) + (g_1^* + l_1 k_1) f_1^* \\
\|A_1 y\| &\leq r_1(m_1 g_1^* + m_1 l_1 k_1) + (g_1^* + l_1 k) f_1^* = r_1.
\end{aligned}$$

Similar to the above calculation, we can conclude that

$$\|A_2 x\| \leq r_2(m_2 g_2^* + m_2 l_2 k_2) + (g_2^* + l_2 k_2) f_2^* = r_2.$$

Therefore,

$$\|Au\|_U = \|A(x, y)\|_U = \|(A_1 y, A_2 x)\|_U \ = \ max\{\|A_1 y\|, \ \|A_2 x\|\} \ = \ r.$$

Then, the operator A is bounded on U and $Au \in B_r$ and

$$\|A(x)\| \leq 2(Cr + A^*) = r, \ \ r = \frac{2A^*}{1 - 2C}$$

where $A^* = (g^* + lk) f^* < \infty$. This proves that the operator $A : B_r \to B_r$.

Now, we show that A is continuous on the ball B_r.

Let $\epsilon > 0$ be given, take $y_1, \ y_2 \in Y$ such that $\|y_1 - y_2\| \leq \epsilon$, then

$$
\begin{aligned}
& |A_1 y_1(t) - A_1 y_2(t)| \\
= & \left| f_1(t, y_1(t)) g_1\left(t, \int_0^t \nu_1(t,s,y_2(s))\,ds\right) - f_1(t, y_2(t)) g_1\left(t, \int_0^t \nu_1(t,s,y_2(s))\,ds\right)\right| \\
\leq & \left| f_1(t, y_1(t)) g_1\left(t, \int_0^t \nu_1(t,s,y_1(s))\,ds\right) - f_1(t, y_2(t)) g_1\left(t, \int_0^t \nu_1(t,s,y_1(s))\,ds\right)\right| \\
+ & \left| f_1(t, y_2(t)) g_1\left(t, \int_0^t \nu_1(t,s,y_1(s))\,ds\right) - f_1(t, y_2(t)) g_1\left(t, \int_0^t \nu_1(t,s,y_2(s))\,ds\right)\right| \\
\leq & \left| f_1(t, y_1(t)) - f_1(t, y_2(t))\right| \left| g_1\left(t, \int_0^t \nu_1(t,s,y_1(s))\,ds\right)\right| \\
+ & \left| f_1(t, y_2(t))\right| \left| g_1\left(t, \int_0^t \nu_1(t,s,y_1(s))\,ds\right) - g_1\left(t, \int_0^t \nu_1(t,s,y_2(s))\,ds\right)\right| \\
\leq & m_1(t)|y_1(t) - y_2(t)|(g^* + l_1 \int_0^t |k_1(t,s)| ds) \\
& + (m_1(t)|y_2(t)| + f^*) l_1 \int_0^t |\nu_1(t,s,y_1(s)) - \nu_1(t,s,y_2(s))|\,ds
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
\leq & \quad m_1 g^* |y_1(t) - y_2(t)| + m_1 l_1 k_1 |y_1(t) - y_2(t)| \\
+ & \quad (m_1 |y_2(t)| + f_1^*) k_1 \int_0^t |\nu_1(t,s,y_1(s)) - \nu_1(t,s,y_2(s))|\,ds \\
\leq & \quad (m_1 g_1^* + m_1 l_1 k_1)\epsilon \\
+ & \quad (m_1 r_1 + f_1^*) l_1 \int_0^t |\nu_1(t,s,y_1(s)) - \nu_1(t,s,y_2(s))|\,ds.
\end{aligned}
$$

$$
\leq (m_1 g_1^* + m_1 l_1 k_1)\epsilon + 2(m_1 r_1 + f_1^*) l_1 \int_0^t k_1(t,s)\,ds. \tag{4}
$$

Select $T > 0$ such that the following inequality holds for $t > T$.

$$
\begin{aligned}
2 r_1 l_1 m_1 \int_0^t k_1(t,s) ds &\leq (1 - C)\frac{\epsilon}{2} \\
2 f^* k_1 \int_0^t k(t,s) ds &\leq (1 - C)\frac{\epsilon}{2}.
\end{aligned} \tag{5}
$$

Take into account the following two situations.

