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Fixed Point Results for Generalized θ-Contraction

Umar Ishtiaq [1], Fahim Ud Din [2], Khaleel Ahmad [3], Doha A. Kattan [4] and Ioannis K. Argyros [5,*]

1 Office of Research, Innovation and Commercialization, University of Management and Technology, Lahore 54770, Pakistan; umar.ishtiaq@umt.edu.pk
2 Abdus Salam School of Mathematical Sciences, Government College University, Lahore 54600, Pakistan; fahim@sms.edu.pk
3 Department of Mathematics, University of Management and Technology, Lahore 54770, Pakistan; s2023265011@umt.edu.pk
4 Department of Mathematics, Faculty of Sciences and Arts, King Abdulaziz University, Rabigh 21589, Saudi Arabia; dakattan@kau.edu.sa
5 Department of Computing and Mathematical Sciences, Cameron University, Lawton, OK 73505, USA
* Correspondence: iargyros@cameron.edu

Abstract: Any two points are close together in a θ-contraction by a factor of θ. The function Δ is implied to be a contraction under this condition, but with a tighter bound on the contraction factor. In this paper, we introduce the notions of orthogonal θ-contraction and orthogonal $\alpha-\theta$-contraction and prove several fixed point results by utilizing these contraction mappings in the context of orthogonal metric spaces. Further, we provide several non-trivial examples to show the validity of our results.

Keywords: orthogonal set; metric space; θ-contraction; admissibility

1. Introduction

In various mathematical and applied contexts, the fixed point (FP) theory offers an effective tool to demonstrate the existence and uniqueness of solutions to different problems in Stability analysis, Optimization, economics, game theory, social sciences, Topology, geometry, Numerical analysis, and functional analysis. These are only a few examples illustrating the significance of FP theory. Its widespread use makes it an essential and adaptable tool for mathematical analysis and its applications in the sciences and engineering. It proves the existence of FPs under specific assumptions, which frequently leads to finding solutions to numerous mathematical problems.

In 1922, Banach [1] introduced the Banach FP theorem, and it was proved by Caccioppoli [2] in 1931. The Banach and Caccioppoli FP theorem guarantees that the function must have an FP under some conditions. Branciari [3] proved the Banach–Caccioppoli FP theorem using a class of generalized metric space. In 2014, Jleli and Samet [4] formulated the new idea of θ-contraction and proved several FP theorems for similar mappings in complete metric spaces (MSs). Samet et al. [5] established FP theorems for α-$\varkappa$-contractive mappings. Ahmad et al. [6] demonstrated FP consequence for generalized θ-contractions. Arshed et al. [7] established some FP consequences by utilizing a universal contraction with triangular α-orbital admissible mappings in the context of Branciari metric spaces.

Goradji et al. [8] provided the notion of an orthogonal set (OS) and generalized the Banach FP theorem. Diminnie [9] presented a new orthogonality relation for normed linear spaces. Further, several FP results for orthogonal (generalized) metric spaces have been proved by Javed et al. [10]. Uddin et al. [11,12] presented several FP results in the framework of orthogonal metric spaces (OMSs). Aydi et al. [13,14] modified F-contractions via α-admissible mappings and generalized admissible- Meir–Keeler contractions in the context of generalized metric spaces. Karapınar and Samet [15] generalized α-ψ contractive-type mappings and related FP theorems using other applications (see [16–19] for related

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results). Ahmad et al. [20,21] proved several FP results for generalized θ-contractions and $\{\alpha,\ \Delta\}-$ expansive locally contractive mappings. Ciric [22] and Jleli et al. [23] presented the generalization of Banach's contraction principle by utilizing different ideas. Naeem et al. [24] and Aljahdaly et al. [25] worked on different fractional operators. For an applications point of view, see the works provided by Manafian [26] and Manafian and Allahverdiyeva [27].

Inspired by [4], in this article, we present the notion of an orthogonal alpha–theta-contraction ($\alpha_\perp - \theta_\perp$-contraction) and provided several generalized FP theorems in the context of orthogonal complete metric spaces (OCMS).

2. Preliminaries

In this part, we provide several definitions from the existing literature that are helpful to understand the main section.

Definition 1 ([17]). *Let Ξ be a non-empty set and $\perp$ defined be a binary relation on $\Xi \times \Xi$. If $\varkappa_1 \in \Xi$ exists, the following condition is true*

$$(\forall \omega \in \Xi\ \varkappa_1 \perp \omega) or\ (\forall \omega \in \Xi\ \omega \perp \varkappa_1).$$

Then, an element $\varkappa_1$ is called an orthogonal element, and (Ξ, D) is an orthogonal set (briefly OS), and an OS may have more than one orthogonal element.

Definition 2 ([13]). *Let $(\lambda,\ \perp)$ be an OS. A sequence $\{\varkappa_n\}$ is said to be an orthogonal sequence (O-Sequence) if*

$$(for\ all\ n \in \mathbb{N},\ \varkappa_n \perp \varkappa_{n+1})\ or\ (\ for\ all\ n \in \mathbb{N},\ \varkappa_{n+1} \perp \varkappa_n).$$

Likewise, a Cauchy sequence $\{\varkappa_{\mathrm{n}}\}$ is called a Cauchy O-sequence (COS) if

$$(\text{for all } n \in \mathbb{N},\ \varkappa_n \perp \varkappa_{n+1}) \text{ or } (\text{for all } n \in \mathbb{N},\ \varkappa_{n+1} \perp \varkappa_n).$$

Definition 3 ([19]). *Let Ξ be an OS and (Ξ, D) be an MS; then, $(\Xi, D, \perp)$ is called an OMS.*

Definition 4 ([14]). *Let $\theta : (0,\ \infty) \to (1,\ \infty)$ be a function verifying the following axioms:*

(1) (θ_1) θ is non-decreasing (ND);
(θ_2) for each sequence $\{\varkappa_n\} \subseteq \mathbb{R}^+$;

$$\lim_{n\to\infty}\theta(\varkappa_n) = 1\ \Leftrightarrow lim_{n\to\infty}(\varkappa_n) = 0,$$

(θ_3) $\exists\, k \in (0,1)$ and $l \in (0, \infty]$. Then,

$$\lim_{t\to\infty}\frac{\theta(t) - 1}{(t)^k} = l.$$

(2) The mapping $\Delta : \Xi \to \Xi$ is named the θ-contraction if the function θ exists, satisfying θ_1 to θ_3 and $k \in (0,1)$, such that

$$D(\Delta\varkappa,\ \Delta\omega) \neq 0 \Rightarrow \theta(D(\Delta\varkappa,\ \Delta\omega)) \leq [\theta(\mathrm{d}(\varkappa, \omega)]^k\ \forall \varkappa, \omega\ \in \Xi,$$

(3) θ is continuous on $(0, \infty)$.
Ω denotes the set of all θ fulfilled in the above 1–3 conditions.

