foundations

Review

Comparison of Harmonic Oscillator Model in Classical and Quantum Theories of Light-Matter Interaction

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Abstract: A brief review of the classical and quantum description of the interaction of electromagnetic radiation with matter based on the model of a harmonic oscillator is presented. This review includes the generalized Bohr correspondence principle, the excitation of a quantum oscillator by electromagnetic pulses including saturation effect, the harmonic limit of the Bloch equations, and a phenomenological account of the damping of the quantum oscillator. In all cases, at the mathematical level, the relationship between the classical and quantum descriptions of the electromagnetic interaction is established and the conditions for such compliance are identified.

Keywords: classical and quantum harmonic oscillators; photoexcitation; Bohr correspondence principle; Schwinger formula; saturation effect; Bloch equations

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1. Introduction

The model of a harmonic oscillator (HO) is the most important model, both in classical and quantum physics [1,2]. It is well-known that a wide range of mechanical systems near the equilibrium position can be described by HO [3]. The quantum HO is a unique model in quantum physics that allows an analytical solution for any external force [4]. It describes a number of the most important quasi-particles, such as photons, phonons, vibrons, etc. In addition, this model applies to an electron in a magnetic field (Landau levels [5]), charge carriers in quantum dots in the case of a parabolic potential [6], etc.

An important field of application of the HO model is the theory of the interaction of electro-magnetic radiation with matter, in particular, with atomic particles. The wide use of this model in the latter case is explained by the large value of the atomic electric field ($E_a \approx 5.14{\cdot}10^9$ V/cm). As a result, the effect of electromagnetic radiation with an amplitude less than the atomic one on atomic electrons is weak, and the HO model is applicable [7].

In particular, in connection with the recent development of the technology for generating ultrashort electromagnetic pulses, the quantum oscillator model is used to describe their interaction with matter [8,9]. The use of this model makes it possible to consistently take into account nonlinear effects in the interaction of a strong electromagnetic field with matter.

It is essential that the harmonic oscillator is a kind of "bridge" between quantum and classical physics of the electromagnetic interaction, which manifests itself, for example, in the Bohr correspondence principle [10]. The Bohr correspondence principle initiated the widespread use of the HO model in describing the electromagnetic interaction, which includes both various photoprocesses (excitation, ionization, dissociation, scattering, etc.) and the interaction of matter with charged particles [11,12]. In the latter case, the concept of photons (equivalent photons of the field of charged particles) can be used, as was demonstrated in the paper of E. Fermi [13].

Although quantum HO has an infinite number of energy levels, its classical analog also appears when describing the interaction of a two-level quantum system with a resonant electromagnetic field using the optical Bloch equations [14]. Indeed, in this case, the dipole moment of a two-level system upon its weak excitation is described by the equation for forced oscillations of a classical oscillator. *Note that a two-level system with a dipole-allowed transition can be considered as a qubit whose control by electromagnetic pulses [15] is promising for applications in quantum optics [16] and quantum informatics.*

The purpose of this paper is to briefly review the most significant manifestations of the HO model in describing the interaction of radiation with matter in classical and quantum approaches and mathematically show the relationship between these descriptions.

2. Classical Harmonic Oscillator

2.1. Model of Harmonic Oscillator

Let *us* consider a one-dimensional system near equilibrium position at point x_0. Then, the following decomposition of its potential energy $U(x)$ is valid [3]

$$U(x \approx x_0) \cong U(x_0) + \frac{1}{2} U''_{x^2}(x_0)(x - x_0)^2. \tag{1}$$

Here, U''_{x^2} is the second derivative of potential energy with respect to coordinate.

Quadratic dependence of potential energy on coordinate *presented* in (1) is the characteristic feature of HO. Total energy of HO is given by *the* well-known expression

$$\mathrm{E} = \frac{m\dot{x}^2}{2} + \frac{m\omega_0^2 (x - x_0)^2}{2}. \tag{2}$$

Here, m is mass of HO and ω_0 is its own frequency which can be expressed via potential energy from (1) as follows:

$$\omega_0 = \sqrt{U''_{x^2}(x_0)/m}. \tag{3}$$

We suppose that $U''_{x^2}(x_0) \neq 0$.