$(i*)$ $t \geq T$. In light of (4) and (5), we obtain

$$
|A_1 y_1(t) - A_1 y_2(t)| \leq (m_1 g_1^* + m_1 l_1 k_1)\epsilon + (1 - C)\frac{\epsilon}{2} + (1 - C)\frac{\epsilon}{2} = \epsilon.
$$

$(ii*)$ $t \leq T$. In this instance, let us take a look at the function w = $w = w(\epsilon)$ given by

$$
w(\epsilon) = \sup\{|\nu_1(t,s,y_1) - \nu_1(t,s,y_2)| : t,s \in [0,T],\ x,y \in [-r,r],\ |y_1 - y_2| < \epsilon\}.
$$

Then, from the uniform continuity of the function $\nu_1 = \nu_1(t,s,x)$ on the set $[0,T] \times [0,T] \times [-r,r]$, we deduce that $w(\epsilon) \to 0$ as $\epsilon \to 0$.

Thus, from (3), we obtain

$$
|A_1 y_1(t) - A_1 y_2(t)| \leq (m_1 g_1^* + l_1 k_1)\epsilon + ({}_1 r m_1 + f_1^*) T w(\epsilon). \tag{6}
$$

Finally, from the two cases $(i*)$, $(ii*)$ and the above established facts, we can deduce that the operator A_1 is continuous on Y.

Similarly, we can conclude that the operator A_1 is continuous and for any $x_1, x_2 \in X$

$$|A_2x_1(t) - A_2x_2(t)| \leq (m_1g_1^* + l_1k_1)\epsilon + (r_1m_1 + f_1^*)Tw(\epsilon). \tag{7}$$

Therefore,

$$\begin{aligned}
\|A(x_1,y_1) - A(x_2,y_2)\|_U &= \|(A_1y_1, A_2x_1) - (A_1y_2, A_2x_2)\| \\
= \|(A_1y_1 - A_1y_2, A_2x_1 - A_2x_2)\| &= max\{\|A_1y_1(t) - A_1y_2(t)\|,\ \|A_2x_1(t) - A_2x_2(t)\|\} \\
&\leq (m_1g_1^* + l_1k_1)\epsilon + (r_1m_1 + f_1^*)Tw(\epsilon).
\end{aligned}$$

Then, the operator A is continuous on the ball B_r.
Now, for any $u_1,\ u_2 \in U$ and fixed $t \geq 0$, we obtain

$$\begin{aligned}
|A_1y_1(t) - A_1y_2(t)| &\leq m_1g_1^*|y_1(t) - y_2(t)| + m_1l_1k_1|y_1(t) - y_2(t)| \\
&+ (m_1|y_2(t)| + f_1^*)l_1 \int_0^t |v_1(t,s,y_1(s)) - v_1(t,s,y_2(s))|\, ds \\
&\leq m_1g_1^*|y_1(t) - y_2(t)| + m_1l_1k_1|y_1(t) - y_2(t)| \\
&+ 2(m_1r_1 + f_1^*)l_1 \int_0^t k_1(t,s)\, ds. \\
&\leq (m_1g_1^* + m_1k_1l_1) diam\ Y(t) + 2(m_1r_1 + f_1^*)l_1 \int_0^t k_1(t,s)\, ds.
\end{aligned}$$

Hence, we obtain

$$diam(A_1Y)(t) \leq (m_1g_1^* + m_1k_1l_1) diam\ Y(t) + 2(m_1r_1 + f_1^*)l_1 \int_0^t k_1(t,s)\, ds.$$