Definition 5 ([7]). *If $\Delta : \Xi \to \Xi$ is a mapping and function $\alpha : \Xi \times \Xi \to [0, \infty)$, we say that Δ is an α-admissible if $\forall \varkappa, \omega \in \Xi\ \alpha(\varkappa, \omega) \geq 1 \Rightarrow\ \alpha(\Delta\varkappa, \Delta\omega) \geq 1$.*

Definition 6 ([19]). *Let $(\Xi, \perp)$ be an OS. The mapping $\Delta_{\perp} : \Xi \to \Xi$ is said to be an orthogonal preserving (briefly OP) if $\Delta_{\perp}\varkappa \perp \Delta_{\perp}\omega$ when $\varkappa \perp \omega$.*

Definition 7 ([19]). *Let $(\Xi, \perp, D)$ be an OMS. Then, $\Delta_{\perp} : \Xi \to \Xi$ is called an orthogonal continuous (OC) at $\varkappa \in \Xi$ if, for each O-sequence $\{\varkappa_n\}$ in Ξ with $\{\varkappa_n\} \to \varkappa$, we have $\Delta_{\perp}\varkappa_n \to \Delta_{\perp}\varkappa$. Also, $\Delta_{\perp}$ is said to be an $\perp$-continuous on Ξ if $\Delta_{\perp}$ is $\perp$- continuous at each $\varkappa \in \Xi$.*

3. Fixed Point Results for $\theta_{\perp}$-Contraction

In this section, we introduce the notion of the $\theta_{\perp}$-contraction in OCMS and prove several FP results.

Definition 8. *Let (Ξ, D) be an OCMS and $\Delta_{\perp} : \Xi \to \Xi$ be a mapping. Then, $\Delta_{\perp}$ is called an $\theta_{\perp}$-contraction if $\exists\, k \in (0,1)$, such that*

$$\forall \varkappa, \omega \in \Xi \text{ with } \varkappa \perp \omega \ \theta(D(\Delta_{\perp}\varkappa, \Delta_{\perp}\omega) \leq [\theta(D(\varkappa, \omega)]^k.$$

Theorem 1. *Let $(\Xi, D, \perp)$ be an OCMS and $\Delta_{\perp} : \Xi \to \Xi$ be the orthogonal complete (OC), OP, and $\theta_{\perp}$-contraction, such that*

$$\frac{1}{2}D(\varkappa, \Delta_{\perp}\varkappa) < D(\varkappa, \omega) \Rightarrow \theta(D(\Delta_{\perp}\varkappa, \Delta_{\perp}\omega) \leq [\theta(D(\varkappa, \omega)]^k$$

$$\forall\, \varkappa, \omega \in \Xi \text{ with } \varkappa \perp \omega \text{ and } k \in (0, 1).$$

Then, $\Delta_{\perp}$ has a unique fixed point (UFP) $e \in \Xi$.

Proof. Let $\varkappa_0 \in \Xi$, such that $(\forall \omega \in \Xi\ \varkappa_0 \perp \omega)$ *or* $(\forall \omega \in \Xi\ \omega \perp \varkappa_0)$ for $n \in \mathbb{N}$. Define a sequence $\{\varkappa_n\}$; it follows that $\varkappa_0 \perp \Delta_{\perp}\varkappa_0$ or $\Delta_{\perp} \perp \varkappa_0$. Assume that $\varkappa_1 = \Delta_{\perp}\varkappa_0$ and $\varkappa_2 = \Delta_{\perp}\varkappa_1 = \Delta_{\perp}{}^2\varkappa_o \ldots \varkappa_{n+1} = \Delta_{\perp}{}^n\varkappa_o$; if $\Delta_{\perp}\varkappa_n = \Delta_{\perp}\varkappa_{n+1}$ for some $n \in \mathbb{N}$, then $\Delta_{\perp}\varkappa_n$ is an FP of $\Delta_{\perp}$ and we are done. Let $\varkappa_n \neq \varkappa_{n+1}$ for all $n \in \mathbb{N}$. Since $\Delta_{\perp}$ is OP, then $(\varkappa_n \perp \varkappa_{n+1})$ *or* $(\varkappa_{n+1} \perp \varkappa_n)$. Hence $\{\varkappa_n\}$ is O-sequence. Then, we have

$$0 < D(\varkappa_n, \Delta_{\perp}\varkappa_n)\ \forall\, n \in \mathbb{N}.$$

So, we obtain

$$\frac{1}{2}D(\varkappa_{n-1}, \Delta_{\perp}\varkappa_{n-1}) < D(\varkappa_{n-1}, \Delta_{\perp}\varkappa_n) = D(\varkappa_n, \varkappa_{n+1}),\ \forall\, n \in \mathbb{N}. \tag{1}$$

This assumption follows

$$\begin{aligned} 1 < D(\varkappa_n, \varkappa_{n+1}) = \theta(D(\Delta_{\perp}\varkappa_{n-1}, \Delta_{\perp}\varkappa_n) \leq [\theta(D(\varkappa_{n-1}, \varkappa_n))]^k = [\theta(D(\Delta_{\perp}\varkappa_{n-2}, \Delta_{\perp}\varkappa_{n-1}))]^k \\ \leq [\theta(D(\varkappa_{n-2}, \varkappa_{n-1}))]^{k^2} \leq \cdots \leq [\theta(D(\varkappa_0, \varkappa_1))]^{k^n}\ \forall\, n \in \mathbb{N}. \end{aligned} \tag{2}$$

Letting $n \to \infty$, we obtain

$$\theta(D(\varkappa_n, \varkappa_{n+1})) \to 1.$$

Using the definition of (θ_2), we have

$$lim_{n\to\infty} D(\varkappa_n, \varkappa_{n+1}) = 0. \tag{3}$$

Now, we examine that $\{\varkappa_n\}$ is a Cauchy sequence. Suppose that $\{\varkappa_n\}$ is not a Cauchy sequence. Then, $\exists\, \varepsilon > 0$ and the sequences $\{p_n\}$ and $\{q_n\}$ of $\mathbb{N}$ are such that for $\forall\, p_n > q_n > n$, we have

$$\left(\varkappa_{p_n},\ \varkappa_{q_n}\right) \geq \varepsilon \text{ and } \left(\varkappa_{p_{n-1}},\ \varkappa_{q_n}\right) < \varepsilon, \tag{4}$$

for $n \in \mathbb{N}$. Using triangle inequality, we obtain

$$\varepsilon \leq D\left(\varkappa_{p(_n)},\ \varkappa_{(q)_n}\right) \leq D\left(\varkappa_{p(_n)},\ \varkappa_{p(_n)-1}\right) + D\left(\varkappa_{p(_n)-1},\ \varkappa_{q_n}\right) < D\left(\varkappa_{p(_n)-1}, \Delta_\perp \varkappa_{p(_n)-1}\right) + \varepsilon\,. \tag{5}$$

Taking the limit as $n \to \infty$ in Equation (5) and using Equation (3), we obtain

$$lim_{n \to \infty} D\left(\varkappa_{p(_n)},\ \varkappa_{q(_n)}\right) = \varepsilon. \tag{6}$$

