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Three-Step Derivative-Free Method of Order Six

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Abstract: Derivative-free iterative methods are useful to approximate the numerical solutions when the given function lacks explicit derivative information or when the derivatives are too expensive to compute. Exploring the convergence properties of such methods is crucial in their development. The convergence behavior of such approaches and determining their practical applicability require conducting local as well as semi-local convergence analysis. In this study, we explore the convergence properties of a sixth-order derivative-free method. Previous local convergence studies assumed the existence of derivatives of high order even when the method itself was not utilizing any derivatives. These assumptions imposed limitations on its applicability. In this paper, we extend the local analysis by providing estimates for the error bounds of the method. Consequently, its applicability expands across a broader range of problems. Moreover, the more important and challenging semi-local convergence not investigated in earlier studies is also developed. Additionally, we survey recent advancements in this field. The outcomes presented in this paper can be proved valuable to practitioners and researchers engaged in the development and analysis of derivative-free numerical algorithms. Numerical tests illuminate and validate further the theoretical results.

Keywords: divided difference; Banach space; convergence; convergence order

MSC: 65Y20; 65H10; 47H17; 41A58

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1. Introduction

There are several numerical methods such as Newton, Broyden, secant and Steffensen [1–8] that can be used to approximate the solution x^* of

$$F(x) = 0, \tag{1}$$

with $F : \Omega \subset B \to B$, F denoting a continuous operator, mapping a Banach space B into itself. Newton's method is a popular iterative method to solve Equation (1). Iterative solution methods are mainly utilized when it is not possible to obtain solution x^* in analytical or closed forms. Instead, these methods generate a sequence of approximate solutions that converge towards x^*.

Steffensen's method [1,2], developed for $n = 0, 1, 2, \ldots$ as

$$x_{n+1} = x_n - B^{-1}F(x_n), \tag{2}$$

where $B = B_n = [w_n, x_n; F]$ and $w_n = x_n + F(x_n)$, is often employed to provide a sequence of iterates converging quadratically to x^*.

Many iterative approaches have been developed to improve efficiency and order convergence (see [9–13]). An approach established in [13] is given for $x_0 \in \Omega$ as

$$\begin{aligned} u_1 &= x_n - aF(x_n), \;\; u_2 = x_n + bF(x_n), \\ y_n &= x_n - [u_1, u_2; F]^{-1}F(x_n), \\ v_n &= I - [u_1, u_2; F]^{-1}[y_n, x_n; F], \\ A_n &= (I + 2v_n - 2(c-2)v_n^2)[u_1, u_2; F]^{-1}, \\ z_n &= y_n - A_nF(y_n), \\ x_{n+1} &= z_n - A_nF(z_n), \end{aligned} \tag{3}$$

where $a, b, c \in \mathbb{R}$, $[., .; F] : \Omega \times \Omega \longrightarrow \mathfrak{L}(B)$. Note that the inversion of the same linear operator and a function evaluation are required per step. The local analysis (LA) of the method is shown to be of the order of six in [13] for $a \neq b$ under conditions of $F^{(1)}, F^{(2)}, \ldots, F^{(7)}$ not present in the method provided, $B = \mathbb{R}^k$. Moreover, the Taylor series expansion is used. We consider as a motivational scalar example solving $h(t) = 0$, where

$$h(t) = \begin{cases} t^6 \log(t) + 7t^7 - 7t^6, \; t \neq 0 \\ 0, \quad t = 0. \end{cases} \tag{4}$$

We let $\Omega = [-1.4, 1.3]$. We notice that equation $h(t) = 0$ is solvable by $t^* = 1 \in \Omega$. We notice that the results in [13] require the existence and boundedness of the seventh derivative $F^{(7)}$ about the solution. But this derivative is not bounded. Thus, the findings in [13] do not imply that $\lim_{n\to\infty} x_n = 1$. However, the sequence $\{x_n\}$ is convergent to 1. That is why it is useful to weaken the convergence criteria in [13] and not rely on derivatives of high order such that $F^{(j)}$, $j = 1, 2, \ldots, 7$ which are also not used in the method. These limitations constitute the reason for writing this paper.

Motivational Limitations

(a) The existence of high-order derivatives not present in the method.
(b) $B = \mathbb{R}$.
(c) A priori error analysis is not provided for $\|x_n - x^*\|$.
(d) Results on the isolation of the solution case not present, either.
(e) The more challenging and important semi-local analysis (SLA) is not given.

We notice also that concerns *(a)*–*(e)* are usually present in the study of other methods using the Taylor series approach [2–14].

The present paper positively addresses all these concerns, *(a)*–*(e)* as follows:

Novelty of the paper

$(a)'$ The analysis of convergence uses only divided differences of order one that are present on the method and not the seventh-order derivative, or even higher, used in [13] and other methods [2–14] utilizing the Taylor series expansion approach.
$(b)'$ The analysis of convergence is carried out in the more general setting of Banach space valued operators, not only on $\mathbb{R}$.
$(c)'$ An a priori error analysis is provided to determine upper error bounds on $\|x_n - x^*\|$. This allows the determination of the number of iterations in advance to be carried out in order to achieve a predecided error tolerance.
$(d)'$ Computational results on the isolation of solutions are developed based on generalized continuity conditions [3–8] on the divided differences (see conditions (C_1) and (C_2) in Section 2).
It is also worth noting that the usual conditions in the convergence analysis of this and the other methods mentioned in the aforementioned references require that $F'(x^*)$ is invertible. That is, x^* must be a simple solution of equation $F(x) = 0$, although derivative $F'(x^*)$ is not present in Method (3). Thus, the earlier results in [13] cannot assure the convergence of Method (3) in cases operator F is a nondifferentiable operator, although the method may converge. But conditions (C_1) and (C_2) under our approach do not require $F'(x^*)$ to exist or be invertible.

Thus, our approach can be utilized to solve equations like (1) in cases the operator is nondifferentiable.

$(e)'$ The semi-local analysis is developed and requires the usage of sequences that are majorizing [3] for Method (3).