As performed above, we can conclude that for any $u_1,\ u_2 \in U$, and fixed $t \geq 0$, we obtain

$$diam(A_2X)(t) \leq (m_1g_2^* + m_2k_2l_2) diam\ X(t) + 2(m_2r_2 + f_2^*)l_2 \int_0^t k_2(t,s)\, ds. \tag{8}$$

Therefore,

$$diam A(X,Y)(t) \quad = \quad max\{diam(A_1Y)(t),\ diam(A_2X)(t)\}.$$

Hence,

$$\begin{aligned}
& diam AU(t) \\
\leq\ & (mg^* + mkk_1) max\{(diam Y(t),\ diam X(t))\} + 2mr + f^*)l[\int_0^t k_1(t,s)\, ds + \int_0^t k_2(t,s)\, ds] \\
\leq\ & (mg^* + mkk_1) diam U(t) + 2(mr + f^*)l[\int_0^t k_1(t,s)\, ds + \int_0^t k_2(t,s)\, ds]
\end{aligned}$$

and

$$\lim_{t\to\infty} \sup diam AU(t) \leq (mg^* + mkk_1) \lim_{t\to\infty} \sup diam U(t).$$

Then,

$$\lim_{t\to\infty} \sup diam AU(t) \leq C \lim_{t\to\infty} \sup diam U(t). \tag{9}$$

Let $T > 0$ and $\epsilon > 0$ be given. Let $(x,y) \in U$ and $t,\ s \in [0,T]$ such that $s \leq t$ and $|t - s| \leq \epsilon$, then

$$
\begin{aligned}
& |A_1y(t) - A_1y(s)| \\
= \ & \left| f_1(t,y(t))g_1\left(t, \int_0^t \nu_1(t,\tau,y(\tau))\, d\tau\right) - f_1(s,y(s))g_1\left(s, \int_0^s \nu_1(s,\tau,y(\tau))\, d\tau\right)\right| \\
\leq \ & \left| f_1(t,y(t))g_1\left(t, \int_0^t \nu_1(t,\tau,y(\tau))\, d\tau\right) - f_1(s,y(s))g_1\left(t, \int_0^t \nu_1(t,\tau,y(\tau))\, d\tau\right)\right. \\
+ \ & \left. f_1(s,y(s))g_1\left(t, \int_0^t \nu_1(t,\tau,y(\tau))\, d\tau\right) - f_1(s,y(s))g_1\left(s, \int_0^s \nu_1(s,\tau,y(\tau))\, d\tau\right)\right| \\
\leq \ & |f_1(t,y(t)) - f_1(s,y(s))|\left|g_1\left(t, \int_0^t \nu_1(t,\tau,y(\tau))\, d\tau\right)\right| \\
+ \ & |f_1(s,y(s))|\left|g_1\left(t, \int_0^t \nu_1(t,\tau,y(\tau))d\tau\right) - g_1\left(s, \int_0^s \nu_1(s,\tau,y(\tau))\, d\tau\right)\right| \\
\leq \ & (|f_1(t,y(t)) - f_1(t,y(s))| + |f_1(t,y(s)) - f_1(s,y(s))|)(g^* + l_1 \int_0^t k_1(t,\tau)d\tau) \\
+ \ & (m_1(s)|y| + f_1^*)|(l_1(\int_0^t |\nu_1(t,\tau,y(\tau)) - \nu_1(s,\tau,y(\tau))|d\tau) \\
+ \ & \int_s^t |\nu_1(t,\tau,y(\tau))|) \\
\leq \ & (m_1(t)|y(t) - y(s)| + |f_1(t,y(s)) - f_1(s,y(s))|)(g_1^* + l_1 \int_0^t k_1(t,\tau)d\tau) \\
+ \ & (m_1(s)|y(s)| + f_1^*)(l_1(\int_0^t |\nu_1(t,\tau,y(\tau)) - \nu_1(s,\tau,y(\tau))|d\tau) + \int_s^t |\nu_1(t,\tau,y(\tau))d\tau)
\end{aligned}
$$