From (1) and (4) and $n_0 \in \mathbb{N}$, we have

$$\frac{1}{2} D\left(\varkappa_{p(n)}, \Delta_\perp \varkappa_{p(n)}\right) < \frac{\varepsilon}{2} < D\left(\varkappa_{p(n)}, \varkappa_{q(n)}\right).$$

For each $n \geq n_0$, we obtain

$$\theta(D\left(\Delta_\perp \varkappa_{p(n)}, \Delta_\perp \varkappa_{q(n)}\right) \leq \left[\theta\left(D\left(\varkappa_{p(n)}, \varkappa_{q(n)}\right)\right)\right]^k. \tag{7}$$

Letting $n \to \infty$ in (7) and using (6), we obtain

$$\theta(\varepsilon) \leq [\theta(\varepsilon)]^k.$$

Which is a contradiction for $k \in (0,\ 1)$. That is, $\{\varkappa_n\}$ is a Cauchy sequence. So, we have OCMS; then, $\exists\, e \in \Xi$ such that $\varkappa_n \to e$ as $n \to \infty$, so that

$$lim_{n \to \infty} D(\varkappa_n, e) = 0.$$

We claim that

$$\frac{1}{2} D(\varkappa_n, \Delta_\perp \varkappa_n) < D(\varkappa_n, e) or \frac{1}{2} D\left(\Delta_\perp \varkappa_n, \Delta_\perp{}^2 \varkappa_n\right) < D(\Delta_\perp \varkappa_n, e),\ \forall\, n \in \mathbb{N}. \tag{8}$$

Let $m \in \mathbb{N}$ such that

$$\frac{1}{2} D(\varkappa_m, \Delta_\perp \varkappa_m) \geq D(\varkappa_n, e) \text{ and}$$

$$\frac{1}{2} D\left(\Delta_\perp \varkappa_m, \Delta_\perp{}^2 \varkappa_m\right) \geq D(\Delta_\perp \varkappa_n, e). \tag{9}$$

Then, we obtain

$$2D(\varkappa_n, e) \leq D(\varkappa_m, \Delta_\perp \varkappa_m) \leq D(\varkappa_m, e) + D(e, \Delta_\perp \varkappa_m).$$

This implies that

$$D(\varkappa_n, e) \leq D(e, \Delta_\perp \varkappa_m). \tag{10}$$

Using Equations (9) and (10), we obtain

$$D(\varkappa_m, e) \leq D(e, \Delta_\perp \varkappa_m) \leq \frac{1}{2} D\left(\Delta_\perp \varkappa_m, \Delta_\perp{}^2 \varkappa_m\right).$$

Since

$$\frac{1}{2} D\left(\Delta_\perp \varkappa_m, \Delta_\perp{}^2 \varkappa_m\right) < D(\varkappa_m, \Delta_\perp \varkappa_m).$$

We obtain

$$\theta D\left(\Delta_{\perp}\varkappa_m, \Delta_{\perp}{}^2\varkappa_m\right) \le [\theta D(\varkappa_m, \Delta_{\perp}\varkappa_m)]^k.$$

Using (θ_1), we have

$$D\left(\Delta_{\perp}\varkappa_m, \Delta_{\perp}{}^2\varkappa_m\right) < D(\varkappa_m, \Delta_{\perp}\varkappa_m). \tag{11}$$

Using Equations (9)–(11), we obtain

$$\begin{aligned} D\left(\Delta_{\perp}\varkappa_m, \Delta_{\perp}{}^2\varkappa_m\right) < D(\varkappa_m, \Delta_{\perp}\varkappa_m) \le D(\varkappa_m, e) + D(e, \Delta_{\perp}\varkappa_m) \le \tfrac{1}{2}D\left(\Delta_{\perp}\varkappa_m, \Delta_{\perp}{}^2\varkappa_m\right) + \tfrac{1}{2}D\left(\Delta_{\perp}\varkappa_m, \Delta_{\perp}{}^2\varkappa_m\right) \\ = D\left(\Delta_{\perp}\varkappa_m, \Delta_{\perp}{}^2\varkappa_m\right). \end{aligned}$$

This is a contradiction. If we let Equation (8) hold, then $\forall\, n \in \mathbb{N}$, and we have

$$1 < \theta D(\Delta_{\perp}\varkappa_m, \Delta_{\perp}e) \le [\theta D(\varkappa_m, e)]^k.$$

By letting $n \to \infty$, we obtain

$$\theta D(\Delta_{\perp}\varkappa_m, \Delta_{\perp}e) \to 1,$$

by (θ_2), we obtain

$$D(\Delta_{\perp}\varkappa_m, \Delta_{\perp}e) = 0.$$

Therefore,

$$D(e, \Delta_{\perp}e) = lim_{n\to\infty}D(\varkappa_{n+1}, \Delta_{\perp}e) = lim_{n\to\infty}D(\Delta_{\perp}\varkappa_n, \Delta_{\perp}e).$$

So e is an FP of $\Delta_{\perp}$. Now, we show that e is a UFP of $\Delta_{\perp}$. We suppose contrary that there is another fixed u of $\Delta_{\perp}$. If $e \neq u$, we can obtain $\varkappa_0 \perp e$ and $\varkappa_0 \perp u$. Since $\Delta_{\perp}$ is OP, we can write $\Delta_{\perp}\varkappa_0 \perp \Delta_{\perp}e$ and $\Delta_{\perp}\varkappa_0 \perp \Delta_{\perp}u$; then

$$\Delta_{\perp}e = e \neq u = \Delta_{\perp}u.$$

So, we have

$$\frac{1}{2}(D(e,\ \Delta_{\perp}e\ )) < D(e\ ,\ u\ ).$$

By the assumption

$$\theta(D(e\ ,\ u\ )) = \theta(D(\Delta_{\perp}e\ ,\ \Delta_{\perp}u\ )) \le [\theta(D(e\ ,\ u))]^k = (D(e\ ,\ u)).$$

Which is contradiction, since $k \in (0\ ,\ 1)$. Thus e is the UFP of $\Delta_{\perp}$. □

Corollary 1. Let $(\Xi,\ D,\ \perp)$ be an OCMS and $\Delta_{\perp} : \Xi \to \Xi$ is OC, OP, and $\theta_{\perp}$-contraction. If $a, k \in (0\ ,\ 1)$ exists, such that

$\forall\ \varkappa, \omega\ \in\ \Xi$ and $k \in (0,\ 1)$, with $\varkappa \perp \omega$ *or* $\omega \perp \varkappa$

$$D(\Delta_{\perp}\varkappa\ , \Delta_{\perp}\omega) \neq 0 \Rightarrow 2 - \frac{2}{\omega}\arctan\frac{1}{D(\Delta_{\perp}\varkappa\ ,\ \Delta_{\perp}\omega)^a} \le \left[2 - \frac{2}{\omega}\arctan\frac{1}{D(\varkappa\ , \omega)^a}\right]^k.$$

Then, $\Delta_{\perp}$ has a UFP $e \in \Xi$ for every $\varkappa_0 \in \Xi$, and the sequence $\{\Delta_{\perp}{}^n\varkappa_o\}$ converges to the point e.