Thus, for small deviations from the equilibrium position, each physical system can be associated with a harmonic oscillator *of* frequency ω_0 *as given by (3) for the one-dimensional case*.

2.2. Interaction of Charged Oscillator with Electromagnetic Radiation

Let us consider the interaction of charged HO with electromagnetic radiation. The Equation of HO motion with account for damping has the form

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{q}{m} E(t). \tag{4}$$

Here, q and γ are *the* charge and relaxation constant of *the* oscillator, and $E(t)$ is electric field strength in the radiation, which can be written as follows:

$$E(t) = E_0 \widetilde{E}(t). \tag{5}$$

Here, $\widetilde{E}(t)$ is dimensionless field strength, and E_0 is amplitude of electric field. We assume that $E(t \to \pm\infty) \to 0$ and $\dot{x}(t \to -\infty) \to 0$.

Note that Equation (6) is valid in dipole approximation when the dependence of the electric field on the coordinate can be neglected. *When interacting with electromagnetic radiation, the dipole approximation is valid in the case when the radiation wavelength significantly exceeds the spatial size of the oscillator. Then, the greatest contribution to the electromagnetic interaction comes from the term corresponding to the dipole distribution of charges in the target.*

The solution of (6) is equal to

$$x(t) = \frac{qE_0}{m} \int\limits_{-\infty}^{\infty} \frac{\widetilde{E}(\omega)\exp(-i\omega t)}{\omega_0^2 - \omega^2 - 2i\omega\gamma} \frac{d\omega}{2\pi}. \tag{6}$$

Here, $\widetilde{E}(\omega)$ is Fourier transform of the dimensionless electric field strength.

2.3. Excitation Energy of Oscillator by Electromagnetic Pulse

The excitation energy of the HO at a given moment of time t is equal to the work carried out on the oscillator under the action of radiation electric field by the time t:

$$\Delta\varepsilon_{clas}(t) = qE_0 \int_{-\infty}^{t} \dot{x}(t')\widetilde{E}(t')dt'. \tag{7}$$

Substituting derivative of HO coordinate (8) in Equation (9) we obtain

$$\Delta\varepsilon_{clas}(t) = \frac{q^2 E_0^2}{m} \int_{-\infty}^{t} dt'\widetilde{E}(t') \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\widetilde{E}(\omega)\frac{i\omega\exp(-i\omega t')}{\omega^2 - {\omega_0}^2 + 2i\omega\gamma}. \tag{8}$$

For oscillator without damping ($\gamma \to 0$), the Formula (10) simplifies to the form [17]:

$$\Delta\varepsilon_{clas}(t) = \frac{q^2 E_0^2}{2m}\left|\int_{-\infty}^{t} dt'\widetilde{E}(t')\exp(i\omega_0 t')\right|^2. \tag{9}$$

In the derivation of Formula (11), we used the Fourier decomposition of $\widetilde{E}(t')$ and the relation

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\frac{i\omega\exp(-i\omega(t'-t''))}{\omega^2-\omega_0^2-2i\omega\gamma} \underset{\gamma\to 0}{\to} \theta(t'-t'')\cos[\omega_0(t'-t'')], \tag{10}$$

where $\theta(\tau)$ is the Heaviside theta function.

