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Review

Ostrowski-Type Fractional Integral Inequalities: A Survey

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Abstract: This paper presents an extensive review of some recent results on fractional Ostrowski-type inequalities associated with a variety of convexities and different kinds of fractional integrals. We have taken into account the classical convex functions, quasi-convex functions, (ζ, m)-convex functions, s-convex functions, (s, r)-convex functions, strongly convex functions, harmonically convex functions, h-convex functions, Godunova-Levin-convex functions, MT-convex functions, P-convex functions, m-convex functions, (s, m)-convex functions, exponentially s-convex functions, (β, m)-convex functions, exponential-convex functions, $\bar{\zeta}, \beta, \gamma, \delta$-convex functions, quasi-geometrically convex functions, $s-e$-convex functions and n-polynomial exponentially s-convex functions. Riemann–Liouville fractional integral, Katugampola fractional integral, k-Riemann–Liouville, Riemann–Liouville fractional integrals with respect to another function, Hadamard fractional integral, fractional integrals with exponential kernel and Atagana-Baleanu fractional integrals are included. Results for Ostrowski-Mercer-type inequalities, Ostrowski-type inequalities for preinvex functions, Ostrowski-type inequalities for Quantum-Calculus and Ostrowski-type inequalities of tensorial type are also presented.

Keywords: Ostrowski inequality; Hadamard fractional integral; Katugampola fractional integral; Riemann–Liouville fractional integral

MSC: 05A30; 26A33; 26A51; 26D07; 26D10; 26D15

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1. Introduction

The theory of convex analysis offers robust ideas and methodologies to address an extensive spectrum of issues in applied sciences. Numerous mathematicians and researchers have been striving to implement innovative ideas of convexity theory to handle the real world problems arising in nonlinear programming, statistics, control theory, optimization, etc. The theory of convexity also plays a leading role in establishing a wide class of inequalities. The theory of inequalities in the framework of fractional operators gives rise to integral inequalities. The Ostrowski-type inequalities have been developed in the literature for various types of convex functions. Ostrowski derived the following remarkable and amazing integral inequality in 1938.

Theorem 1 ([1]). *Let $\Pi : I \to \mathbb{R}$ be a differentiable function in the interior I° of I, and let $\varsigma_1, \varsigma_2 \in I^\circ$ with $\varsigma_1 < \varsigma_2$. If $|\Pi'(x)| \leq M$ for all $x \in [\varsigma_1, \varsigma_2]$, then*

$$\left|\Pi(x) - \frac{1}{\varsigma_2 - \varsigma_1}\int_{\varsigma_1}^{\varsigma_2} \Pi(t)dt\right| \leq M(\varsigma_2 - \varsigma_1)\left[\frac{1}{4} + \frac{\left(x - \frac{\varsigma_1+\varsigma_2}{2}\right)^2}{(\varsigma_2 - \varsigma_1)^2}\right]$$

$\forall\, x \in [\varsigma_1, \varsigma_2]$. *In the above famous integral inequality the constant value $\frac{1}{4}$ is an amazing choice in the aspect that it cannot be substituted by a smaller one.*

The Ostrowski-type inequality is found to be an exalted and applicable tool in several branches of mathematics. Integral inequalities, which are used to determine the bounds of physical quantities, find extensive applications in operator theory, statistics, probability theory, numerical integration, nonlinear analysis, information theory, stochastic analysis, approximation theory, biological sciences, physics and technology. Many researchers have shown a keen interest in developing several variants and aspects of this inequality.

During the past few decades, fractional calculus has evolved as a fast-emerging and prominent area of investigation due to the nonlocal nature of fractional order integral and derivative operators. The tools of fractional calculus have been widely applied to formulate the mathematical models associated with various phenomena and processes occurring in engineering and scientific disciplines. The importance and applications of fractional calculus are eminent in the related literature. In the realm of inequalities, fractional order operators have played a fundamental role in the advancement of the topic. In particular, fractional integral operators are found to be of exceptional value in generalizing the standard integral inequalities. Here, we recall that certain inequalities are quite helpful in investigating optimization problems.

The main aim of this manuscript is to present an up-to-date review of Ostrowski-type inequalities involving different convexities and fractional integral operators. In each section/subsection of the paper, we first describe the fractional integral operators and convexities, related to the results collected for Ostrowski-type fractional integral inequalities. We provide comprehensive details for each Ostrowski-type inequality collected in this survey (without proof) for the convenience of the reader. Our survey paper contains the state-of-the-art literature review on fractional Ostrowski-type inequalities and serves as an excellent platform for the researchers who wish to initiate/develop new work on such inequalities.

The structure of this review paper is designed as follows. Section 2 summarizes Ostrowski-type fractional integral inequalities for different families of convexities, including classical convex functions, quasi-convex functions, (ζ, m)-convex functions, s-convex functions, (s, r)-convex functions, strongly convex functions, harmonically convex functions, h-convex functions, Godunova-Levin-convex functions, MT-convex functions, P-convex, m-convex functions, (s, m)-convex functions, exponentially s-convex functions, (β, m)-convex functions, exponential-convex functions, $\bar{\bar{\zeta}}, \beta, \gamma, \delta$-convex functions, quasi-geometrically convex functions, $s-e$-convex functions and n-polynomial exponentially s-convex functions. Section 3 consists of Ostrowski-type fractional integral inequalities for Katugampola fractional integral operators. In Section 4, we present Ostrowski-type fractional integral inequalities involving k-Riemann–Liouville fractional integrals. Section 5 is concerned with Ostrowski-type fractional integral inequalities for preinvex functions, while Ostrowski-type fractional integral inequalities involving fractional integrals with respect to another function are described in Section 6. Mercer-Ostrowski-type fractional integral inequalities for Riemann–Liouville fractional integral operators are included in Section 7. Ostrowski-type fractional integral inequalities obtained via Hadamard fractional integral are discussed in Section 8. We collect Ostrowski-type fractional integral inequalities for integrals with exponential kernel function in Section 9. Section 10 deals with Ostrowski-type fractional integral inequalities for Atangana-Baleanu-type fractional integral operators, while Section 11 contains Ostrowski-type inequalities in terms of generalized fractional integral operators. In Section 12, we discuss Ostrowski-type fractional integral inequalities obtained via operators of quantum-calculus and Ostrowski-type inequalities of tensorial type are presented in Section 13.

2. Ostrowski-Type Inequalities via Riemann–Liouville Fractional Integral

First, we add the definitions of fractional operators, namely Riemann–Liouville, in the left and right aspects.

Definition 1 ([2]). *Let $\Pi \in L[\varsigma_1, \varsigma_2]$. Then, the Riemann–Liouville integrals (left and right aspect) $J^{\zeta}_{\varsigma_1+}\Pi$ and $J^{\zeta}_{\varsigma_2-}\Pi$, $\zeta > 0$, $\varsigma_1 \geq 0$ are stated by*

$$J^{\zeta}_{\varsigma_1+}\Pi(x) = \frac{1}{\Gamma(\zeta)} \int_{\varsigma_1}^{x} (x-t)^{\zeta-1}\Pi(t)dt, \;\; x > \varsigma_1,$$

and

$$J^{\zeta}_{\varsigma_2-}\Pi(x) = \frac{1}{\Gamma(\zeta)} \int_{x}^{\varsigma_2} (t-x)^{\zeta-1}\Pi(t)dt, \;\; x < \varsigma_2,$$

respectively. Here, $\Gamma(\zeta)$ represent the Euler Gamma function and $J^{0}_{\varsigma_1+}\Pi(x) = J^{0}_{\varsigma_2-}\Pi(x) = \Pi(x)$.

2.1. Ostrowski-Type Fractional Integral Inequalities for Functions with Bounded Derivative

In this subsection, we present results on Ostrowski-type fractional integral inequalities for functions with bounded derivatives. We have the following results that provide lower and upper bounds for the Ostrowski differences.

Theorem 2 ([3]). *Let $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be a differentiable mapping on $(\varsigma_1, \varsigma_2)$ and $|\Pi'(x)| \leq M$ for all $x \in [\varsigma_1, \varsigma_2]$. Then*

$$\begin{aligned} &\left| \frac{(x-\varsigma_1)^{\zeta} + (\varsigma_2 - x)^{\zeta}}{\varsigma_2 - \varsigma_1}\Pi(x) - \frac{\Gamma(\zeta+1)}{\varsigma_2 - \varsigma_1}\left[J^{\zeta}_{x-}\Pi(\varsigma_1) + J^{\zeta}_{x+}\Pi(\varsigma_2)\right] \right. \\ \leq \;& \frac{M\left[(x-\varsigma_1)^{\zeta+1} + (\varsigma_2 - x)^{\zeta+1}\right]}{\zeta+1}, \end{aligned}$$

for all $x \in [\varsigma_1, \varsigma_2]$ and $\zeta \geq 0$.

Theorem 3 ([3]). *Let the assumptions of this theorem be as stated in Theorem 2 and $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\begin{aligned} &\left| \frac{(x-\varsigma_1)^{\zeta} + (\varsigma_2 - x)^{\zeta}}{\varsigma_2 - \varsigma_1}\Pi(x) - \frac{\Gamma(\zeta+1)}{\varsigma_2 - \varsigma_1}\left[J^{\zeta}_{x-}\Pi(\varsigma_1) + J^{\zeta}_{x+}\Pi(\varsigma_2)\right] \right. \\ \leq \;& \frac{1}{(\zeta q + 1)^{\frac{1}{q}}}\left[(x-\varsigma_1)^{\zeta+\frac{1}{q}}\|\Pi'\|_{p,[\varsigma_1,x]} + (\varsigma_2 - x)^{\zeta+\frac{1}{q}}\|\Pi'\|_{p,[x,\varsigma_2]}\right], \end{aligned}$$

$\forall\, x \in [\varsigma_1, \varsigma_2]$ and $\zeta \geq 0$ where $\|\Pi'\|_{p,[\varsigma_1,x]} = \left(\int_{\varsigma_1}^{x} |\Pi'(y)|^p dy \right)^{\frac{1}{p}}$.

Theorem 4 ([4]). *Let the assumptions of this theorem be as stated in Theorem 2. Then*

$$\left| \frac{(x-\varsigma_1)^{\zeta} + (\varsigma_2 - x)^{\zeta}}{\Gamma(\zeta+1)}\Pi(x) - \left[J^{\zeta}_{x-}\Pi(\varsigma_1) + J^{\zeta}_{x+}\Pi(\varsigma_2)\right] \right| \leq \frac{M\left[(x-\varsigma_1)^{\zeta+1} + (\varsigma_2 - x)^{\zeta+1}\right]}{\Gamma(\zeta+2)},$$

for all $x \in [\varsigma_1, \varsigma_2]$ and $\zeta \geq 0$.

Theorem 5 ([4]). *Let the assumptions of this theorem be as stated in Theorem 2 and $\Pi' \in L^2[\varsigma_1, \varsigma_2]$. If $m \leq |\Pi'(x)| \leq M$ for all $x \in [\varsigma_1, \varsigma_2]$, then*

$$\left|\frac{\zeta\Pi(x)+\Pi(\varsigma_1)}{\Gamma(\zeta)(\zeta+1)}(x-\varsigma_1)^{\zeta-1}-\frac{\zeta}{x-\varsigma_1}J^{\zeta}_{x-}\Pi(\varsigma_1)\right.$$
$$\left.+\frac{\zeta\Pi(x)+\Pi(\varsigma_2)}{\Gamma(\zeta)(\zeta+1)}(\varsigma_2-x)^{\zeta-1}-\frac{\zeta}{\varsigma_2-x}J^{\zeta}_{x-}\Pi(\varsigma_2)\right|$$
$$\leq \sqrt{\frac{1}{2\zeta+1}-\frac{1}{(\zeta+1)^2}}\frac{(x-\varsigma_1)^{\zeta}K_1+(\varsigma_2-x)^{\zeta}K_2}{\Gamma(\zeta)}$$
$$\leq \sqrt{\frac{1}{2\zeta+1}-\frac{1}{(\zeta+1)^2}}\frac{(x-\varsigma_1)^{\zeta}+(\varsigma_2-x)^{\zeta}}{2\Gamma(\zeta)}(M-m)$$

for all $x\in[\varsigma_1,\varsigma_2]$ *and* $\zeta\geq 1$, *where*

$$K_1^2=\mathcal{M}(\Pi'^2;\varsigma_1,x)-\mathcal{M}^2(\Pi';\varsigma_1,x),\quad K_2^2=\mathcal{M}(\Pi'^2;x\varsigma_2)-\mathcal{M}^2(\Pi';x,\varsigma_2)$$

and $\mathcal{M}(\Pi;\varsigma_1,\varsigma_2)=\dfrac{1}{\varsigma_2-\varsigma_1}\displaystyle\int_{\varsigma_1}^{\varsigma_2}\Pi(x)dx.$

In the next result, fractional integral inequalities of Ostrowski-Grüss type are presented.

Theorem 6 ([5]). *Suppose* $\Pi: I\to\mathbb{R}$ *is a differentiable function and* $\varsigma_1,\varsigma_2\in I^{\circ}$ *with* $\varsigma_1<\varsigma_2$. *If* $\Pi':(\varsigma_1,\varsigma_2)\to\mathbb{R}$ *is bounded on* $(\varsigma_1,\varsigma_2)$ *with* $m\leq\Pi'(x)\leq M$, *for all* $x\in[\varsigma_1,\varsigma_2]$, *then*

$$\left|\frac{\Pi(x)}{\Gamma(\zeta)}-\frac{(\varsigma_2-x)^{1-\zeta}}{\varsigma_2-\varsigma_1}J^{\zeta}_{\varsigma_1}\Pi(\varsigma_2)+(\varsigma_2-x)^{1-\zeta}J^{\zeta-1}_{\varsigma_1}\Pi(\varsigma_2)\right.$$
$$\left.-\Big(\frac{\Pi(\varsigma_2)-\Pi(\varsigma_1)}{\varsigma_2-\varsigma_1}\Big)\Big(\frac{(\varsigma_2-x)^{1-\zeta}(\varsigma_2-\varsigma_1)^{\zeta}}{\Gamma(\zeta+2)}-\frac{\varsigma_2-x}{\Gamma(\zeta+1)}\Big)\right|$$
$$\leq (\varsigma_2-\varsigma_1)(K(x))^{\frac{1}{2}}\Big(\frac{1}{(\varsigma_2-\varsigma_1)\Gamma^2(\zeta)}\|\Pi'\|_2^2-\Big(\frac{\Pi(\varsigma_2)-\Pi(\varsigma_1)}{(\varsigma_2-\varsigma_1)\Gamma(\zeta)}\Big)^2\Big)^{\frac{1}{2}}$$
$$\leq \frac{(K(x))^{\frac{1}{2}}}{2\Gamma(\zeta)}(\varsigma_2-\varsigma_1)(M-m),$$

for all $x\in[\varsigma_1,\varsigma_2]$ *and* $\zeta\geq 1$, *where*

$$K(x) = (\varsigma_2-x)^{1-\zeta}(\varsigma_2-\varsigma_1)^{2\zeta-2}\Big(\frac{1}{2\zeta+1}+\frac{1}{2\zeta-1}-\frac{1}{\zeta}\Big)$$
$$+\frac{(\mathrm{v}_2-x)^{\zeta}}{(\varsigma_2-\varsigma_1)^2}\Big(\frac{\varsigma_2-x}{\zeta}-\frac{\varsigma_2-\varsigma_1}{2\zeta-1}\Big)-\Big(\frac{(\varsigma_2-x)^{1-\zeta}(\varsigma_2-\varsigma_1)^{\zeta-1}}{\zeta(\zeta+1)}-\frac{\varsigma_2-x}{\zeta(\varsigma_2-\varsigma_1)}\Big)^2.$$

Now we give one more Ostrowski-Grüss-type inequality of fractional type.

Theorem 7 ([6]). *Let the assumptions of this theorem be stated in Theorem 2. Then*

$$\left|\frac{1}{2}\Pi(x)-(\zeta+1)\Gamma(\zeta)\frac{(\varsigma_2-x)^{1-\zeta}}{2(\varsigma_2-\varsigma_1)}J^{\zeta}_{\varsigma_1}\Pi(\varsigma_2)+\frac{1}{2}(\varsigma_2-x)^{1-\zeta}\Gamma(\zeta)J^{\zeta-1}_{\varsigma_1}\Pi(\varsigma_2)\right.$$
$$\left.+\frac{(\varsigma_2-x)^{2-\zeta}}{2(\varsigma_2-\varsigma_1)}\Gamma(\zeta)J^{\zeta-1}_{\varsigma_1}\Pi(\varsigma_2)+\frac{(\varsigma_2-x)^{1-\zeta}(x-\varsigma_1)}{2(\varsigma_2-\varsigma_1)^{2-\zeta}}\Pi(\varsigma_1)\right|$$
$$\leq \frac{M(\varsigma_2-x)^{1-\zeta}}{\varsigma_2-\varsigma_1}\Big[\frac{(\varsigma_2-\varsigma_1)^{\zeta}(x-\varsigma_1)+(\varsigma_2-x)^{\zeta}(\varsigma_1+\varsigma_1-2x)}{2\zeta}\Big],$$

where $\varsigma_1 \leq x < \varsigma_2$.

2.2. Ostrowski-Type Fractional Integral Inequalities for Convex Functions

Definition 2 ([7])**.** *A real-valued function* Π *is convex on an interval* I*, if*

$$\Pi(\lambda\varsigma_1 + (1-\lambda)\varsigma_2) \leq \lambda\Pi(\varsigma_1) + (1-\lambda)\Pi(\varsigma_2), \tag{1}$$

holds for all $\varsigma_1, \varsigma_2 \in I$ *and* $\lambda \in [0,1]$.

In the following theorems, we show the Ostrowski-type inequalities in the frame of Riemann–Liouville fractional integrals for absolutely continuous and convex functions.

Theorem 8 ([8])**.** *Let* $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ *be an absolutely continuous function on* $[\varsigma_1, \varsigma_2]$*. If* $x \in (\varsigma_1, \varsigma_2)$ *and there exist real numbers* $m_1(x), M_1(x), m_2(x), M_2(x)$ *such that*

$$m_1(x) \leq \Pi'(t) \leq M_1(x), \quad \text{for all } \ t \in (\varsigma_1, x)$$

and

$$m_2(x) \leq \Pi'(t) \leq M_2(x), \quad \text{for all } \ t \in (x, \varsigma_2).$$

Then

$$\begin{aligned}
& \frac{1}{\Gamma(\zeta+2)}\Big[m_2(x)(\varsigma_2 - x)^{\zeta+1} - M_1(x)(x-\varsigma_1)^{\zeta+1}\Big] \\
\leq\ & \frac{1}{\Gamma(\zeta+1)}\Big[(x-\varsigma_1)^{\zeta}\Pi(\varsigma_1) + (\varsigma_2 - x)^{\zeta}\Pi(\varsigma_2)\Big] - J^{\zeta}_{\varsigma_1+}\Pi(x) - J^{\zeta}_{\varsigma_2-}\Pi(x) \\
\leq\ & \frac{1}{\Gamma(\zeta+2)}\Big[M_2(x)(\varsigma_2 - x)^{\zeta+1} - m_1(x)(x-\varsigma_1)^{\zeta+1}\Big]
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{\Gamma(\zeta+2)}\Big[m_2(x)(\varsigma_2 - x)^{\zeta+1} - M_1(x)(x-\varsigma_1)^{\zeta+1}\Big] \\
\leq\ & J^{\zeta}_{x-}\Pi(\varsigma_1) + J^{\zeta}_{x+}\Pi(\varsigma_2) - \frac{1}{\Gamma(\zeta+1)}\Big[(x-\varsigma_1)^{\zeta}\Pi(x) + (\varsigma_2 - x)^{\zeta}\Pi(x)\Big] \\
\leq\ & \frac{1}{\Gamma(\zeta+2)}\Big[M_2(x)(\varsigma_2 - x)^{\zeta+1} - m_1(x)(x-\varsigma_1)^{\zeta+1}\Big].
\end{aligned}$$

Theorem 9 ([8])**.** *Let* $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ *be a convex function and* $x \in (\varsigma_1, \varsigma_2)$*. Then*

$$\begin{aligned}
& \frac{1}{\Gamma(\zeta+2)}\Big[\Pi'_{+}(x)(\varsigma_2 - x)^{\zeta+1} - \Pi'_{-}(x)(x-\varsigma_1)^{\zeta+1}\Big] \\
\leq\ & \frac{1}{\Gamma(\zeta+1)}\Big[(x-\varsigma_1)^{\zeta}\Pi(\varsigma_1) + (\varsigma_2 - x)^{\zeta}\Pi(\varsigma_2)\Big] - J^{\zeta}_{\varsigma_1+}\Pi(x) - J^{\zeta}_{\varsigma_2-}\Pi(x) \\
\leq\ & \frac{1}{\Gamma(\zeta+2)}\Big[\Pi'(\varsigma_2)(\varsigma_2 - x)^{\zeta+1} - \Pi'(\varsigma_1)(x-\varsigma_1)^{\zeta+1}\Big]
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{\Gamma(\zeta+2)}\Big[\Pi'_{+}(x)(\varsigma_2 - x)^{\zeta+1} - \Pi'_{-}(x)(x-\varsigma_1)^{\zeta+1}\Big] \\
\leq\ & J^{\zeta}_{x-}\Pi(\varsigma_1) + J^{\zeta}_{x+}\Pi(\varsigma_2) - \frac{1}{\Gamma(\zeta+1)}\Big[(x-\varsigma_1)^{\zeta}\Pi(x) + (\varsigma_2 - x)^{\zeta}\Pi(x)\Big] \\
\leq\ & \frac{1}{\Gamma(\zeta+2)}\Big[\Pi'_{-}(\varsigma_2)(\varsigma_2 - x)^{\zeta+1} - \Pi'_{+}(\varsigma_1)(x-\varsigma_1)^{\zeta+1}\Big],
\end{aligned}$$

where $\Pi'_{\pm}(\cdot)$ *are the lateral derivatives of* Π.

Theorem 10 ([9]). *Let $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be a function which is differentiable on $(\varsigma_1, \varsigma_2)$ with $\varsigma_1 < \varsigma_2$ such that $\Pi' \in L[\varsigma_1, \varsigma_2]$. If $|\Pi'|$ is convex on $[\varsigma_1, \varsigma_2]$ and $x \in [\varsigma_1, \varsigma_2]$, then*

$$\begin{aligned}
&\left|\left[\frac{(x-\varsigma_1)^{\zeta}+(\varsigma_2-x)^{\zeta}}{(\varsigma_2-\varsigma_1)^{\zeta+1}}\right]\Pi(x)-\frac{\Gamma(\zeta+1)}{(\varsigma_2-\varsigma_1)^{\zeta+1}}\left[J^{\zeta}_{x+}\Pi(\varsigma_2)+J^{\zeta}_{x-}\Pi(\varsigma_1)\right]\right| \\
\leq\ & \frac{1}{\zeta+2}\left\{\left(\frac{(\varsigma_2-x)^{\zeta+2}}{(\varsigma_2-\varsigma_1)^{\zeta+2}}+\frac{(x-\varsigma_1)^{\zeta+2}}{(\varsigma_2-\varsigma_1)^{\zeta+2}}\left[\frac{1}{\zeta+1}+\frac{\varsigma_2-x}{\varsigma_2-\varsigma_1}\right]\right)|\Pi'(\varsigma_1)|\right. \\
&\left.+\left(\frac{(x-\varsigma_1)^{\zeta+2}}{(\varsigma_2-\varsigma_1)^{\zeta+2}}+\frac{(\varsigma_2-x)^{\zeta+2}}{(\varsigma_2-\varsigma_1)^{\zeta+2}}\left[\frac{1}{\zeta+1}+\frac{x-\varsigma_1}{\varsigma_2-\varsigma_1}\right]\right)|\Pi'(\varsigma_2)|\right\}.
\end{aligned}$$

Theorem 11 ([9]). *Let Π be as in Theorem 10. If $|\Pi'|^q, q > 1$ is convex on $[\varsigma_1, \varsigma_2]$, and $x \in [\varsigma_1, \varsigma_2]$, then*

$$\begin{aligned}
&\left|\left[\frac{(x-\varsigma_1)^{\zeta}+(\varsigma_2-x)^{\zeta}}{(\varsigma_2-\varsigma_1)^{\zeta+1}}\right]\Pi(x)-\frac{\Gamma(\zeta+1)}{(\varsigma_2-\varsigma_1)^{\zeta+1}}\left[J^{\zeta}_{x+}\Pi(\varsigma_2)+J^{\zeta}_{x-}\Pi(\varsigma_1)\right]\right| \\
\leq\ & \frac{1}{(\varsigma_2-\varsigma_1)^{\zeta+1}(\zeta p+1)^{\frac{1}{p}}}\left[(\varsigma_2-x)^{\zeta+1}\left(\frac{|\Pi'(\varsigma_1)|^q+|\Pi'(\varsigma_2)|^q}{2}\right)^{\frac{1}{q}}\right. \\
&\left.+(x-\varsigma_1)^{\zeta+1}\left(\frac{|\Pi'(\varsigma_1)|^q+|\Pi'(\varsigma_2)|^q}{2}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $\zeta > 0$.