Proof. If we take $\theta(t) = 2 - \frac{2}{\omega}\arctan\frac{1}{(t)^a}$; then, by using Theorem 1, we obtain the solution. □

Example 1. *Consider $\Xi = (0, \infty)$. Define an OCMS by*

$$D(\varkappa\, , \omega) = |\varkappa - \omega|, \ \forall\ \varkappa, \omega \in \Xi,$$

where $\varkappa \perp \omega \iff \omega < \varkappa$ and orthogonal Cauchy sequence $\{\varkappa_n\}$ by

$$\varkappa_n = \frac{n(n+1)}{2},\ \forall n \in \mathbb{N}.$$

Define a mapping $\Delta_\perp : \Xi \to \Xi$ by

$$\Delta_\perp(\varkappa_n) = \begin{cases} \varkappa_1 & \text{if } \ n = 1, \\ \varkappa_{n-1} & \text{if } \ n > 1. \end{cases}$$

If $\omega < \varkappa$ then, it is easy to see that $\Delta_\perp \omega < \Delta_\perp \varkappa$. Since $\Delta_\perp$ is an OP and OC. For each $n > 1$, $\Delta_\perp$ does not fulfill Banach's contraction. We can quickly examine that

$$lim_{n\to\infty} \frac{D(\Delta_\perp \varkappa_n\, , \, \Delta_\perp \varkappa_1)}{D(\varkappa_n\, , \varkappa_{1)}} = lim_{n\to\infty} \frac{D(\varkappa_{n-1}\, , \, \varkappa_1)}{D(\varkappa_n\, , \varkappa_{1)}} = lim_{n\to\infty} \frac{n(n-1)-2}{n(n+1)-2} = 1.$$

Assume that $\theta : (0, \infty) \to (1, \infty)$ be non-decreasing function defined by

$$\theta(t) = e^{te^t},\ \forall\ t > 0.$$

We, prove that $\Delta_\perp$ is an $\theta_\perp$-contraction. Without the loss of generality $(\varkappa_n \perp \varkappa_m) \Leftrightarrow (n < m\ \forall\ n, m \in \mathbb{N})$, we obtain

$$D(\Delta_\perp \varkappa_n\, , \, \Delta_\perp \varkappa_m) \neq 0 \Rightarrow e^{D(\Delta_\perp \varkappa_n\, , \, \Delta_\perp \varkappa_m) e^{D(\Delta_\perp \varkappa_n\, , \, \Delta_\perp \varkappa_m)}} \leq e^{k[(D(\varkappa_n\, , \, \varkappa_m) e^{D(\varkappa_n\, , \, \varkappa_m)}]}.$$

For some $k \in (0,\ 1)$, we have the following cases:
Case 1: If every $n, m\ \in \mathbb{N}$, take $n = 1$ *and* $m > 1$, we have

$$\begin{aligned} e^{D(\Delta_\perp \varkappa_1\, , \, \Delta_\perp \varkappa_m) e^{D(\Delta_\perp \varkappa_1\, , \, \Delta_\perp \varkappa_m)}} &\leq e^{k[(D(\varkappa_1\, , \, \varkappa_m) e^{D(\varkappa_1\, , \, \varkappa_m)}]}, \\ e^{D(\varkappa_1\, , \, \varkappa_{m-1}) e^{D(\varkappa_1\, , \, \varkappa_{m-1})}} &\leq e^{k[(D(\varkappa_1\, , \, \varkappa_m) e^{D(\varkappa_1\, , \, \varkappa_m)}]}, \\ D(\varkappa_1\, , \, \varkappa_{m-1}) e^{D(\varkappa_1\, , \, \varkappa_{m-1})} &\leq k\Big[\Big(D(\varkappa_1\, , \, \varkappa_m) e^{D(\varkappa_1\, , \, \varkappa_m)}\Big]. \\ D(\Delta_\perp \varkappa_n\, , \, \Delta_\perp \varkappa_m) \neq 0 \Rightarrow \frac{D(\varkappa_1\, , \, \varkappa_{m-1})}{(D(\varkappa_1\, , \, \varkappa_m)} e^{D(\varkappa_1\, , \, \varkappa_{m-1}) - D(\varkappa_1\, , \, \varkappa_m)} &\leq k \\ \frac{D(\varkappa_1\, , \, \varkappa_{m-1})}{(D(\varkappa_1\, , \, \varkappa_m)} e^{D(\varkappa_1\, , \, \varkappa_{m-1}) - D(\varkappa_1\, , \, \varkappa_m)} & \\ = \frac{m(m-1)-2}{m(m+1)-2} e^{\frac{m(m-1)-m(m+1)}{2}} &< e^{-1}. \end{aligned} \tag{12}$$

Case 2: If for every $n, m\ \in \mathbb{N}$ and $m > n > 1$, we have

$$\begin{aligned} e^{D(\Delta_\perp \varkappa_n\, , \, \Delta_\perp \varkappa_m) e^{D(\Delta_\perp \varkappa_n\, , \, \Delta_\perp \varkappa_m)}} &\leq e^{k[(D(\varkappa_n\, , \, \varkappa_m) e^{D(\varkappa_n\, , \, \varkappa_m)}]} \\ \frac{D(\varkappa_{m-1}\, , \, \varkappa_{n-1})}{D(\varkappa_m\, , \, \varkappa_n)} e^{D(\varkappa_{m-1}\, , \, \varkappa_{n-1}) - D(\varkappa_m\, , \, \varkappa_n)} & \\ = \frac{m(m-1)-n(n-1)}{m(m+1)-n(n+1)} e^{\frac{m(m-1)-n(n-1)-m(m+1)+n(n+1)}{2}} &< e^{-1}, \end{aligned}$$

for $k = e^{-1} \in (0, 1)$ Equation (12) is satisfied. Hence, $\Delta_\perp$ is a $\theta_\perp$-contraction of Theorem 1 $\Delta_\perp$ and has a UFP $\varkappa_1$.

Theorem 2. *Let $(\Xi,\ D,\ \perp)$ be an OCMS and $\Delta_\perp : \Xi \to \Xi$ be OC, OP and $\theta_\perp$-contraction, such that*

$$\begin{aligned} &\tfrac{1}{2} D(\varkappa\, , \, \Delta_\perp \varkappa) < D(\varkappa, \omega) \Rightarrow \theta(D(\Delta_\perp \varkappa\, , \Delta_\perp \omega) \\ &\leq [\theta max\{(D(\varkappa,\ \omega), D(\varkappa, \Delta_\perp \varkappa), D(\omega, \Delta_\perp \omega)]^k + H\min\{D(\varkappa\, , \, \Delta_\perp \varkappa),\ D(\omega\, , \, \Delta_\perp \varkappa), D(\varkappa,\ \Delta_\perp \omega)\}, \end{aligned}$$

$\forall\, \varkappa, \omega \ \in \Xi$ with $\varkappa \perp \omega,\ H \geq 0$ and $k \in (0,\ 1)$. Then, $\Delta_\perp$ has a UFP $e \in \Xi$.