$$
\begin{aligned}
\leq \ & (m_1|y(t) - y(s)| + w_r^T(f_1,\epsilon))(g_1^* + l_1 \int_0^t k_1(t,\tau)d\tau) \\
+ \ & mr_1l_1 \int_s^t k_1(s,\tau)d\tau + f_1^*l_1 \int_s^t k_1(t,\tau)d\tau + l_1T[m_1r_1 + f_1^*])w_r^{-T}(\nu_1,\epsilon)
\end{aligned}
$$

where

$$w_r^T(f_i,\epsilon) = \sup\{|f_it,y) - f_i(s,y) : t,\ s \in [0,T],\ |t-s| \leq \epsilon,\ |y| \leq r,\ \ i = 1,2.\}$$

$$w_r^{-T}(\nu_i,\epsilon) = \sup\{|\nu_i(t,\tau,y) - \nu_i(s,\tau,y) :\ |t-s| \leq \epsilon,\ \tau \in [0,T],\ |y| \leq r,\ \ i = 1,2.\}.$$

Hence, we deduce that

$$
\begin{aligned}
w(A_1Y,\epsilon) \ \leq \ & (mw(Y,\epsilon) + w_r^T(f_1,\epsilon))(g_1^* + l_1 \int_0^\tau |k_1(s,\tau)|d\tau) \\
+ \ & (\epsilon\, m_1\, r\, l_1\ \sup\{|k_1(t,\tau)| : \tau \in [0,T]\} \\
+ \ & \epsilon\, l_1 f_1^* \sup\{|k_1(t,\tau)| : \tau \in [0,T]\} \\
+ \ & k_1T(m_1r_1 + f_1^*)w_r^{-T}(\nu_1,\epsilon)
\end{aligned}
$$

Through a similar method, we obtain

$$
\begin{aligned}
w(A_2X,\epsilon) \ \leq \ & (m_2w^T(X,\epsilon) + w_r^T(f_2,\epsilon))(g_2^* + l_2 \int_0^\tau |k_2(s,\tau)|d\tau) \\
+ \ & (\epsilon\, m_2\, r_2\, l_2\ \sup\{|k_2(t,\tau)| : \tau \in [0,T]\} \\
+ \ & \epsilon\, k_2 f_2^* \sup\{|k_2(t,\tau)| : \tau \in [0,T]\} \\
+ \ & l_2T(m_2r_2 + f_2^*)w_r^{-T}(\nu_2,\epsilon).
\end{aligned}
$$

Therefore,

$$\begin{aligned}
w(AU,\epsilon) &= max\{w(A_1Y,\epsilon),\ w(A_2X,\epsilon)\} \\
&= (mw(U,\epsilon)+w_r^T(f,\epsilon))(g^*+l\int_0^\tau |k(s,\tau)|d\tau) \\
&+ (\epsilon\, m\, r\, l\ \sup\{|k(t,\tau)| : \tau \in [0,T]\} \\
&+ \epsilon\, l f^* \sup\{|k(t,\tau)| : \tau \in [0,T]\} \\
&+ lT(mr+f^*)w_r^{-T}(\nu_1,\epsilon) \\
&+ (\epsilon\, m\, r\, l\ \sup\{|k(t,\tau)| : \tau \in [0,T]\} \\
&+ \epsilon\, l f^* \sup\{|k(t,\tau)| : \tau \in [0,T]\} \\
&+ lT(mr+f^*)w_r^{-T}(\nu_2,\epsilon).
\end{aligned}$$