Therefore, advantages *(a)′–(e)′* extend the applicability of Method (3). Moreover, the methodology of this paper can be used on other methods [2–14] utilizing inverses of linear operators along the same lines in order to also achieve advantages *(a)′–(e)′*.

The concepts of local convergence and semi-local convergence are crucial for analyzing the behavior and effectiveness of iterative algorithms in the fields of mathematical optimization and numerical analysis. These ideas assist us in comprehending the behavior of optimization techniques and how they might be used in practical situations. We gain a deeper knowledge of how iterative algorithms converge towards optimal solutions and how they move close to these solutions by researching (LA, SLA). The definitions, characteristics and practical implications of (LA, SLA) are examined in this paper, shedding light on their importance from an application perspective. The brief description of LA and SLA is given below.

Definition 1. *Local convergence analysis uses information about the actual solution to determine the rate and radius of convergence of the method. This typically involves estimating the size of the region around the true solution where the method is guaranteed to converge. This type of analysis also usually involves deriving upper bounds on the error norms, which provide an estimate of how close the iterates of the method are to the true solution.*

Definition 2. *Semi-local convergence behavior of the method is studied using the information from the initial point, typically by deriving sufficient conditions that guarantee convergence of the method. This analysis is usually carried out without any knowledge of the actual solution of the problem.*

Generalized Lipschitz-type conditions are often used in both semi-local and local convergence analyses. These conditions involve bounding the difference between the iterates of the method and the true solution using a Lipschitz constant or a related quantity. These conditions can be used to derive sufficient conditions for convergence, as well as to estimate the rate and radius of convergence of the method.

It is crucial to examine how Technique (3) converges in both the LA (Section 2) and the SLA (Section 3) cases. Moreover, our approach offers prior error analysis estimates and the isolation of x^* results not provided before and in the Banach space. This approach also enables a comparison of the convergence criteria of a method. If the approach is examined separately, the new conditions may be weaker than those that were provided. The numerical examples are included in Section 4. This section contains nonlinear equations and systems of equations as well as integral equations as a sample of where Method (3) can be applied to solve equations. Finally, concluding remarks are discussed in Section 5.

2. Local Analysis

The conditions are described below.

(C_1) There exist continuous as well as nondecreasing functions $f_1 : M = [0, \infty) \to M$, $f_2 : M \to M$, and $\varphi_0 : M \times M \to M$ such that equation

$$\varphi_0(f_1(t), f_2(t)) - 1 = 0$$

has a positive solution, and the smallest (PSS) is denoted by δ. Let $M_1 = [0, \delta)$.

(C_2) There exists an invertible linear operator L and $x^* \in \Omega$ with $F(x^*) = 0$ so that for $x \in \Omega$,

$$||L^{-1}([u_1, u_2; F] - L)|| \leq \varphi_0(||u_1 - x^*||, ||u_2 - x^*||),$$
$$||u_1 - x^*|| \leq f_1(||d||),$$
$$||u_2 - x^*|| \leq f_2(||d||),$$

where $d = x - x^*$. We let $\Omega_0 = \Omega \cap u(x^*, \delta)$.

(C_3) There exist continuous as well as nondecreasing functions $\varphi_1 : M_1 \to M$, $\varphi : M_1 \times M_1 \times M_1 \to M$, and $\varphi_2 : M_1 \times M_1 \times M_1 \times M_1 \to M$ for each $x \in \Omega_0$,

$$||L^{-1}([x, x^*; F] - L)|| \leq \varphi_1(||d||),$$
$$||L^{-1}([u_1, u_2; F] - [x, x^*; F])|| \leq \varphi(||d||, ||u_1 - x^*||, ||u_2 - x^*||),$$
$$||L^{-1}([u_1, u_2; F] - [y, x; F])|| \leq \varphi_2(||d||, ||y - x^*||, ||u_1 - x^*||, ||u_2 - x^*||).$$

(C_4) Equation $g_1(t) - 1 = 0$ has a PSS denoted by $r_1 \in M - \{0\}$, where $g_1 : M \times M \times M \longrightarrow M$,

$$g_1(t) = \frac{\varphi(t, f_1(t), f_2(t))}{1 - \varphi_0(f_1(t), f_2(t))}.$$

We define function $P : M_1 \times M_1 \times M_1 \longrightarrow M$ by

$$p(t) = \frac{\varphi(t, g_1(t), f_2(t))}{1 - \varphi_0(f_1(t), f_2(t))}.$$

(C_5) Equations $g_k(t) - 1 = 0$, $k = 2, 3$ have PSS denoted by $r_2, r_3 \in M - \{0\}$, respectively, where $g_2 : M_1 \longrightarrow M$, $g_3 : M_1 \longrightarrow M$ are

$$g_2(t) = \Big[1 + (1 + 2p(t) + 2(c-2)p^2(t))\frac{1 + \int_0^1 \varphi_1(\theta g_1(t)t)d\theta}{1 - \varphi_0(f_1(t), f_2(t))}\Big]g_1(t)$$

and

$$g_3(t) = \Big[1 + (1 + 2p(t) + 2(c-2)p^2(t))\frac{1 + \int_0^1 \varphi_1(\theta g_2(t)t)d\theta}{1 - \varphi_0(f_1(t), f_2(t))}\Big]g_2(t).$$

(C_6) $u[x^*, r] \subset \Omega$, with $r = min\{r_i\}$, $i = 1, 2, 3$.
It is implied, if $t \in [0, r)$, that

$$0 \leq \varphi_0(f_1(t), f_2(t)) < 1 \tag{5}$$

and

$$0 \leq g_i(t) < 1. \tag{6}$$

Theorem 1. *Assume conditions $(C_1) - (C_6)$ are validated and pick $x_0 \in u(x^*, r) - \{x^*\}$. Then, the following items hold:*

$$\{x_n\} \subset \Omega, \tag{7}$$

$$\|y_n - x^*\| \leq g_1(\|d_n\|)\|d_n\| \leq \|\|d_n\| < r, \tag{8}$$
$$\|z_n - x^*\| \leq g_2(\|d_n\|)\|d_n\| \leq \|\|d_n\|, \tag{9}$$
$$\|x_{n+1} - x^*\| \leq g_3(\|d_n\|)\|d_n\| \leq \|\|d_n\|, \tag{10}$$

where $d_n = x_n - x^*$ with the radius r as defined in condition (C_6) and functions φ_i are as previously given.