Let us consider the excitation of HO by electric pulse with duration τ. Then, in the long time limit $t >> \tau$, we have instead of (10)

$$\Delta\varepsilon_{clas}(t >> \tau) = \frac{q^2 E_0^2}{2m}\int_{0}^{\infty} d\omega\left|\widetilde{E}(\omega)\right|^2 \frac{4\omega^2\gamma/\pi}{(\omega^2-\omega_0^2)^2 + 4\omega^2\gamma^2}. \tag{11}$$

In the case of HO without damping, we obtain from (11) and (13) the following simple expression for excitation energy:

$$\Delta\varepsilon_{clas}(t >> \tau) = \frac{q^2 E_0^2}{2m}\left|\widetilde{E}(\omega_0,\tau,\omega_c)\right|^2. \tag{12}$$

Here, we explicitly show the dependence of the Fourier transform of electric field strength on duration and carrier frequency ω_c of exciting electromagnetic pulse. Note that the resulting expression (14) coincides with the formula for energy transfer to a classical harmonic oscillator under the action of an external force $F(t) = qE(t)$, which is given in the textbook [3].

Thus, when oscillator damping is absent (γ = 0), the instant value of the excitation energy at the time moment t is determined by incomplete Fourier transform of electric field strength calculated at the own frequency of HO (11) while in the limit of long time $t >> \tau$, it is determined by complete Fourier transform (14).

The resulting expressions for the excitation energy can be used to analyze its dependence on the pulse parameters (duration, carrier frequency, and envelope). So in paper [18], it is shown that the dependence of the excitation energy on the pulse duration (τ-dependence for short) is essentially determined by its envelope. In the case of a pulse with an exponential envelope, this dependence has a monotonically increasing character for any carrier frequencies of the pulse. For the Gaussian envelope, the τ-dependence can have extrema at sufficiently large detunings of the carrier frequency from the own frequency of the oscillator. In the case of long quasi-monochromatic pulses, the τ-dependence is linear for all pulse envelopes.

3. Bohr Correspondence Principle and Its Generalization

3.1. Original Version of Bohr Correspondence Principle

Bohr's so-called old quantum theory correctly describes the hydrogen atom and, using the postulates formulated by N. Bohr, lays the foundations for the theory of interaction between atoms and electromagnetic radiation.

The next step towards the phenomenological description of this interaction was the Bohr correspondence principle (BCP). This principle states that an atom, when interacting with an electromagnetic field, can be represented by a set of harmonic oscillators corresponding to dipole-allowed transitions between stationary states of electrons in an atom [10]. The own frequencies of these oscillators are equal to the frequencies of transitions between atomic energy levels, and the strength of the electromagnetic interaction is determined by a dimensionless parameter called the oscillator strength. Each dipole-allowed transition can be associated with a two-level system (TLS), which, within the framework of this principle, is described by a harmonic oscillator.

Note that BCP can, in particular, describe the dynamical polarizability of an atom without resorting to the quantum mechanical formalism [11]. Dynamic polarizability is included in the expression for the cross section of Rayleigh scattering of radiation on an atom [12], and also determines the constants of the van der Waals interaction and some other atomic characteristics.

3.2. Generalization of Bohr Correspondence Principle

It is instructive to consider the generalization of the BCP to the time dependence of the process of excitation of dipole-allowed transitions in an atom [17]. This generalization can be obtained by comparing the time dependence of the excitation energy of TLS associated with the dipole-allowed transition and the harmonic oscillator corresponding to this TLS according to BCP. Excitation energy of classical HO is described by the expression (10). Temporal dependence of the excitation energy of TLS is given by the equality,

$$\varepsilon_{quant}(t) = \frac{q^2 E_0^2}{2m} \int_0^\infty d\omega G(\omega) \left| \int_{-\infty}^{t} \widetilde{E}(t') \exp(i\omega t') dt' \right|^2. \tag{13}$$

Here, $G(\omega)$ is the spectral profile of TLS photoexcitation cross section. This formula can be derived in the framework of conventional quantum-mechanical approach in the first order of perturbation theory by analogy with the consideration carried out in the paper [19].

In the case of spectral profile with zero width, when $G(\omega) \to \delta(\omega - \omega_0)$ equality (13) coincides with (11) and there is a complete correspondence between the classical and quantum results.