Theorem 12 ([9]). *Let Π be as in Theorem 10. If $|\Pi'|^q, q \geq 1$ is convex on $[\varsigma_1, \varsigma_2]$, and $x \in [\varsigma_1, \varsigma_2]$, then*

$$\begin{aligned}
&\left|\left[\frac{(x-\varsigma_1)^{\zeta}+(\varsigma_2-x)^{\zeta}}{(\varsigma_2-\varsigma_1)^{\zeta+1}}\right]\Pi(x)-\frac{\Gamma(\zeta+1)}{(\varsigma_2-\varsigma_1)^{\zeta+1}}\left[J^{\zeta}_{x+}\Pi(\varsigma_2)+J^{\zeta}_{x-}\Pi(\varsigma_1)\right]\right| \\
\leq\ & \left(\frac{1}{\zeta+1}\right)^{1-\frac{1}{q}}\left(\frac{1}{\zeta+2}\right)^{\frac{1}{q}} \\
&\times\left\{\left(\frac{\varsigma_2-x}{\varsigma_2-\varsigma_1}\right)^{\zeta+1}\left[\left(\frac{\varsigma_2-x}{\varsigma_2-\varsigma_1}\right)|\Pi'(\varsigma_1)|^q+\left(\frac{1}{\zeta+1}+\frac{x-\varsigma_1}{\varsigma_2-\varsigma_1}\right)|\Pi'(\varsigma_2)|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{x-\varsigma_1}{\varsigma_2-\varsigma_1}\right)^{\zeta+1}\left[\left(\frac{1}{\zeta+1}+\frac{\varsigma_2-x}{\varsigma_2-\varsigma_1}\right)|\Pi'(\varsigma_1)|^q+\left(\frac{x-\varsigma_1}{\varsigma_2-\varsigma_1}\right)|\Pi'(\varsigma_2)|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Theorem 13 ([9]). *Let Π be as in Theorem 10. If $|\Pi'|^q, q \geq 1$ is convex on $[\varsigma_1, \varsigma_2]$, and $x \in [\varsigma_1, \varsigma_2]$, then*

$$\begin{aligned}
&\left|\left[\frac{(x-\varsigma_1)^{\zeta}+(\varsigma_2-x)^{\zeta}}{(\varsigma_2-\varsigma_1)^{\zeta+1}}\right]\Pi(x)-\frac{\Gamma(\zeta+1)}{(\varsigma_2-\varsigma_1)^{\zeta+1}}\left[J^{\zeta}_{x+}\Pi(\varsigma_2)+J^{\zeta}_{x-}\Pi(\varsigma_1)\right]\right| \\
\leq\ & \frac{1}{(\varsigma_2-\varsigma_1)^{\zeta+1}(\zeta p+1)^{\frac{1}{p}}}\left[(\varsigma_2-x)^{\zeta+1}\left|\Pi'\left(\frac{\varsigma_2+x}{2}\right)\right|+(x-\varsigma_1)^{\zeta+1}\left|\Pi'\left(\frac{\varsigma_1+x}{2}\right)\right|\right].
\end{aligned}$$

Now we give some weighted fractional Ostrowski-type integral inequalities.

Theorem 14 ([10])**.** *Let $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be a function which is differentiable on $(\varsigma_1, \varsigma_2)$ with $\varsigma_1 < \varsigma_2$ and $\Pi' \in L[\varsigma_1, \varsigma_2]$. If $g : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ is continuous and $|\Pi'|$ is convex on $[\varsigma_1, \varsigma_2]$, then*

$$\begin{aligned}
&\left|\left[J^{\zeta}_{\varsigma_1+}g(x) + J^{\zeta}_{\varsigma_2-}g(x)\right]\Pi(x) - \left[J^{\zeta}_{\varsigma_1+}(\Pi g)(x) + J^{\zeta}_{\varsigma_2-}(\Pi g)(x)\right]\right| \\
\leq\ & \frac{\|g\|_\infty}{(\varsigma_2 - \varsigma_1)\Gamma(\zeta+1)}\Big[(x-\varsigma_1)^{\zeta+1}\Big(\Big(\varsigma_2 - \frac{\varsigma_1 + x}{2}\Big) - \frac{(\zeta+2)(\varsigma_2 - x) + (\zeta+1)(x-\varsigma_1)}{(\zeta+1)(\zeta+2)}\Big) \\
&+(\varsigma_2 - x)^{\zeta+2}\Big(\frac{1}{2} - \frac{1}{(\zeta+1)(\zeta+2)}\Big)\Big]|\Pi'(\varsigma_1)| \\
&+\Big[(\varsigma_2 - x)^{\zeta+1}\Big(\Big(\frac{\varsigma_2 + x}{2} - \varsigma_1\Big) - \frac{(\zeta+1)(\varsigma_2 - x) + (\zeta+2)(x-\varsigma_1)}{(\zeta+1)(\zeta+2)}\Big) \\
&+(x-\varsigma_1)^{\zeta+2}\Big(\frac{1}{2} - \frac{1}{(\zeta+1)(\zeta+2)}\Big)\Big]|\Pi'(\varsigma_2)|,
\end{aligned}$$

for all $x \in [\varsigma_1, \varsigma_2]$, where $\|g\|_\infty = \sup\{|g(x)| : x \in [\varsigma_1, \varsigma_2]\}$.

Theorem 15 ([10])**.** *Let Π and g be as in Theorem 14. If $|\Pi'|^q, q > 1$ is convex on $[\varsigma_1, \varsigma_2]$, then*

$$\begin{aligned}
&\left|\left[J^{\zeta}_{\varsigma_1+}g(x) + J^{\zeta}_{\varsigma_2-}g(x)\right]\Pi(x) - \left[J^{\zeta}_{\varsigma_1+}(\Pi g)(x) + J^{\zeta}_{\varsigma_2-}(\Pi g)(x)\right]\right| \\
\leq\ & \frac{\|g\|_\infty}{(\varsigma_2 - \varsigma_1)^{\frac{1}{q}}\Gamma(\zeta+1)}\Big(1 - \frac{1}{p\zeta+1}\Big)^{\frac{1}{p}}\Big[(x-\varsigma_1)^{\zeta+1}\Big(\Big(\varsigma_2 - \frac{\varsigma_1 + x}{2}\Big)|\Pi'(\varsigma_1)|^q \\
&+\frac{x-\varsigma_1}{2}|\Pi'(\varsigma_2)|^q\Big)^{\frac{1}{q}} + (\varsigma_2 - x)^{\zeta+1}\Big(\frac{\varsigma_2 - x}{2}|\Pi'(\varsigma_1)|^q + \Big(\frac{x+\varsigma_2}{2} - \varsigma_1\Big)|\Pi'(\varsigma_2)|^q\Big)^{\frac{1}{q}}\Big],
\end{aligned}$$

for all $x \in [\varsigma_1, \varsigma_2]$, where $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

2.3. Ostrowski-Type Fractional Integral Inequalities for Quasi-Convex Functions

Definition 3 ([11])**.** *A real-valued Π is quasi-convex, if*

$$\Pi(\lambda x + (1-\lambda)y) \leq \max\{\Pi(x), \Pi(y)\}, \tag{2}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

In the following theorems, we explore some weighted Ostrowski-type inequalities in the frame of fractional operator for quasi-convex functions.

Theorem 16 ([12])**.** *Let $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be a function which is differentiable on $(\varsigma_1, \varsigma_2)$ where $0 \leq \varsigma_1 < \varsigma_2$ and $g : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be a continuous function. If $|\Pi'|$ is quasi-convex, then*

$$\begin{aligned}
&\left|J^{\zeta}_{x-}(\Pi g)(\varsigma_1) + J^{\zeta}_{x+}(\Pi g)(\varsigma_2) - \left[J^{\zeta}_{x-}\Pi(\varsigma_1) + J^{\zeta}_{x+}\Pi(\varsigma_2)\right]\right| \\
\leq\ & \frac{(\varsigma_2 - x)^{\zeta+1}}{\Gamma(\zeta+2)}\max\{|\Pi'(x)|, |\Pi'(\varsigma_2)|\}\|g\|_{[x,\varsigma_2],\infty} \\
&+\frac{(x-\varsigma_1)^{\zeta+1}}{\Gamma(\zeta+2)}\max\{|\Pi'(x)|, |\Pi'(\varsigma_1)|\}\|g\|_{[\varsigma_1,x],\infty},
\end{aligned}$$

for all $x \in [\varsigma_1, \varsigma_2]$.

Theorem 17 ([12])**.** *Let* Π *be as in Theorem 16. If* $|\Pi'|^q$ *is quasi-convex,* $q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*, then*

$$\begin{aligned}
&\left| J^{\zeta}_{x-}g\Pi(\varsigma_1) + J^{\zeta}_{x+}g\Pi(\varsigma_2) - \left[J^{\zeta}_{x-}\Pi(\varsigma_1) + J^{\zeta}_{x+}\Pi(\varsigma_2) \right] \right| \\
\leq\ & \frac{(\varsigma_2 - x)^{\zeta+1}}{(\zeta p+1)^{\frac{1}{p}}\Gamma(\zeta+1)} \Big(\max\{|\Pi'(x)|^q, |\Pi'(\varsigma_2)|^q\} \Big)^{\frac{1}{q}} \|g\|_{[x,\varsigma_2],\infty} \\
&+ \frac{(x - \varsigma_1)^{\zeta+1}}{(\zeta p+1)^{\frac{1}{p}}\Gamma(\zeta+2)} \Big(\max\{|\Pi'(x)|^q, |\Pi'(\varsigma_1)|^q\} \Big)^{\frac{1}{q}} \|g\|_{[\varsigma_1,x],\infty},
\end{aligned}$$

for all $x \in [\varsigma_1, \varsigma_2]$*.*

Theorem 18 ([12])**.** *Let* Π *be as in Theorem 16. If* $|\Pi'|^q$ *is quasi-convex,* $q \geq 1$ *then*

$$\begin{aligned}
&\left| J^{\zeta}_{x-}g\Pi(\varsigma_1) + J^{\zeta}_{x+}g\Pi(\varsigma_2) - \left[J^{\zeta}_{x-}\Pi(\varsigma_1) + J^{\zeta}_{x+}\Pi(\varsigma_2) \right] \right| \\
\leq\ & \frac{(\varsigma_2 - x)^{\zeta+1}}{\Gamma(\zeta+1)} \Big(\max\{|\Pi'(x)|^q, |\Pi'(\varsigma_2)|^q\} \Big)^{\frac{1}{q}} \|g\|_{[x,\varsigma_2],\infty} \\
&+ \frac{(x - \varsigma_1)^{\zeta+1}}{\Gamma(\zeta+2)} \Big(\max\{|\Pi'(x)|^q, |\Pi'(\varsigma_1)|^q\} \Big)^{\frac{1}{q}} \|g\|_{[\varsigma_1,x],\infty},
\end{aligned}$$

for all $x \in [\varsigma_1, \varsigma_2]$*.*

A further result for functions with a bounded first derivative is given in the next theorem.

Theorem 19 ([12])**.** *Let the assumptions of this theorem be as stated in Theorem 16. If there exist constants* $m < M$ *such that* $-\infty < m \leq \Pi'(x) \leq M < +\infty$ *for all* $x \in [\varsigma_1, \varsigma_2]$*, then*

$$\begin{aligned}
&\left| J^{\zeta}_{x-}g\Pi(\varsigma_1) + J^{\zeta}_{x+}g\Pi(\varsigma_2) - \left[J^{\zeta}_{x-}\Pi(\varsigma_1) + J^{\zeta}_{x+}\Pi(\varsigma_2) \right] \right. \\
&\left. - \frac{(M+m)\left[(\varsigma_2 - x)^{\zeta+1} - (x-\varsigma_1)^{\zeta+1} \right]}{2\Gamma(\zeta)} \int_0^1 (p_1(\tau) + p_2(\tau)) d\tau \right| \\
\leq\ & \frac{(M-m)(\varsigma_2 - x)^{\zeta+1}}{2\Gamma(\zeta+2)} \|g\|_{[x,\varsigma_2],\infty} + \frac{(M-m)(x-\varsigma_1)^{\zeta+1}}{2\Gamma(\zeta+2)} \|g\|_{[\varsigma_1,x],\infty},
\end{aligned}$$

where

$$\begin{aligned}
p_1(\tau) &= \int_\tau^1 (1-\sigma)^{\zeta-1} g(\sigma\varsigma_2 + (1-\sigma)x) d\sigma, \\
p_2(\tau) &= \int_\tau^1 (1-\sigma)^{\zeta-1} g(\sigma\varsigma_1 + (1-\sigma)x) d\sigma.
\end{aligned}$$

Definition 4 ([13])**.** *A function* $\Pi : \mathbb{I} \to \mathbb{R}$ *is said to be a strongly quasi-convex function with modulus* $c \geq 0$*, if*

$$\Pi(tx + (1-t)y) \leq \max\{\Pi(x), \Pi(y)\} - ct(1-t)(y-x)^2, \quad \forall x, y \in \mathbb{I}, \ \ t \in [0,1].$$

The aim of this subsection is to give some Ostrowski-type fractional integral inequalities for strongly quasi-convex functions.

Theorem 20 ([14])**.** *Let* $\Pi : [\varsigma_1, \varsigma_2] \subset [0, \infty) \to \mathbb{R}$ *be a differentiable mapping on* $(\varsigma_1, \varsigma_2)$ *such that* $\Pi' \in L[\varsigma_1, \varsigma_2]$*. If* $|\Pi'|$ *is a strongly quasi-convex function with modulus* $c \geq 0$*, on* $[\varsigma_1, \varsigma_2]$*, then*

$$
\begin{aligned}
&\left|\Pi(x)-\frac{1}{\varsigma_2-\varsigma_1}\int_{\varsigma_1}^{\varsigma_2}\Pi(s)ds\right|\\
\le\ & \frac{(\varsigma_2-x)^2}{2(\varsigma_2-\varsigma_1)}\max\{|\Pi'(x)|,|\Pi'(\varsigma_2)|\}-c\Big(\frac{(\varsigma_2-x)^3}{3(\varsigma_2-\varsigma_1)^2}-\frac{(\varsigma_2-x)^4}{4(\varsigma_2-\varsigma_1)^3}\Big)(x-\varsigma_2)^2\\
&+\frac{(x-\varsigma_1)^2}{2(\varsigma_2-\varsigma_1)}\max\{|\Pi'(x)|,|\Pi'(\varsigma_1)|\}c(x-\varsigma_1)^2\Big(\frac{1}{12}-\frac{(\varsigma_2-x)^2}{2(\varsigma_2-\varsigma_1)}+\frac{2(\varsigma_2-x)^3}{3(\varsigma_2-\varsigma_1)^2}-\frac{(\varsigma_2-x)^4}{4(\varsigma_2-\varsigma_1)^3}\Big),
\end{aligned}
$$

for each $x\in[\varsigma_1,\varsigma_2]$.

Theorem 21 ([14]). *Let* Π *be as in Theorem 20. If* $|\Pi'|^q$ *is a strongly quasi-convex function with modulus* $c\ge 0$, *on* $[\varsigma_1,\varsigma_2]$, *then*

$$
\begin{aligned}
&\left|\Pi(x)-\frac{1}{\varsigma_2-\varsigma_1}\int_{\varsigma_1}^{\varsigma_2}\Pi(s)ds\right|\\
\le\ & \frac{(\varsigma_2-x)^{p+1}}{(\varsigma_2-\varsigma_1)(p+1)}\Big(\max\{|\Pi'(x)|^q,|\Pi'(\varsigma_2)|^q\}-c\Big(\frac{(\varsigma_2-x)^2}{2(\varsigma_2-\varsigma_1)^2}-\frac{(\varsigma_2-x)^3}{3(\varsigma_2-\varsigma_1)^3}\Big)(x-\varsigma_2)^2\Big)^{\frac{1}{q}}\\
&+\frac{(x-\varsigma_1)^{p+1}}{(\varsigma_2-\varsigma_1)(p+1)}\Big(\max\{|\Pi'(x)|^q,|\Pi'(\varsigma_1)|^q\}-c\Big(\frac{1}{6}-\frac{(\varsigma_2-x)^2}{2(\varsigma_2-\varsigma_1)^2}+\frac{(\varsigma_2-x)^3}{3(\varsigma_2-\varsigma_1)^3}\Big)(x-\varsigma_1)^2\Big)^{\frac{1}{q}},
\end{aligned}
$$

for each $x\in[\varsigma_1,\varsigma_2]$, $q>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 22 ([14]). *Let* Π *be as in Theorem 20. If* $|\Pi'|^q$ *is a strongly quasi-convex function with modulus* $c\ge 0$, *on* $[\varsigma_1,\varsigma_2]$, *then*

$$
\begin{aligned}
&\left|\Pi(x)-\frac{1}{\varsigma_2-\varsigma_1}\int_{\varsigma_1}^{\varsigma_2}\Pi(s)ds\right|\\
\le\ & \frac{(\varsigma_2-x)^2}{2(\varsigma_2-\varsigma_1)}\Big(\max\{|\Pi'(x)|^q,|\Pi'(\varsigma_2)|^q\}-c\Big(\frac{2(\varsigma_2-x)}{3(\varsigma_2-\varsigma_1)}-\frac{(\varsigma_2-x)^2}{2(\varsigma_2-\varsigma_1)^2}\Big)(x-\varsigma_2)^2\Big)^{\frac{1}{q}}\\
&+\frac{(x-\varsigma_1)^2}{2(\varsigma_2-\varsigma_1)}\Big(\max\{|\Pi'(x)|^q,|\Pi'(\varsigma_1)|^q\}-c\Big(\frac{(\varsigma_2-x)^2}{6(x-\varsigma_1)^2}-\frac{(\varsigma_2-x)^2}{(x-\varsigma_1)^2}+\frac{4(\varsigma_2-x)^3}{3(\varsigma_2-\varsigma_1)(x-\varsigma_1)^2}\\
&-\frac{(\varsigma_2-x)^4}{2(\varsigma_2-\varsigma_1)^2(x-\varsigma_1)^2}\Big)(x-\varsigma_1)^2\Big)^{\frac{1}{q}},
\end{aligned}
$$

for each $x\in[\varsigma_1,\varsigma_2]$.

In the next, we present fractional weighted Ostrowski-type fractional integral inequalities via a strongly quasi-convex function.

Theorem 23 ([14]). *Let* Π *be as in Theorem 20 and* $g:[\varsigma_1,\varsigma_2]\to\mathbb{R}$ *be a continuous function. If* $|\Pi'|$ *is a strongly quasi-convex function with modulus* $c\ge 0$, *on* $[\varsigma_1,\varsigma_2]$, *then*

$$
\begin{aligned}
&\left|J_{x-}^{\zeta}g\Pi(\varsigma_1)+J_{x+}^{\zeta}g\Pi(\varsigma_2)-\Big[J_{x-}^{\zeta}g\Pi(\varsigma_1)+J_{x+}^{\zeta}g\Pi(\varsigma_2)\Big]\Pi(\varsigma_1)\right|\\
\le\ & \frac{(\varsigma_2-x)^{\zeta+1}}{\Gamma(\zeta+1)}\Big(\max\{|\Pi'(x)|,|\Pi'(\varsigma_2)|\}\|g\|_{[x,\varsigma_2],\infty}-\Big(\frac{(\varsigma_2-x)^{\zeta+1}}{\Gamma(\zeta+3)}-\frac{(\varsigma_2-x)^{\zeta+1}}{\Gamma(\zeta+4)}\Big)c(x-\varsigma_2)^2\Big)\|g\|_{[x,\varsigma_2],\infty}\\
&+\frac{(x-\varsigma_1)^{\zeta+1}}{\Gamma(\zeta+2)}\max\{|\Pi'(x)|,|\Pi'(\varsigma_1)|\}\|g\|_{[\varsigma_1,x],\infty}-\Big(\frac{(x-\varsigma_1)^{\zeta+1}}{\Gamma(\zeta+3)}-\frac{(x-\varsigma_1)^{\zeta+1}}{\Gamma(\zeta+4)}c(x-\varsigma_1)^2\Big)\|g\|_{[x,\varsigma_2],\infty},
\end{aligned}
$$

for each $x\in[\varsigma_1,\varsigma_2]$.

Theorem 24 ([14]). *Let Π be as in Theorem 20 and $g : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be a continuous function. If $|\Pi'|^q$ is a strongly quasi-convex function with modulus $c \geq 0$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\begin{aligned} &\left| J^{\zeta}_{x-} g\Pi(\varsigma_1) + J^{\zeta}_{x+} g\Pi(\varsigma_2) - \left[J^{\zeta}_{x-} g\Pi(\varsigma_1) + J^{\zeta}_{x+} g\Pi(\varsigma_2) \right] \Pi(\varsigma_1) \right| \\ \leq\ & \frac{(\varsigma_2 - x)^{\zeta+1}}{(\zeta p + 1)^{\frac{1}{p}} \Gamma(\zeta+1)} \|g\|_{[x,\varsigma_2],\infty} \left(\max\{|\Pi'(x)|^q, |\Pi'(\varsigma_2)|^q\} - \frac{c}{6}(x-\varsigma_2)^2 \right)^{\frac{1}{q}} \\ &+ \frac{(x - \varsigma_1)^{\zeta+1}}{(\zeta p + 1)^{\frac{1}{p}} \Gamma(\zeta+1)} \|g\|_{[\varsigma_1,x],\infty} \left(\max\{|\Pi'(x)|^q, |\Pi'(\varsigma_1)|^q\} - \frac{c}{6}(x-\varsigma_1)^2 \right)^{\frac{1}{q}}, \end{aligned}$$

for each $x \in [\varsigma_1, \varsigma_2]$.

2.4. Ostrowski-Type Fractional Integral Inequalities for (ζ, m)-Convex Functions

Definition 5 ([15]). *The function $\Pi : [0, b] \to \mathbb{R}$, $b > 0$ is said to be (ζ, m)-convex, if*

$$\Pi(tx + (1-t)y) \leq t^{\zeta}\Pi(x) + m(1 - t^{\zeta})\Pi(y),$$

for all $x, y \in [0, b]$, $(\zeta, m) \in [0,1]^2$ and $t \in [0,1]$.

Ostrowski-type fractional integral inequalities pertaining to Riemann–Liouville fractional integral for (ζ, m)-convex functions are presented in the following theorems.