From the uniform continuity of the functions $f_i = f_i(t,x)$ and the functions $\nu_i = \nu_i(t,s,x)$, $i = 1,2$ on the set $[0,T]\times[-r,r]$, we deduce that $w^r(f_i,\epsilon) \to 0$ and $w_r^{-T}(\nu_i,\epsilon) \to 0$ as $\epsilon \to 0$. Consequently, we obtain

$$w_0(AU) \le mw_0(U)(g^*+l\int_s^t k(t,\tau)d\tau),$$

$$w_0(AU) \le (mg^*+mkl)w_0^T(U)$$

and as $T\to\infty$, we have

$$w_0(AU) \le Cw_0(U). \tag{10}$$

Now, from the estimations (9) and (10) and the definition of μ on U, we obtain

$$\mu(AU) \le C\mu(U).$$

Since all the requirements of Theorem 2 are met, then A has a fixed point $(x,y)\in U$. Consequently, the coupled system of quadratic functional integral Equation (1) has at least one solution $(x,y)\in E,\ \ x,y\in BC(\mathbb{R}_+)$. $\square$

3. Asymptotic Stability

We can now deduce from the proof of Theorem 2 the following corollary.

Corollary 1. *The solution $u \in U$ of the coupled system of quadratic functional integral Equation (1) is asymptotically stable; that is to say, $\forall\ \epsilon > 0$, there exists $T(\epsilon) > 0$ and $r > 0$, such that, if any two solutions to the coupled system of the quadratic functional integral Equation (1) are $(x,y),\ (x_1,y_1) \in U$, then*

$$||(x,y)-(x_1,y_1)|| \le \epsilon,\quad t \ge T(\epsilon).$$

This implies that

$$|x(t)-x_1(t)| \le \epsilon \ \ \textit{and}\ \ |y(t)-y_1(t)| \le \epsilon,\quad t\ge T(\epsilon).$$

Proof. Let $(x,y),(x_1,y_1)\in U$ be any two solutions of the coupled system of quadratic functional integral Equation (1). Using assumptions of Theorem 2 and by a similar way to how relations (6) and (7) are estimated, we have

$$
\begin{aligned}
|y(t) - y_1(t)| &= |A_1 y(t) - A_1 y_1(t)| \\
&\leq (m_1 g_1^* + l_1 k_1)\epsilon + (r_1 m_1 + f_1^*) T w(\epsilon),
\end{aligned}
$$

and

$$
\begin{aligned}
|x(t) - x_1(t)| &= |A_2 x(t) - A_2 x_1(t)| \\
&\leq (m_2 g_2^* + l_2 k_2)\epsilon + (r_2 m_2 + f_2^*) T w(\epsilon).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\|(x,y) - (x_1,y_1)\|_U &= \|(A_1 y, A_2 x) - (A_1 y_1, A_2 x_1)\| \\
&= \|(A_1 y - A_1 y_1, A_2 x - A_2 x_1)\| \\
&= max\{\|A_1 y(t) - A_1 y_1(t)\|, \|A_2 x(t) - A_2 x_1(t)\|\} \\
&\leq (mg^* + k_1 k)\epsilon + (rm + f^*) T w(\epsilon),
\end{aligned}
$$

for $t > T(\epsilon)$. $\square$

4. Particular Cases and Example

In this section, we demonstrate some particular systems, which are deduced from Theorem 2.

- Let $f_i(t,x) = 1,\ i = 1,2$; then, the coupled system (1) takes the form

$$
\begin{aligned}
x(t) &= g_1\left(t, \int_0^t v_1(t,s,y(s))\, ds\right),\ \ t \geq 0 \\
y(t) &= g_2\left(t, \int_0^t v_2(t,s,x(s))\, ds\right),\ \ t \geq 0.
\end{aligned}
$$

  Based on conditions (iii)–(v) of Theorem 2, then(1) has at least one asymptotically stable solution $x \in BC(\mathbb{R}_+)$.
  Moreover, when $g_i(t,x) = x,\ i = 1,2$. Then, we have a coupled system of Urysohn integral equations

$$
\begin{aligned}
x(t) &= \int_0^t v_1(t,s,y(s))\, ds,\ \ t \geq 0 \\
y(t) &= \int_0^t v_2(t,s,x(s))\, ds,\ \ t \geq 0,
\end{aligned}
$$