For long time $t >> \tau$, we have from (13)

$$\varepsilon_{quant}(t >> \tau) = \frac{q^2 E_0^2}{2m} \int_0^\infty d\omega G(\omega) \left| \widetilde{E}(\omega, \tau, \omega_c) \right|^2. \tag{14}$$

This expression corresponds to classical analogue (13) if the following relation is valid:

$$\frac{4\omega^2 \gamma / \pi}{\left(\omega^2 - \omega_0^2\right)^2 + 4\omega^2 \gamma^2} \equiv G_{osc}(\omega) \leftrightarrow G(\omega). \tag{15}$$

We introduce here the so-called "oscillator" spectral profile $G_{osc}(\omega)$. It coincides with quantum spectral profile in the case of Lorentz $G(\omega)$ and for sufficiently small spectral width $\gamma << \omega_0$.

Numerical analysis [17] shows that, in the general case, a good agreement between the time dependences of the excitation energy calculated in the framework of the classical and quantum approaches takes place for a small value of the ratio γ/ω_0. As it increases, the correspondence worsens, especially for long times but nevertheless remains good enough for the values of the ratio γ/ω_0 characteristic for dipole-allowed transitions between atomic energy levels.

Thus, the extension of the BCP to the time dependence of electromagnetic processes has been demonstrated via comparison between classical HO and quantum two-level system.

4. Charged Quantum Oscillator in the Electromagnetic Field

4.1. Schwinger Formula for Excitation of Quantum Oscillator between Stationary States

The basic formula describing the probability of the transition of an undamped quantum oscillator between stationary states was obtained in the work of J. Schwinger [4] to describe the interaction of a quantized electromagnetic field with a given electric current. This expression for the probability W_{mn} of transition from the stationary state $|n\rangle$ to the stationary state $|m\rangle$ ($m > n$) is given by the following equality [4]:

$$W_{n\to m} = \frac{n!}{m!}\left(|J|^2\right)^{m-n}\exp\left(-|J|^2\right)\left|L_n^{m-n}\left(|J|^2\right)\right|^2. \tag{16}$$

Here, J is dimensionless Fourier transform of electric current and $L_m^k(x)$ are the generalized Laguerre polynomials. To obtain an expression for the probability of charged HO excitation by electromagnetic pulse, we use Formula (16) and the following relation [20]:

$$|J|^2 \to \Omega_0^2\left|\widetilde{E}(\omega_0,\tau,\omega_c)\right|^2. \tag{17}$$

We introduce here Rabi frequency Ω_0 according to the definition

$$\Omega_0 = \frac{qE_0}{\sqrt{2m\hbar\omega_0}}. \tag{18}$$

Thus, Rabi frequency (18) describes the strength of electromagnetic influence on quantum HO.

In the paper [21], relation (17) was generalized to account the dependence of the excitation probability on current time.

4.2. Classical HO Is a Driver of Quantum One

Taking into account Formula (11) and Definition (18), replacement (17) can be represented as

$$|J|^2 \to \frac{\Delta\varepsilon_{clas}(t)}{\hbar\omega_0} \equiv \nu(t). \tag{19}$$

Here, we introduce key dimensionless parameter $\nu(t)$, which determines dynamics of quantum HO under external action [21]. An expression for this parameter in terms of the excitation energy of a classical oscillator associated (having the same parameters) with a quantum oscillator was first obtained by Husimi [22].

The Schwinger Formula (16) takes into account all nonlinear effects in the interaction of an undamped quantum oscillator with an electromagnetic pulse. It is important that in this way an exact description of the dynamics of a quantum oscillator is carried out for any strength of external influence. This circumstance singles out the harmonic oscillator as a unique quantum system, for which a fully analytical description is possible for any external perturbation.

An important consequence of the foregoing is the fact that the associated classical oscillator, according to replacement (19), is a quantum oscillator "driver", determining the dynamics of the latter in accordance with Expression (16). Thus, there is a correspondence between the model of a harmonic oscillator in classical and quantum physics of light–matter interactions.