Proof. Items (7)–(10) are validated through mathematical induction. By hypothesis $x_0 \in \mu(x^*, r) - \{x^*\}$, (C_1), (C_2) and (C_6), it follows that

$$\begin{aligned} \|L^{-1}([x_0 - aF(x_0), x_0 + bF(x_0); F] - L)\| &\leq \varphi_0(\|x_0 - aF(x_0) - x^*\|, \|x_0 + bF(x_0) - x^*\|) \\ &\leq \varphi_0(f_1(\|d_0\|), f_2(\|d_0\|)) \leq \varphi_0(r, r) < 1. \end{aligned}$$

Thus, $[x_0 - aF(x_0), x_0 + bF(x_0); F]^{-1}$ exists and

$$\|[x_0 - aF(x_0), x_0 + bF(x_0); F]^{-1}L\| \leq \frac{1}{1 - \varphi_0(f_1(\|d_0\|), f_2(\|d_0\|)}, \tag{11}$$

by the standard Banach perturbation Lemma [3] involving inverses of linear operators. Then, from the first substep of (3), y_0 exists, and

$$\begin{aligned} y_0 - x^* &= d_0 - [u_1, u_2; F]^{-1}F(x_0). \\ &= [u_1, u_2; F]^{-1}([u_1, u_2; F] - [x_0, x^*; F])(d_0). \end{aligned} \tag{12}$$

Using (C_3), (C_6), (6) (for $i = 1$), (11) and (12),

$$\begin{aligned} \|y_0 - x^*\| &\leq \|[u_1, u_2; F]^{-1}L\|\|L^{-1}([u_1, u_2; F] - [x_0, x^*; F])\|\|d_0\| \\ &\leq g_1(\|d_0\|)\|d_0\| \leq \|d_0\| < r. \end{aligned} \tag{13}$$

Hence, the iterate $y_0 \in u(x^*, r)$, and item (8) is validated if $n = 0$.

Notice that iterates z_0 and x_1 are also well defined by the invertibility of linear operator $[u_1, u_2; F]^{-1}$. Some estimates are needed:

$$\begin{aligned} \|v_0\| &= \|I - [u_1, u_2; F]^{-1}[y_0, x_0; F]\| \\ &\leq \|[u_1, u_2; F]^{-1}L\|\|L^{-1}([u_1, u_2; F] - [y_0, x_0; F])\| \\ &\leq \frac{\varphi_2(\|d_0\|, \|y_0 - x^*\|, \|u_1 - x^*\|, \|u_2 - x^*\|)}{1 - \varphi_0(f_1(\|d_0\|), f_2(\|d_0\|)} = P_0, \end{aligned}$$

$$\begin{aligned} F(y_0) &= F(y_0) - F(x^*) = \int_0^1 F'(x^* + a(y - x^*))da(y_0 - x^*), \\ &\leq \int_0^1 [F'(x^* + a(y - x^*))da - L + L](y_0 - x^*), \end{aligned}$$

so

$$\|L^{-1}F(y_0)\| \leq \ (1 + \int_0^1 \varphi_0(a\|(y - x^*)\|)da)\|y_0 - x^*\|. \tag{14}$$

Then, by the second substep, (6) (for $i = 2$), (11) and (14),

$$z_0 - x^* = y_0 - x^* - A_0^{-1}F(y_0).$$

It follows that

$$\begin{aligned} \|z_0 - x^*\| &\leq \Big[1 + (1 + 2P_0 + 2|c - 2|P_0^2)\frac{1 + \int_0^1 \varphi_1(\theta\|y_0 - x^*\|)d\theta}{1 - \varphi_0(\|u_1 - x^*\|, \|u_2 - x^*\|)}\|y_0 - x^*\| \\ &\leq g_2(\|d_0\|)\|d_0\| \leq \|d_0\|. \end{aligned} \tag{15}$$

Therefore, iterate $z_0 \in u(x^*, r)$. Moreover, Item (9) is validated if $n = 0$. Similarly, from the last substep, we have

$$\begin{aligned}\|d_1\| &\leq \left[1+(1+2P_0+2|c-2|P_0^2)\frac{1+\int_0^1\varphi_1(\theta\|z_0-x^*\|)d\theta}{1-\varphi_0(\|u_1-x^*\|,\|u_2-x^*\|)}\right]\|z_0-x^*\| \\ &\leq g_3(\|d_0\|)\|d_0\| \leq \|d_0\|.\end{aligned} \tag{16}$$

Therefore, iterate $x_1 \in u(x^*, r)$ and item (7) are true if $n = 1$. These calculations are repeatable for x_0, y_0, z_0, x_1, switched with x_m, y_m, z_m, x_{m+1} terminating the induction for Items (7)–(10). Then, from estimation

$$\|d_{m+1}\| \leq \xi\|d_m\| < r,$$

where $\xi = g_3(\|d_0\|) \in [0,1)$, it follows that $\lim_{m\to\infty} x_m = x^*$. □

Remark 1.

(*i*) *The real functions f_1 and f_2 are left uncluttered in Theorem 1. But some choices are motivated by calculations*

$$\begin{aligned}u_1 - x^* &= d - bF(x) = (I - a[x, x^*; F])(d) \\ &= [(1-a\,L) - aLL^{-1}([x, x^*; F] - L)](d), \\ \|u_1 - x^*\| &\leq [\|I - a\,L\| + |a|\|L\|\varphi_1(\|(d)\|)]\|(d)\|.\end{aligned}$$

Thus, we can choose

$$f_1(t) = \big(\|I - a\,L\| + |a|\|L\|\varphi_1(t)\big)t,$$

and similarly

$$f_2(t) = \big(\|I + b\,L\| + |b|\|L\|\varphi_1(t)\big)t.$$

(*ii*) *Conditions can be expressed without u_1 and u_2 like, for example,*

$$\|L^{-1}([x, y; F] - L)\| \leq \bar{\varphi}_0(\|d\|, \|y - x^*\|).$$

$(C_2)'$ *For all $x, y \in \Omega$ where $\bar{\varphi}_0$ is as φ_0. But then, we must require r in (C_6) to be*

$$\bar{r} = max\{r, f_1(r), f_2(r)\}.$$

Notice, however, that condition $(C_6)'$ is stronger than (C_2) and function $\bar{\varphi}_0$ is less tight than φ_0.