Proof. Easy to show on the lines of Theorem 1. □

Theorem 3. *Let* $(\Xi,\ D,\ \perp)$ *be an OCMS and* $\Delta_\perp : \Xi \to \Xi$ *be OC, OP and* $\theta_\perp$*-contraction, such that*

$$D(\Delta_\perp \varkappa\,, \Delta_\perp \omega) \neq 0 \Rightarrow \theta(D(\Delta_\perp \varkappa\,, \Delta_\perp \omega) \leq [\theta U(\varkappa, \omega)]^k, \tag{13}$$

$\forall\, \varkappa, \omega \ \in \ \Xi$ *with* $\varkappa \perp \omega$ *and* $k \in (0,\ 1)$, where

$$U(\varkappa, \omega) = \max\left\{D(\varkappa, \omega), D(\varkappa, \Delta_\perp \varkappa), D(\omega, \Delta_\perp \omega),\ \frac{D(\varkappa, \Delta_\perp \varkappa)D(\omega, \Delta_\perp \omega)}{1 + D(\varkappa, \omega)}\right\}. \tag{14}$$

Then, $\Delta_\perp$ has a UFP $e \in \Xi$.

Proof. It is easy to show on the lines of Theorem 1. □

Corollary 2. Let $(\Xi, D, \perp)$ be an OCMS and $\Delta_\perp : \Xi \to \Xi$ be a self-mapping. Assume that $\Gamma \in (0, 1)$, such that

$$D(\Delta_\perp \varkappa, \Delta_\perp \omega) \leq \Gamma \max\{D(\varkappa, \omega), D(\varkappa, \Delta_\perp \varkappa), D(\omega, \Delta_\perp \omega)\},\ \forall \varkappa, \omega \in \Xi \text{ with } \varkappa \perp \omega.$$

Then, $\Delta_\perp$ has a UFP.

Proof. The function $\theta : (0, \infty) \to (1, \infty)$ defined by $\theta(t) = e^{\sqrt{t}}$, so

$$e^{\sqrt{D(\Delta_\perp \varkappa, \Delta_\perp \omega)}} \leq \left[e^{\sqrt{\max\{D(\varkappa,\omega), D(\varkappa,\Delta_\perp \varkappa), D(\omega,\Delta_\perp \omega)\}}}\right]^{\sqrt{\Gamma}} \ \forall \varkappa, \omega \in \Xi.$$

Using Theorem 1, $\Delta_\perp$ has a UFP. □

Corollary 3. Let $(\Xi,\ D,\ \perp)$ be an OCMS, and $\Delta_\perp$ is a self-mapping and is the OC, OP, and $\theta_\perp$-contraction. If these constants exist, $a, k \in (0\ ,\ 1)$, such that

$$D(\Delta_\perp \varkappa\,, \Delta_\perp \omega) \neq 0 \Rightarrow 2 - \frac{2}{\omega} \arctan \frac{1}{D(\Delta_\perp \varkappa, \Delta_\perp \omega)^a}$$

$$\leq \left[2 - \frac{2}{\omega} \arctan \frac{1}{\left[\max\left\{D(\varkappa, \omega), D(\varkappa, \Delta_\perp \varkappa), D(\omega, \Delta_\perp \omega),\ \frac{D(\varkappa, \Delta_\perp \varkappa)D(\omega, \Delta_\perp \omega)}{1 + D(\varkappa, \omega)}\right\}\right]^a}\right]^k,$$

$\forall\, \varkappa, \omega \in \Xi$ *and* $k \in (0,\ 1)$, where $\varkappa \perp \omega$ *or* $\omega \perp \varkappa$. Then, $\Delta_\perp$ has a UPF $e \in \Xi$ for every $\varkappa_0 \in \Xi$, and the sequence $\{{\Delta_\perp}^n \varkappa_o\}$ converges to point e.

Proof. Let $\theta(t) = 2 - \frac{2}{\omega} \arctan \frac{1}{(t)^a}$ and Theorem 3 gives the proof. □

4. Fixed Point Theorems for $\alpha_\perp - \theta_\perp$-Contraction

In this part, we prove several FP results for $\alpha_\perp - \theta_\perp$-contraction in OCMS.

Definition 9. *Let $(\Xi,\ D)$ be an OCMS and $\Delta_\perp : \Xi \to \Xi$ be a mapping. We say that $\Delta_\perp$ is an orthogonal $\alpha_\perp - \theta_\perp$-contraction if two functions $\alpha_\perp : \Xi \times \Xi \to [0,\infty)$ and $\theta \in \omega$ exist, such that*

$$\alpha_\perp(\varkappa,\omega)\theta(D(\Delta_\perp\ \varkappa\ ,\Delta_\perp\ \omega) \leq [\theta(D(\varkappa,\ \omega)]^k,\ \forall\varkappa,\omega \in \Xi \text{ with } \varkappa\perp\omega \text{ and } k(0,\ 1).$$

Definition 10. *Let $\Delta_\perp : \Xi \to \Xi$ and $\alpha_\perp : \Xi \times \Xi \to [0,\infty)$. Then, $\Delta_\perp$ is called an $\alpha_\perp$-admissible if $\varkappa,\omega \in \Xi$ with $\varkappa\perp\omega$, such that*

$$\alpha_\perp(\varkappa,\omega) \geq 1 \Rightarrow \alpha_\perp(\Delta_\perp\varkappa,\Delta_\perp\omega) \geq 1.$$

Example 2. *Let $\Xi = (0,\ 1]) = A \cup B = (0,1]\left\{\frac{1}{4},\frac{1}{3},\frac{1}{2}\right\} \cup \left\{\frac{1}{4},\frac{1}{3},\frac{1}{2}\right\}$. Define $\Delta_\perp : \Xi \to \Xi$ and $\alpha_\perp : \Xi \times \Xi \to [0,\infty)$ by $\Delta_\perp(\varkappa) = \frac{5}{3}\varkappa,\ \forall\ \varkappa \in \Xi$. Define an orthogonal relation $\perp$ by $\{\varkappa\perp\omega \Leftrightarrow \varkappa \leq \omega\}$ and*

$$\alpha_\perp(\varkappa,\omega) = \frac{1}{max\{\varkappa,\omega\}}\ \forall\varkappa \in A,\omega \in B.$$

Then, $\Delta_\perp$ is an $\alpha_\perp$-admissible.