4.3. Average Number of Excited Quanta

The average number of excited quanta ($\overline{n}$) under the action of electromagnetic pulse on quantum HO can be easily obtained using Formulas (16) and (17) and the definition of this quantity. When HO is excited from ground state, $\overline{n}$ is equal to

$$\overline{n} = \Omega_0^2 \left| \widetilde{E}(\omega_0, \tau, \omega_c) \right|^2 = \frac{\Delta\varepsilon_{clas}}{\hbar\omega_0}. \quad (20)$$

Obviously, this expression can be rewritten in the form

$$\Delta\varepsilon_{clas} = \overline{n}\hbar\omega_0. \quad (21)$$

This equality "returns" us from the quantum oscillator to the classical one; it has a clear physical meaning, which is intuitively easy to understand. Formula (21) also demonstrates the relationship between the harmonic oscillator model in classical and quantum physics, namely, the quantum oscillator after averaging gives the classical result for the excitation energy.

4.4. Saturation Effect upon Pulsed Excitation of a Quantum Oscillator

One of the important nonlinear phenomena arising during the interaction of resonant electromagnetic radiation with a quantum system is the saturation effect. In the case of excitation of TLS, the saturation effect leads to the equalization of the populations of both energy levels and to the field broadening of a homogeneous spectral absorption profile or the burning of the spectral dips in the inhomogeneous spectral profile. When a quantum oscillator is excited by electromagnetic pulse, other specific features of this effect appear, due to an infinite number of energy levels and a finite duration of the exciting pulse.

These "quantum HO features" of the saturation effect can be analyzed using Expressions (16) and (17), as well as the explicit form of the Fourier transform of an electric field strength in a pulse. Such an analysis was carried out in [20] for the Gaussian envelope and the envelope in the form of a hyperbolic secant, and in paper [23] for a double exponential envelope.

In the papers cited, the conditions for weak and strong excitation modes were determined depending on the parameters of the exciting pulse. In the strong excitation mode, the saturation effect manifests itself both in the spectrum (dependence of the excitation probability on the carrier frequency of the pulse) and in the τ-dependence. This manifestation is in the transformation of the maximum to a minimum and the appearance of new maxima in the spectral and τ-dependencies. The Rabi frequency, which determines the effect of "spectral" saturation, is inversely proportional to the pulse duration, while the saturation Rabi frequency for the τ-dependence of the excitation probability is proportional to the detuning of the pulse carrier frequency from the own frequency of the oscillator.

The essential difference between the saturation effect upon excitation of a quantum oscillator and the saturation effect upon excitation of TLS is that in the case of an oscillator, the populations of nearby levels do not equalize with increasing field amplitude (Rabi frequency). Instead, the population maximum shifts to higher energy levels due to their infinite number. The latter circumstance can be seen from Formula (20), which implies an increase in the average number of excited quanta with increasing Rabi frequency.

5. Other Correspondences between Quantum Descriptions of Light–Matter Interaction and Classical HO Model

5.1. HO Limit of Bloch Equations

The dynamics of a two-level system with a dipole-allowed transition in the electric field can be described in terms of the optical Bloch vector R [7]. The first and third components of the this vector are defined by the equalities

$$d(t) = d_0 R_1(t) \text{ and } R_3(t) = N_1(t) - N_2(t). \tag{22}$$

Here, $d(t)$ is dipole moment of TLS (d_0 is matrix element of dipole moment) and $N_{1,2}(t)$ are populations of TLS levels. The second component of the optical Bloch vector is related to the quadrature component of the dipole moment. It is shifted in phase by 90 degrees with respect to the first component.