(*iii*) *Linear operator L is chosen so that functions "φ" are as tight as possible. Some popular choices are: $L = F^T(x^*)$ (the differentiable case) or $L = [x_{-1}, x_0; F]$, $x_{-1}, x_0 \in \Omega$ (the non-differentiable case). It is worth noticing that the invertibility of $F'(x^*)$ is not assumed or implied.*

The next result discusses the isolation of solution $x*$.

Proposition 1. *Suppose that $\delta \in u(x^*, \delta_1)$ is solvable by equation $F(x) = 0$ with some $\delta_1 > 0$; (C_2) is valid in ball $u(x^*, \delta_1)$ and there exists $\delta_2 \geq \delta_1$ such that*

$$\varphi_1(\delta_2) < 1. \tag{17}$$

Define the set $\Omega_1 = \Omega \cap u[x^*, \delta_2]$. Then, x^* in collection Ω_1 can only solve equation $F(x) = 0$.

Proof. Suppose $\delta \neq x^*$. Then, the divided difference $[\delta, x^*; F]$ exists. Then, (C_2) and (17) offer

$$\|L^{-1}([\delta, x^*; F] - L)\| \leq \varphi_0(\|\delta - x^*\|) \leq \varphi_0(\delta_2) < 1.$$

Hence, $[\delta, x^*; F]^{-1}$ exists and

$$\delta - x^* = [\delta, x^*; F]^{-1}(F(\delta) - F(x^*)) = [\delta, x^*; F]^{-1}(0) = 0.$$

Therefore, we deduce $\delta = x^*$. □

3. Semi-Local Analysis

The role of x^* is exchanged by x_0 in this analysis. In particular, we suppose the items described below.

(H_1) There exist continuous as well as nondecreasing functions $\psi_0 : M \times M \to M$, $f_3 : M \to M$ and $f_4 : M \to M$ such that equation

$$\psi_0(f_3(t), f_4(t)) - 1 = 0$$

has a PSS denoted by δ_3. We let $\Omega_2 = \Omega \cap u[x_0, \delta_3)$ and $M_2 = [0, \delta_3)$.

(H_2) There exist an initial point $x_0 \in \Omega$ and a linear operator L which is invertible such that

$$\begin{aligned} \|L^{-1}([u_1, u_2; F] - L)\| &\leq \varphi_0(\|u_1 - x_0\|, \|u_2 - x_0\|), \\ \|u_1 - x_0\| &\leq f_3(\|x - x_0\|) \end{aligned}$$

and

$$\|u_2 - x_0\| \leq f_4(\|x - x_0\|)$$

for each $x \in \Omega$.

Notice that conditions (H_1) and (H_2) offer, for $x = x_0$,

$$\|L^{-1}([u_1, u_2; F] - L)\| \leq \varphi_0(f_3(0), f_4(0)) < 1.$$

Thus, $[u_1, u_2; F]^{-1}$ is invertible and we can set $\|[u_1, u_2; F]^{-1}F(x_0)\| \leq b_0$ for some $b_0 \geq 0$.

(H_3) There exists continuous as well as nondecreasing function $\psi : M_2 \times M_2 \times M_2 \times M_2 \to M$, so for $x, y \in \Omega_2$;

$$\|L^{-1}([u_1, u_2; F] - [x, y; F])\| \leq \psi(\|x - x_0\|, \|y - x_0\|, \|u_1 - x_0\|, \|u_2 - x_0\|).$$

We define the scalar sequence $\{a_r\}$, $\{b_r\}$ and $\{c_r\}$ for $a_0 = 0$, $b_0 \in [0, \delta_3)$ and each $r = 0, 1, 2, \ldots$ by

$$\begin{aligned} q_r &= \frac{\psi(a_r, b_r, f_3(a_r), f_4(a_r))}{1 - \psi_0(f_3(a_r), f_4(a_r))}, \\ \lambda_r &= \psi(a_r, b_r, f_3(a_r), f_4(a_r)), \\ c_r &= b_r + \frac{(1 + 2q_r + 2|c - 2|q_r^2)\lambda_r}{1 - \psi_0(f_3(a_r), f_4(a_r))}, \\ \mu_r &= \int_0^1 (1 + \psi_0(b_r + \theta(c_r - b_r))d\theta)(c_r - b_r) + \lambda_r, \\ a_{r+1} &= c_r + \frac{(1 + 2q_r + 2|c - 2|q_r^2)\mu_r}{1 - \psi_0(f_3(a_r), f_4(a_r))} \\ \delta_{r+1} &= \psi(a_r, a_{r+1}, f_3(a_r), f_4(a_r))(a_{r+1} - a_r) + (1 + \psi_0(f_3(a_r), f_4(a_r))(a_{r+1} - a_r) \end{aligned} \tag{18}$$

and

$$b_{r+1} = a_{r+1} + \frac{\delta_{r+1}}{1 - \psi_0(f_3(a_{r+1}), f_4(a_{r+1}))}.$$

(H_4) There exists $\bar{\delta} \in [0, \delta_3)$, so

$$\psi_0(f_3(a_r), f_4(a_r)) < 1 \quad \text{and } a_r \leq \bar{\delta} \text{ for all } r = 0, 1, 2, \ldots.$$

Then, by Formula (18) and this condition that

$$\begin{aligned} F(y_r) &= F(y_r) - F(x_r) - [u_1, u_2; F](y_r - x_r), \\ &= ([y_r, x_r; F] - [u_1, u_2; F])(y_r - x_r), \end{aligned}$$

we obtain

$$\begin{aligned} \|L^{-1}F(y_r)\| &\leq \psi(\|x_r - x_0\|, \|y_r - x_0\|, \|u_1 - x_0\|, \|u_2 - x_0\|)\|y_r - x_r\|, \\ &\leq \psi(a_r, b_r, f_3(a_r), f_4(a_r))(b_r - a_r) = \lambda_r. \end{aligned}$$