Theorem 25 ([16]). *Let I be an open real interval such that $[0, \infty) \subset I$ and $\Pi : I \to \mathbb{R}$ be a differentiable mapping on I such that $\Pi' \in L[m\varsigma_1, m\varsigma_2]$, where $m\varsigma_1, m\varsigma_2 \in I$ with $\varsigma_1 < \varsigma_2$, $m \in (0,1]$. If $|\Pi'|$ is (ζ, m)-convex on $[m\varsigma_1, m\varsigma_2]$ for $(\zeta, m) \in [0,1]^2$ and $|\Pi'(x)| \leq M$, then*

$$\begin{aligned} &\left| \left(\frac{(x - m\varsigma_1)^{\zeta} + (m\varsigma_2 - x)^{\zeta}}{\varsigma_2 - \varsigma_1} \right) \Pi(x) - \frac{\Gamma(\zeta+1)}{\varsigma_2 - \varsigma_1} \left[I^{\zeta}_{x-}\Pi(m\varsigma_1) + I^{\zeta}_{x-}\Pi(m\varsigma_1) \right] \right. \\ \leq\ & M \left[\frac{(x - m\varsigma_1)^{\zeta+1} + (m\varsigma_2 - x)^{\zeta+1}}{\varsigma_2 - \varsigma_1} \right] \left[\frac{1 + m\zeta}{1 + 2\zeta} \right], \end{aligned}$$

for all $x \in [m\varsigma_1, m\varsigma_2]$.

Theorem 26 ([16]). *Let Π be as in Theorem 25. If $|\Pi'|^q, q > 1$ is (ζ, m)-convex on $[m\varsigma_1, m\varsigma_2]$ for $(\zeta, m) \in [0,1]^2$ and $|\Pi'(x)| \leq M$, then*

$$\begin{aligned} &\left| \left(\frac{(x - m\varsigma_1)^{\zeta} + (m\varsigma_2 - x)^{\zeta}}{\varsigma_2 - \varsigma_1} \right) \Pi(x) - \frac{\Gamma(\zeta+1)}{\varsigma_2 - \varsigma_1} \left[I^{\zeta}_{x-}\Pi(m\varsigma_1) + I^{\zeta}_{x-}\Pi(m\varsigma_1) \right] \right. \\ \leq\ & M \left(\frac{1}{\zeta p + 1} \right)^{\frac{1}{p}} \left[\frac{(x - m\varsigma_1)^{\zeta+1} + (m\varsigma_2 - x)^{\zeta+1}}{\varsigma_2 - \varsigma_1} \right] \left[\frac{1 + m\zeta}{1 + \zeta} \right]^{\frac{1}{q}}, \end{aligned}$$

with $\frac{1}{p} + \frac{1}{q} = 1$ and $x \in [m\varsigma_1, m\varsigma_2]$.

Theorem 27 ([16]). *Let Π be as in Theorem 25. If $|\Pi'|^q, q \geq 1$ is (ζ, m)-convex on $[m\varsigma_1, m\varsigma_2]$ for $(\zeta, m) \in [0,1]^2$ and $|\Pi'(x)| \leq M$, then*

$$\begin{aligned} &\left| \left(\frac{(x - m\varsigma_1)^{\zeta} + (m\varsigma_2 - x)^{\zeta}}{\varsigma_2 - \varsigma_1} \right) \Pi(x) - \frac{\Gamma(\zeta+1)}{\varsigma_2 - \varsigma_1} \left[I^{\zeta}_{x-}\Pi(m\varsigma_1) + I^{\zeta}_{x-}\Pi(m\varsigma_1) \right] \right. \\ \leq\ & M \left[\frac{(x - m\varsigma_1)^{\zeta+1} + (m\varsigma_2 - x)^{\zeta+1}}{(\varsigma_2 - \varsigma_1)(\zeta+1)} \right] \left[\frac{1 + \zeta(m+1)}{2\zeta + 1} \right]^{\frac{1}{q}}, \end{aligned}$$

for all $x \in [m\varsigma_1, m\varsigma_2]$.

2.5. Ostrowski-Type Fractional Integral Inequalities for s-Convex Functions

Definition 6 ([17]). *A function* $\Pi : [0,\infty) \to \mathbb{R}$ *is said to be s-convex in the second sense, if*

$$\Pi(\lambda\varsigma_1 + (1-\lambda)\varsigma_2) \leq \lambda^s\Pi(\varsigma_1) + (1-\lambda)^s\Pi(\varsigma_2), \tag{3}$$

holds for all $\varsigma_1, \varsigma_2 \in [0,\infty)$, $\lambda \in [0,1]$ *and for some fixed* $s \in (0,1]$.

Ostrowski-type inequalities pertaining to Riemann–Liouville fractional integral for s-convex functions are presented.

Theorem 28 ([18]). *Let* $\Pi : [\varsigma_1, \varsigma_2] \subset [0,\infty) \to \mathbb{R}$ *be a function which is differentiable on* $(\varsigma_1, \varsigma_2)$ *with* $\varsigma_1 < \varsigma_2$ *such that* $\Pi' \in L[\varsigma_1, \varsigma_2]$. *Suppose* $|\Pi'|$ *is s-convex in the second sense on* $[\varsigma_1, \varsigma_2]$ *for* $s \in (0,1]$ *and* $|\Pi'(x)| \leq M,\ x \in [\varsigma_1, \varsigma_2]$. *Then, for all* $x \in [\varsigma_1, \varsigma_2]$,

$$\begin{aligned} &\left|\left(\frac{(x-\varsigma_1)^\zeta + (\varsigma_2 - x)^\zeta}{\varsigma_2 - \varsigma_1}\right)\Pi(x) - \frac{\Gamma(\zeta+1)}{\varsigma_2 - \varsigma_1}\left[J^\zeta_{x-}\Pi(\varsigma_1) + J^\zeta_{x+}\Pi(\varsigma_2)\right]\right| \\ \leq\ & \frac{M}{\varsigma_2 - \varsigma_1}\left(1 + \frac{\Gamma(\zeta+1)\Gamma(s+1)}{\Gamma(\zeta+s+1)}\right)\left[\frac{(x-\varsigma_1)^{\zeta+1} + (\varsigma_2 - x)^{\zeta+1}}{\zeta+s+1}\right]. \end{aligned}$$

Theorem 29 ([18]). *Let* Π *be as in Theorem 28. If* $|\Pi'|^q, q > 1$ *is s-convex in the second sense on* $[\varsigma_1, \varsigma_2]$ *for* $s \in (0,1]$ *and* $|\Pi'(x)| \leq M,\ x \in [\varsigma_1, \varsigma_2]$, *then*

$$\begin{aligned} &\left|\left(\frac{(x-\varsigma_1)^\zeta + (\varsigma_2 - x)^\zeta}{\varsigma_2 - \varsigma_1}\right)\Pi(x) - \frac{\Gamma(\zeta+1)}{\varsigma_2 - \varsigma_1}\left[J^\zeta_{x-}\Pi(\varsigma_1) + J^\zeta_{x+}\Pi(\varsigma_2)\right]\right| \\ \leq\ & \frac{M}{(1+\zeta p)^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left[\frac{(x-\varsigma_1)^{\zeta+1} + (\varsigma_2 - x)^{\zeta+1}}{\varsigma_2 - \varsigma_1}\right],\ for all x \in [\varsigma_1, \varsigma_2], \end{aligned}$$

where $\zeta > 0$ *and* $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Theorem 30 ([18]). *Let* Π *be as in Theorem 28. If* $|\Pi'|^q, q \geq 1$ *is s-convex in the second sense on* $[\varsigma_1, \varsigma_2]$ *for* $s \in (0,1]$ *and* $|\Pi'(x)| \leq M,\ x \in [\varsigma_1, \varsigma_2]$, *then*

$$\begin{aligned} &\left|\left(\frac{(x-\varsigma_1)^\zeta + (\varsigma_2 - x)^\zeta}{\varsigma_2 - \varsigma_1}\right)\Pi(x) - \frac{\Gamma(\zeta+1)}{\varsigma_2 - \varsigma_1}\left[J^\zeta_{x-}\Pi(\varsigma_1) + J^\zeta_{x+}\Pi(\varsigma_2)\right]\right| \\ \leq\ & M\left(\frac{1}{1+\zeta}\right)^{1-\frac{1}{q}}\left(\frac{1}{\zeta+s+1}\right)^{\frac{1}{q}}\left(1 + \frac{\Gamma(\zeta+1)\Gamma(s+1)}{\Gamma(\zeta+s+1)}\right)\left[\frac{(x-\varsigma_1)^{\zeta+1} + (\varsigma_2 - x)^{\zeta+1}}{\varsigma_2 - \varsigma_1}\right], \end{aligned}$$

for all $x \in [\varsigma_1, \varsigma_2]$, *with* $\zeta > 0$.

Theorem 31 ([19]). *Let* Π *be as in Theorem 28. If* $|\Pi'(x)| \leq M,\ x \in [\varsigma_1, \varsigma_2]$, *then*

$$\begin{aligned} &\left|\left(\frac{(x-\varsigma_1)^\zeta + (\varsigma_2 - x)^\zeta}{(\varsigma_2 - \varsigma_1)^{\zeta+1}}\right)\Pi(x) - \frac{\Gamma(\zeta+1)}{(\varsigma_2 - \varsigma_1)^{\zeta+1}}\left[J^\zeta_{x-}\Pi(\varsigma_1) + J^\zeta_{x+}\Pi(\varsigma_2)\right]\right| \\ \leq\ & \left[\frac{1}{\zeta+s+1}\left(\frac{\varsigma_2 - x}{\varsigma_2 - \varsigma_1}\right)^{\zeta+s+1} + B\left(\frac{\varsigma_2 - x}{\varsigma_2 - \varsigma_1}; \zeta+1, s+1\right)\right]|\Pi'(\varsigma_1)| \\ & + \left[\frac{1}{\zeta+s+1}\left(\frac{x - \varsigma_1}{\varsigma_2 - \varsigma_1}\right)^{\zeta+s+1} + B\left(\frac{\varsigma_2 - x}{\varsigma_2 - \varsigma_1}; \zeta+1, s+1\right)\right]|\Pi'(\varsigma_2)|, \end{aligned}$$

for all $x \in (\varsigma_1, \varsigma_2)$.

Theorem 32 ([19]). *Let Π be as in Theorem 28. If $|\Pi'|^q, q > 1, \frac{1}{p} + \frac{1}{q} = 1$ is s-convex in the second sense on $[\varsigma_1, \varsigma_2]$ for $s \in (0,1]$ and $|\Pi'(x)| \le M,\ x \in [\varsigma_1, \varsigma_2]$, then*

$$\begin{aligned}
&\left|\left(\frac{(x-\varsigma_1)^\zeta + (\varsigma_2 - x)^\zeta}{(\varsigma_2-\varsigma_1)^{\zeta+1}}\right)\Pi(x) - \frac{\Gamma(\zeta+1)}{(\varsigma_2-\varsigma_1)^{\zeta+1}}\left[J^\zeta_{x-}\Pi(\varsigma_1) + J^\zeta_{x+}\Pi(\varsigma_2)\right]\right| \\
\le\ & \left(\frac{1}{\zeta p+1}\right)^{\frac{1}{p}}\left(\frac{\varsigma_2-x}{\varsigma_2-\varsigma_1}\right)^{\zeta+\frac{1}{p}}\left[\frac{1}{s+1}\left\{\left(\frac{\varsigma_2-x}{\varsigma_2-\varsigma_1}\right)^{s+1}|\Pi'(\varsigma_1)|^q + \left[1-\left(\frac{x-\varsigma_1}{\varsigma_2-\varsigma_1}\right)^{s+1}\right]|\Pi'(\varsigma_2)|^q\right\}\right]^{\frac{1}{q}} \\
&+\left(\frac{1}{\zeta p+1}\right)^{\frac{1}{p}}\left(\frac{x-\varsigma_1}{\varsigma_2-\varsigma_1}\right)^{\zeta+\frac{1}{p}}\left[\frac{1}{s+1}\left\{\left[1-\left(\frac{\varsigma_2-x}{\varsigma_2-\varsigma_1}\right)^{s+1}\right]|\Pi'(\varsigma_1)|^q + \left(\frac{x-\varsigma_1}{\varsigma_2-\varsigma_1}\right)^{s+1}|\Pi'(\varsigma_2)|^q\right\}\right]^{\frac{1}{q}},
\end{aligned}$$

for all $x \in (\varsigma_1, \varsigma_2)$.

Theorem 33 ([19]). *Let Π be as in Theorem 28. If $|\Pi'|^q, q > 1, \frac{1}{p} + \frac{1}{q} = 1$ is s-convex in the second sense on $[\varsigma_1, \varsigma_2]$ for $s \in (0,1]$ and $|\Pi'(x)| \le M,\ x \in [\varsigma_1, \varsigma_2]$, then*

$$\begin{aligned}
&\left|\left(\frac{(x-\varsigma_1)^\zeta + (\varsigma_2 - x)^\zeta}{(\varsigma_2-\varsigma_1)^{\zeta+1}}\right)\Pi(x) - \frac{\Gamma(\zeta+1)}{(\varsigma_2-\varsigma_1)^{\zeta+1}}\left[J^\zeta_{x-}\Pi(\varsigma_1) + J^\zeta_{x+}\Pi(\varsigma_2)\right]\right| \\
\le\ & \left(\frac{1}{\zeta p+1}\right)^{\frac{1}{p}}\left\{\left(\frac{\varsigma_2-x}{\varsigma_2-\varsigma_1}\right)^{\zeta+1}\left(\frac{|\Pi'(x)|^q + |\Pi'(\varsigma_2)|^q}{2}\right)^{\frac{1}{q}} + \left(\frac{x-\varsigma_1}{\varsigma_2-\varsigma_1}\right)^{\zeta+1}\left(\frac{|\Pi'(x)|^q + |\Pi'(\varsigma_1)|^q}{2}\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

for all $x \in (\varsigma_1, \varsigma_2)$.

2.6. Ostrowski-Type Fractional Integral Inequalities for (s,r)-Convex Functions

Definition 7 ([20]). *A function $\Pi : I \subset [0,\infty) \to [0,\infty)$ is said to be (s,r)-convex in mixed kind, if*

$$\Pi(\lambda x + (1-\lambda)y) \le \lambda^{rs}\Pi(x) + (1-\lambda^r)^s\Pi(y), \tag{4}$$

holds for all $x, y \in I$, $\lambda \in [0,1]$ and $(s,r) \in [0,1]^2$.

Now, we state the generalization of the classical Ostrowski inequality via fractional integrals, which is obtained for (s,r)-convex function in mixed kind.

Theorem 34 ([20]). *Let $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be a function which is differentiable on $(\varsigma_1, \varsigma_2)$ with $\varsigma_1 < \varsigma_2$ and $\Pi \in L[\varsigma_1, \varsigma_2]$. If $|\Pi'|$ is (s,r)-convex on $[\varsigma_1, \varsigma_2]$ and $|\Pi'(x)| \le M,\ x \in [\varsigma_1, \varsigma_2]$, then*

$$\begin{aligned}
&\left|\left(\frac{(x-\varsigma_1)^\zeta + (\varsigma_2 - x)^\zeta}{\varsigma_2-\varsigma_1}\right)\Pi(x) - \frac{\Gamma(\zeta+1)}{\varsigma_2-\varsigma_1}\left[J^\zeta_{x-}\Pi(\varsigma_1) + J^\zeta_{x+}\Pi(\varsigma_2)\right]\right| \\
\le\ & M\left(\frac{1}{\zeta+rs+1} + \frac{B\left(\frac{\zeta+1}{r}, s+1\right)}{r}\right)\left[\frac{(x-\varsigma_1)^{\zeta+1} + (\varsigma_2-x)^{\zeta+1}}{\zeta+s+1}\right],
\end{aligned}$$

for all $x \in (\varsigma_1, \varsigma_2)$.

Theorem 35 ([20]). *Let Π be as in Theorem 34. If $|\Pi|^q$ is (s,r)-convex on $[\varsigma_1, \varsigma_2]$, $q \ge 1$ and $|\Pi'(x)| \le M,\ x \in [\varsigma_1, \varsigma_2]$, then*

$$\begin{aligned}
&\left|\left(\frac{(x-\varsigma_1)^\zeta + (\varsigma_2 - x)^\zeta}{\varsigma_2-\varsigma_1}\right)\Pi(x) - \frac{\Gamma(\zeta+1)}{\varsigma_2-\varsigma_1}\left[J^\zeta_{x-}\Pi(\varsigma_1) + J^\zeta_{x+}\Pi(\varsigma_2)\right]\right| \\
\le\ & \frac{M}{(\zeta+1)^{1-\frac{1}{q}}}\left(\frac{1}{\zeta+rs+1} + \frac{B\left(\frac{\zeta+1}{r}, s+1\right)}{r}\right)^{\frac{1}{q}}\left[\frac{(x-\varsigma_1)^{\zeta+1} + (\varsigma_2-x)^{\zeta+1}}{\zeta+s+1}\right],
\end{aligned}$$

for all $x \in (\varsigma_1, \varsigma_2)$.

Theorem 36 ([20]). *Let Π be as in Theorem 34. If $|\Pi|^q$ be (s,r)-convex on $[\varsigma_1,\varsigma_2]$, $q>1$ and $|\Pi'(x)|\leq M$, $x\in[\varsigma_1,\varsigma_2]$, then*

$$\begin{aligned} &\left|\left(\frac{(x-\varsigma_1)^\zeta+(\varsigma_2-x)^\zeta}{\varsigma_2-\varsigma_1}\right)\Pi(x)-\frac{\Gamma(\zeta+1)}{\varsigma_2-\varsigma_1}\left[J^\zeta_{x-}\Pi(\varsigma_1)+J^\zeta_{x+}\Pi(\varsigma_2)\right]\right| \\ \leq\; & \frac{M}{(\zeta p+1)^{\frac{1}{p}}}\left(\frac{1}{rs+1}+\frac{B\left(\frac{1}{r},s+1\right)}{r}\right)^{\frac{1}{q}}\left[\frac{(x-\varsigma_1)^{\zeta+1}+(\varsigma_2-x)^{\zeta+1}}{\zeta+s+1}\right], \end{aligned}$$

for all $x\in(\varsigma_1,\varsigma_2)$ and $\frac{1}{p}+\frac{1}{q}=1$.

2.7. Ostrowski-Type Fractional Integral Inequalities for Harmonically-Convex Functions

Definition 8 ([21]). *Let $\mathbb{I}\subset\mathbb{R}\setminus\{0\}$ be a real interval. A function $\Pi:\mathbb{I}\to\mathbb{R}$ is harmonically convex, if*

$$\Pi\left(\frac{xy}{tx+(1-t)y}\right)\leq t\Pi(y)+(1-t)\Pi(x)$$

for all $x,y\in\mathbb{I}$ and $t\in[0,1]$.

Some new Ostrowski's-type fractional integral inequalities for functions whose first derivatives are harmonically convex, via Riemann–Liouville fractional integrals are given in the next theorems.

Theorem 37 ([22]). *Let $\Pi:[\varsigma_1,\varsigma_2]\subset(0,\infty)\to\mathbb{R}$ be a differentiable mapping on $(\varsigma_1,\varsigma_2)$ with $\varsigma_1<\varsigma_2$ such that $\Pi\in L[\varsigma_1,\varsigma_2]$. If $|\Pi'|$ is harmonically convex on $[\varsigma_1,\varsigma_2]$, then, for all $x\in[\varsigma_1,\varsigma_2]$,*

$$\begin{aligned} &\left|\frac{\Gamma(\zeta+1)}{2}\left(\frac{\varsigma_1\varsigma_2}{\varsigma_2-\varsigma_1}\right)^\zeta\left[J^\zeta_{\frac{1}{\varsigma_1}-}(\Pi\circ h)\left(\frac{1}{\varsigma_2}\right)+J^\zeta_{\frac{1}{\varsigma_2}+}(\Pi\circ h)\left(\frac{1}{\varsigma_1}\right)\right]-\Pi(x)\right| \\ \leq\; & \frac{\varsigma_1\varsigma_2(\varsigma_2-\varsigma_1)}{2}\left[\mu_1(\varsigma_1,\varsigma_2,\zeta)|\Pi'(\varsigma_1)|+\mu_2(\varsigma_1,\varsigma_2,\zeta)|\Pi'(\varsigma_2)|\right], \end{aligned}$$

where

$$\begin{aligned} \mu_1(\varsigma_1,\varsigma_2,\zeta) &= \frac{1}{(\varsigma_2-\varsigma_1)^2}\left(\frac{\varsigma_1}{\varsigma_2}-1+\ln\frac{\varsigma_2}{\varsigma_1}\right)+\frac{{}_2F_1\left(2,2;\zeta+3;\frac{1}{2}\left(1-\frac{\varsigma_2}{\varsigma_1}\right)\right)}{4\varsigma_1^2(\zeta+1)(\zeta+2)} \\ &\quad+\frac{{}_2F_1\left(2,1;\zeta+2;\frac{1}{2}\left(1-\frac{\varsigma_1}{\varsigma_2}\right)\right)}{2\varsigma_2^2(\zeta+1)}-\frac{{}_2F_1\left(2,2;\zeta+3;\frac{1}{2}\left(1-\frac{\varsigma_1}{\varsigma_2}\right)\right)}{4\varsigma_2^2(\zeta+1)(\zeta+2)}, \\ \mu_2(\varsigma_1,\varsigma_2,\zeta) &= \frac{1}{(\varsigma_2-\varsigma_1)^2}\left(\frac{\varsigma_2}{\varsigma_1}-1+\ln\frac{\varsigma_1}{\varsigma_2}\right)+\frac{{}_2F_1\left(2,1;\zeta+2;\frac{1}{2}\left(1-\frac{\varsigma_2}{\varsigma_1}\right)\right)}{2\varsigma_1^2(\zeta+1)} \\ &\quad+\frac{{}_2F_1\left(2,2;\zeta+3;\frac{1}{2}\left(1-\frac{\varsigma_1}{\varsigma_2}\right)\right)}{4\varsigma_2^2(\zeta+1)(\zeta+2)}-\frac{{}_2F_1\left(2,2;\zeta+3;\frac{1}{2}\left(1-\frac{\varsigma_2}{\varsigma_1}\right)\right)}{4\varsigma_2^2(\zeta+1)(\zeta+2)}, \end{aligned}$$

with ${}_2F_1(.,.;.;.)$ the hypergeometric function and $h(x)=\frac{1}{x}, x\in\left[\frac{1}{\varsigma_2},\frac{1}{\varsigma_1}\right]$.

Theorem 38 ([22]). *Let $\Pi : [\varsigma_1, \varsigma_2] \subset (0,\infty) \to \mathbb{R}$ be a differentiable mapping on $(\varsigma_1, \varsigma_2)$ with $\varsigma_1 < \varsigma_2$ such that $\Pi \in L[\varsigma_1, \varsigma_2]$. If $|\Pi'|^q$ is harmonically convex on $[\varsigma_1, \varsigma_2]$, where $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then, for all $x \in [\varsigma_1, \varsigma_2]$,*

$$\begin{aligned}
&\left|\frac{\Gamma(\zeta+1)}{2}\Big(\frac{\varsigma_1\varsigma_2}{\varsigma_2-\varsigma_1}\Big)^{\zeta}\Big[J^{\zeta}_{\frac{1}{\varsigma_1}-}(\Pi\circ h)\Big(\frac{1}{\varsigma_2}\Big)+J^{\zeta}_{\frac{1}{\varsigma_2}+}(\Pi\circ h)\Big(\frac{1}{\varsigma_1}\Big)\Big]-\Pi(x)\right| \\
\leq\ & \frac{\varsigma_1\varsigma_2(\varsigma_2-\varsigma_1)}{2}\Big[\frac{1}{(\varsigma_2-\varsigma_1)^{\frac{1}{p}}(2p-1)^{\frac{1}{p}}}\Big(\frac{1}{\varsigma_1^{2p-1}}-\frac{1}{\varsigma_2^{2p-1}}\Big)^{\frac{1}{p}}\Big(\frac{|\Pi'(\varsigma_1)|^q+|\Pi'(\varsigma_2)|^q}{2}\Big)^{\frac{1}{q}} \\
&+\frac{1}{2^{\frac{1}{p}}\varsigma_1^2}\Bigg(\frac{{}_2F_1\Big(2p,1;\zeta p+2;\frac{1}{2}\Big(1-\frac{\varsigma_2}{\varsigma_1}\Big)\Big)}{\zeta p+1}\Bigg)^{\frac{1}{p}}\Big(\frac{|\Pi'(\varsigma_1)|^q+3|\Pi'(\varsigma_2)|^q}{4}\Big)^{\frac{1}{q}} \\
&+\frac{1}{2^{\frac{1}{p}}\varsigma_2^2}\Bigg(\frac{{}_2F_1\Big(2p,1;\zeta p+2;\frac{1}{2}\Big(1-\frac{\varsigma_1}{\varsigma_2}\Big)\Big)}{\zeta p+1}\Bigg)^{\frac{1}{p}}\Big(\frac{3|\Pi'(\varsigma_1)|^q+|\Pi'(\varsigma_2)|^q}{4}\Big)^{\frac{1}{q}}\Big].
\end{aligned}$$

2.8. Ostrowski-Type Fractional Integral Inequalities for h-Convex Functions

Definition 9 ([23]). *Suppose h is a non-negative and real-valued function. Then $\Pi : \mathbb{I} \to \mathbb{R}$ is an h-convex, if Π is non-negative and for all $x, y \in \mathbb{I}$, $\lambda \in (0,1)$ we have*

$$\Pi(\lambda x + (1-\lambda)y) \leq h(\lambda)\Pi(x) + h(1-\lambda)\Pi(y).$$

Some Ostrowski-type inequalities via Riemann–Liouville fractional integrals for h-convex are given in the next theorems.