The system of equations for the components of the optical Bloch vector has the form [7]

$$\dot{R}_1 = \omega_0 R_2 \tag{23}$$

$$\dot{R}_2 = -\omega_0 R_1 + 2\Omega(t) R_3 \tag{24}$$

$$\dot{R}_3 = -2\Omega(t) R_2. \tag{25}$$

The time-dependent Rabi frequency introduced here is defined by

$$\Omega(t) = d_0 E(t)/\hbar. \tag{26}$$

It is convenient to represent the matrix element of the TLS dipole moment in the form

$$d_0 = q x_0 \tag{27}$$

where x_0 is characteristic length dimension parameter equal to the matrix element of the coordinate calculated between the wave functions of the TLS. Using this parameter, one can determine the TLS coordinate in terms of the first component of the Bloch vector by the equality

$$x(t) = x_0 R_1(t). \tag{28}$$

The system of Equations (23) and (24) is a consequence of the Schrödinger equation. It is written in neglect of the relaxation of the Bloch vector, which is valid at times $t < T_{1,2}$ ($T_{1,2}$ are the relaxation times of the populations and the dipole moment of the TLS), which is what we are assuming here.

Eliminating the second component of the Bloch vector from the system of Equations (23)–(26), we find

$$\ddot{R}_1 + \omega_0^2 R_1 = 2\omega_0 \frac{d_0 E(t)}{\hbar} R_3(t). \tag{29}$$

Taking into account equalities (27) and (28), we arrive at the following equation for the TLS coordinate:

$$\ddot{x} + \omega_0^2 x = f_0 R_3(t) \frac{qE(t)}{m}. \tag{30}$$

Here, the oscillator strength of the TLS is introduced by the formula

$$f_0 = \frac{2m\omega_0 x_0^2}{\hbar}. \tag{31}$$

This equality coincides with the standard definition of the oscillator strength of a dipole-allowed transition [24].

So, the derived Equation (30) describes the forced oscillations of a classical harmonic oscillator corresponding to TLS coordinate under the action of a driving force $F(t) = qE(t)$. In addition to the oscillator strength, the coupling between the TLS and the electric field is also determined by the third component of the Bloch vector $R_3(t)$. In the case of a weak perturbation, $R_3(t) \cong 1$ and Equation (30) coincides completely with the equation for HO without damping in an external field. If the populations of the TLS levels are equal, then $R_3(t) = 0$, and there is no coupling between the TLS and the electric field. This is a distinctive feature of the equation for the TLS coordinate, due to the quantum nature of the optical Bloch vector.

The established correspondence of the TLS to the classical HO can also be considered as a substantiation of the Bohr correspondence principle from the point of view of the quantum approach. It follows from the above consideration that exact correspondence takes place when the TLS is weakly excited, when $R_3(t) \cong 1$.

5.2. Accounting for the Damping of HO in the Framework of the Classical and Quantum Approaches

The basic Equation (6) of a damped classical harmonic oscillator in electric field can be rewritten via dimensionless coordinate

$$Q = \frac{x}{x_0} = \sqrt{\frac{2m\omega_0}{\hbar}}x,\ x_0 = \sqrt{\frac{\hbar}{2m\omega_0}} \tag{32}$$

in the form

$$\ddot{Q} + 2\gamma\dot{Q} + \omega_0^2 Q = 2\omega_0\Omega_0\widetilde{E}(t); \tag{33}$$

Rabi frequency Ω_0 is given by Equation (18).

The Hamiltonian of a harmonic oscillator in an electric field is defined as follows [25]:

$$\hat{H} = \hbar\omega_0\hat{a}^+\hat{a} - \hat{d}E_0\widetilde{E}(t), \tag{34}$$

where $\hat{a}^+$ is the creation operator and $\hat{a}$ is the annihilation operator of oscillator quanta.

The operator of the dipole moment of a quantum oscillator $\hat{d}$ has the following form:

$$\hat{d} = qx_0\left(\hat{a} + \hat{a}^+\right) \tag{35}$$

and

$$\hat{H} = \hbar\omega_0\hat{a}^+\hat{a} - \hbar\Omega_0\left(\hat{a} + \hat{a}^+\right)\widetilde{E}(t). \tag{36}$$

A consistent description of the damping of a quantum oscillator is a difficult quantum mechanical problem [26]. Here, we use the phenomenological approach, in which damping is taken into account using the following substitution [27]:

$$\omega_0 \to \omega_0 - i\gamma. \tag{37}$$

This substitution is used, in particular, to take into account the finite lifetime of the stationary states of an electron in an atom [1,5].