Moreover,

$$\begin{aligned} F(z_r) &= F(z_r) - F(y_r) + F(y_r), \\ \|L^{-1}F(z_r)\| &\leq \left(1 + \int_0^1 \psi_0(\|y_r - x_0\| + \theta\|z_r - y_r\|)d\theta\right)\|z_r - y_r\| + \lambda_r, \\ &\leq \left(1 + \int_0^1 \psi_0(b_r + \theta\|c_r - b_r\|)d\theta\right)\|c_r - b_r\| + \lambda_r = \mu_r, \\ \|x_{r+1} - z_r\| &\leq \|A_r L\|\|L^{-1}F(z_r)\|, \\ &\leq \frac{(1 + 2q_r + 2|c - 2|q_r^2)\mu_r}{1 - \psi_0(f_3(a_r), f_4(a_r))}, \\ &= a_{r+1} - a_r, \\ \|x_{r+1} - x_0\| &\leq \|x_{r+1} - z_r\| + \|z_r - x_0\|, \\ &\leq a_{r+1} - c_r + c_r - a_0 = a_{r+1} < a^*, \end{aligned}$$

$$\begin{aligned} F(x_{r+1}) &= F(x_{r+1}) - F(x_r) - [u_1, u_2; F](y_r - x_r), \\ &= ([x_{r+1}, x_r; F] - [u_1, u_2; F])(x_{r+1} - x_r) + [u_1, u_2; F](x_{r+1} - y_r), \end{aligned}$$

so

$$\begin{aligned} \|L^{-1}F(x_{r+1})\| &\leq \psi(a_r, a_{r+1}, f_3(a_r), f_4(a_r))(a_{r+1} - a_r) \\ &\quad + (1 + \psi_0(f_3(a_r), f_4(a_r)))(a_{r+1} - b_r) = \delta_{r+1}. \end{aligned} \tag{19}$$

Consequently, we obtain

$$\begin{aligned} \|y_{r+1} - x_{r+1}\| &\leq \|[u_1, u_2; F]^{-1}L\|\|L^{-1}F(a_{r+1})\| \\ &\leq \frac{\delta_{r+1}}{1 - \psi_0(f_3(a_{r+1}), f_4(a_{r+1}))} = b_{r+1} - a_{r+1} \end{aligned}$$

and

$$\begin{aligned} \|y_{r+1} - x_0\| &\leq \|y_{r+1} - x_{r+1}\| + \|x_{r+1} - x_0\| \\ &\leq b_{r+1} - a_{r+1} + a_{r+1} - a_0 = b_{r+1} < a^*. \end{aligned}$$

Hence, $\{x_r\}$ is fundamental in B, so it converges to $x^* \in u[x_0, a^*]$. Letting $r \to +\infty$ in (19), we obtain $F(x^*) = 0$. Hence, we provide the semi-local analysis of Method (3).

Theorem 2. *Assume conditions $(H_1) - (H_4)$ are validated. Then, the following items hold:*

$$\{x_r\} \subset u(x_0, a^*),$$
$$\|y_r - x_r\| \le b_r - a_r,$$
$$\|z_r - y_r\| \le c_r - b_r,$$
$$\|x_{r+1} - z_r\| \le a_{r+1} - b_r$$

and there exists $x^* \in u[x_0, a^*]$ with $F(x^*) = 0$. Moreover, for $r = 0, 1, 2,$

$$\|x^* - x_r\| \le a^* - a_r. \tag{20}$$

Proof. All items except (20) are shown above Theorem 2. By the estimate

$$\|x_{i+m} - x_i\| \le a_{i+m} - a_i, \tag{21}$$

we deduce (20) by letting $m \to +\infty$ in (21). □

Remark 2. *Comments as in Remark 1 can be offered. In particular, choices for functions f_3 and f_4 can be*

$$f_3(t) = (\|I - a\,L\| + |a|\|L\|\psi_1(t))t + |a|\|F(x_0)\|$$

and

$$f_4(t) = (\|I + b\,L\| + |b|\|L\|\psi_1(t))t + |b|\|F(x_0)\|$$

provided

$$\|L^{-1}([x, x_0; F] - L)\| \le \psi_1(\|x - x_0\|),$$

for each $x \in \Omega_1$ and some continuous as well as nondecreasing function $\psi_1 : M_1 \longrightarrow M$. The computation for the derivation of the function f_3 is

$$u_1 - x_0 = (I - a\,L + a\,L\,L^{-1}([x, x_0; F] - L))(x - x_0) + a\,F(x_0),$$

so

$$\|u_1 - x_0\| \le (\|I - a\,L\| + |a|\ \|L\|\ \psi_1(\|x - x_0\|))\|x - x_0\| + |a|\|F(x_0)\|.$$

Similar computations lead to the definition of function f_4.

The isolation of the solution results is specified in the next result.

Proposition 2. *Assume that equation $F(x) = 0$ is solvable by some $h \in u(x_0, \delta_4)$ for $\delta_4 > 0$; (H_2) holds ball $u(x_0, \delta_4)$ and there exists $\delta_5 \ge \delta_4$ so that*

$$w_0(\delta_4, \delta_5) < 1. \tag{22}$$

Consider the set $\Omega_3 = \Omega \cap u[x_0, \delta_5]$. Then, h in the set Ω_3 can only solve $F(x) = 0$.

Proof. Let $h_1 \in \Omega_3$ for $F(h_1) = 0$. Define operator $T = \int_0^1 F'(h + \theta(h_1 - h))d\theta$. It follows that, in view of condition (H_2) and (22),

$$\begin{aligned}\|L^{-1}(T - L)\| &\le \psi_0(\|h - x_0\|, \|h_1 - x_0\|)\\ &\le \psi_0(\delta_4, \delta_5) < 1.\end{aligned}$$

Thus, we conclude that $h_1 = h$. □

4. Numerical Tests

In the following numerical examples, we estimate the real parameters defined in the preceding sections.