- Let $g_i(t,x) = x,\ i = 1,2$, then, the coupled system (1) takes the form

$$
\begin{aligned}
x(t) &= f_1(t,y(t)) \cdot \int_0^t v_1(t,s,y(s))\, ds,\ \ t \geq 0 \\
y(t) &= f_2(t,x(t)) \cdot \int_0^t v_2(t,s,x(s))\, ds,\ \ t \geq 0,
\end{aligned}
$$

  under the conditions of Theorem 2, then the coupled system of quadratic integral Equation (1) has at least one asymptotically stable solution $(x,y) \in BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$.
- Let $g(t,x) = 1 + x$, in (1), we have

$$
\begin{aligned}
x(t) &= f_1(t,y(t))\left(1 + \int_0^t v_1(t,s,y(s))\, ds\right),\ \ t \geq 0 \\
y(t) &= f_2(t,x(t))\left(1 + \int_0^t v_2(t,s,x(s))\, ds\right),\ \ t \geq 0,
\end{aligned}
$$

which under the assumptions of Theorem 2, has at least one asymptotically stable solution $(x, y) \in BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$.

Example: Consider the following coupled system of functional equations

$$x(t) = \frac{\arctan(t + x(t))}{(1+t)^2} \ \sin(t + \int_0^t \frac{t(2t-s)\sin(|y(s)|)}{1+t^4} ds), \quad t \geq 0 \tag{11}$$

$$y(t) = \frac{\operatorname{arccot}(t + x(t))}{(1+t)^2} \ \cos(t + \int_0^t \frac{t\sin(|x(s)|)}{4\pi(t^2+1)(s+1)} ds), \quad t \geq 0.$$

Now, we study the solvability of a coupled system of functional Equation (11) on the space $BC(R_+) \times BC(R_+)$. Take into account that this coupled system of functional equations is a specific instance of system (1) with

$$\begin{aligned} g_1(t, x(t)) &= \frac{1}{1+t} \ \sin(t + x(t)), \quad g_2(t, x(t)) = \frac{1}{1+t} \ \cos(t + x(t)), \\ f_1(t, x(t)) &= \frac{\arctan(t + x(t))}{1+t}, \qquad f_2(t, x(t)) = \frac{\operatorname{arccot}(t + x(t))}{1+t}, \\ \nu_1(t, s, x(s)) &= \frac{t(2t-s)\sin(|x(t)|)}{1+t^4}, \ \nu_2(t, s, x(s)) = \frac{\sin(|x(t)|)}{4\pi(t^{\frac{2}{3}}+1)(\sqrt{t-s}+1)}. \end{aligned}$$

Obviously, functions f_i, $(i = 1, 2)$ are mutually continuous. Currently, for any $x, y \in R_+$ and $t \in R_+$

$$|f_i(t, x(t)) - f_i(t, y(t))| \quad \leq \quad \frac{1}{2}|x(t) - y(t)|.$$

This indicates that condition (v) is satisfied with $f^* = \frac{\pi}{2}$ and $m = \frac{1}{2}$, where $f_1(t, 0) = \frac{t}{1+t^2} \ \arctan(t)$, $f_2(t, 0) = \frac{1}{1+t} \operatorname{arccot}(t)$. However, we also have

$$|g_i(t, x(t)) - g_i(t, y(t))| \leq \frac{|x(t) - y(t)|}{2},$$

with $l = \frac{1}{2}$, $g_1(t, 0) = \frac{1}{1+t} \ \sin(t)$, $g_2(t, 0) = \frac{1}{1+t} \ \cos(t)$ and $g^* = \frac{1}{2}$. Further, observe that $\nu_i(t, s, x)(i = 1, 2)$ fulfills condition (iv), with