Theorem 39 ([24]). *Let $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be a function which is differentiable on $(\varsigma_1, \varsigma_2)$ with $\varsigma_1 < \varsigma_2$ such that $\Pi' \in L[\varsigma_1, \varsigma_2$. If $|\Pi'|$ is h-convex on $[\varsigma_1, \varsigma_2]$ and $|\Pi'(x)| \leq M$, $x \in [\varsigma_1, \varsigma_2]$, then*

$$\begin{aligned}
&\left|\Pi(x)-\Gamma(\zeta+1)\Big[\frac{1}{2(x-\varsigma_1)^{\zeta}}J^{\zeta}_{x-}\Pi(\varsigma_1)+\frac{1}{2(\varsigma_2-x)^{\zeta}}J^{\zeta}_{x+}\Pi(\varsigma_2)\Big]\right| \\
\leq\ & \frac{M(\varsigma_2-\varsigma_1)}{2}\int_0^1[h(1-t)+h(t)]t^{\zeta}dt,
\end{aligned}$$

for each $x \in [\varsigma_1, \varsigma_2]$.

Theorem 40 ([24]). *Let Π be as in the Theorem 39. If $|\Pi'|^q$ is h-convex on $[\varsigma_1, \varsigma_2]$, $p, q > 1, \frac{1}{p} + \frac{1}{q} = 1$, then*

$$\begin{aligned}
&\left|\Pi(x)-\Gamma(\zeta+1)\Big[\frac{1}{2(x-\varsigma_1)^{\zeta}}J^{\zeta}_{x-}\Pi(\varsigma_1)+\frac{1}{2(\varsigma_2-x)^{\zeta}}J^{\zeta}_{x+}\Pi(\varsigma_2)\Big]\right| \\
\leq\ & \frac{M(\varsigma_2-\varsigma_1)}{2(\zeta p+1)^{\frac{1}{p}}}\Big(2\int_0^1 h(t)dt\Big)^{\frac{1}{q}},
\end{aligned}$$

for each $x \in [\varsigma_1, \varsigma_2]$.

Theorem 41 ([24]). *Let Π be as in the Theorem 39. If $|\Pi'|^q$ is h-convex on $[\varsigma_1, \varsigma_2]$, $q \geq 1$, then*

$$\left|\Pi(x) - \Gamma(\zeta+1)\left[\frac{1}{2(x-\varsigma_1)^\zeta} J^\zeta_{x-}\Pi(\varsigma_1) + \frac{1}{2(\varsigma_2 - x)^\zeta} J^\zeta_{x+}\Pi(\varsigma_2)\right]\right|$$
$$\leq \frac{M(\varsigma_2-\varsigma_1)}{2}\left(\frac{1}{\zeta+1}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^\zeta [h(t) + h(1-t)]dt\right)^{\frac{1}{q}},$$

for each $x \in [\varsigma_1, \varsigma_2]$.

Ostrowski-type fractional integral inequalities for super-multiplicative functions pertaining to Riemann–Liouville fractional integrals are given now.

Definition 10 ([25]). *We say that $h : J \to \mathbb{R}$ is a super-multiplicative function, if for all $x, y \in J$, one has*

$$h(x,y) \geq h(x)h(y).$$

Theorem 42 ([26]). *Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ ($[0,1] \subseteq J$) be a super-multiplicative and non-negative function, $h(t) \geq t$ for $0 \leq t \leq 1$, $\Pi : [\varsigma_1, \varsigma_2] \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on $(\varsigma_1, \varsigma_2)$ with $\varsigma_1 < \varsigma_2$ such that $\Pi' \in L[\varsigma_1, \varsigma_2]$. If $|\Pi'|$ is a h-convex function on $[\varsigma_1, \varsigma_2]$ and $|\Pi'(x)| \leq M$, $x \in [\varsigma_1, \varsigma_2]$, then for $\zeta > 0$ and $x \in [\varsigma_1, \varsigma_2]$ we have:*

$$\left|\left(\frac{(x-\varsigma_1)^\zeta + (\varsigma_2 - x)^\zeta}{\varsigma_2 - \varsigma_1}\right)\Pi(x) - \frac{\Gamma(\zeta+1)}{\varsigma_2-\varsigma_1}\left[J^\zeta_{x-}\Pi(\varsigma_1) + J^\zeta_{x+}\Pi(\varsigma_2)\right]\right|$$
$$\leq \frac{M\left[(x-\varsigma_1)^{\zeta+1} + (\varsigma_2-x)^{\zeta+1}\right]}{\varsigma_2-\varsigma_1}\int_0^1 [t^\zeta h(t) + t^\zeta h(1-t)]dt$$
$$\leq \frac{M\left[(x-\varsigma_1)^{\zeta+1} + (\varsigma_2-x)^{\zeta+1}\right]}{\varsigma_2-\varsigma_1}\int_0^1 [h(t^{\zeta+1}) + h(t^\zeta(1-t)]dt.$$

Theorem 43 ([26]). *Let Π be as in Theorem 42. If $|\Pi'|^q$ is a h-convex function on $[\varsigma_1, \varsigma_2]$, $p, q > 1$ $\frac{1}{p} + \frac{1}{q} = 1$ and $|\Pi'(x)| \leq M$, $x \in [\varsigma_1, \varsigma_2]$, then for $\zeta > 0$ and $x \in [\varsigma_1, \varsigma_2]$ we have:*

$$\left|\left(\frac{(x-\varsigma_1)^\zeta + (\varsigma_2 - x)^\zeta}{\varsigma_2 - \varsigma_1}\right)\Pi(x) - \frac{\Gamma(\zeta+1)}{\varsigma_2-\varsigma_1}\left[J^\zeta_{x-}\Pi(\varsigma_1) + J^\zeta_{x+}\Pi(\varsigma_2)\right]\right|$$
$$\leq \frac{M\left[(x-\varsigma_1)^{\zeta+1} + (\varsigma_2-x)^{\zeta+1}\right]}{(1+p\zeta)^{\frac{1}{p}}(\varsigma_2-\varsigma_1)}\left(\int_0^1 [h(t) + h(1-t)]dt\right)^{\frac{1}{q}}$$
$$\leq \frac{M\left[(x-\varsigma_1)^{\zeta+1} + (\varsigma_2-x)^{\zeta+1}\right]}{(1+p\zeta)^{\frac{1}{p}}(\varsigma_2-\varsigma_1)}h^{\frac{1}{q}}(1).$$

Theorem 44 ([26]). *Let Π be as in Theorem 42. If $|\Pi'|^q, q \geq 1$ is a h-convex function on $[\varsigma_1, \varsigma_2]$, and $|\Pi'(x)| \leq M$, $x \in [\varsigma_1, \varsigma_2]$, then for $\zeta > 0$ and $x \in [\varsigma_1, \varsigma_2]$ we have:*

$$\left|\left(\frac{(x-\varsigma_1)^\zeta + (\varsigma_2 - x)^\zeta}{\varsigma_2 - \varsigma_1}\right)\Pi(x) - \frac{\Gamma(\zeta+1)}{\varsigma_2-\varsigma_1}\left[J^\zeta_{x-}\Pi(\varsigma_1) + J^\zeta_{x+}\Pi(\varsigma_2)\right]\right|$$
$$\leq \frac{M}{(1+\zeta)^{1-\frac{1}{q}}}\frac{(x-\varsigma_1)^{\zeta+1} + (\varsigma_2-x)^{\zeta+1}}{\varsigma_2-\varsigma_1}\left(\int_0^1 [t^\zeta h(t) + t^\zeta h(1-t)]dt\right)^{\frac{1}{q}}$$
$$\leq \frac{M}{(1+\zeta)^{1-\frac{1}{q}}}\frac{(x-\varsigma_1)^{\zeta+1} + (\varsigma_2-x)^{\zeta+1}}{\varsigma_2-\varsigma_1}\left(\int_0^1 [h(t^{\zeta+1}) + h(t^\zeta(1-t))]dt\right)^{\frac{1}{q}}.$$

2.9. Ostrowski-Type Fractional Integral Inequalities for Godunova-Levin Functions

Definition 11 ([27])**.** *A function* $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ *is a Godunova–Levin function if*

$$\Pi(tx + (1-t)y) \leq \frac{\Pi(x)}{t} + \frac{\Pi(y)}{1-t},$$

for all $x, y \in [\varsigma_1, \varsigma_2]$ *and* $t \in [0,1]$.

Definition 12 ([28])**.** *A function* $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ *is an s-Godunova-Levin function of the first kind, where* $s \in (0,1]$, *if*

$$\Pi(tx + (1-t)y) \leq \frac{\Pi(x)}{t^s} + \frac{\Pi(y)}{1-t^s},$$

for all $x, y \in [\varsigma_1, \varsigma_2]$ *and* $t \in (0,1)$.

Definition 13 ([28])**.** *A function* $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ *is said to be an s-Godunova-Levin function of the second kind, where* $s \in (0,1]$, *if*

$$\Pi(tx + (1-t)y) \leq \frac{\Pi(x)}{t^s} + \frac{\Pi(y)}{(1-t)^s},$$

for all $x, y \in [\varsigma_1, \varsigma_2]$ *and* $t \in (0,1)$.

In this subsection, we show some Ostrowski-type inequalities pertaining to Riemann–Liouville fractional integrals for s-Godunova-Levin functions.

Theorem 45 ([29])**.** *Suppose* $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ *is a differentiable function on* $(\varsigma_1, \varsigma_2)$ *with* $\varsigma_1 < \varsigma_2$ *and* $\Pi' \in L[\varsigma_1, \varsigma_2]$. *If* $|\Pi'|$ *is an s-Godunova-Levin function of the second kind on* $[\varsigma_1, \varsigma_2]$ *and* $|\Pi'(x)| \leq M,\ x \in [\varsigma_1, \varsigma_2]$, *then*

$$\begin{aligned} &\left| \left(\frac{(x-\varsigma_1)^\zeta + (\varsigma_2 - x)^\zeta}{\varsigma_2 - \varsigma_1} \right) \Pi(x) - \frac{\Gamma(\zeta+1)}{\varsigma_2 - \varsigma_1} \left[J^\zeta_{x-}\Pi(\varsigma_1) + J^\zeta_{x+}\Pi(\varsigma_2) \right] \right| \\ \leq\ & M\left(\frac{1}{1+\zeta-s} + \frac{\Gamma(1-s)\Gamma(\zeta+1)}{\Gamma(2+\zeta-s)} \right) \left[\frac{(x-\varsigma_1)^{\zeta+1} + (\varsigma_2 - x)^{\zeta+1}}{\varsigma_2 - \varsigma_1} \right], \end{aligned}$$

for all $x \in (\varsigma_1, \varsigma_2)$.

Theorem 46 ([29])**.** *Let* Π *be as in Theorem 45. If* $|\Pi'|^q$ *is an s-Godunova-Levin function of the second kind on* $[\varsigma_1, \varsigma_2]$, $p, q > 1, \dfrac{1}{p} + \dfrac{1}{q} = 1$ *and* $|\Pi'(x)| \leq M,\ x \in [\varsigma_1, \varsigma_2]$, *then:*

$$\begin{aligned} &\left| \left(\frac{(x-\varsigma_1)^\zeta + (\varsigma_2 - x)^\zeta}{\varsigma_2 - \varsigma_1} \right) \Pi(x) - \frac{\Gamma(\zeta+1)}{\varsigma_2 - \varsigma_1} \left[J^\zeta_{x-}\Pi(\varsigma_1) + J^\zeta_{x+}\Pi(\varsigma_2) \right] \right| \\ \leq\ & M\left(\frac{1}{1-s} \right)^{\frac{1}{q}} \left(\frac{1}{1+p\zeta} \right)^{\frac{1}{p}} \left[\frac{(x-\varsigma_1)^{\zeta+1} + (\varsigma_2 - x)^{\zeta+1}}{\varsigma_2 - \varsigma_1} \right], \end{aligned}$$

for all $x \in (\varsigma_1, \varsigma_2)$.

Now, we present some new family of s-Godunova-Levin functions, which are called (s,m)-Godunova-Levin functions of the second kind. Next, we present some new Ostrowski-type integral inequalities for (s,m)-Godunova-Levin functions via fractional integrals.

Definition 14 ([30]). *A function $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ is said to be an (s, m)-Godunova-Levin function of the second kind, where $s \in [0, 1]$, $m \in (0, 1]$, if*

$$\Pi(tx + (1-t)y) \le \frac{\Pi(x)}{t^s} + \frac{m\Pi(y)}{(1-t)^s},$$

for all $x, y \in [\varsigma_1, \varsigma_2]$ and $t \in (0, 1)$.

Theorem 47 ([30]). *Suppose $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ is a differentiable function on open interval $(\varsigma_1, \varsigma_2)$ with $\varsigma_1 < \varsigma_2$ such that $\Pi' \in L[\varsigma_1, \varsigma_2]$. If $|\Pi'|$ is an (s, m)-Godunova-Levin function of the second kind on $[\varsigma_1, \varsigma_2]$ and $|\Pi'(x)| \le M$, $x \in [\varsigma_1, \varsigma_2]$, then, for all $x \in [\varsigma_1, \varsigma_2]$,*

$$\begin{aligned}&\left|\left(\frac{(x-\varsigma_1)^\zeta + (\varsigma_2 - x)^\zeta}{\varsigma_2 - \varsigma_1}\right)\Pi(x) - \frac{\Gamma(\zeta+1)}{\varsigma_2 - \varsigma_1}\left[J^\zeta_{x-}\Pi(\varsigma_1) + J^\zeta_{x+}\Pi(\varsigma_2)\right]\right| \\ \le\ & \min\{\vartheta_1(\varsigma_1, \varsigma_2; m, \zeta; x), \vartheta_2(\varsigma_1, \varsigma_2; m, \zeta; x)\},\end{aligned}$$

where

$$\begin{aligned}&\vartheta_1(\varsigma_1, \varsigma_2; m, \zeta; x) \\ =\ & \frac{M}{1+\zeta-s}\left[\frac{(x-\varsigma_1)^{\zeta+1} + (\varsigma_2 - x)^{\zeta+1}}{\varsigma_2 - \varsigma_1}\right] \\ & + \frac{m\Gamma(1-s)\Gamma(\zeta+1)}{\Gamma(2+\zeta-s)}\left[\frac{(x-\varsigma_1)^{\zeta+1}\left|\Pi'\left(\frac{\varsigma_2}{m}\right)\right| + (\varsigma_2 - x)^{\zeta+1}\left|\Pi'\left(\frac{\varsigma_1}{m}\right)\right|}{\varsigma_2 - \varsigma_1}\right],\end{aligned}$$

$$\begin{aligned}&\vartheta_2(\varsigma_1, \varsigma_2; m, \zeta; x) \\ =\ & \left[\frac{m}{1+\zeta-s}\left|\Pi'\left(\frac{x}{m}\right)\right| + M\frac{\Gamma(1-s)\Gamma(\zeta+1)}{\Gamma(2+\zeta-s)}\right]\left[\frac{(x-\varsigma_1)^{\zeta+1} + (\varsigma_2 - x)^{\zeta+1}}{\varsigma_2 - \varsigma_1}\right].\end{aligned}$$

Theorem 48 ([30]). *Let Π be as in Theorem 47. Then, for all $x \in [\varsigma_1, \varsigma_2]$,*

$$\begin{aligned}&\left|\left(\frac{(x-\varsigma_1)^\zeta + (\varsigma_2 - x)^\zeta}{\varsigma_2 - \varsigma_1}\right)\Pi(x) - \frac{\Gamma(\zeta+1)}{\varsigma_2 - \varsigma_1}\left[J^\zeta_{x-}\Pi(\varsigma_1) + J^\zeta_{x+}\Pi(\varsigma_2)\right]\right| \\ \le\ & \min\{\varphi_1(\varsigma_1, \varsigma_2; m, \zeta; x), \varphi_2(\varsigma_1, \varsigma_2; m, \zeta; x)\},\end{aligned}$$

where

$$\begin{aligned}\varphi_1(\varsigma_1, \varsigma_2; m, \zeta; x) &= \left(\frac{1}{p\zeta+1}\right)^{\frac{1}{p}}\left[\left(\frac{M^q}{1-s} + \frac{m}{1-s}\left|\Pi'\left(\frac{\varsigma_1}{m}\right)\right|^q\right)^{\frac{1}{q}}\frac{(x-\varsigma_1)^{\zeta+1}}{\varsigma_2 - \varsigma_1}\right. \\ &\quad \left. + \left(\frac{M^q}{1-s} + \frac{m}{1-s}\left|\Pi'\left(\frac{\varsigma_2}{m}\right)\right|^q\right)^{\frac{1}{q}}\frac{(\varsigma_2 - x)^{\zeta+1}}{\varsigma_2 - \varsigma_1}\right], \\ \varphi_2(\varsigma_1, \varsigma_2; m, \zeta; x) &= \left(\frac{1}{p\zeta+1}\right)^{\frac{1}{p}}\left[\frac{m}{1-s}\left|\Pi'\left(\frac{x}{m}\right)\right|^q + \frac{M^q}{1-s}\right]^{\frac{1}{q}}\left[\frac{(x-\varsigma_1)^{\zeta+1} + (\varsigma_2 - x)^{\zeta+1}}{\varsigma_2 - \varsigma_1}\right].\end{aligned}$$

Theorem 49 ([30]). *Let the assumptions of this theorem be stated in Theorem 47. Then, for all $x \in [\varsigma_1, \varsigma_2]$,*

$$\begin{aligned}&\left|\left(\frac{(x-\varsigma_1)^\zeta + (\varsigma_2 - x)^\zeta}{\varsigma_2 - \varsigma_1}\right)\Pi(x) - \frac{\Gamma(\zeta+1)}{\varsigma_2 - \varsigma_1}\left[J^\zeta_{x-}\Pi(\varsigma_1) + J^\zeta_{x+}\Pi(\varsigma_2)\right]\right| \\ \le\ & \min\{\rho_1(\varsigma_1, \varsigma_2; m, \zeta; x), \rho_2(\varsigma_1, \varsigma_2; m, \zeta; x)\},\end{aligned}$$

where

$$\begin{aligned}
&\rho_1(\varsigma_1,\varsigma_2;m,\zeta;x)\\
=\;&\Big(\frac{1}{\zeta+1}\Big)^{1-\frac{1}{q}}\Big[\frac{(x-\varsigma_1)^{\zeta+1}}{\varsigma_2-\varsigma_1}\Big(\frac{M^q}{1+\zeta-s}+\frac{m\Gamma(1-s)\Gamma(\zeta+1)}{\Gamma(2+\zeta-s)}\Big|\Pi'\Big(\frac{\varsigma_1}{m}\Big)\Big|^q\Big)^{\frac{1}{q}}\\
&+\frac{(\varsigma_2-x)^{\zeta+1}}{\varsigma_2-\varsigma_1}\Big(\frac{M^q}{1+\zeta-s}+\frac{m\Gamma(1-s)\Gamma(\zeta+1)}{\Gamma(2+\zeta-s)}\Big|\Pi'\Big(\frac{\varsigma_2}{m}\Big)\Big|^q\Big)^{\frac{1}{q}}\Big],\\
&\rho_2(\varsigma_1,\varsigma_2;m,\zeta;x)\\
=\;&\Big(\frac{1}{\zeta+1}\Big)^{1-\frac{1}{q}}\Big[\Big(\frac{(x-\varsigma_1)^{\zeta+1}+(\varsigma_2-x)^{\zeta+1}}{\varsigma_2-\varsigma_1}\Big)\\
&\times\Big(\frac{m}{1+\zeta-s}\Big|\Pi'\Big(\frac{x}{m}\Big)\Big|^q+\frac{M^q\Gamma(1-s)\Gamma(\zeta+1)}{\Gamma(2+\zeta-s)}\Big)^{\frac{1}{q}}\Big].
\end{aligned}$$

2.10. Ostrowski-Type Fractional Integral Inequalities for MT-Convex Function

Definition 15 ([31])**.** *A real-valued and non-negative function* Π *is MT-convex function, if*

$$\Pi(tx+(1-t)y)\le\frac{\sqrt{t}}{2\sqrt{1-t}}\Pi(x)+\frac{\sqrt{1-t}}{2\sqrt{t}}\Pi(y),$$

for all $x,y\in\mathbb{I}$ *and* $t\in(0,1)$.

In this subsection, we give some Ostrowski-type fractional integral inequalities for MT-convex functions via Riemann–Liouville fractional integrals.