Example 3. *Let $\Xi = (-2,\ 2]$ and define a relation $\perp$ by $\varkappa\perp\omega \iff \varkappa + \omega \geq 0$. Define a mapping $\alpha_\perp : \Xi \times \Xi \to [0,\infty)$ by*

$$\alpha_\perp(\varkappa,\omega) = \begin{cases} \frac{\min\{\varkappa,\omega\}}{1+\max\{\varkappa,\ \omega\}} & if\ \ \varkappa,\omega \in (0,2], \\ e^{-\varkappa} & if\ \ \varkappa,\omega \in [0,-2), \\ 0 & Otherwise. \end{cases}$$

Define a mapping $\Delta_\perp : \Xi \to \Xi$ by

$$\Delta_\perp(\varkappa) = \begin{cases} 1 & if\ \ \varkappa \in \left[\frac{-1}{2},\frac{1}{2}\right], \\ \frac{min\{1,\varkappa\}}{1+max\{1,\varkappa\}} & if\ \ Otherwise. \end{cases}$$

Then, $\Delta_\perp$ is an $\alpha_\perp$-admissible, but not an α-admissible mapping. Let $\varkappa = 0$ and $\omega = -1$; then,

$$\alpha(0,-1) = e^0 = 1,$$

But,

$$\alpha(\Delta_\perp(0),\Delta_\perp(-1)) = \alpha\left(1,-\frac{1}{2}\right) = 0.$$

Remark 1. *Every α-admissible mapping is an $\alpha_\perp$-admissible, but the converse is not true in general.*

Theorem 4. *Suppose $(\Xi,\ D,\ \perp)$ is an OCMS. Let $\Delta_\perp$ is a self mapping and $\alpha_\perp : \Xi \times \Xi \to [0,\infty)$ be a mapping. Suppose that the below conditions verify:*

(i) $\forall\varkappa,\omega \in \Xi$ *with* $\varkappa\perp\omega$ and $k \in (0,\ 1)$

$$\alpha_\perp(\varkappa,\omega)\theta(D(\Delta_\perp\varkappa\ ,\Delta_\perp\omega) \leq [\theta(D(\varkappa,\ \omega)]^k, \tag{15}$$

(ii) $\Delta_\perp$ is an $\alpha_\perp$-admissible;
(iii) $\varkappa_0 \in \Xi$ exists, such that $\varkappa_0\perp\Delta_\perp\varkappa_0$ and $\alpha_\perp(\varkappa_0,\Delta_\perp\varkappa_0) \geq 1$;
(iv) $\Delta_\perp$ is an OP;

(v) $\Delta_{\perp}$ is an OC.

Then, $\Delta_{\perp}$ has a UFP $e \in \Xi$.

Proof. Let $\varkappa_0 \in \Xi$, such that $(\forall \omega \in \Xi\ \varkappa_0 \perp \omega)$ *or* $(\forall \omega \in \Xi\ \omega \perp \varkappa_0)$ for $n \in \mathbb{N}$. Define a sequence $\{\varkappa_n\}$, it follows that $\varkappa_0 \perp \Delta_{\perp}\varkappa_0$ or $\Delta_{\perp}\varkappa_0 \perp \varkappa_0$. Assume that $\varkappa_1 = \Delta_{\perp}\varkappa_0$ and $\varkappa_2 = \Delta_{\perp}\varkappa_1 = \Delta_{\perp}{}^2\varkappa_o \ldots \varkappa_{n+1} = \Delta_{\perp}{}^n\varkappa_o$ for each $\Delta_{\perp}\varkappa_n = \Delta_{\perp}\varkappa_{n+1}$ for some $n \in \mathbb{N}$; then, $\Delta_{\perp}\varkappa_n$ is an FP of $\Delta_{\perp}$, and so, the proof is completed. Let $\Delta_{\perp}\varkappa_n \neq \Delta_{\perp}\varkappa_{n+1}$ for all $n \in \mathbb{N}$. Since $\Delta_{\perp}$ is OP, then $(\Delta_{\perp}\varkappa_n \perp \Delta_{\perp}\varkappa_{n+1})$ *or* $(\Delta_{\perp}\varkappa_{n+1} \perp \Delta_{\perp}\varkappa_n)$. Hence, $\{\varkappa_n\}$ is an O-sequence. Then, by condition (iii), we have

$$\alpha_{\perp}(\varkappa_n, \Delta_{\perp}\varkappa_n) = \alpha_{\perp}(\varkappa_n, \varkappa_{n+1}) \geq 1,\ \forall\, n \in \mathbb{N}. \tag{16}$$

From (15) and (16), we obtain

$$1 < \theta D(\varkappa_n\ , \varkappa_{n+1}) = \theta(D(\Delta_{\perp}\varkappa_{n-1}, \Delta_{\perp}\varkappa_n) \leq \alpha_{\perp}(\varkappa_{n-1}, \varkappa_n)\theta(D(\Delta_{\perp}\varkappa_{n-1}, \Delta_{\perp}\varkappa_n) \leq [\theta(D(\varkappa_{n-1}\ , \varkappa_n))]^k. \tag{17}$$

Using θ_1, we have

$$D(\varkappa_n\ , \varkappa_{n+1}) < D(\varkappa_{n-1}\ , \varkappa_n).$$

Hence, the sequence $\{D(\varkappa_n\ , \varkappa_{n+1})\}$ is decreasing and $\{D(\varkappa_n\ , \varkappa_{n+1})\}$ converges to a non-negative real number. $r \geq 0$ exists, such that

$$lim_{n\to\infty}\mathrm{d}(\varkappa_n, \varkappa_{n+1}) = r \text{ and } D(\varkappa_n, \varkappa_{n+1}) \geq r. \tag{18}$$

Now, we show that $r = 0$. By assuming that $r > 0$ utilizing θ_1 and Equations (17) and (18), we obtain

$$1 < \theta(r) = \theta D(\varkappa_n\ , \varkappa_{n+1}) \leq [\theta(D(\varkappa_{n-1}\ , \varkappa_n))]^k \leq \cdots \leq [\theta(D(\varkappa_0\ , \varkappa_1))]^{k^n} \forall n \in \mathbb{N}. \tag{19}$$

By letting $n \to \infty$, we obtain $\theta(r) = 1$ and by using θ_2, we have $r = 0$; therefore,

$$lim_{n\to\infty}D(\varkappa_n, \varkappa_{n+1}) = 0. \tag{20}$$

Assume that $\exists\, n, p \in \mathbb{N}$ such that $\varkappa_n = \varkappa_{n+p}$, and we show that $p = 1$. Assume that $p > 1$; then, by using (15) and (16), we obtain

$$\begin{aligned} \theta D(\varkappa_n\ , \varkappa_{n+1}) &= \theta D\big(\varkappa_{n+p}\ , \varkappa_{n+p+1}\big) = \theta(D\big(\Delta_{\perp}\varkappa_{n+p-1}, \Delta_{\perp}\varkappa_{n+p}\big) \\ &\leq \alpha_{\perp}\big(\varkappa_{n+p-1}, \varkappa_{n+p}\big)\theta(D\big(\Delta_{\perp}\varkappa_{n+p-1}, \Delta_{\perp}\varkappa_{n+p}\big) \\ &\leq \big[\theta\big(D\big(\varkappa_{n+p-1}\ , \varkappa_{n+p}\big)\big)\big]^k. \end{aligned} \tag{21}$$