$$|\nu_1(t, s, x(s))| \leq \frac{t}{2\pi(t^2+1)(s+1)},$$

and

$$|\nu_2(t, s, x(s))| \leq \frac{1}{4\pi(t^{\frac{2}{3}}+1)(\sqrt{t-s}+1)}$$

This indicates that we can insert $k_1(t, s) = \frac{t}{2\pi(t^2+1)(s+1)}$, and $k_2(t, s) = \frac{1}{4\pi(t^{\frac{3}{4}}+1)(\sqrt{t-s}+1)}$. To verify the assumption (iv), notice that

$$\lim_{t \to \infty} \int_0^t k_1(t, s) = \lim_{t \to \infty} \int_0^t \frac{t}{2\pi(t^2+1)(s+1)} ds = \lim_{t \to \infty} \frac{t \ln(t+1)}{2\pi \cdot (t^2+1)} = 0,$$

and

$$\lim_{t \to \infty} \int_0^t k_2(t, s) = \lim_{t \to \infty} \int_0^t \frac{1}{4\pi(t^{\frac{2}{3}}+1)(\sqrt{t-s}+1)} ds = \lim_{t \to \infty} \frac{\sqrt{t} - \ln\left(\sqrt{t}+1\right)}{2\pi \cdot (t^{\frac{2}{3}}+1)} = 0.$$

Moreover, we have $k = 0.0906987$.

Finally, let us pay attention to the fact that the inequality of Theorem 2 has the form

$$C = mg^* + mkl \simeq 0.272674675 < 1,$$

consequently, all the requirements of Theorem 2 have been met. As a result, the coupled system (11) has at least one asymptotically stable solution in the space $BC(R_+) \times BC(R_+)$.

5. Conclusions

Coupled systems of differential and integral equations have been addressed by many authors and in different classes of functions; for example, see [16–23].

The investigations in this work continue those contained in papers [11–13]. In particular, in this paper, we use a technique associated with measures of noncompactness in $BC(R_+)$ by a given modulus of continuity, to establish the solvability of a coupled system of integral equations.

We discussed the solvability and asymptotic stability of that coupled system of functional integral equation on the real half-axis. Our investigation is lying in the space of bounded continuous functions on R_+ $(BC(R_+))$. We started by applying the technique associated with measures of noncompactness on a coupled system of functional integral equation in $BC(R_+)$. Finally, some particular coupled systems of the well-known Uryshon integral equations, a coupled system of functional equations and an example are illustrated.

Author Contributions: These authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors are thankful to the referees for remarks and suggestions for the improvement of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Akhmerov, R.R.; Kamenskii, M.I.; Potapov, A.S.; Rodkina, A.E.; Sadovskii, B.N. Measure of noncompactness and condensing operators. In *Operator Theory: Advanced and Applications*; Brikhäuser Verlag: Basel, Switzerland, 1992; p. 51.
2. Toledano, J.M.A.; Benavides, T.D.; Acedo, G.L. *Measures of Noncompactness in Metric Fixed Point Theory*; Brikhäuser Verlag: Basel, Switzerland, 1997.
3. Banaś, J.; Merentes, N.; Rzepka, B. Measures of noncompactness in the space of continuous and bounded functions defined on the real half-axis. In *Advances in Nonlinear Analysis via the Concept of Measure of Noncompactness*, Banas, J., Jleli, M., Mursaleen, M., Samet, B., Vetro, C., Eds.; Springer: Singapore; New Delhi, India, 2017; pp. 1–58.
4. Banaś, J.; Goebel, K. Measure of noncompactnees in Banach spaces. In *Lecture Notes in Pure and Aplied Mathimatices*; Marcel Dekker: New York, NY, USA, 1980; Volume 60.
5. Banaś, J., Jleli, M., Mursaleen, M., Samet, B., Vetro, C. (Eds.) *Advances in Nonlinear Analysis via the Concept of Measure of Noncompactness*; Springer: Singapore; New Delhi, India, 2017.
6. Banaś, J.; Mena, J.C. Some Properties of Nonlinear Volterra-Stieltjes Integral Operators. *Comput. Math. Appl.* **2015**, *49*, 1565–1573. [CrossRef]
7. Banaś, J.; O'Regan, D. Volterra-Stieltjes integral operators. *Math. Comput. Model.* **2005**, *41*, 335–344. [CrossRef]
8. Goldenštein, L.S.; Gohberg, I.T.; Markus, A.S. Investigations of some properties of bounded linear operators with their q-norms. *Ucen. Zap. Kishinevsk. Univ.* **1957**, *29*, 29–36.
9. Aghajani, A.; Allahyari, R.; Mursaleen, M. A generalization of Darbo's theorem with application to the solvability of systems of integral equations. *J. Comput. Appl. Math.* **2014**, *260*, 68–77. [CrossRef]
10. Banaś, J.; Chlebowicz, A. On solutions of an infinite system of nonlinear integral equations on the real half-axis. *Banach J. Math. Anal.* **2019**, *13*, 944–968. [CrossRef]
11. Banaś, J.; Chlebowicz, A.; Woxsx, W. On measures of noncompactness in the space of functions defined on the half-axis with values in a Banach space. *J. Math. Anal. Appl.* **2020**, *489*, 124–187. [CrossRef]