Example 1. *Let $B = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and $\Omega = u(\xi^*, 1)$ with $\xi^* = (0,0,0)^T$. Define mapping H for $\xi = (\xi_1, \xi_2, \xi_3)^T$, $\xi_i \in \mathbb{R}$ by*

$$H(\xi) = \left(\xi_1, e^{\xi_2} - 1, \frac{(e-1)}{2}\xi_3^2 + \xi_3\right)^T.$$

This definition provides the possibility to assert that the H' of mapping H is the Jacobian matrix

$$H'(\xi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{\xi_2} & 0 \\ 0 & 0 & (e-1)\xi_3 + 1 \end{bmatrix}.$$

Notice that $H(\xi^*) = O$ and $H'(\xi^*) = I$. Then, conditions $(C_1) - (C_5)$ are validated if

$$\begin{aligned} \varphi_0(\theta_1, \theta_2) &= \frac{1}{2}(e-1)(\theta_1 + \theta_2), \\ \varphi(\theta_1, \theta_2, \theta_3) &= \frac{1}{2}(e-1)(\theta_1 + \theta_2 + \theta_3), \\ \varphi_1(\theta) &= \frac{1}{2}(e-1)(\theta). \end{aligned}$$

Next, we obtain the radius of convergence r by using (C_6) as $r = min\{r_i\}$, $i = 1,2,3$. Parameter r_1 is the smallest root of $g_1(t) - 1 = 0$, which, on solving, offers $r_1 \approx 0.14976$. Parameter r_2 is the smallest root of $g_2(t) - 1 = 0$, which, when solved, offers $r_2 \approx 0.05704$. Parameter r_3 is the smallest root of $g_3(t) - 1 = 0$, which offers $r_3 \approx 0.03913$. Therefore,

$$r = min\{0.14976, 0.05704, 0.03913\} = 0.03913.$$

Figure 1 shows with graph that r_3 is the radius of convergence in Example 1.

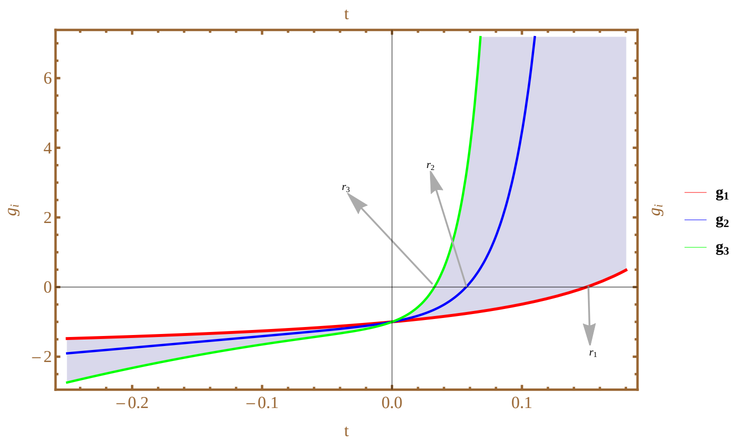

Figure 1. Graph for radius of convergence of Example 1.

Example 2. *Consider $B = C[0,1]$ to be the space of functions in $[0,1]$ which are continuous and $\Omega = u[l^*, 1]$. Assume the nonlinear integral [14] given*

$$l(d) = \int_0^1 T(d,\omega)\left(l(\omega)^{3/2} + \frac{l(\omega)^2}{2}\right)d\omega,$$

$$T(d,\omega) = \begin{cases} (1-d)\omega, & \omega \le d, \\ d(1-\omega), & d \le \omega. \end{cases}$$

Notice that $l^*(d) = 0$. Define $H : \Omega \subseteq [0,1] \to C[0,1]$ as

$$H(l)(d) = l(d) - \int_0^1 T(d,\omega)\left(l(\omega)^{3/2} + \frac{l(\omega)^2}{2}\right)d\omega.$$

Derivative H' is given by

$$H'(l)q(d) = q(d) - \int_0^1 T(d,\omega)\Big(\frac{3}{2}\, l(\omega)^{1/2} + l(\omega)\Big)d\omega;$$

since $H'(l^*(d)) = 1$, it follows that

$$\|H'(\alpha)^{-1}(H'(l) - H'(q))\| \leq \frac{5}{16}\|l - q\|. \tag{23}$$

In (23), switch q by l_0:

$$\|H'(\alpha)^{-1}(H'(l) - H'(l_0))\| \leq \frac{5}{16}\|l - l_0\|.$$

Thus, we take

$$\begin{aligned} \varphi_0(\theta_1,\theta_2) &= \theta_1 + \theta_2, \\ \varphi(\theta_1,\theta_2,\theta_3) &= \theta_1 + \theta_2 + \theta_3, \\ \varphi_1(\theta) &= \theta. \end{aligned}$$

Parameters r_i, $i = 1,2,3$ are the PSS of $g_i(t) - 1 = 0$, and on solving, we have $r_1 \approx 0.12867$, $r_2 \approx 0.04784$, and $r_3 \approx 0.02734$. Then, radius r is

$$r = min\{0.0.12867, 0.04784, 0.02734\} = 0.02734.$$

Figure 2 shows with graph that r_3 is the radius of convergence in Example 2.

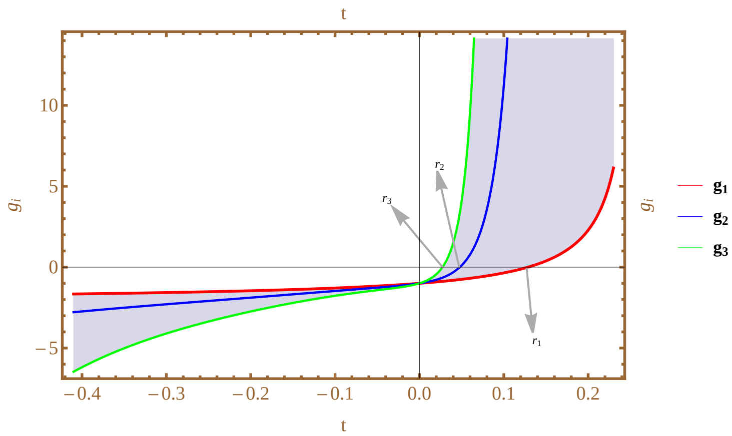

Figure 2. Graph for radius of convergence of Example 2.