Theorem 50 ([32])**.** *Suppose* $\Pi:[\varsigma_1,\varsigma_2]\subset(0,\infty)\to\mathbb{R}$ *is a mapping which is differentiable on* $(\varsigma_1,\varsigma_2)$ *with* $\varsigma_1<\varsigma_2$ *such that* $\Pi'\in L[\varsigma_1,\varsigma_2]$. *If* $|\Pi'|$ *is MT-convex function on* $[\varsigma_1,\varsigma_2]$ *and* $|\Pi'(x)|\le M,\ x\in[\varsigma_1,\varsigma_2]$, *then for* $\zeta>0$ *and* $x\in[\varsigma_1,\varsigma_2]$ *we have:*

$$\begin{aligned}
&\Big|\Big(\frac{(x-\varsigma_1)^{\zeta}+(\varsigma_2-x)^{\zeta}}{\varsigma_2-\varsigma_1}\Big)\Pi(x)-\frac{\Gamma(\zeta+1)}{\varsigma_2-\varsigma_1}\Big[J^{\zeta}_{x-}\Pi(\varsigma_1)+J^{\zeta}_{x+}\Pi(\varsigma_2)\Big]\Big|\\
\le\;& M\frac{\Gamma\Big(\zeta+\frac{1}{2}\Big)\Gamma\Big(\frac{1}{2}\Big)}{2\Gamma(\zeta+1)}\frac{(x-\varsigma_1)^{\zeta+1}+(\varsigma_2-x)^{\zeta+1}}{\varsigma_2-\varsigma_1}.
\end{aligned}$$

Theorem 51 ([32])**.** *Let* Π *be as in Theorem 50. If* $|\Pi'|^q$ *is MT-convex function on* $[\varsigma_1,\varsigma_2]$, $q>1,\frac{1}{p}+\frac{1}{q}=1$ *and* $|\Pi'(x)|\le M,\ x\in[\varsigma_1,\varsigma_2]$, *then for* $\zeta>0$ *and* $x\in[\varsigma_1,\varsigma_2]$ *we have:*

$$\begin{aligned}
&\Big|\Big(\frac{(x-\varsigma_1)^{\zeta}+(\varsigma_2-x)^{\zeta}}{\varsigma_2-\varsigma_1}\Big)\Pi(x)-\frac{\Gamma(\zeta+1)}{\varsigma_2-\varsigma_1}\Big[J^{\zeta}_{x-}\Pi(\varsigma_1)+J^{\zeta}_{x+}\Pi(\varsigma_2)\Big]\Big|\\
\le\;&\frac{M}{(1+p\zeta)^{\frac{1}{p}}}\Big(\frac{\pi}{2}\Big)^{\frac{1}{q}}\frac{(x-\varsigma_1)^{\zeta+1}+(\varsigma_2-x)^{\zeta+1}}{\varsigma_2-\varsigma_1}.
\end{aligned}$$

Theorem 52 ([32])**.** *Let* Π *be as in Theorem 50. If* $|\Pi'|^q, q\ge1$ *is MT-convex function on* $[\varsigma_1,\varsigma_2]$ *and* $|\Pi'(x)|\le M,\ x\in[\varsigma_1,\varsigma_2]$, *then for* $\zeta>0$ *and* $x\in[\varsigma_1,\varsigma_2]$ *we have:*

$$\begin{aligned}
&\Big|\Big(\frac{(x-\varsigma_1)^{\zeta}+(\varsigma_2-x)^{\zeta}}{\varsigma_2-\varsigma_1}\Big)\Pi(x)-\frac{\Gamma(\zeta+1)}{\varsigma_2-\varsigma_1}\Big[J^{\zeta}_{x-}\Pi(\varsigma_1)+J^{\zeta}_{x+}\Pi(\varsigma_2)\Big]\Big|\\
\le\;&\frac{M}{(1+p\zeta)^{\frac{1}{p}}}\left(\frac{\Gamma(\zeta+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\zeta+1)}\right)^{\frac{1}{q}}\frac{(x-\varsigma_1)^{\zeta+1}+(\varsigma_2-x)^{\zeta+1}}{\varsigma_2-\varsigma_1}.
\end{aligned}$$

Theorem 53 ([33]). *Let the assumptions of this theorem be stated in Theorem 50. Then for $\zeta > 0$ and $x \in [\varsigma_1, \varsigma_2]$ we have:*

$$\begin{aligned}
&\left|\left(\frac{(x-\varsigma_1)^\zeta + (\varsigma_2 - x)^\zeta}{\varsigma_2 - \varsigma_1}\right)\Pi(x) - \frac{\Gamma(\zeta+1)}{\varsigma_2 - \varsigma_1}\left[J^\zeta_{x-}\Pi(\varsigma_1) + J^\zeta_{x+}\Pi(\varsigma_2)\right]\right| \\
\leq\ & 2MB\left(\zeta + \frac{1}{2}, \frac{1}{2}\right)\frac{(x-\varsigma_1)^{\zeta+1} + (\varsigma_2 - x)^{\zeta+1}}{\varsigma_2 - \varsigma_1}.
\end{aligned}$$

Theorem 54 ([33]). *Let Π be as in Theorem 50. If $|\Pi'|^q$ is MT-convex function on $[\varsigma_1, \varsigma_2]$, $q > 1, \frac{1}{p} + \frac{1}{q} = 1$ and $|\Pi'(x)| \leq M,\ x \in [\varsigma_1, \varsigma_2]$, then for $\zeta > 0$ and $x \in [\varsigma_1, \varsigma_2]$ we have:*

$$\begin{aligned}
&\left|\left(\frac{(x-\varsigma_1)^\zeta + (\varsigma_2 - x)^\zeta}{\varsigma_2 - \varsigma_1}\right)\Pi(x) - \frac{\Gamma(\zeta+1)}{\varsigma_2 - \varsigma_1}\left[J^\zeta_{x-}\Pi(\varsigma_1) + J^\zeta_{x+}\Pi(\varsigma_2)\right]\right| \\
\leq\ & \frac{M}{(1+p\zeta)^{\frac{1}{p}}}\left(\frac{\pi}{4}\right)^{\frac{1}{q}}\frac{(x-\varsigma_1)^{\zeta+1} + (\varsigma_2 - x)^{\zeta+1}}{\varsigma_2 - \varsigma_1}.
\end{aligned}$$

Theorem 55 ([34]). *Let the assumptions of this theorem be stated in Theorem 50. Then for $\zeta > 0, \lambda \in [0,1]$ and $x \in [\varsigma_1, \varsigma_2]$ we have:*

$$\begin{aligned}
&\left|(1-\lambda)\left(\frac{(x-\varsigma_1)^\zeta + (\varsigma_2 - x)^\zeta}{\varsigma_2 - \varsigma_1}\right)\Pi(x) + \lambda\left(\frac{(x-\varsigma_1)^\zeta\Pi(\varsigma_1) + (\varsigma_2 - x)^\zeta\Pi(\varsigma_2)}{\varsigma_2 - \varsigma_1}\right)\right. \\
&\left.-\frac{\Gamma(\zeta+1)}{\varsigma_2 - \varsigma_1}\left[J^\zeta_{x-}\Pi(\varsigma_1) + J^\zeta_{x+}\Pi(\varsigma_2)\right]\right| \\
\leq\ & M\frac{1}{2(\varsigma_2 - \varsigma_1)}\left[(x-\varsigma_1)^{\zeta+1} + (\varsigma_2 - x)^{\zeta+1}\right]A(\zeta, \lambda),
\end{aligned}$$

where

$$\begin{aligned}
A(\zeta,\lambda) \ =\ & 2\lambda\left(B\left(\lambda^{\frac{1}{\zeta}}; \frac{3}{2}, \frac{1}{2}\right) + B\left(\lambda^{\frac{1}{\zeta}}; \frac{1}{2}, \frac{3}{2}\right) + B\left(\zeta + \frac{1}{2}, \frac{1}{2}\right) - \lambda\pi\right. \\
& -2\left(B\left(\lambda^{\frac{1}{\zeta}}; \zeta + \frac{3}{2}, \frac{1}{2}\right) + B\left(\lambda^{\frac{1}{\zeta}}; \zeta + \frac{1}{2}, \frac{3}{2}\right)\right),
\end{aligned}$$

and $B(a,;x,y) = \int_0^a t^{x-1}(1-t)^{y-1}dt,\ \ 0 < a \leq 1,\ x,y > 0$ the incomplete Beta function.

Theorem 56 ([34]). *Let Π be as in Theorem 50. If $|\Pi'|^q$ is MT-convex function on $[\varsigma_1, \varsigma_2]$, $q > 1, \frac{1}{p} + \frac{1}{q} = 1$ and $|\Pi'(x)| \leq M,\ x \in [\varsigma_1, \varsigma_2]$, then for $\zeta > 0$ and $x \in [\varsigma_1, \varsigma_2]$ we have:*

$$\begin{aligned}
&\left|(1-\lambda)\left(\frac{(x-\varsigma_1)^\zeta + (\varsigma_2 - x)^\zeta}{\varsigma_2 - \varsigma_1}\right)\Pi(x) + \lambda\left(\frac{(x-\varsigma_1)^\zeta\Pi(\varsigma_1) + (\varsigma_2 - x)^\zeta\Pi(\varsigma_2)}{\varsigma_2 - \varsigma_1}\right)\right. \\
&\left.-\frac{\Gamma(\zeta+1)}{\varsigma_2 - \varsigma_1}\left[J^\zeta_{x-}\Pi(\varsigma_1) + J^\zeta_{x+}\Pi(\varsigma_2)\right]\right| \\
\leq\ & M\frac{1}{2(\varsigma_2 - \varsigma_1)}\left[(x-\varsigma_1)^{\zeta+1} + (\varsigma_2 - x)^{\zeta+1}\right]\left(\frac{\pi}{2}\right)^{\frac{1}{q}}B(\zeta,\lambda)^{\frac{1}{p}},
\end{aligned}$$

where

$$B(\zeta,\lambda) = \frac{2}{\zeta}\int_0^\lambda (\lambda - s)^p s^{\frac{1}{\zeta}-1}ds - \frac{1}{\zeta}\int_0^1 (\lambda - s)^p s^{\frac{1}{\zeta}-1}ds.$$

Theorem 57 ([34]). *Let Π be as in Theorem 50. If $|\Pi'|^q, q \geq 1$ is MT-convex function on $[\varsigma_1, \varsigma_2]$ and $|\Pi'(x)| \leq M,\ x \in [\varsigma_1, \varsigma_2]$, then for $\zeta > 0$ and $x \in [\varsigma_1, \varsigma_2]$ we have:*

$$\begin{aligned} &\Bigg|(1-\lambda)\Big(\frac{(x-\varsigma_1)^\zeta + (\varsigma_2 - x)^\zeta}{\varsigma_2 - \varsigma_1}\Big)\Pi(x) + \lambda\Big(\frac{(x-\varsigma_1)^\zeta\Pi(\varsigma_1) + (\varsigma_2 - x)^\zeta\Pi(\varsigma_2)}{\varsigma_2 - \varsigma_1}\Big) \\ &-\frac{\Gamma(\zeta+1)}{\varsigma_2 - \varsigma_1}\Big[J^\zeta_{x-}\Pi(\varsigma_1) + J^\zeta_{x+}\Pi(\varsigma_2)\Big]\Bigg| \\ \leq\ & M\frac{1}{\varsigma_2 - \varsigma_1}\Big[(x-\varsigma_1)^{\zeta+1} + (\varsigma_2 - x)^{\zeta+1}\Big]\Big(\frac{A(\zeta,\lambda)}{2}\Big)^{\frac{1}{q}}\Big(\frac{2\zeta\lambda^{1+\frac{1}{\zeta}}+1}{\zeta+1} - \lambda\Big)^{1-\frac{1}{q}}, \end{aligned}$$

where $A(\zeta, \lambda)$ is given in Theorem 55.

2.11. Ostrowski-Type Fractional Integral Inequalities for P-Convex, m-Convex and (s, m)-Convex Functions

In this subsection, we show results on Ostrowski-type fractional integral inequalities for twice differentiable functions and different kinds of convexity.

Definition 16 ([35]). *The function $\Pi : I \subset \mathbb{R} \to \mathbb{R}$ is said to be P-convex, if is nonnegative and*

$$\Pi(tx + (1-t)y) \leq \Pi(x) + \Pi(y),$$

$\forall\, x, y \in I$ and $t \in [0, 1]$.

Definition 17 ([36]). *A real-valued function Π is m-convex, if*

$$\Pi(tx + m(1-t)y) \leq t\Pi(x) + m(1-t)\Pi(y),$$

$\forall\, x, y \in (0, b]$ $t \in [0, 1]$ and $m \in (0, 1]$.

Definition 18 ([15]). *A real-valued function Π is (s, m)-convex, if*

$$\Pi(tx + m(1-t)y) \leq t^s\Pi(x) + m(1-t^s)\Pi(y),$$

$\forall\, x, y \in (0, b]$ $t \in [0, 1]$ and $(s, m) \in (0, 1]^2$.

Theorem 58 ([37]). *Let $\Pi : I \to \mathbb{R}$ be a twice differentiable function on I° such that $|\Pi''| \in L[\varsigma_1, \varsigma_2]$, where $\varsigma_1, \varsigma_2 \in I$, with $\varsigma_1 < \varsigma_2$. If $|\Pi''|$ is a convex function on $[\varsigma_1, \varsigma_2]$, and Π'' is bounded, i.e., $\|\Pi''\|_\infty = \sup_{x\in[\varsigma_1,\varsigma_2]} |\Pi''(x)| < \infty$, for any $x \in [\varsigma_1, \varsigma_2]$, then*

$$\begin{aligned} &\Big|(\zeta+1)(\varsigma_2 - x)^\zeta(x-\varsigma_1)^\zeta(\varsigma_2 - \varsigma_1)\Pi(x) \\ &-\Gamma(\zeta+2)\Big[(\varsigma_2 - x)^{\zeta+1}J^\zeta_{x-}\Pi(\varsigma_1) + (x-\varsigma_1)^{\zeta+1}J^\zeta_{x+}\Pi(\varsigma_2)\Big| \\ \leq\ & \frac{(x-\varsigma_1)^{\zeta+1}(\varsigma_2 - x)^{\zeta+1}(\varsigma_2 - \varsigma_1)}{\zeta+2}\|\Pi''\|_\infty. \end{aligned}$$

Theorem 59 ([37]). *Let Π be as in Theorem 58. If $|\Pi''|$ is a P-convex function on $[\varsigma_1, \varsigma_2]$, and Π'' is bounded, i.e., $\|\Pi''\|_\infty = \sup_{x\in[\varsigma_1,\varsigma_2]} |\Pi''(x)| < \infty$, for any $x \in [\varsigma_1, \varsigma_2]$, then*

$$\begin{aligned} &\Big|(\zeta+1)(\varsigma_2 - x)^\zeta(x-\varsigma_1)^\zeta(\varsigma_2 - \varsigma_1)\Pi(x) \\ &-\Gamma(\zeta+2)\Big[(\varsigma_2 - x)^{\zeta+1}J^\zeta_{x-}\Pi(\varsigma_1) + (x-\varsigma_1)^{\zeta+1}J^\zeta_{x+}\Pi(\varsigma_2)\Big| \\ \leq\ & \frac{2(x-\varsigma_1)^{\zeta+1}(\varsigma_2 - x)^{\zeta+1}(\varsigma_2 - \varsigma_1)}{\zeta+2}\|\Pi''\|_\infty. \end{aligned}$$

Theorem 60 ([37]). *Let Π be as in Theorem 58. If $|\Pi''|$ is s-convex on $[\varsigma_1, \varsigma_2]$, and Π'' is bounded, i.e., $\|\Pi''\|_\infty = \sup_{x\in[\varsigma_1,\varsigma_2]} |\Pi''(x)| < \infty$, for any $x \in [\varsigma_1, \varsigma_2]$, then*

$$\begin{aligned} &\Big|(\zeta+1)(\varsigma_2-x)^{\zeta}(x-\varsigma_1)^{\zeta}(\varsigma_2-\varsigma_1)\Pi(x) \\ &-\Gamma(\zeta+2)\Big[(\varsigma_2-x)^{\zeta+1}J^{\zeta}_{x-}\Pi(\varsigma_1)+(x-\varsigma_1)^{\zeta+1}J^{\zeta}_{x+}\Pi(\varsigma_2)\Big| \\ \le\ &(\varsigma_2-\varsigma_1)(x-\varsigma_1)^{\zeta+1}(\varsigma_2-x)^{\zeta+1}\Big[\frac{1}{\zeta+s+2}+B(\zeta+2,s+1)\Big]\|\Pi''\|_\infty. \end{aligned}$$

Theorem 61 ([37]). *Let Π be as in Theorem 58. If $|\Pi''|$ is h-convex on $[\varsigma_1, \varsigma_2]$, and Π'' is bounded, i.e., $\|\Pi''\|_\infty = \sup_{x\in[\varsigma_1,\varsigma_2]} |\Pi''(x)| < \infty$, for any $x \in [\varsigma_1, \varsigma_2]$, then*

$$\begin{aligned} &\Big|(\zeta+1)(\varsigma_2-x)^{\zeta}(x-\varsigma_1)^{\zeta}(\varsigma_2-\varsigma_1)\Pi(x) \\ &-\Gamma(\zeta+2)\Big[(\varsigma_2-x)^{\zeta+1}J^{\zeta}_{x-}\Pi(\varsigma_1)+(x-\varsigma_1)^{\zeta+1}J^{\zeta}_{x+}\Pi(\varsigma_2)\Big| \\ \le\ &\|\Pi''\|_\infty(\varsigma_2-\varsigma_1)(x-\varsigma_1)^{\zeta+1}(\varsigma_2-x)^{\zeta+1}\int_0^1 (t^{\zeta+1}+(1-t)^{\zeta+1})h(t)dt. \end{aligned}$$

Theorem 62 ([37]). *Let Π be as in Theorem 58. If $|\Pi''|$ is m-convex on $[\varsigma_1, \varsigma_2]$, and Π'' is bounded, i.e., $\|\Pi''\|_\infty = \sup_{x\in[\varsigma_1,\varsigma_2]} |\Pi''(x)| < \infty$, for any $x \in [\varsigma_1, \varsigma_2]$, then*

$$\begin{aligned} &\Big|(\zeta+1)(\varsigma_2-x)^{\zeta}(x-\varsigma_1)^{\zeta}(\varsigma_2-\varsigma_1)\Pi(x) \\ &-\Gamma(\zeta+2)\Big[(\varsigma_2-x)^{\zeta+1}J^{\zeta}_{x-}\Pi(\varsigma_1)+(x-\varsigma_1)^{\zeta+1}J^{\zeta}_{x+}\Pi(\varsigma_2)\Big| \\ \le\ &\|\Pi''\|_\infty(1-m)(\varsigma_2-\varsigma_1)(x-\varsigma_1)^{\zeta+1}(\varsigma_2-x)^{\zeta+1}\Big\{(x-\varsigma_1)\Big[\frac{1}{\zeta+3}\Big(\frac{x-\varsigma_1}{x-m\varsigma_1}\Big) \\ &+\frac{1}{\zeta+2}\Big(\frac{(1-m)\varsigma_1}{x-m\varsigma_1}+\frac{m}{1-m}\Big)\Big]+(\varsigma_2-x)\Big[-\frac{1}{\zeta+3}\Big(\frac{\varsigma_2-x}{\varsigma_2-mx}\Big)+\frac{1}{\zeta+2}\Big(\frac{1}{1-m}\Big)\Big]\Big\}. \end{aligned}$$

Theorem 63 ([37]). *Let Π be as in Theorem 58. If $|\Pi''|$ is (s, m)-convex on $[\varsigma_1, \varsigma_2]$, $(s, m) \in (0, 1]^2$ and Π'' is bounded, i.e., $\|\Pi''\|_\infty = \sup_{x\in[\varsigma_1,\varsigma_2]} |\Pi''(x)| < \infty$, for any $x \in [\varsigma_1, \varsigma_2]$, then*

$$\begin{aligned} &\Big|(\zeta+1)(\varsigma_2-x)^{\zeta}(x-\varsigma_1)^{\zeta}(\varsigma_2-\varsigma_1)\Pi(x) \\ &-\Gamma(\zeta+2)\Big[(\varsigma_2-x)^{\zeta+1}J^{\zeta}_{x-}\Pi(\varsigma_1)+(x-\varsigma_1)^{\zeta+1}J^{\zeta}_{x+}\Pi(\varsigma_2)\Big| \\ \le\ &\|\Pi''\|_\infty(1-m)\Big[\frac{m}{(1-m)(\zeta+2)}(\varsigma_2-x)^{\zeta+1}(x-\varsigma_1)^{\zeta+1}(\varsigma_2-\varsigma_1) \\ &+(\varsigma_2-x)^{\zeta+1}(x-m\varsigma_1)^{-s}((1-m)\varsigma_1)^{\zeta+s+2}B(\zeta+2,-s-\zeta-2) \\ &+(x-\varsigma_1)^{\zeta+1}(\varsigma_2-mx)^{\zeta+2}B(\zeta+2,s+1)\Big]. \end{aligned}$$

2.12. Ostrowski-Type Fractional Integral Inequalities for n-Polynomial Exponentially s-Convex Functions

Now, we present some Ostrowski-type inequalities for differentiable exponentially s-convex functions.

Definition 19 ([38]). *Let $s \in [\ln 2.4, 1]$. Then the real-valued function Π is an exponentially s-convex function if*

$$\Pi(tx+(1-t)y) \le (e^{st}-1)\Pi(x)+(e^{s(1-t)}-1)\Pi(y),$$

$\forall\ x, y \in I$ and $t \in [0, 1]$.

Theorem 64 ([38]). *Let $\Pi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $\varsigma_1, \varsigma_2 \in I$ with $\varsigma_1 < \varsigma_2$. If $|\Pi'|$ is an exponentially s-convex function on $[\varsigma_1, \varsigma_2]$ for some $s \in [\ln 2.4, 1]$, $\Pi' \in L[\varsigma_1, \varsigma_2]$ and $|\Pi'(x)| \le M$, for all $x \in [\varsigma_1, \varsigma_2]$, then*

$$\begin{aligned}
&\left|\Pi(x) - \frac{1}{\varsigma_2 - \varsigma_1}\int_{\varsigma_1}^{\varsigma_2} \Pi(z)dz\right| \\
\le\ & \frac{M}{(\varsigma_2 - \varsigma_1)}\Big[(x-\varsigma_1)^2\Big\{\Big(\frac{2+2(s-1)e^s - s^2)}{2s^2}\Big) + \Big(\frac{2e^s - s^2 - 2s - 2}{2s^2}\Big)\Big\} \\
& +(\varsigma_2 - x)^2\Big\{\Big(\frac{2+2(s-1)e^s - s^2)}{2s^2}\Big) + \Big(\frac{2e^s - s^2 - 2s - 2}{2s^2}\Big)\Big\}\Big],
\end{aligned}$$

for all $x \in [\varsigma_1, \varsigma_2]$.

Theorem 65 ([38]). *Let $\Pi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $\varsigma_1, \varsigma_2 \in I$ with $\varsigma_1 < \varsigma_2$. If $|\Pi'|^q$ is an exponentially s-convex function on $[\varsigma_1, \varsigma_2]$ for some $s \in [\ln 2.4, 1]$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\Pi' \in L[\varsigma_1, \varsigma_2]$ and $|\Pi'(x)| \le M$, for all $x \in [\varsigma_1, \varsigma_2]$, then*

$$\begin{aligned}
&\left|\Pi(x) - \frac{1}{\varsigma_2 - \varsigma_1}\int_{\varsigma_1}^{\varsigma_2} \Pi(z)dz\right| \\
\le\ & \frac{2^{\frac{1}{q}} M}{(\varsigma_2 - \varsigma_1)\sqrt[q]{n}}\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big[(x-\varsigma_1)^2\Big\{\Big(\frac{e^s - s - 1)}{s}\Big)\Big\}^{\frac{1}{q}} + (\varsigma_2 - x)^2\Big\{\Big(\frac{e^s - s - 1)}{s}\Big)\Big\}^{\frac{1}{q}}\Big],
\end{aligned}$$

for all $x \in [\varsigma_1, \varsigma_2]$.

Theorem 66 ([38]). *Let $\Pi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $\varsigma_1, \varsigma_2 \in I$ with $\varsigma_1 < \varsigma_2$. If $|\Pi'|^q, q \ge 1$ is an exponentially s-convex function on $[\varsigma_1, \varsigma_2]$ for some $s \in [\ln 2.4, 1]$, $\Pi' \in L[\varsigma_1, \varsigma_2]$ and $|\Pi'(x)| \le M$, for all $x \in [\varsigma_1, \varsigma_2]$, then*

$$\begin{aligned}
&\left|\Pi(x) - \frac{1}{\varsigma_2 - \varsigma_1}\int_{\varsigma_1}^{\varsigma_2} \Pi(z)dz\right| \\
\le\ & \frac{M}{(\varsigma_2 - \varsigma_1)2^{1-\frac{1}{q}}}\Big[(x-\varsigma_1)^2\Big\{\Big(\frac{2+2(s-1)e^s - s^2)}{2s^2}\Big) + \Big(\frac{2e^s - s^2 - 2s - 2}{2s^2}\Big)\Big\}^{\frac{1}{q}} \\
& +(\varsigma_2 - x)^2\Big\{\Big(\frac{2+2(s-1)e^s - s^2)}{2s^2}\Big) + \Big(\frac{2e^s - s^2 - 2s - 2}{2s^2}\Big)\Big\}^{\frac{1}{q}}\Big],
\end{aligned}$$

for all $x \in [\varsigma_1, \varsigma_2]$.

Some enhancements of the Ostrowski-type inequality for differentiable n-polynomial exponentially s-convex functions are presented in the next theorems.

Definition 20 ([39]). *Let $n \in \mathbb{N}$ and $s \in [\ln 2.4, 1]$. Then $\Pi : I \subset \mathbb{R} \to \mathbb{R}$ is an n-polynomial exponentially s-convex function if*

$$\Pi(tx + (1-t)y) \le \frac{1}{n}\sum_{i=1}^{n}(e^{st} - 1)^i \Pi(x) + \frac{1}{n}\sum_{i=1}^{n}(e^{s(1-t)} - 1)^i \Pi(y),$$

for all $x, y \in I$ and $t \in [0, 1]$.