12. Chlebowicz, A. Solvability of an infinite system of nonlinear integral equations of Volterra Hammerstein type. *Adv. Nonlinear Anal.* **2020**, *9*, 1187–1204. [CrossRef]
13. Chlebowicz, A. Existence of solutions to infinite systems of nonlinear integral equations on the real half-axis. *Electron. J. Differ. Equ.* **2021**, *2021*, 1–20.
14. Hazarika, B.; Arab, R.; Mursaleen, M. Application Measure of Noncompactness and Operator Type Contraction for Solvability of an Infinite System of Differential Equations in l_p-Space. *Filomat* **2019**, *33*, 2181–2189. [CrossRef]
15. Darbo, G. Punti in transformazationi a condominio non compatto, *Rend. Sem. Math. Univ. Padova* **1955**, *4*, 84–92.
16. Ahmad, B.; Nieto, J.J. Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions. *Comput. Math. Appl.* **2009**, *58*, 1838–1843. [CrossRef]
17. El-Sayed, A.M.A.; Hashem, H.H.G.; Ziada, E.A.A. Picard and Adomian Methods decomposition methods for a coupled system of quadratic integral equations of fractional order. *J. Nonlinear Anal. Optim.* **2012**, *3*, 171–183.
18. Hashem, H.H.G.; El-Sayed, A.M.A. Solvability of coupled systems of fractional order integro-differential equations. *J. Indones. Math. Soc.* **2013**, *19*, 111–121. [CrossRef]
19. Hashem, H.H.G.; El-Sayed, A.M.A.; Baleanu, D. Existence results for block matrix operator of fractional orders in Banach algebras. *Mathematics* **2019**, *7*, 856. [CrossRef]
20. Minhós, F.; de Sousa, R. Solvability of second-order coupled systems on the half-line. *Lith. Math. J.* **2019**, *59*, 198–212. [CrossRef]
21. Minhós, F.; de Sousa, R. Solvability of Coupled Systems of Generalized Hammerstein-Type Integral Equations in the Real Line. *Mathematics* **2020**, *8*, 111. [CrossRef]
22. Banaś, J.; Dubiel, A. Solvability of a Volterra-Stieltjes integral equation in the class of functions having limits at infinity. *Electron. J. Qual. Theory Differ. Equ.* **2017**, *2017*, 1–17. [CrossRef]
23. Banaś, J.; Rzepka, B. On existence and asymptotic stability of solutions of non linear integral equation. *Math. Anal. Appl.* **2003**, *284*, 165–173. [CrossRef]