Using (θ_1), we obtain

$$\theta D(\varkappa_n\ , \varkappa_{n+1}) < D\big(\varkappa_{n+p-1}\ , \varkappa_{n+p}\big),$$

by using (15), we obtain

$$\begin{aligned} \theta D\big(\varkappa_{n+p-1}\ , \varkappa_{n+p}\big) &= \theta(D\big(\Delta_{\perp}\varkappa_{n+p-2}, \Delta_{\perp}\varkappa_{n+p-1}\big) \\ &\leq \alpha_{\perp}\big(\varkappa_{n+p-2}, \varkappa_{n+p-1}\big)\theta(D\big(\Delta_{\perp}\varkappa_{n+p-2}, \Delta_{\perp}\varkappa_{n+p-1}\big) \\ &\leq \big[\theta\big(D\big(\varkappa_{n+p-2}\ , \varkappa_{n+p-1}\big)\big)\big]^k < \big(D\big(\varkappa_{n+p-1}\ , \varkappa_{n+p}\big)\big). \end{aligned} \tag{22}$$

Since by (θ_1), we obtain

$$D\big(\varkappa_{n+p-1}\ , \varkappa_{n+p}\big) < D\big(\varkappa_{n+p-1}\ , \varkappa_{n+p}\big),$$

continuing this process, we obtain

$$D(\varkappa_n\ , \varkappa_{n+1}) < D\big(\varkappa_{n+p-1}, \varkappa_{n+p}\big) < D\big(\varkappa_{n+p-2}, \varkappa_{n+p-1}\big) < \ldots < D(\varkappa_n, \varkappa_{n+1}).$$

Which is contradiction, and hence, $p = 1$. We assume that $\Delta_\perp$ has an FP. Now, we show that $\{\varkappa_n\}$ is a Cauchy sequence. Assume that $\{\varkappa_n\}$ is not a Cauchy sequence. So, $\varepsilon > 0$ exists, and we consider two subsequences of $\{\varkappa_n\}$, which are $\{\varkappa_{n_k}\}$ and $\{\varkappa_{m_k}\}$ with $n_k > m_k > k$, for which

$$D(\varkappa_{n_k}, \varkappa_{m_k}) \geq \varepsilon\, D(\varkappa_{n_k}, \varkappa_{m_k-1}) < \varepsilon. \tag{23}$$

Using the triangular inequality, we have

$$\varepsilon \leq D(\varkappa_{n_k}, \varkappa_{m_k}) \leq D(\varkappa_{n_k}, \varkappa_{m_k-1}) + D(\varkappa_{m_k-1}, \varkappa_{m_k}). \tag{24}$$

Letting $k \to \infty$ and using (22), (18), and then (24), we have

$$lim_{n\to\infty} D(\varkappa_{n_k}, \varkappa_{m_k}) = \varepsilon. \tag{25}$$

Using Equation (15) there exist a positive integer k_0, such that

$$D(\varkappa_{n_k}, \varkappa_{m_k}) > 0\, \forall n_k > m_k > k \geq k_0.$$

Then, we have

$$\begin{aligned} \theta(\varepsilon) \leq \theta\big(D(\varkappa_{n_k+1}, \varkappa_{m_k+1})\big) = \theta\big(D(\Delta_\perp \varkappa_{n_k}, \Delta_\perp \varkappa_{m_k})\big) \\ \leq \alpha_\perp(\varkappa_{n_k}, \varkappa_{m_k})\theta(D(\Delta_\perp \varkappa_{n_k}, \Delta_\perp \varkappa_{m_k}) \leq \big[\theta\big(D(\varkappa_{n_k}, \varkappa_{m_k})\big)\big]^k = [\theta(\varepsilon)]^k. \end{aligned}$$

Which is a contradiction because $k \in (0,1)$; hence, $\{\varkappa_n\}$ is Cauchy sequence. Thus, we have OCMS; then, $\exists\, e \in \Xi$ such that $\varkappa_n \to e$ as $n \to \infty$, so that

$$e = lim_{n\to\infty}\varkappa_{n+1} = lim_{n\to\infty}\Delta_\perp \varkappa_n = \Delta_\perp e.$$

So, e is an FP of $\Delta_\perp$. □

Theorem 5. *Let $(\Xi,\ D,\ \perp)$ be an OCMS. Let $\Delta_\perp$ is a self mapping and $\alpha_\perp : \Xi \times \Xi \to [0,\infty))$ be a mapping. Assume that the below conditions are verified:*

(i) $\forall \varkappa, \omega \in \Xi$ *with* $\varkappa \perp \omega$ and $k \in (0,\ 1)$

$$\alpha_\perp(\varkappa, \omega)\theta(D(\Delta\varkappa\,, \Delta\omega) \leq [\theta(D(\varkappa,\ \omega)]^k, \tag{26}$$

(ii) $\Delta_\perp$ is an $\alpha_\perp$-admissible;
(iii) there exist $\varkappa_0 \in \Xi$, such that $\alpha_\perp(\varkappa_0, \Delta_\perp \varkappa_0) \geq 1$;
(iv) $\Delta_\perp$ is an OP;
(v) If $\{\varkappa_n\}$ is an orthogonal sequence in Ξ such that $\alpha(\varkappa_n, \varkappa_{n+1}) \geq 1$ for each n and
(vi) $\varkappa_n \to \infty$. Then, there exists an orthogonal subsequence $\{\varkappa_{n_k}\}$ of $\{\varkappa_n\}$ such that $\alpha(\varkappa_{n_k}, \varkappa) \geq 1$ for each k.

Then, $\Delta_\perp$ has a UFP $e \in \Xi$.

Proof. Easy to show on the lines of Theorem 4. □

Theorem 6. *Let $(\Xi,\ \perp,\ D)$ be an OCMS. Let $\Delta_\perp : \Xi \to \Xi$ be a self-mapping and $\alpha_\perp : \Xi \times \Xi \to [0,\infty)$ be a mapping then the below conditions hold:*

(i) Suppose that $\exists\, \theta \in \omega$ and $k \in (0,\ 1)$ such that

$$D(\Delta_{\perp\varkappa}\,, \Delta_{\perp\omega}) \neq 0 \Rightarrow \theta(D(\Delta_{\perp\varkappa}\,, \Delta_{\perp\omega}) \leq [\theta U(\varkappa, \omega)]^k,$$

$\forall\, \varkappa, \omega \in \Xi$ *with* $\varkappa \perp \omega$ *and* $k \in (0, 1)$, where

$$U(\varkappa, \omega) = \max\left\{ D(\varkappa, \omega), D(\varkappa, \Delta_\perp \varkappa), D(\omega, \Delta_\perp \omega), \frac{D(\varkappa, \Delta_\perp \varkappa) D(\omega, \Delta_\perp \omega)}{1 + D(\varkappa, \omega)} \right\},$$

(ii) $\Delta_\perp$ is an $\alpha_\perp$-admissible;
(iv) there exist $\varkappa_0 \in \Xi$, such that $\varkappa_0 \perp \Delta_\perp \varkappa_0$ and $\alpha_\perp(\varkappa_0, \Delta_\perp \varkappa_0) \geq 1$;
(iv) $\Delta_\perp$ is an OP;
(v) $\Delta_\perp$ is an OC.

Then, $\Delta_\perp$ has a UFP $e \in \Xi$.