Theorem 67 ([39]). *Let $\Pi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $[\varsigma_1, \varsigma_2] \in I$ with $\varsigma_1 < \varsigma_2$. If $|\Pi'|$ is an n-polynomial exponentially s-convex function on $[\varsigma_1, \varsigma_2]$ for some $s \in [\ln 2.4, 1]$, $\Pi' \in L[\varsigma_1, \varsigma_2]$ and $|\Pi'(x)| \le M$, for all $x \in [\varsigma_1, \varsigma_2]$, then*

$$\begin{aligned}
&\left|\Pi(x) - \frac{1}{\varsigma_2 - \varsigma_1}\int_{\varsigma_1}^{\varsigma_2} \Pi(z)dz\right| \\
\le\ & \frac{M}{(\varsigma_2 - \varsigma_1)n}\Bigg[(x-\varsigma_1)^2\Bigg\{\sum_{i=1}^{n}\left(\frac{2+2(s-1)e^s - s^2)}{2s^2}\right)^i + \sum_{i=1}^{n}\left(\frac{2e^s - s^2 - 2s - 2}{2s^2}\right)^i\Bigg\} \\
&+(\varsigma_2 - x)^2\Bigg\{\sum_{i=1}^{n}\left(\frac{2+2(s-1)e^s - s^2)}{2s^2}\right)^i + \sum_{i=1}^{n}\left(\frac{2e^s - s^2 - 2s - 2}{2s^2}\right)^i\Bigg\}\Bigg],
\end{aligned}$$

for all $x \in [\varsigma_1, \varsigma_2]$.

Theorem 68 ([39]). *Let Π be as in Theorem 67. If $|\Pi'|^q$ is an n-polynomial exponentially s-convex function on $[\varsigma_1, \varsigma_2]$ for some $s \in [\ln 2.4, 1]$, $q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$, $\Pi' \in L[\varsigma_1, \varsigma_2]$ and $|\Pi'(x)| \le M$, for all $x \in [\varsigma_1, \varsigma_2]$, then:*

$$\begin{aligned}
&\left|\Pi(x) - \frac{1}{\varsigma_2 - \varsigma_1}\int_{\varsigma_1}^{\varsigma_2} \Pi(z)dz\right| \\
\le\ & \frac{2^{\frac{1}{q}} M}{(\varsigma_2 - \varsigma_1)\sqrt[q]{n}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\Bigg[(x-\varsigma_1)^2\Bigg\{\sum_{i=1}^{n}\left(\frac{e^s - s - 1)}{s}\right)^i\Bigg\}^{\frac{1}{q}} \\
&+(\varsigma_2 - x)^2\Bigg\{\sum_{i=1}^{n}\left(\frac{e^s - s - 1)}{s}\right)^i\Bigg\}^{\frac{1}{q}}\Bigg],
\end{aligned}$$

for all $x \in [\varsigma_1, \varsigma_2]$.

Theorem 69 ([39]). *Let Π be as in Theorem 67. If $|\Pi'|^q, q \ge 1$ is an n-polynomial exponentially s-convex function on $[\varsigma_1, \varsigma_2]$ for some $s \in [\ln 2.4, 1]$, $\Pi' \in L[\varsigma_1, \varsigma_2]$ and $|\Pi'(x)| \le M$, for all $x \in [\varsigma_1, \varsigma_2]$, then*

$$\begin{aligned}
&\left|\Pi(x) - \frac{1}{\varsigma_2 - \varsigma_1}\int_{\varsigma_1}^{\varsigma_2} \Pi(z)dz\right| \\
\le\ & \frac{M}{(\varsigma_2 - \varsigma_1)\sqrt[q]{n}2^{1-\frac{1}{q}}}\Bigg[(x-\varsigma_1)^2\Bigg\{\sum_{i=1}^{n}\left(\frac{2+2(s-1)e^s - s^2)}{2s^2}\right)^i + \sum_{i=1}^{n}\left(\frac{2e^s - s^2 - 2s - 2}{2s^2}\right)^i\Bigg\}^{\frac{1}{q}} \\
&+(\varsigma_2 - x)^2\Bigg\{\sum_{i=1}^{n}\left(\frac{2+2(s-1)e^s - s^2)}{2s^2}\right)^i + \sum_{i=1}^{n}\left(\frac{2e^s - s^2 - 2s - 2}{2s^2}\right)^i\Bigg\}^{\frac{1}{q}}\Bigg],
\end{aligned}$$

for all $x \in [\varsigma_1, \varsigma_2]$.

Theorem 70 ([40]). *Let the assumptions of this theorem be stated in Theorem 67. Then*

$$\begin{aligned}&\left|\left(\frac{(x-\varsigma_1)^{\zeta}+(\varsigma_2-x)^{\zeta}}{\varsigma_2-\varsigma_1}\right)\Pi(x)-\frac{\Gamma(\zeta+1)}{\varsigma_2-\varsigma_1}\left[J^{\zeta}_{x-}\Pi(\varsigma_1)+J^{\zeta}_{x+}\Pi(\varsigma_2)\right|\right.\\ \leq\ &\frac{M}{n(\varsigma_2-\varsigma_1)}\left[(x-\varsigma_1)^{\zeta+1}\left\{\sum_{i=1}^{n}\left(\frac{B(\zeta+1,-s)-\Gamma(\zeta+1)}{(-s)^{\zeta}s}-\frac{1}{\zeta+1}\right)^{i}\right.\right.\\ &\left.-\sum_{i=1}^{n}\left(\frac{(B(\zeta+1,s)-\Gamma(\zeta+1))e^{s}}{s^{\zeta+1}}+\frac{1}{\zeta+1}\right)^{i}\right\}\\ &+(\varsigma_2-x)^{\zeta+1}\left\{\sum_{i=1}^{n}\left(\frac{B(\zeta+1,-s)-\Gamma(\zeta+1)}{(-s)^{\zeta}s}-\frac{1}{\zeta+1}\right)^{i}\right.\\ &\left.\left.-\sum_{i=1}^{n}\left(\frac{(B(\zeta+1,s)-\Gamma(\zeta+1))e^{s}}{s^{\zeta+1}}+\frac{1}{\zeta+1}\right)^{i}\right\}\right],\end{aligned}$$

for all $x\in(\varsigma_1,\varsigma_2)$.

Theorem 71 ([40]). *Let* Π *be as in Theorem 67. If* $|\Pi'|^{q}$ *is an n-polynomial exponentially s-convex function on* $[\varsigma_1,\varsigma_2]$ *for some* $s\in(0,1)$, $q>1$, *and* $\frac{1}{p}+\frac{1}{q}=1$, $\Pi'\in L[\varsigma_1,\varsigma_2]$ *and* $|\Pi'(x)|\leq M$, *for all* $x\in[\varsigma_1,\varsigma_2]$, *then:*

$$\begin{aligned}&\left|\left(\frac{(x-\varsigma_1)^{\zeta}+(\varsigma_2-x)^{\zeta}}{\varsigma_2-\varsigma_1}\right)\Pi(x)-\frac{\Gamma(\zeta+1)}{\varsigma_2-\varsigma_1}\left[J^{\zeta}_{x-}\Pi(\varsigma_1)+J^{\zeta}_{x+}\Pi(\varsigma_2)\right|\right.\\ \leq\ &\frac{2^{\frac{1}{q}}M}{\sqrt[q]{n}(\varsigma_2-\varsigma_1)}\left(\frac{1}{\zeta p+1}\right)^{\frac{1}{p}}\left[(x-\varsigma_1)^{\zeta+1}\left\{\sum_{i=1}^{n}\left(\frac{e^{s}-s-1}{s}\right)^{i}\right\}^{\frac{1}{q}}\right.\\ &\left.+(\varsigma_2-x)^{\zeta+1}\left\{\sum_{i=1}^{n}\left(\frac{e^{s}-s-1}{s}\right)^{i}\right\}^{\frac{1}{q}}\right],\end{aligned}$$

for all $x\in(\varsigma_1,\varsigma_2)$.

Theorem 72 ([40]). *Let* Π *be as in Theorem 67. If* $|\Pi'|^{q}, q\geq 1$ *is an n-polynomial exponentially s-convex function on* $[\varsigma_1,\varsigma_2]$ *for some* $s\in(0,1)$, $\Pi'\in L[\varsigma_1,\varsigma_2]$ *and* $|\Pi'(x)|\leq M$, *for all* $x\in[\varsigma_1,\varsigma_2]$, *then:*

$$\begin{aligned}&\left|\left(\frac{(x-\varsigma_1)^{\zeta}+(\varsigma_2-x)^{\zeta}}{\varsigma_2-\varsigma_1}\right)\Pi(x)-\frac{\Gamma(\zeta+1)}{\varsigma_2-\varsigma_1}\left[J^{\zeta}_{x-}\Pi(\varsigma_1)+J^{\zeta}_{x+}\Pi(\varsigma_2)\right|\right.\\ \leq\ &\frac{M}{\sqrt[q]{n}(\varsigma_2-\varsigma_1)}\left(\frac{1}{\zeta+1}\right)^{1-\frac{1}{p}}\left[(x-\varsigma_1)^{\zeta+1}\left\{\sum_{i=1}^{n}\left(\frac{B(\zeta+1,-s)-\Gamma(\zeta+1)}{(-s)^{\zeta}s}-\frac{1}{\zeta+1}\right)^{i}\right.\right.\\ &\left.-\sum_{i=1}^{n}\left(\frac{(B(\zeta+1,s)-\Gamma(\zeta+1))e^{s}}{s^{\zeta+1}}-\frac{1}{\zeta+1}\right)^{i}\right\}^{\frac{1}{q}}\\ &+(\varsigma_2-x)^{\zeta+1}\left\{\sum_{i=1}^{n}\left(\frac{B(\zeta+1,-s)-\Gamma(\zeta+1)}{(-s)^{\zeta}s}-\frac{1}{\zeta+1}\right)^{i}\right.\\ &\left.\left.-\sum_{i=1}^{n}\left(\frac{(B(\zeta+1,s)-\Gamma(\zeta+1))e^{s}}{s^{\zeta+1}}-\frac{1}{\zeta+1}\right)^{i}\right\}^{\frac{1}{q}}\right],\end{aligned}$$

for all $x\in(\varsigma_1,\varsigma_2)$.

3. Ostrowski-Type Inequalities for Katugampola Fractional Integral Operator

Here, we present some Ostrowski-type inequalities via the Katugampola fractional integral operator.

Definition 21 ([41]). *Let $[\varsigma_1, \varsigma_2] \subset \mathbb{R}$ be a finite interval. Then, the left- and right-side Katugampola fractional integral of order $\zeta > 0$ of $\Pi \in X_c^p(\varsigma_1, \varsigma_2)$ are defined by*

$$^{\rho}I^{\zeta}_{\varsigma_1+}\Pi(x) = \frac{\rho^{1-\zeta}}{\Gamma(\zeta)} \int_{\varsigma_1}^{x} \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-\zeta}} \Pi(t)dt \text{ and } ^{\rho}I^{\zeta}_{\varsigma_2-}\Pi(x) = \frac{\rho^{1-\zeta}}{\Gamma(\zeta)} \int_{x}^{\varsigma_2} \frac{t^{\rho-1}}{(t^\rho - x^\rho)^{1-\zeta}} \Pi(t)dt,$$

with $\varsigma_1 < x < \varsigma_2$ and $\rho > 0$, if the integrals exist. Here, $X_c^p(\varsigma_1, \varsigma_2)$, $c \in \mathbb{R}$, $1 \leq p \leq \infty$ denote the space of those complex-valued Lebesque measurable functions Π on $[\varsigma_1, \varsigma_2]$ for which $\|\Pi\|_{X_c^p} < \infty$, where $\|\Pi\|_{X_c^p} = \left(\int_{\varsigma_1}^{\varsigma_2} |t^c\Pi(t)|^p \frac{dt}{t}\right)^{1/p} < \infty$ for $1 \leq p < \infty$ and $\|\Pi\|_{X_c^p} =$ ess $\sup_{x_1 \leq t \leq x_2}[t^c|\Pi(t)|]$, if $p = \infty$.

Theorem 73 ([42]). *Let $\Pi : [\varsigma_1^\rho, \varsigma_2^\rho] \subseteq [0, \infty) \to \mathbb{R}$ be a differentiable function on $(\varsigma_1^\rho, \varsigma_2^\rho)$ with $\varsigma_1^\rho < \varsigma_2^\rho$ such that $\Pi' \in L[\varsigma_1^\rho, \varsigma_2^\rho]$. If Π' is h-convex on $[\varsigma_1^\rho, \varsigma_2^\rho]$ and $|\Pi'(x^\rho)| \leq M$, $x \in [\varsigma_1, \varsigma_2]$, then*

$$\begin{aligned}
&\left|\Pi(x^\rho) - \frac{(\zeta\rho + \rho - 1)\Gamma(\zeta)}{\rho^{1-\zeta}}\left[\frac{^{\rho}I^{\zeta}_{x-}\Pi(\varsigma_1^\rho)}{2(x^\rho - \varsigma_1^\rho)^\zeta} + \frac{^{\rho}I^{\zeta}_{x+}\Pi(\varsigma_2^\rho)}{2(\varsigma_2^\rho - x^\rho)^\zeta}\right]\right| \\
\leq\ & \frac{M\rho(\varsigma_2^\rho - \varsigma_1^\rho)}{2} \int_0^1 t^{\zeta\rho+\rho-1}[h(t^\rho) + h(1 - t^\rho)]dt,
\end{aligned}$$

with $\zeta, \rho > 0$ and $x \in (\varsigma_1^\rho, \varsigma_2^\rho)$.

Theorem 74 ([42]). *Let Π be as in Theorem 73. If $|\Pi'|^q, q > 1$ is h-convex on $[\varsigma_1^\rho, \varsigma_2^\rho]$ and $|\Pi'(x^\rho)| \leq M$, $x \in [\varsigma_1, \varsigma_2]$, then:*

$$\begin{aligned}
&\left|\Pi(x^\rho) - \frac{(\zeta\rho + \rho - 1)\Gamma(\zeta)}{\rho^{1-\zeta}}\left[\frac{^{\rho}I^{\zeta}_{x-}\Pi(\varsigma_1^\rho)}{2(x^\rho - \varsigma_1^\rho)^\zeta} + \frac{^{\rho}I^{\zeta}_{x+}\Pi(\varsigma_2^\rho)}{2(\varsigma_2^\rho - x^\rho)^\zeta}\right]\right| \\
\leq\ & \frac{M\rho(\varsigma_2^\rho - \varsigma_1^\rho)}{2(p(\zeta\rho + \rho - 1) + 1)^{\frac{1}{p}}} \left(\int_0^1 [h(t^\rho) + h(1 - t^\rho)]dt\right)^{\frac{1}{q}},
\end{aligned}$$

with $\zeta, \rho > 0$, $x \in (\varsigma_1^\rho, \varsigma_2^\rho)$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 75 ([42]). *Let Π be as in Theorem 73. If $|\Pi'|^q, q \geq 1$ is h-convex on $[\varsigma_1^\rho, \varsigma_2^\rho]$ and $|\Pi'(x^\rho)| \leq M$, $x \in [\varsigma_1, \varsigma_2]$, then:*

$$\begin{aligned}
&\left|\Pi(x^\rho) - \frac{(\zeta\rho + \rho - 1)\Gamma(\zeta)}{\rho^{1-\zeta}}\left[\frac{^{\rho}I^{\zeta}_{x-}\Pi(\varsigma_1^\rho)}{2(x^\rho - \varsigma_1^\rho)^\zeta} + \frac{^{\rho}I^{\zeta}_{x+}\Pi(\varsigma_2^\rho)}{2(\varsigma_2^\rho - x^\rho)^\zeta}\right]\right| \\
\leq\ & \frac{M\rho(\varsigma_2^\rho - \varsigma_1^\rho)}{2}\left(\frac{1}{\rho(\zeta + 1)}\right)^{1-\frac{1}{q}} \left(\int_0^1 t^{\zeta\rho+\rho-1}[h(t^\rho) + h(1 - t^\rho)]dt\right)^{\frac{1}{q}},
\end{aligned}$$

with $\zeta, \rho > 0$ and $x \in (\varsigma_1^\rho, \varsigma_2^\rho)$.

Some Ostrowski-type inequalities pertaining to Katugampola fractional integral for s-Godunova-Levin functions are presented.

Theorem 76 ([43]). *Let $\Pi : [\varsigma_1^\rho, \varsigma_2^\rho] \subseteq [0, \infty) \to \mathbb{R}$ be a function which is differentiable on $(\varsigma_1^\rho, \varsigma_2^\rho)$ with $\varsigma_1 < \varsigma_2$ such that $\Pi' \in L[\varsigma_1, \varsigma_2]$. If $|\Pi'|$ is an s-Godunova-Levin function of the second kind on $[\varsigma_1^\rho, \varsigma_2^\rho]$ and $|\Pi'(x^\rho)| \le M,\ x \in [\varsigma_1, \varsigma_2]$, then:*

$$\begin{aligned}&\left|\Big(\frac{(x^\rho-\varsigma_1^\rho)^\zeta+(\varsigma_2^\rho-x)^\zeta}{\varsigma_2-\varsigma_1}\Big)\Pi(x^\rho)-\frac{(\zeta\rho+\rho-1)\Gamma(\zeta)}{\rho^{1-\zeta}(\varsigma_2-\varsigma_1)}\Big[{}^\rho I_{x-}^\zeta\Pi(\varsigma_1^\rho+{}^\rho I_{x-}^\zeta\Pi(\varsigma_1^\rho\Big]\right|\\ \le\ & M\Big[\frac{(x^\rho-\varsigma_1^\rho)^{\zeta+1}+(\varsigma_2^\rho-x)^{\zeta+1}}{\varsigma_2-\varsigma_1}\Big]\Big[\frac{1}{\zeta+1-s}+\frac{\Gamma(\zeta+1)\Gamma(1-s)}{\Gamma(\zeta+2-s)}\Big],\end{aligned}$$

with $\zeta, \rho > 0$ and $x \in (\varsigma_1^\rho, \varsigma_2^\rho)$.

Theorem 77 ([43]). *Let Π be as in Theorem 76. If $|\Pi'|^q, q > 1$ is an s-Godunova-Levin function of the second kind on $[\varsigma_1^\rho, \varsigma_2^\rho]$ and $|\Pi'(x^\rho)| \le M,\ x \in [\varsigma_1, \varsigma_2]$, then:*

$$\begin{aligned}&\left|\Big(\frac{(x^\rho-\varsigma_1^\rho)^\zeta+(\varsigma_2^\rho-x)^\zeta}{\varsigma_2-\varsigma_1}\Big)\Pi(x^\rho)-\frac{(\zeta\rho+\rho-1)\Gamma(\zeta)}{\rho^{1-\zeta}(\varsigma_2-\varsigma_1)}\Big[{}^\rho I_{x-}^\zeta\Pi(\varsigma_1^\rho+{}^\rho I_{x-}^\zeta\Pi(\varsigma_1^\rho\Big]\right|\\ \le\ & M\rho\Big[\frac{(x^\rho-\varsigma_1^\rho)^{\zeta+1}+(\varsigma_2^\rho-x)^{\zeta+1}}{(\varsigma_2-\varsigma_1)(1+p(\zeta\rho+\rho-1))^{\frac{1}{p}}}\Big]\Big[\frac{1}{1-\rho s}\Big]^{\frac{1}{q}},\end{aligned}$$

with $\zeta, \rho > 0$, $x \in (\varsigma_1^\rho, \varsigma_2^\rho)$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 78 ([43]). *Let Π be as in Theorem 76. If $|\Pi'|^q, q \ge 1$ is an s-Godunova-Levin function of the second kind on $[\varsigma_1^\rho, \varsigma_2^\rho]$ and $|\Pi'(x^\rho)| \le M,\ x \in [\varsigma_1, \varsigma_2]$, then*

$$\begin{aligned}&\left|\Big(\frac{(x^\rho-\varsigma_1^\rho)^\zeta+(\varsigma_2^\rho-x)^\zeta}{\varsigma_2-\varsigma_1}\Big)\Pi(x^\rho)-\frac{(\zeta\rho+\rho-1)\Gamma(\zeta)}{\rho^{1-\zeta}(\varsigma_2-\varsigma_1)}\Big[{}^\rho I_{x-}^\zeta\Pi(\varsigma_1^\rho+{}^\rho I_{x-}^\zeta\Pi(\varsigma_1^\rho\Big]\right|\\ \le\ & \frac{M\rho}{(\zeta\rho+\rho)^{1-\frac{1}{q}}}\Big[\frac{(x^\rho-\varsigma_1^\rho)^{\zeta+1}+(\varsigma_2^\rho-x)^{\zeta+1}}{\varsigma_2-\varsigma_1}\Big]\Big(\frac{1}{\rho(\zeta-s+1)}+\frac{\Gamma(\zeta+1)\Gamma(1-s)}{\rho\Gamma(\zeta-s+2)}\Big)^{\frac{1}{q}},\end{aligned}$$

with $\zeta, \rho > 0$ and $x \in (\varsigma_1^\rho, \varsigma_2^\rho)$.

Theorem 79 ([43]). *Let the assumptions of this theorem be as stated in Theorem 76. Then:*

$$\begin{aligned}&\left|\Pi(x^\rho)-\frac{(\zeta\rho+\rho-1)\Gamma(\zeta)}{\rho^{1-\zeta}}\Big[\frac{{}^\rho I_{x-}^\zeta\Pi(\varsigma_1^\rho}{2(x^\rho-\varsigma_1^\rho)^\zeta}+\frac{{}^\rho I_{x+}^\zeta\Pi(\varsigma_2^\rho}{2(\varsigma_2^\rho-x^\rho)^\zeta}\Big]\right|\\ \le\ & \frac{M(\varsigma_2-\varsigma_1)}{2}\Big(\frac{1}{\zeta-s+1}+\frac{\Gamma(\zeta+1)\Gamma(1-s)}{\Gamma(\zeta-s+2)}\Big),\end{aligned}$$

with $\zeta, \rho > 0$ and $x \in (\varsigma_1^\rho, \varsigma_2^\rho)$.

Theorem 80 ([43]). *Let Π be as in Theorem 76. If $|\Pi'|^q, q > 1$ is an s-Godunova-Levin function of the second kind on $[\varsigma_1^\rho, \varsigma_2^\rho]$ and $|\Pi'(x^\rho)| \le M,\ x \in [\varsigma_1, \varsigma_2]$, then:*

$$\begin{aligned}&\left|\Pi(x^\rho)-\frac{(\zeta\rho+\rho-1)\Gamma(\zeta)}{\rho^{1-\zeta}}\Big[\frac{{}^\rho I_{x-}^\zeta\Pi(\varsigma_1^\rho}{2(x^\rho-\varsigma_1^\rho)^\zeta}+\frac{{}^\rho I_{x+}^\zeta\Pi(\varsigma_2^\rho}{2(\varsigma_2^\rho-x^\rho)^\zeta}\Big]\right|\\ \le\ & \frac{M\rho(\varsigma_2^\rho-\varsigma_1^\rho)}{2(1+p(\zeta\rho+\rho-1))^{\frac{1}{p}}}\Big(\frac{1}{1-\rho s}\Big)^{\frac{1}{q}},\end{aligned}$$

with $\zeta, \rho > 0$, $x \in (\varsigma_1^\rho, \varsigma_2^\rho)$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Using (ζ, m)-convex function with the aid of Katugampola fractional integral, some Ostrowski-type inequalities are obtained, which are given in the next theorems.