Example 3. *Let $B = C[0,1]$ be as in Example 2 and $\Omega = \bar{u}[0,1]$. Consider H on Ω as*

$$H(\varphi)(l) = \varphi(l) - 10\int_0^1 l\rho\varphi(\rho)^3 d\rho.$$

The definition gives

$$H'(\varphi(\xi))(l) = \xi(l) - 30\int_0^1 l\rho\varphi(\rho)^2\xi(\rho)d\rho, \ \ for \ \ each \ \xi \in \Omega.$$

Since $l^* = 0$, we can set

$$\begin{aligned} \varphi_0(\theta_1,\theta_2) &= 2(\theta_1 + \theta_2), \\ \varphi(\theta_1,\theta_2,\theta_3) &= 2(\theta_1 + \theta_2 + \theta_3), \\ \varphi_1(\theta) &= \frac{\theta}{5}. \end{aligned}$$

Then, using (C_6), *we have*

$$r = min\{0.07057, 0.02136, 0.01192\} = 0.01192.$$

Figure 3 shows with graph that r_3 *is the radius of convergence in Example 3.*

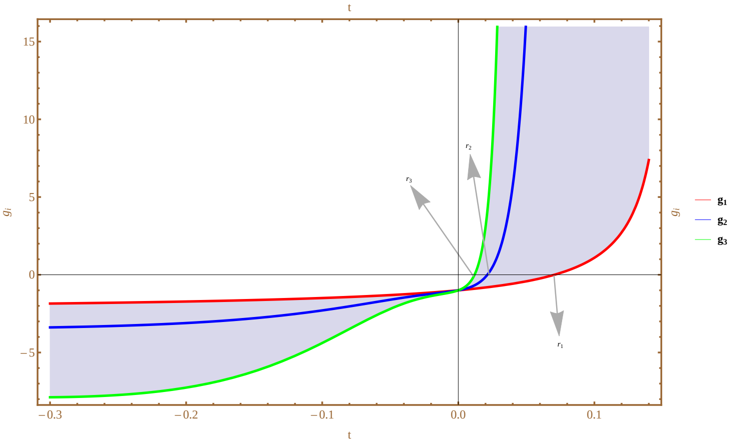

Figure 3. Graph for radius of convergence of Example 3.

Example 4. *Let* $X = Y = \mathbb{R}$, $\Omega = (-1, 1)$ *and define* Θ *on* Ω *by*

$$\Theta(x) = e^x - 1.$$

Then, it is obvious that $x^* = 0$ *and* $\Theta'(x^*) = 1$. *Then, for* $x, r, s, u, v, w \in D_0$, *it follows that*

$$\begin{aligned}
|\Theta'(x^*)^{-1}([x, r; \Theta] - \Theta'(x^*))| &= \Big| \int_0^1 \Theta'(\Psi x + (1 - \Psi)r)d\Psi - \Theta'(x^*)) \Big| \\
&= \Big| \int_0^1 (e^{\Psi x + (1-\Psi)r} - 1)d\Psi \Big| \\
&= \Big| \int_0^1 (\Psi x + (1 - \Psi)r)(1 + \frac{\Psi x + (1 - \Psi)r}{2!} + \frac{(\Psi x + (1 - \Psi)r)^2}{3!} + \cdots)d\Psi \Big| \\
&\leq \Big| \int_0^1 (\Psi x + (1 - \Psi)r)(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots)d\Psi \Big| \\
&\leq \frac{e - 1}{2}(|d| + |r - x^*|),
\end{aligned}$$

$$\begin{aligned}
|\Theta'(x^*)^{-1}([x, r; \Theta] - [x, s; \Theta])| &= \Big| \int_0^1 (\Theta'(\Psi x + (1 - \Psi)r) - \Theta'(\Psi x + (1 - \Psi)s))d\Psi) \Big| \\
&= \Big| \int_0^1 \int_0^1 (\Theta''(\gamma(\Psi x + (1 - \Psi)r) + (1 - \gamma)(\Psi x + (1 - \Psi)s)) \\
&\quad \times (\Psi x + (1 - \Psi)r - (\Psi x + (1 - \Psi)s))d\Psi d\Psi \Big| \\
&= \Big| \int_0^1 \int_0^1 e^{(\Psi(\Psi x + (1-\Psi)r) + (1-\Psi)(\Psi x + (1-\Psi)s)}(\Psi x + (1 - \Psi)r \\
&\quad - (\Psi x + (1 - \Psi)s)d\gamma d\Psi \Big| \\
&\leq \int_0^1 e|(1 - \Psi)(r - s)|d\Psi \\
&\leq \frac{e^{\frac{1}{e-1}}}{2}(|r - s|)
\end{aligned}$$

and

$$\begin{aligned}|\Theta'(x^*)^{-1}([u,v;\Theta]-[w,v;\Theta])| &= \left|\int_0^1 (\Theta'(\Psi v+(1-\Psi)u)-\Theta'(\Psi v+(1-\Psi)w))d\Psi)\right| \\ &= \left|\int_0^1\int_0^1 (\Theta''(\gamma(\Psi v+(1-\Psi)u)+(1-\gamma)(\Psi v+(1-\Psi)w)) \right. \\ &\quad \left. \times (\Psi v+(1-\Psi)u-(\Psi v+(1-\Psi)w))d\gamma d\Psi \right| \\ &= \left|\int_0^1\int_0^1 e^{(\gamma(\Psi v+(1-\Psi)u)+(1-\gamma)(\Psi v+(1-\Psi)w)}(\Psi v+(1-\Psi)u \right. \\ &\quad \left. -(\Psi v+(1-\Psi)w)d\gamma d\Psi\right| \\ &\le \int_0^1 e|(1-\Psi)(u-w)|d\Psi \\ &\le \frac{e^{\frac{1}{e-1}}}{2}(|u-w|).\end{aligned}$$