Proof. It is easy to show on the lines of Theorem 4. □

Corollary 4. Let $(\Xi, \perp, D)$ be an OCMS. Let $\Delta_\perp : \Xi \to \Xi$ be a self-mapping. Assume that $\theta \in \omega$ and $k \in (0, 1)$ exist, such that

$$\varkappa, \omega \in \Xi \textit{ and } \varkappa \perp \omega \ D(\Delta_\perp \varkappa, \Delta_\perp \omega) \neq 0 \ \theta D(\Delta_\perp \varkappa, \Delta_\perp \omega) \leq [\theta \Xi(\varkappa, \omega)]^k,$$

where $\Xi(\varkappa, \omega) = \max\{D(\varkappa, \omega), D(\varkappa, \Delta_\perp \varkappa), D(\omega, \Delta_\perp \omega)\}$. Then, $\Delta_\perp$ has a UFP.

Corollary 5. Let $(\Xi, \perp, D)$ be an OCMS. Suppose $\Delta_\perp : \Xi \to \Xi$ is a self-mapping and $\alpha_\perp : \Xi \times \Xi \to [0, \infty)$ is a mapping; if the below conditions hold:

(1) Suppose that $\exists\, \theta \in \omega$ and $k \in (0, 1)$ such that

$$\varkappa, \omega \in \Xi \textit{ and } \varkappa \perp \omega \ \alpha_\perp(\varkappa, \omega) \ D(\Delta_\perp \varkappa, \Delta_\perp \omega) \neq 0 \ \theta D(\Delta_\perp \varkappa, \Delta_\perp \omega \leq [\theta U(\varkappa, \omega)]^k,$$

where $U(\varkappa, \omega) = \max\{D(\varkappa, \omega), D(\varkappa, \Delta_\perp \varkappa), D(\omega, \Delta_\perp \omega), \frac{D(\varkappa, \Delta_\perp \varkappa), D(\omega, \Delta_\perp \omega)}{1 + D(\varkappa, \omega)}, \frac{D(\varkappa, \Delta_\perp \varkappa) + D(\omega, \Delta_\perp \omega)}{2}\}$,
(2) $\Delta_\perp$ is an $\alpha_\perp$-admissible;
(3) $\varkappa_0 \in \Xi$ exists, such that $\varkappa_0 \perp \Delta_\perp \varkappa_0$ and $\alpha_\perp(\varkappa_0, \Delta_\perp \varkappa_0) \geq 1$;
(4) $\Delta_\perp$ is an OP;
(5) $\Delta_\perp$ is an OC.

Then, $\Delta_\perp$ has a UFP $e \in \Xi$, and $\{\Delta_\perp \varkappa_n\}$ converges to e.

Example 4. *Consider* $\Xi = (-2, 2]$ *and*

$$D(\varkappa, \omega) = |\varkappa - \omega| \qquad \forall\, \varkappa, \omega \in \Xi,$$

where $\varkappa \perp \omega \Leftrightarrow \omega + \varkappa \geq 0$, then (Ξ, D) is an OCMS.

Define a mapping $\alpha_\perp : \Xi \times \Xi \to [0, \infty)$ by

$$\alpha_\perp(\varkappa, \omega) = \begin{cases} \frac{\min\{\varkappa, \omega\}}{1 + \max\{\varkappa, \omega\}}, & if \quad \varkappa, \omega \in (0, 2] \\ e^{-\varkappa}, & if \quad \varkappa, \omega \in [0, -2) \\ 1, & if \quad \varkappa = \omega = 1 \\ 0, & Otherwise. \end{cases}$$

Define a mapping $\Delta_\perp : \Xi \to \Xi$ by

$$\Delta_\perp(\varkappa) = \begin{cases} 1, & if \quad \varkappa \in \left[\frac{-1}{2}, \frac{1}{2}\right] \cup \{1\} \\ \frac{\min\{1, \varkappa\}}{1 + \max\{1, \varkappa\}}, & if \ \varkappa \in \left(-2, \frac{-1}{2}\right) \cup \left(\frac{1}{2}, 2\right) \setminus \{1\}. \end{cases}$$

If $\omega \perp \varkappa \Leftrightarrow \omega + \varkappa \geq 0$, then it is easy to observe that $\Delta_\perp \omega \perp \Delta_\perp \varkappa \Leftrightarrow \Delta_\perp \omega, \Delta_\perp \varkappa \geq 0$. So, $\Delta_\perp$ is an OP. Assume $\{\varkappa_n\}$ be an O-sequence that converges to $\varkappa$; then,

$$\lim_{n\to\infty} D(\varkappa_n,\ \varkappa) = \lim_{n\to\infty} |\varkappa_n, -\varkappa| = 0,$$

which implies that

$$\lim_{n\to\infty} D(\Delta_\perp \varkappa_n, \Delta_\perp \varkappa) = \lim_{n\to\infty} |\Delta_\perp \varkappa_n, -\Delta_\perp \varkappa| = 0.$$

Consider $\varkappa_n = \frac{1}{n}$, we have

$$\lim_{n\to\infty} D\left(\frac{1}{n},\ 0\right) = 0,$$

Then

$$\lim_{n\to\infty} D\left(\Delta_\perp\left(\frac{1}{n}\right), \Delta_\perp\ (0)\right) = D(\Delta_\perp(0), \Delta_\perp\ (0)) = D(1,1) = 0.$$

Which shows that $\Delta_\perp$ is an OC. Also, $\Delta_\perp$ is an $\alpha_\perp$-admissible, but not α-admissible mapping. Let Ξ is not an OS, $\varkappa = 0$ and $\omega = -1$; then,

$$\alpha(0, -1) = e^0 = 1$$

and

$$\alpha(\Delta_\perp(0), \Delta_\perp(-1)) = \alpha\left(1, -\frac{1}{2}\right) = 0 \ngeq 1.$$

Assume that $\theta : (0, \infty) \to (1, \infty)$ be ND function defined by

$$\theta(t) = e^t,\ \forall\ t > 0,$$

Now we show that $\Delta_\perp$ is not an $\alpha - \theta$-contraction, but $\Delta_\perp$ is an $\alpha_\perp - \theta_\perp$-contraction. For this, let $\varkappa = -1$ and $\omega = -2$; then, $\alpha(-1, -2) = e^{-(-1)} = e^1$, and

$$\alpha(-1, -2) e^{(D(-\frac{1}{2}, -\frac{2}{2}))} = e^1 e^{(0.5)} \nleq e^{k\ D(-1,-2)} = e^k.$$

So, $\Delta_\perp$ is not an $\alpha - \theta$-contraction, but $\Delta_\perp$ is an $\alpha_\perp - \theta_\perp$-contraction for each $k \in \left[\frac{1}{2}, 1\right)$. Hence, all the conditions of Theorem 4 are satisfied, and $\Delta_\perp$ has a UFP $e = 1$.

5. Conclusions

In this manuscript, we introduced the notions of orthogonal θ-contraction and orthogonal $\alpha - \theta$-contraction and proved several FP results by utilizing these contraction mappings in the context of OCMSs. Further, we provided several non-trivial examples to support our main results. This work can be extended to include orthogonal controlled metric spaces, orthogonal S-metric spaces, orthogonal *bvs*-metric spaces, and many other generalized spaces.

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