Theorem 81 ([44]). *Suppose $\Pi : I \to \mathbb{R}$ is a differentiable function on I such that $\Pi' \in L[m\varsigma_1, m\varsigma_2]$, where $m\varsigma_1, m\varsigma_2 \in I$ with $\varsigma_1 < \varsigma_2$, $m \in (0,1]$. If $|\Pi'|$ is (ζ, m)-convex on $[m\varsigma_1, m\varsigma_2]$ and $|\Pi'(x^\rho)| \leq M$, then*

$$\begin{aligned} &\Bigg| \Big(\frac{(x^\rho - m^\rho \varsigma_1^\rho)^\zeta + (m^\rho \varsigma_2^\rho - x)^\zeta}{\varsigma_2 - \varsigma_1} \Big) \Pi(x^\rho) \\ &- \frac{(\zeta\rho + \rho - 1)\Gamma(\zeta)}{\rho^{1-\zeta}(\varsigma_2 - \varsigma_1)} \Big[{}^\rho I^\zeta_{x-} \Pi(m^\rho \varsigma_1^\rho) + {}^\rho I^\zeta_{x-} \Pi(m^\rho \varsigma_1^\rho) \Big] \\ \leq \quad & M \Big[\frac{(x^\rho - m^\rho \varsigma_1^\rho)^{\zeta+1} + (m^\rho \varsigma_2^\rho - x)^{\zeta+1}}{\varsigma_2 - \varsigma_1} \Big] \Big[\frac{1 + m^\rho \zeta}{1 + 2\zeta} \Big], \end{aligned}$$

with $\zeta, \rho > 0$ and $x \in [m\varsigma_1, m\varsigma_2]$.

Theorem 82 ([44]). *Let Π be as in Theorem 81. If $|\Pi'|^q, q > 1$ is (ζ, m)-convex on $[m\varsigma_1, m\varsigma_2]$ and $|\Pi'(x^\rho)| \leq M$, then:*

$$\begin{aligned} &\Bigg| \Big(\frac{(x^\rho - m^\rho \varsigma_1^\rho)^\zeta + (m^\rho \varsigma_2^\rho - x)^\zeta}{\varsigma_2 - \varsigma_1} \Big) \Pi(x^\rho) \\ &- \frac{(\zeta\rho + \rho - 1)\Gamma(\zeta)}{\rho^{1-\zeta}(\varsigma_2 - \varsigma_1)} \Big[{}^\rho I^\zeta_{x-} \Pi(m^\rho \varsigma_1^\rho) + {}^\rho I^\zeta_{x-} \Pi(m^\rho \varsigma_1^\rho) \Big] \\ \leq \quad & M\rho \Big[\frac{(x^\rho - m^\rho \varsigma_1^\rho)^{\zeta+1} + (m^\rho \varsigma_2^\rho - x)^{\zeta+1}}{(\varsigma_2 - \varsigma_1)(p(\zeta\rho + \rho - 1) + 1)^{\frac{1}{p}}} \Big] \Big[\frac{1 + m^\rho \zeta\rho}{1 + \zeta\rho} \Big]^{\frac{1}{q}}, \end{aligned}$$

with $\zeta, \rho > 0$, $\frac{1}{p} + \frac{1}{q} = 1$ and $x \in [m\varsigma_1, m\varsigma_2]$.

Theorem 83 ([44]). *Let Π be as in Theorem 81. If $|\Pi'|^q, q \geq 1$ is (ζ, m)-convex on $[m\varsigma_1, m\varsigma_2]$ and $|\Pi'(x^\rho)| \leq M$, then:*

$$\begin{aligned} &\Bigg| \Big(\frac{(x^\rho - m^\rho \varsigma_1^\rho)^\zeta + (m^\rho \varsigma_2^\rho - x)^\zeta}{\varsigma_2 - \varsigma_1} \Big) \Pi(x^\rho) \\ &- \frac{(\zeta\rho + \rho - 1)\Gamma(\zeta)}{\rho^{1-\zeta}(\varsigma_2 - \varsigma_1)} \Big[{}^\rho I^\zeta_{x-} \Pi(m^\rho \varsigma_1^\rho) + {}^\rho I^\zeta_{x-} \Pi(m^\rho \varsigma_1^\rho) \Big] \\ \leq \quad & M\rho \Big[\frac{(x^\rho - m^\rho \varsigma_1^\rho)^{\zeta+1} + (m^\rho \varsigma_2^\rho - x)^{\zeta+1}}{(\varsigma_2 - \varsigma_1)(\rho(\zeta + 1))^{1-\frac{1}{q}}} \Big] \Big[\frac{1 + m^\rho \zeta}{\rho(2\zeta + 1)} \Big]^{\frac{1}{q}}, \end{aligned}$$

with $\zeta, \rho > 0$, and $x \in [m\varsigma_1, m\varsigma_2]$.

Theorem 84 ([44]). *Let Π be as in Theorem 81. If $|\Pi'|^q, q \geq 1$ is (ζ, m)-convex on $[m\varsigma_1, m\varsigma_2]$ and $|\Pi'(x^\rho)| \leq M$, then:*

$$\begin{aligned} &\Bigg| \Pi(x^\rho) - \frac{(\zeta\rho + \rho - 1)\Gamma(\zeta)}{\rho^{1-\zeta}} \Big[\frac{{}^\rho I^\zeta_{x-} \Pi(m^\rho \varsigma_1^\rho}{2(x^\rho - m^\rho \varsigma_1^\rho)^\zeta} + \frac{{}^\rho I^\zeta_{x+} \Pi(m^\rho \varsigma_2^\rho}{2(m^\rho \varsigma_2^\rho - x^\rho)^\zeta} \Big] \Bigg| \\ \leq \quad & \frac{M m^\rho [\varsigma_2^\rho - \varsigma_1^\rho]}{2} \Big[\frac{1 + m^\rho \zeta}{2\zeta + 1} \Big], \quad x \in [m\varsigma_1, m\varsigma_2], \end{aligned}$$

with $\zeta, \rho > 0$.

Theorem 85 ([44]). *Let Π be as in Theorem 81. If $|\Pi'|^q, q > 1$ is (ζ, m)-convex on $[m\varsigma_1, m\varsigma_2]$ and $|\Pi'(x^\rho)| \le M$, then:*

$$\left|\Pi(x^\rho) - \frac{(\zeta\rho + \rho - 1)\Gamma(\zeta)}{\rho^{1-\zeta}}\left[\frac{{}^\rho I^\zeta_{x-}\Pi(m^\rho\varsigma_1^\rho}{2(x^\rho - m^\rho\varsigma_1^\rho)^\zeta} + \frac{{}^\rho I^\zeta_{x+}\Pi(m^\rho\varsigma_2^\rho}{2(m^\rho\varsigma_2^\rho - x^\rho)^\zeta}\right]\right|$$
$$\le \frac{M\rho[m^\rho\varsigma_2^\rho - m^\rho\varsigma_1^\rho]}{2(p(\zeta\rho + \rho - 1) + 1)^{\frac{1}{p}}}\left[\frac{1 + m^\rho\zeta\rho}{\zeta\rho + 1}\right]^{\frac{1}{q}}, \quad x \in [m\varsigma_1, m\varsigma_2],$$

with $\zeta, \rho > 0$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

We continue by giving some Ostrowski-type inequalities for p-convex functions pertaining to the Katugampola fractional integral.

Definition 22 ([45]). *A function $\Pi : I \to \mathbb{R}$ is p-convex, if*

$$\Pi\Big([tx^p + (1-t)y^p]^{1/p}\Big) \le t\Pi(x) + (1-t)\Pi(y), \tag{5}$$

$\forall\, x, y \in I$ and $t \in [0,1]$.

Theorem 86 ([46]). *Let $\Pi : [\varsigma_1, \varsigma_2] \subset (0, \infty) \to \mathbb{R}$ be a function which is differentiable on $(\varsigma_1, \varsigma_2)$ with $\varsigma_1 < \varsigma_2$ such that $\Pi' \in L[\varsigma_1, \varsigma_2]$. Let $|\Pi'|$ be a p-convex function, $|\Pi'(x)| \le M, \forall x \in [\varsigma_1, \varsigma_2]$ and $\zeta > 0$.*

(i) If $p \in (-\infty, 0) \cup (1, \infty)$, then

$$\left|\frac{(x^p - \varsigma_1^p)^\zeta\Pi(\varsigma_1) + (\varsigma_2^p - x^p)^\zeta\Pi(\varsigma_2)}{p^\zeta(\varsigma_2 - \varsigma_1)} - \frac{\Gamma(\zeta+1)}{\varsigma_2 - \varsigma_1}\left[({}^p I^\zeta_{x-}\Pi)(\varsigma_1) + ({}^p I^\zeta_{x+}\Pi)(\varsigma_2)\right]\right|$$
$$\le \frac{\varsigma_1^{1-p}M}{p^{1+\zeta}(\zeta+1)}\left[\frac{(x^p - \varsigma_1^p)^{\zeta+1} + (\varsigma_2^p - x^p)^{\zeta+1}}{(\varsigma_2 - \varsigma_1)}\right], \quad x \in (\varsigma_1, \varsigma_2).$$

(ii) If $p \in (0,1)$, then we have:

$$\left|\frac{(x^p - \varsigma_1^p)^\zeta\Pi(\varsigma_1) + (\varsigma_2^p - x^p)^\zeta\Pi(\varsigma_2)}{p^\zeta(\varsigma_2 - \varsigma_1)} - \frac{\Gamma(\zeta+1)}{\varsigma_2 - \varsigma_1}\left[({}^p I^\zeta_{x-}\Pi)(\varsigma_1) + ({}^p I^\zeta_{x+}\Pi)(\varsigma_2)\right]\right|$$
$$\le \frac{\varsigma_2^{1-p}M}{p^{1+\zeta}(\zeta+1)}\left[\frac{(x^p - \varsigma_1^p)^{\zeta+1} + (\varsigma_2^p - x^p)^{\zeta+1}}{(\varsigma_2 - \varsigma_1)}\right], \quad x \in (\varsigma_1, \varsigma_2).$$

Theorem 87 ([46]). *Let Π be as in Theorem 86. Let $|\Pi'|^q$ be a p-convex function, $|\Pi'(x)| \le M, \forall x \in [\varsigma_1, \varsigma_2]$, $\zeta > 0$ and $r > 1$.*

(i) If $p \in (-\infty, 0) \cup (1, \infty)$, then

$$\left|\frac{(x^p - \varsigma_1^p)^\zeta\Pi(\varsigma_1) + (\varsigma_2^p - x^p)^\zeta\Pi(\varsigma_2)}{p^\zeta(\varsigma_2 - \varsigma_1)} - \frac{\Gamma(\zeta+1)}{\varsigma_2 - \varsigma_1}\left[({}^p I^\zeta_{x-}\Pi)(\varsigma_1) + ({}^p I^\zeta_{x+}\Pi)(\varsigma_2)\right]\right|$$
$$\le \frac{\varsigma_1^{1-p}M}{p^{1+\zeta}(1 + r\zeta)^{1/r}}\left[\frac{(x^p - \varsigma_1^p)^{\zeta+1} + (\varsigma_2^p - x^p)^{\zeta+1}}{\varsigma_2 - \varsigma_1}\right], \quad x \in (\varsigma_1, \varsigma_2).$$

(ii) *If $p \in (0,1)$, then*

$$\left|\frac{(x^p-\varsigma_1^p)^\zeta \Pi(\varsigma_1)+(\varsigma_2^p-x^p)^\zeta \Pi(\varsigma_2)}{p^\zeta(\varsigma_2-\varsigma_1)}-\frac{\Gamma(\zeta+1)}{\varsigma_2-\varsigma_1}\left[({}^pI^\zeta_{x-}\Pi)(\varsigma_1)+({}^pI^\zeta_{x+}\Pi)(\varsigma_2)\right]\right|$$
$$\leq \frac{\varsigma_2^{1-p}M}{p^{1+\zeta}(1+r\zeta)^{1/r}}\left[\frac{(x^p-\varsigma_1^p)^{\zeta+1}+(\varsigma_2^p-x^p)^{\zeta+1}}{\varsigma_2-\varsigma_1}\right], \quad x\in(\varsigma_1,\varsigma_2).$$

Theorem 88 ([46]). *Let the assumptions of this theorem be as stated in Theorem 87, and $r,q>1$ such that $1/r+1/q=1$.*

(i) *Suppose $p\in(-\infty,0)\cup(1,\infty)$, then*

$$\left|\frac{(x^p-\varsigma_1^p)^\zeta \Pi(\varsigma_1)+(\varsigma_2^p-x^p)^\zeta \Pi(\varsigma_2)}{p^\zeta(\varsigma_2-\varsigma_1)}-\frac{\Gamma(\zeta+1)}{\varsigma_2-\varsigma_1}\left[({}^pI^\zeta_{x-}\Pi)(\varsigma_1)+({}^pI^\zeta_{x+}\Pi)(\varsigma_2)\right]\right|$$
$$\leq \frac{(x^p-\varsigma_1^p)^{\zeta+1}+(\varsigma_2^p-x^p)^{\zeta+1}}{p^{1+\zeta}(\varsigma_2-\varsigma_1)}\left[\frac{(\varsigma_1^{1-p})^r}{r(\zeta r+1)}+\frac{M^q}{q}\right], \quad x\in(\varsigma_1,\varsigma_2).$$

(ii) *Suppose $p\in(0,1)$, then*

$$\left|\frac{(x^p-\varsigma_1^p)^\zeta \Pi(\varsigma_1)+(\varsigma_2^p-x^p)^\zeta \Pi(\varsigma_2)}{p^\zeta(\varsigma_2-\varsigma_1)}-\frac{\Gamma(\zeta+1)}{\varsigma_2-\varsigma_1}\left[({}^pI^\zeta_{x-}\Pi)(\varsigma_1)+({}^pI^\zeta_{x+}\Pi)(\varsigma_2)\right]\right|$$
$$\leq \frac{(x^p-\varsigma_1^p)^{\zeta+1}+(\varsigma_2^p-x^p)^{\zeta+1}}{p^{1+\zeta}(\varsigma_2-\varsigma_1)}\left[\frac{(\varsigma_2^{1-p})^r}{r(\zeta r+1)}+\frac{M^q}{q}\right], \quad x\in(\varsigma_1,\varsigma_2).$$

Theorem 89 ([46]). *Let the assumptions of this theorem be as stated in Theorem 86 and $r,q>0$ such that $r+q=1$.*

(i) *Suppose $p\in(-\infty,0)\cup(1,\infty)$, then*

$$\left|\frac{(x^p-\varsigma_1^p)^\zeta \Pi(\varsigma_1)+(\varsigma_2^p-x^p)^\zeta \Pi(\varsigma_2)}{p^\zeta(\varsigma_2-\varsigma_1)}-\frac{\Gamma(\zeta+1)}{\varsigma_2-\varsigma_1}\left[({}^pI^\zeta_{b-}\Pi)(\varsigma_1)+({}^pI^\zeta_{a+}\Pi)(\varsigma_2)\right]\right|$$
$$\leq \frac{(x^p-\varsigma_1^p)^{\zeta+1}+(\varsigma_2^p-x^p)^{\zeta+1}}{p^{1+\zeta}(\varsigma_2-\varsigma_1)}\left[\frac{r\varsigma_1^{1-p}}{\zeta+1}+Mq\right], \quad x\in(\varsigma_1,\varsigma_2).$$

(ii) *If $p\in(0,1)$, then*

$$\left|\frac{(x^p-\varsigma_1^p)^\zeta \Pi(\varsigma_1)+(\varsigma_2^p-x^p)^\zeta \Pi(\varsigma_2)}{p^\zeta(\varsigma_2-\varsigma_1)}-\frac{\Gamma(\zeta+1)}{\varsigma_2-\varsigma_1}\left[({}^pI^\zeta_{b-}\Pi)(\varsigma_1)+({}^pI^\zeta_{a+}\Pi)(\varsigma_2)\right]\right|$$
$$\leq \frac{(x^p-\varsigma_1^p)^{\zeta+1}+(\varsigma_2^p-x^p)^{\zeta+1}}{p^{1+\zeta}(\varsigma_2-\varsigma_1)}\left[\frac{r\varsigma_2^{1-p}}{\zeta+1}+Mq\right], \quad x\in(\varsigma_1,\varsigma_2).$$

Theorem 90 ([47]). *Let $\Pi: I\subset(0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° and $\varsigma_1,\varsigma_2\in I$ with $\varsigma_1<\varsigma_2$ such that $\Pi\in L[\varsigma_1,\varsigma_2]$. If $|\Pi'|$ is p-convex on I and $|\Pi'(x)|\leq M$, for all $x\in[\varsigma_1,2^{\frac{1}{\rho}}\varsigma_1]$ (if $2^{\frac{1}{\rho}}\varsigma_1<\varsigma_2$, otherwise $x\in[\varsigma_1,\varsigma_2]$), then*

$$\left|\frac{\rho\Pi(x)}{\varsigma_2-\varsigma_1}\left[(x^\rho-\varsigma_1^\rho)^\zeta+(\varsigma_2^\rho-x^\rho)^\zeta\right]-\frac{\rho^{\zeta+1}\Gamma(\zeta+1)}{\varsigma_2-\varsigma_1}\left[{}^\rho I^\zeta_{x-}\Pi(\varsigma_1)+{}^\rho I^\zeta_{x+}\Pi(\varsigma_2)\right]\right|$$
$$\leq M\frac{(x^\rho-\varsigma_1^\rho)^{\zeta+1}}{\varsigma_2-\varsigma_1}[R(\varsigma_1)+S(\varsigma_1)]+M\frac{(\varsigma_1^\rho-x^\rho)^{\zeta+1}}{\varsigma_2-\varsigma_1}[R(\varsigma_2)+S(\varsigma_2)],$$

where

$$\begin{aligned}
R(\lambda) &= \frac{\lambda^{1-\rho}}{\zeta+2}\,{}_2F_1\Big(\zeta+2,\frac{\rho-1}{\rho};\zeta+3;1-\frac{x^\rho}{\lambda^\rho}\Big),\\
S(\lambda) &= \frac{\lambda^{1-\rho}}{(\zeta+1)(\zeta+2)}\Big[(\zeta+2){}_2F_1\Big(\zeta+1,\frac{\rho-1}{\rho};\zeta+2;1-\frac{x^\rho}{\lambda^\rho}\Big)\\
&\quad -(\zeta+1){}_2F_1\Big(\zeta+2,\frac{\rho-1}{\rho};\zeta+3;1-\frac{x^\rho}{\lambda^\rho}\Big)\Big],
\end{aligned}$$

and $\rho > 1, \zeta > 0, \lambda \in \{\varsigma_1, \varsigma_2\}$ *and* ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ *is the hypergeometric function.*

Theorem 91 ([47])**.** *Let* Π *be as in Theorem 90. If* $|\Pi'|^q$ *is p-convex on I and* $|\Pi'(x)| \le M$*, for all* $x \in I\setminus\{\varsigma_1,\varsigma_2\}$ *then*

$$\begin{aligned}
&\left|\frac{\rho\Pi(x)}{\varsigma_2-\varsigma_1}\Big[(x^\rho-\varsigma_1^\rho)^\zeta+(\varsigma_2^\rho-x^\rho)^\zeta\Big]-\frac{\rho^{\zeta+1}\Gamma(\zeta+1)}{\varsigma_2-\varsigma_1}\Big[{}^\rho I^\zeta_{x-}\Pi(\varsigma_1)+{}^\rho I^\zeta_{x+}\Pi(\varsigma_2)\Big]\right.\\
\le\;& \frac{M}{\varsigma_2-\varsigma_1}\Big(\frac{1}{\zeta q+1}\Big)^{\frac{1}{q}}\Big[(x^\rho-\varsigma_1^\rho)^{\zeta+1}K^{\frac{1}{r}}(\varsigma_1)+(\varsigma_2^\rho-x^\rho)^{\zeta+1}K^{\frac{1}{r}}(\varsigma_2)\Big],
\end{aligned}$$

where

$$K(\lambda) = \frac{\rho(x^{r(1-\rho)+\rho}-\lambda^{r(1-\rho)+\rho})}{(x^\rho-\lambda^\rho)(r(1-\rho)+\rho)},$$

and $\rho > 0, \zeta > 0, \lambda \in \{\varsigma_1,\varsigma_2\}$ $r > 1$ *and* $\frac{1}{r}+\frac{1}{q} = 1.$

Theorem 92 ([47])**.** *Let* Π *be as in Theorem 90. If* $|\Pi'|$ *is p-convex on I and* $|\Pi'(x)| \le M$*, for all* $x \in [\varsigma_1, 2^{\frac{1}{\rho}}\varsigma_1]$ *(if* $2^{\frac{1}{\rho}}\varsigma_1 < \varsigma_2$*, otherwise* $x \in [\varsigma_1,\varsigma_2]$*), then*

$$\begin{aligned}
&\left|\frac{\rho\Pi(x)}{\varsigma_2-\varsigma_1}\Big[(x^\rho-\varsigma_1^\rho)^\zeta+(\varsigma_2^\rho-x^\rho)^\zeta\Big]-\frac{\rho^{\zeta+1}\Gamma(\zeta+1)}{\varsigma_2-\varsigma_1}\Big[{}^\rho I^\zeta_{x-}\Pi(\varsigma_1)+{}^\rho I^\zeta_{x+}\Pi(\varsigma_2)\Big]\right.\\
\le\;& \frac{M}{\varsigma_2-\varsigma_1}(x^\rho-\varsigma_1^\rho)^{\zeta+1}L^{1-\frac{1}{q}}(\varsigma_1)[R(\varsigma_1)+S(\varsigma_1)]^{\frac{1}{q}}\\
&+\frac{M}{\varsigma_2-\varsigma_1}(\varsigma_1^\rho-x^\rho)^{\zeta+1}L^{1-\frac{1}{q}}(\varsigma_2)[R(\varsigma_2)+S(\varsigma_2)]^{\frac{1}{q}},
\end{aligned}$$

where

$$L(\lambda) = \frac{\lambda^{1-\rho}}{\zeta+1}\,{}_2F_1\Big(\zeta+1,\frac{\rho-1}{\rho};\zeta+2;1-\frac{x^\rho}{\lambda^\rho}\Big),$$

and $\rho > 1, \zeta > 0, \lambda \in \{\varsigma_1,\varsigma_2\}$.

4. Ostrowski-Type Fractional Integral Inequalities via *k*-Riemann–Liouville Fractional Integral

Definition 23 ([48])**.** *Let* $\Pi \in L[\varsigma_1,\varsigma_2]$*,* $\varsigma_1 \ge 0$*, and* $k > 0$*. The k-Riemann–Liouville fractional integrals* $I^{\zeta,k}_{\varsigma_1+}\Pi$ *and* $I^{\zeta,k}_{\varsigma_2-}\Pi$ *of order* $\zeta > 0$ *for a real-valued function* Π *are defined by*

$$I^{\zeta,k}_{\varsigma_1+}\Pi(t) = \frac{1}{k\Gamma_k(\zeta)}\int_{x_1}^t (t-s)^{\frac{\zeta}{k}-1}\Pi(s)ds,\ \ t > \varsigma_1,$$

and

$$I^{\zeta,k}_{\varsigma_2-}\Pi(t) = \frac{1}{k\Gamma_k(\zeta)}\int_t^b (s-t)^{\frac{\zeta}{k}-1}\Pi(s)ds,\ \ t < \varsigma_2,$$

respectively, where Γ_k *is the k-Gamma function* $\Gamma_k(t) = \int_0^\infty s^{t-1}e^{-\frac{s^k}{k}}ds.$

We present some Ostrowski-type inequalities for s-Godunova-Levin of a second kind via the Riemann–Liouville k-fractional integral.