That is to say, condition (C_3) *is true for*

$$\begin{aligned}\varphi_0(\theta_1,\theta_2) &= \frac{e-1}{2}(\theta_1-\theta_2), \\ \varphi(\theta_1,\theta_2,\theta_3) &= \frac{e^{\frac{1}{e-1}}}{2}(\theta_1+\theta_2+\theta_3), \\ \varphi_1(\theta) &= \frac{e^{\frac{1}{e-1}}}{2}\theta.\end{aligned}$$

Then, from condition (C_6)*, it follows that*

$$r = min\{0.53377, 0.10194, 0.04567\} = 0.04567.$$

Figure 4 shows with graph that r_3 *is the radius of convergence in Example 4.*

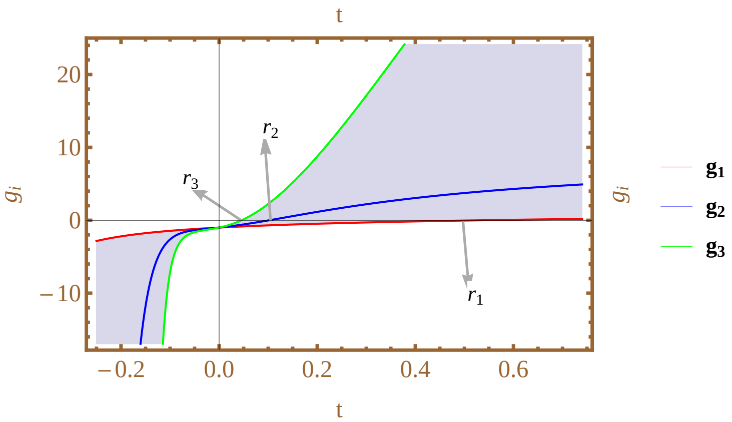

Figure 4. Graph for radius of convergence of Example 4.

Example 5. *According to research in biology, the blood velocity in an artery depends on how far it is located from the central axis of the artery (Figure 5). Then, the Poiseuille's law states that the function is*

$$S(r) = C(R^2 - r^2),$$

where S is the velocity (cm/s) of the blood and r cm is the distance from the central axis of the artery, R is the radius of the artery, C is the constant that depends on the blood viscosity. We say that artery

$$C = 1.76 \times 10^5 \text{ cm/s} \quad (24)$$

and

$$R = 1.2 \times 10^{-2}\ \text{cm}$$

is the case.

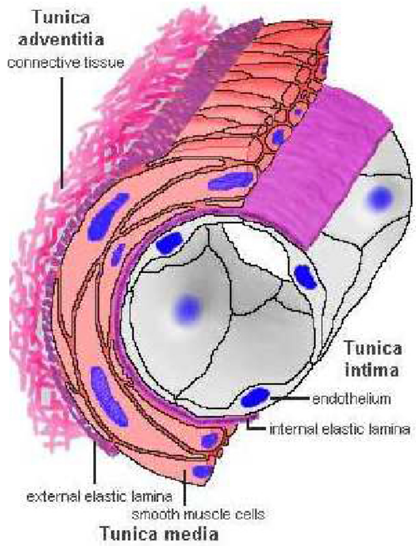

Figure 5. Cut-away view of an artery.

Then, the problem becomes

$$H(x) = 25.344 - 176000x^2 = 0,$$

where $x = r$.

Parameters $x^* = 0$ solve $H(x) = 0$. Thus, we obtain

$$\begin{aligned} \varphi_0(\theta_1, \theta_2) &= 8.42(\theta_1 + \theta_2), \\ \varphi(\theta_1, \theta_2, \theta_3) &= 8.42(\theta_1 + \theta_2 + \theta_3), \\ \varphi_1(\theta) &= 5\theta. \end{aligned}$$

Therefore, $r = min\{r_i\}, i = 1, 2, 3.$ yields

$$r = min\{0.15885 \times 10^{-1}, 0.30577 \times 10^{-2}, 0.16119 \times 10^{-2}\} = 0.16119 \times 10^{-2}.$$

Figure 6 shows with graph that r_3 is the radius of convergence in Example 5.

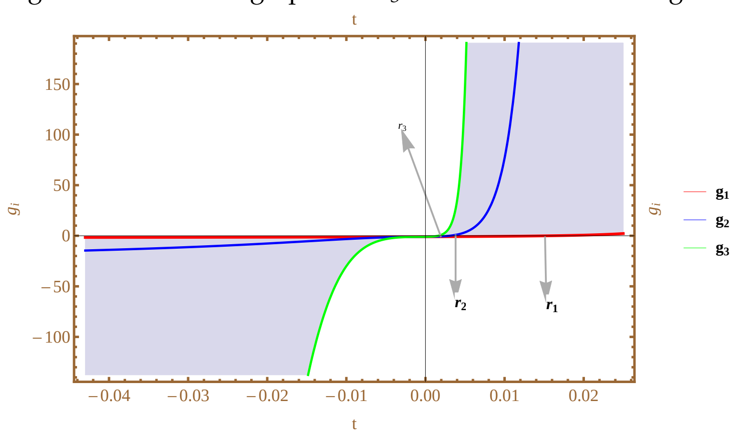

Figure 6. Graph for radius of convergence of Example 5.

5. Conclusions

The focus of this paper was to provide a comprehensive analysis of the LA and SLA of a derivative-free sixth-order method in the Banach space. It is noteworthy that the convergence has been investigated in earlier studies by supposing the existence of some derivatives of high order but which in fact do not appear in the iterative method. Contrary to this, our approach only considers the first-order divided differences that are actually present in the iterative process. This unique feature makes the method applicable to a wider range of functions, thereby expanding its utility. In the analysis, we present an error estimate and a convergence ball that bounds the iterates, providing further benefits to the analysis of convergence. In addition, the sufficient conditions are developed to show the uniqueness of solution in the given domain. To verify the theoretical results, we conducted numerical tests on several problems, demonstrating the effectiveness of this approach. The specific advantages were listed and explained in items *(a)*′–*(e)*′ of the introduction. Moreover, this technique can be extended to other methods, making it a valuable contribution in the field of the theory of iterative functions. This will be the focus of our research on the methods appearing in [2–4,6,8–12,14].

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