Using (37) we have for quantum HO Hamiltonian with phenomenological account for damping

$$\hat{H} = \hbar(\omega_0 - i\gamma)\hat{a}^+\hat{a} - \hbar\Omega_0\left(\hat{a} + \hat{a}^+\right)\widetilde{E}(t). \tag{38}$$

Let us write the Heisenberg equations $i\hbar\dot{\hat{O}} = \left[\hat{O}, \hat{H}\right]$ for the creation and annihilation operators using Hamiltonian (38):

$$\dot{\hat{a}} = -i(\omega_0 - i\gamma)\hat{a} + i\Omega_0\widetilde{E}(t) \tag{39}$$

$$\dot{\hat{a}}^{+} = i(\omega_0 + i\gamma)\hat{a}^{+} - i\Omega_0 \widetilde{E}(t). \tag{40}$$

We introduce dimensionless operators of coordinate and momentum of a quantum oscillator using the formulas

$$\hat{Q} = \hat{a} + \hat{a}^{+} \tag{41}$$

$$\hat{P} = i\left(\hat{a}^{+} - \hat{a}\right). \tag{42}$$

Rewriting Equations (39) and (40) in terms of the variables (41) and (42) we obtain

$$\dot{\hat{Q}} = \omega_0 \hat{P} - \gamma \hat{Q} \tag{43}$$

$$\dot{\hat{P}} = -\omega_0 \hat{Q} - \gamma \hat{P} + 2\Omega_0 \widetilde{E}(t). \tag{44}$$

Eliminating the momentum operator from (43) and (44), we arrive at the equation for the quantum oscillator coordinate operator

$$\ddot{\hat{Q}} + 2\gamma \dot{\hat{Q}} + \left(\omega_0^2 + \gamma^2\right)\hat{Q} = 2\omega_0 \Omega_0 \widetilde{E}(t). \tag{45}$$

Equation (45) coincides with Equation (33) up to the replacement

$$\omega_0^2 \to \omega_0^2 + \gamma^2. \tag{46}$$

Relation (46) means renormalization of the own frequency of the oscillator with damping taken into account.

We see that the phenomenological replacement of the own frequency of a quantum oscillator (37) corresponds to the equation for forced oscillations of damped classical oscillator with renormalization (46) taken into account. Thus, a connection is traced between the classical and quantum models of damped HO.

6. Conclusions

We briefly considered the application of the HO model in the classical and quantum theories of light–matter interaction and on a number of different examples demonstrated mathematically the correspondence between them. It is shown that this correspondence, postulately introduced by N. Bohr to describe atomic radiation processes in the framework of classical physics, can be generalized to time dependence and substantiated in the quantum formalism using the Bloch equations.

The probability of excitation of a quantum HO, derived in the framework of a rigorous quantum approach, is determined by the excitation energy of the associated classical oscillator, which, therefore, is the "driver" of its quantum counterpart. The dependence of the excitation energy of classical HO on the duration of the exciting pulse (τ) is essentially determined by the pulse envelope. This dependence, when the envelope changes from exponential to Gaussian, is transformed from a monotonically increasing function into a function with extrema. The above is true for the τ-dependence of the average number of excited quanta of the quantum HO, which is equal to the ratio of the excitation energy of the associated classical HO to the quantum energy.

The phenomenological account for the damping of a quantum HO leads to an equation for the operator of its coordinates, which coincides with the corresponding classical equation, up to renormalization of the eigenfrequency of the oscillator.

Thus, it is shown that the quantum-mechanical Bloch equations and the Heisenberg equations for quantum GO are conjugate with the equation of forced oscillations of a classical oscillator.

It follows from the analysis carried out in the present paper that the HO model has interconnected "projections" on the classical and quantum theory of electromagnetic processes, thus uniting them into one whole.

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