Theorem 93 ([49])**.** *Let $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be a function which is differentiable on $(\varsigma_1, \varsigma_2)$ with $\varsigma_1 < \varsigma_2$ such that $\Pi' \in L[\varsigma_1, \varsigma_2]$. If $|\Pi'|$ is an s-Godunova-Levin function of the second kind on $[\varsigma_1, \varsigma_2]$ and $|\Pi'(x)| \leq M,\ x \in [\varsigma_1, \varsigma_2]$, then, for all $x \in [\varsigma_1, \varsigma_2]$,*

$$\begin{aligned} &\left|\left(\frac{(x-\varsigma_1)^{\frac{\zeta}{k}} + (\varsigma_2 - x)^{\frac{\zeta}{k}}}{\varsigma_2 - \varsigma_1}\right)\Pi(x) - \frac{\Gamma_k(\zeta + k)}{\varsigma_2 - \varsigma_1}\left[J_{x-}^{\zeta,k}\Pi(\varsigma_1) + J_{x+}^{\zeta,k}\Pi(\varsigma_2)\right]\right| \\ \leq\ & M\left[\frac{(x-\varsigma_1)^{\frac{\zeta}{k}+1} + (\varsigma_2 - x)^{\frac{\zeta}{k}+1}}{\varsigma_2 - \varsigma_1}\right]\left[\frac{1}{\frac{\zeta}{k} + 1 - s} + \frac{\Gamma_k(\zeta + k)\Gamma_k(k - sk)}{\Gamma_k(\zeta + 2k - sk)}\right]. \end{aligned}$$

Theorem 94 ([49])**.** *Let Π be as in Theorem 93. If $|\Pi'|^q, q > 1$ is an s-Godunova-Levin function of the second kind on $[\varsigma_1, \varsigma_2]$ and $|\Pi'(x)| \leq M,\ x \in [\varsigma_1, \varsigma_2]$, then, for all $x \in [\varsigma_1, \varsigma_2]$,*

$$\begin{aligned} &\left|\left(\frac{(x-\varsigma_1)^{\frac{\zeta}{k}} + (\varsigma_2 - x)^{\frac{\zeta}{k}}}{\varsigma_2 - \varsigma_1}\right)\Pi(x) - \frac{\Gamma_k(\zeta + k)}{\varsigma_2 - \varsigma_1}\left[J_{x-}^{\zeta,k}\Pi(\varsigma_1) + J_{x+}^{\zeta,k}\Pi(\varsigma_2)\right]\right| \\ \leq\ & M\left[\frac{(x-\varsigma_1)^{\frac{\zeta}{k}+1} + (\varsigma_2 - x)^{\frac{\zeta}{k}+1}}{(\varsigma_2 - \varsigma_1)(1 + p\frac{\zeta}{k})^{\frac{1}{p}}}\right]\left(\frac{2}{1 - s}\right)^{\frac{1}{q}}, \end{aligned}$$

with $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Theorem 95 ([49])**.** *Let Π be as in Theorem 93. If $|\Pi'|^q, q \geq 1$ is an s-Godunova-Levin function of the second kind on $[\varsigma_1, \varsigma_2]$ and $|\Pi'(x)| \leq M,\ x \in [\varsigma_1, \varsigma_2]$, then, for all $x \in [\varsigma_1, \varsigma_2]$,*

$$\begin{aligned} &\left|\left(\frac{(x-\varsigma_1)^{\frac{\zeta}{k}} + (\varsigma_2 - x)^{\frac{\zeta}{k}}}{\varsigma_2 - \varsigma_1}\right)\Pi(x) - \frac{\Gamma_k(\zeta + k)}{\varsigma_2 - \varsigma_1}\left[J_{x-}^{\zeta,k}\Pi(\varsigma_1) + J_{x+}^{\zeta,k}\Pi(\varsigma_2)\right]\right| \\ \leq\ & \frac{M}{\left(1 + \frac{\zeta}{k}\right)^{1 - \frac{1}{q}}}\left[\frac{(x-\varsigma_1)^{\frac{\zeta}{k}+1} + (\varsigma_2 - x)^{\frac{\zeta}{k}+1}}{\varsigma_2 - \varsigma_1}\right]\left[\frac{1}{\frac{\zeta}{k} + 1 - s} + \frac{\Gamma_k(\zeta + k)\Gamma_k(k - sk)}{\Gamma_k(\zeta + 2k - sk)}\right]^{\frac{1}{q}}. \end{aligned}$$

Theorem 96 ([49])**.** *Let the assumptions of this theorem be stated in Theorem 93. Then, for all $x \in [\varsigma_1, \varsigma_2]$,*

$$\begin{aligned} &\left|\Pi(x) - \Gamma_k(\zeta + k)\left[\frac{1}{2(x-\varsigma_1)^{\frac{\zeta}{k}}}J_{x-}^{\zeta,k}\Pi(\varsigma_1) + \frac{1}{2(\varsigma_2 - x)^{\frac{\zeta}{k}}}J_{x+}^{\zeta,k}\Pi(\varsigma_2)\right]\right| \\ \leq\ & \frac{M(\varsigma_2 - \varsigma_1)}{2}\left[\frac{1}{\frac{\zeta}{k} + 1 - s} + \frac{\Gamma_k(\zeta + k)\Gamma_k(k - sk)}{\Gamma_k(\zeta + 2k - sk)}\right]. \end{aligned}$$

Here, utilizing strongly (β, m)-convex via the k-Riemann–Liouville fractional integral, some Ostrowski-type inequalities are presented.

Definition 24 ([50])**.** *A real-valued function $\Pi : [0, d] \subset \mathbb{R} \to \mathbb{R}$ is strongly (β, m)-convex, if*

$$\Pi(tx + (1-t)y) \leq t^{\beta}\Pi(x) + m(1 - t^{\beta})\Pi(y) - \mu t(1-t)(y-x)^2,$$

for all $x, y \in [0, d]$ and $t \in [0, 1]$.

Theorem 97 ([50]). *Let $\Pi : [0,\infty) \to \mathbb{R}$ be a differentiable function on $(0,\infty)$ such that $\Pi' \in L[\varsigma_1, \varsigma_2]$ with $0 \le \varsigma_1 < \varsigma_2$. If $|\Pi'|$ is strongly (β, m)-convex with modulus $\mu \ge 0$ for $\beta \in [0,1]$ and $m \in (0,1]$, then*

$$\left|\frac{(x-\varsigma_1)^{\frac{\zeta}{k}} + (\varsigma_2 - x)^{\frac{\zeta}{k}}}{\varsigma_2 - \varsigma_1}\Pi(x) - \frac{\Gamma(\zeta+k)}{\varsigma_2-\varsigma_1}\left[J^{\zeta,k}_{x-}\Pi(\varsigma_1) + \frac{1}{2(\varsigma_2-x)^{\frac{\zeta}{k}}} J^{\zeta,k}_{x+}\Pi(\varsigma_2)\right]\right|$$
$$\le \frac{(x-\varsigma_1)^{\frac{\zeta}{k}+1}}{\varsigma_2-\varsigma_1}\left[\frac{|\Pi'(x)|}{\frac{\zeta}{k}+\beta+1} + \frac{\beta m\left|\Pi'\left(\frac{\varsigma_1}{m}\right)\right|}{\left(\frac{\zeta}{k}+1\right)\left(\frac{\zeta}{k}+\beta+1\right)} - \frac{\mu\left(x-\frac{\varsigma_1}{m}\right)^2}{\left(\frac{\zeta}{k}+2\right)\left(\frac{\zeta}{k}+3\right)}\right]$$
$$+\frac{(\varsigma_2-x)^{\frac{\zeta}{k}+1}}{\varsigma_2-\varsigma_1}\left[\frac{|\Pi'(x)|}{\frac{\zeta}{k}+\beta+1} + \frac{\beta m\left|\Pi'\left(\frac{\varsigma_2}{m}\right)\right|}{\left(\frac{\zeta}{k}+1\right)\left(\frac{\zeta}{k}+\beta+1\right)} - \frac{\mu\left(\frac{\varsigma_2}{m}-x\right)^2}{\left(\frac{\zeta}{k}+2\right)\left(\frac{\zeta}{k}+3\right)}\right]$$

for all $x \in [\varsigma_1, \varsigma_2]$ and $\zeta, k > 0$.

Theorem 98 ([50]). *Let Π be as in Theorem 97. If $|\Pi'|^q, q > 1$ is strongly (β, m)-convex with modulus $\mu \ge 0$ for $\beta \in [0,1]$ and $m \in (0,1]$, then:*

$$\left|\frac{(x-\varsigma_1)^{\frac{\zeta}{k}} + (\varsigma_2 - x)^{\frac{\zeta}{k}}}{\varsigma_2 - \varsigma_1}\Pi(x) - \frac{\Gamma(\zeta+k)}{\varsigma_2-\varsigma_1}\left[J^{\zeta,k}_{x-}\Pi(\varsigma_1) + \frac{1}{2(\varsigma_2-x)^{\frac{\zeta}{k}}} J^{\zeta,k}_{x+}\Pi(\varsigma_2)\right]\right|$$
$$\le \frac{(x-\varsigma_1)^{\frac{\zeta}{k}+1}}{(\varsigma_2-\varsigma_1)\left(\frac{\zeta}{k}p+1\right)^{\frac{1}{p}}}\left[\frac{|\Pi'(x)|^q}{\beta+1} + \frac{\beta m\left|\Pi'\left(\frac{\varsigma_1}{m}\right)\right|^q}{\beta+1} - \frac{\mu\left(x-\frac{\varsigma_1}{m}\right)^2}{6}\right]^{\frac{1}{q}}$$
$$+\frac{(\varsigma_2-x)^{\frac{\zeta}{k}+1}}{(\varsigma_2-\varsigma_1)\left(\frac{\zeta}{k}p+1\right)^{\frac{1}{p}}}\left[\frac{|\Pi'(x)|^q}{\beta+1} + \frac{\beta m\left|\Pi'\left(\frac{\varsigma_2}{m}\right)\right|^q}{\beta+1} - \frac{\mu\left(\frac{\varsigma_2}{m}-x\right)^2}{6}\right]^{\frac{1}{q}},$$

for all $x \in [\varsigma_1, \varsigma_2]$, $\zeta, k > 0$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Theorem 99 ([50]). *Let Π be as in Theorem 97. If $|\Pi'|^q, q \ge 1$ is strongly (β, m)-convex with modulus $\mu \ge 0$ for $\beta \in [0,1]$ and $m \in (0,1]$, then:*

$$\left|\frac{(x-\varsigma_1)^{\frac{\zeta}{k}} + (\varsigma_2 - x)^{\frac{\zeta}{k}}}{\varsigma_2 - \varsigma_1}\Pi(x) - \frac{\Gamma(\zeta+k)}{\varsigma_2-\varsigma_1}\left[J^{\zeta,k}_{x-}\Pi(\varsigma_1) + \frac{1}{2(\varsigma_2-x)^{\frac{\zeta}{k}}} J^{\zeta,k}_{x+}\Pi(\varsigma_2)\right]\right|$$
$$\le \frac{(x-\varsigma_1)^{\frac{\zeta}{k}+1}}{(\varsigma_2-\varsigma_1)\left(\frac{\zeta}{k}+1\right)^{\frac{1}{p}}}\left[\frac{|\Pi'(x)|^q}{\frac{\zeta}{k}+\beta+1} + \frac{\beta m\left|\Pi'\left(\frac{\varsigma_1}{m}\right)\right|^q}{\left(\frac{\zeta}{k}+1\right)\left(\frac{\zeta}{k}+\beta+1\right)} - \frac{\mu\left(x-\frac{\varsigma_1}{m}\right)^2}{\left(\frac{\zeta}{k}+2\right)\left(\frac{\zeta}{k}+3\right)}\right]^{\frac{1}{q}}$$
$$+\frac{(\varsigma_2-x)^{\frac{\zeta}{k}+1}}{(\varsigma_2-\varsigma_1)\left(\frac{\zeta}{k}+1\right)^{\frac{1}{p}}}\left[\frac{|\Pi'(x)|^q}{\frac{\zeta}{k}+\beta+1} + \frac{\beta m\left|\Pi'\left(\frac{\varsigma_2}{m}\right)\right|^q}{\left(\frac{\zeta}{k}+1\right)\left(\frac{\zeta}{k}+\beta+1\right)} - \frac{\mu\left(\frac{\varsigma_2}{m}-x\right)^2}{\left(\frac{\zeta}{k}+2\right)\left(\frac{\zeta}{k}+3\right)}\right]^{\frac{1}{q}}$$

for all $x \in [\varsigma_1, \varsigma_2]$ and $\zeta, k > 0$.

Here, we add some Ostrowski-type inequalities for exponentially convex functions via the k-Riemann–Liouville fractional integral.

Definition 25 ([51]). *A function $\Pi : [\varsigma_1, \varsigma_2] \subset \mathbb{R} \to \mathbb{R}$ is said to be an exponential-convex function, if*

$$e^{\Pi(tx+(1-t)y)} \leq te^{\Pi(x)} + (1-t)e^{\Pi(y)},$$

for all $t \in [0,1]$ and all $x, y \in [\varsigma_1, \varsigma_2]$.

Theorem 100 ([52]). *Let $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be a differentiable mapping on $(\varsigma_1, \varsigma_2)$. If Π is an exponential-convex function, then*

$$\begin{aligned} &\Big|\Big((\Pi(x)-\Pi(\varsigma_1))^{\frac{\zeta}{k}}+(\Pi(\varsigma_2)-\Pi(x))^{\frac{\zeta}{k}}\Big)e^{\Pi(x)} \\ &-\Big(\Gamma_k(\zeta+k)J^{\zeta,k}_{\varsigma_1+}e^{\Pi(x)}+\Gamma_k(\zeta+k)J^{\zeta,k}_{\varsigma_2-}e^{\Pi(x)}\Big)\Big| \\ \leq\ & \frac{M\rho}{\rho+k}(\Pi(x)-\Pi(\varsigma_1))^{\frac{\zeta}{k}+1}+\frac{M\rho}{\rho+k}(\Pi(\varsigma_2)-\Pi(x))^{\frac{\zeta}{k}+1} \end{aligned}$$

if $\psi'(x) \geq 1$ and $|e^{\Pi(\lambda)}| \leq M$ for all $x, \lambda \in [\varsigma_1, \varsigma_2]$.

Theorem 101 ([52]). *Let $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be a function which is differentiable on $(\varsigma_1, \varsigma_2)$, and $\psi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be a strictly increasing function such that $\psi'(x) \geq 1$, $|(e^{\Pi(\lambda)})'| \leq M$ and $m \leq (e^{\Pi(\lambda)})' \leq M$ for all $x, \lambda \in [\varsigma_1, \varsigma_2]$, $m \leq 0, M > 0$. If Π is an exponential-convex function, the following inequalities for k-fractional integrals hold:*

$$\begin{aligned} &\Big|\Big((\Pi(x)-\Pi(\varsigma_1))^{\frac{\zeta}{k}}+(\Pi(\varsigma_2)-\Pi(x))^{\frac{\zeta}{k}}\Big)e^{\Pi(x)} \\ &-\Big(\Gamma_k(\zeta+k)J^{\zeta,k}_{\varsigma_1+}e^{\Pi(x)}+\Gamma_k(\zeta+k)J^{\zeta,k}_{\varsigma_2-}e^{\Pi(x)}\Big)\Big| \\ \leq\ & \frac{M\rho}{\rho+k}(\Pi(x)-\Pi(\varsigma_1))^{\frac{\zeta}{k}+1}+\frac{M\rho}{\rho+k}(\Pi(\varsigma_2)-\Pi(x))^{\frac{\zeta}{k}+1}, \end{aligned}$$

and

$$\begin{aligned} &\Big|\Big((\Pi(x)-\Pi(\varsigma_1))^{\frac{\zeta}{k}}+(\Pi(\varsigma_2)-\Pi(x))^{\frac{\zeta}{k}}\Big)e^{\Pi(x)} \\ &-\Big(\Gamma_k(\zeta+k)J^{\zeta,k}_{\varsigma_1+}e^{\Pi(x)}+\Gamma_k(\zeta+k)J^{\zeta,k}_{\varsigma_2-}e^{\Pi(x)}\Big)\Big| \\ \leq\ & \frac{-m\rho}{\rho+k}(\Pi(x)-\Pi(\varsigma_1))^{\frac{\zeta}{k}+1}+\frac{-m\rho}{\rho+k}(\Pi(\varsigma_2)-\Pi(x))^{\frac{\zeta}{k}+1}, \end{aligned}$$

for all $x \in [\varsigma_1, \varsigma_2]$.

Ostrowski-type fractional integral inequalities via k-fractional integral, which are obtained for (s, r)-convex in mixed kind, are presented in the next theorems.

Theorem 102 ([53]). *Let $\Pi : [\varsigma_1, \varsigma_2] \subset (0, \infty) \to \mathbb{R}$ be a function which is absolutely continuous and $\Pi' \in L[\varsigma_1, \varsigma_2]$. If $|\Pi'|$ is an (s, r)-convex function on $[\varsigma_1, \varsigma_2]$ and $|\Pi'(x)| \leq M$, $x \in [\varsigma_1, \varsigma_2]$, then for $\zeta, k > 0$ and $x \in [\varsigma_1, \varsigma_2]$ we have:*

$$\begin{aligned} &\left|\left(\frac{(x-\varsigma_1)^{\frac{\zeta}{k}}+(\varsigma_2-x)^{\frac{\zeta}{k}}}{\varsigma_2-\varsigma_1}\right)\Pi(x)-\frac{k\Gamma(\zeta+1)}{\varsigma_2-\varsigma_1}\Big[I^{\zeta,k}_{x-}\Pi(\varsigma_1)+I^{\zeta,k}_{x+}\Pi(\varsigma_2)\Big]\right| \\ \leq\ & M\Big[\int_0^1 t^{\frac{\zeta}{k}}t^{rs}dt+\int_0^1 t^{\frac{\zeta}{k}}(1-t^r)^s dt\Big]\Big[\frac{(x-\varsigma_1)^{\frac{\zeta}{k}+1}}{\varsigma_2-\varsigma_1}+\frac{(\varsigma_2-x)^{\frac{\zeta}{k}+1}}{\varsigma_2-\varsigma_1}\Big]. \end{aligned}$$

Theorem 103 ([53]). *Let $\Pi : [\varsigma_1, \varsigma_2] \subset (0, \infty) \to \mathbb{R}$ be an absolutely continuous function and $\Pi' \in L[\varsigma_1, \varsigma_2]$. If $|\Pi'|^q, q > 1$ is an (s, r)-convex function on $[\varsigma_1, \varsigma_2]$ and $|\Pi'(x)| \leq M$, $x \in [\varsigma_1, \varsigma_2]$, then for $\zeta, k > 0$ and $x \in [\varsigma_1, \varsigma_2]$ we have:*

$$\left|\Big(\frac{(x-\varsigma_1)^{\frac{\zeta}{k}} + (\varsigma_2 - x)^{\frac{\zeta}{k}}}{\varsigma_2 - \varsigma_1}\Big)\Pi(x) - \frac{k\Gamma(\zeta+1)}{\varsigma_2 - \varsigma_1}\Big[I^{\zeta,k}_{x-}\Pi(\varsigma_1) + I^{\zeta,k}_{x+}\Pi(\varsigma_2)\Big]\right|$$
$$\leq \frac{M}{\left(\int_0^1 t^{\frac{\zeta}{k}}dt\right)^{\frac{1}{q}-1}}\Big[\int_0^1 t^{\frac{\zeta}{k}} t^{rs} dt + \int_0^1 t^{\frac{\zeta}{k}}(1-t^r)^s dt\Big]^{\frac{1}{q}}\Big[\frac{(x-\varsigma_1)^{\frac{\zeta}{k}+1}}{\varsigma_2 - \varsigma_1} + \frac{(\varsigma_2 - x)^{\frac{\zeta}{k}+1}}{\varsigma_2 - \varsigma_1}\Big].$$

Theorem 104 ([53]). *Let $\Pi : [\varsigma_1, \varsigma_2] \subset (0, \infty) \to \mathbb{R}$ be an absolutely continuous function and $\Pi' \in L[\varsigma_1, \varsigma_2]$. If $|\Pi'|^q, p, q > 1$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$ is an (s, r)-convex function on $[\varsigma_1, \varsigma_2]$ and $|\Pi'(x)| \leq M$, $x \in [\varsigma_1, \varsigma_2]$, then for $\zeta, k > 0$ and $x \in [\varsigma_1, \varsigma_2]$ we have:*

$$\left|\Big(\frac{(x-\varsigma_1)^{\frac{\zeta}{k}} + (\varsigma_2 - x)^{\frac{\zeta}{k}}}{\varsigma_2 - \varsigma_1}\Big)\Pi(x) - \frac{k\Gamma(\zeta+1)}{\varsigma_2 - \varsigma_1}\Big[I^{\zeta,k}_{x-}\Pi(\varsigma_1) + I^{\zeta,k}_{x+}\Pi(\varsigma_2)\Big]\right|$$
$$\leq \frac{M\left(\int_0^1 t^{\frac{\zeta p}{k}}dt\right)^{\frac{1}{p}}}{\varsigma_2 - \varsigma_1}\Big(\frac{1}{rs+1} + \frac{1}{r}B\Big(\frac{1}{r}, s+1\Big)\Big)^{\frac{1}{q}}\Big[(x-\varsigma_1)^{\frac{\zeta}{k}+1} + (\varsigma_2 - x)^{\frac{\zeta}{k}+1}\Big].$$

Here, we add some fractional Ostrowski-type inequalities via MT-convex functions.

Theorem 105 ([54]). *Let $\Pi : [\varsigma_1, \varsigma_2] \subset (0, \infty) \to \mathbb{R}$ be a differentiable mapping on $(\varsigma_1, \varsigma_2)$ with $\varsigma_1 < \varsigma_2$ such that $\Pi' \in L[\varsigma_1, \varsigma_2]$. If $|\Pi'|$ is an MT-convex function on $[\varsigma_1, \varsigma_2]$ and $|\Pi'(x)| \leq M$, $x \in [\varsigma_1, \varsigma_2]$, then for $\zeta > 0$ and $x \in [\varsigma_1, \varsigma_2]$ we have:*

$$\left|\Big(\frac{(x-\varsigma_1)^{\frac{\zeta}{k}} + (\varsigma_2 - x)^{\frac{\zeta}{k}}}{\varsigma_2 - \varsigma_1}\Big)\Pi(x) - \frac{\Gamma_k(\zeta+k)}{\varsigma_2 - \varsigma_1}\Big[I^{\zeta,k}_{x-}\Pi(\varsigma_1) + I^{\zeta,k}_{x+}\Pi(\varsigma_2)\Big]\right|$$
$$\leq M\frac{\Gamma\Big(\zeta + \frac{k}{2}\Big)\Gamma_k\Big(\frac{k}{2}\Big)}{2\Gamma_k(\zeta+k)}\frac{(x-\varsigma_1)^{\frac{\zeta}{k}+1} + (\varsigma_2 - x)^{\frac{\zeta}{k}+1}}{\varsigma_2 - \varsigma_1}.$$

Theorem 106 ([54]). *Let Π be as in Theorem 105. If $|\Pi'|^q$ is an MT-convex function on $[\varsigma_1, \varsigma_2]$, $q > 1, \dfrac{1}{p} + \dfrac{1}{q} = 1$, and $|\Pi'(x)| \leq M$, $x \in [\varsigma_1, \varsigma_2]$, then for $\zeta > 0$ and $x \in [\varsigma_1, \varsigma_2]$ we have:*

$$\left|\Big(\frac{(x-\varsigma_1)^{\frac{\zeta}{k}} + (\varsigma_2 - x)^{\frac{\zeta}{k}}}{\varsigma_2 - \varsigma_1}\Big)\Pi(x) - \frac{\Gamma_k(\zeta+k)}{\varsigma_2 - \varsigma_1}\Big[I^{\zeta,k}_{x-}\Pi(\varsigma_1) + I^{\zeta,k}_{x+}\Pi(\varsigma_2)\Big]\right|$$
$$\leq \frac{M}{(1+p\frac{\zeta}{k})^{\frac{1}{p}}}\Big(\frac{\pi}{2}\Big)^{\frac{1}{q}}\frac{(x-\varsigma_1)^{\frac{\zeta}{k}+1} + (\varsigma_2 - x)^{\frac{\zeta}{k}+1}}{\varsigma_2 - \varsigma_1}.$$

Theorem 107 ([54]). *Let Π be as in Theorem 105. If $|\Pi'|^q, q \geq 1$ is an MT-convex function on $[\varsigma_1, \varsigma_2]$, and $|\Pi'(x)| \leq M$, $x \in [\varsigma_1, \varsigma_2]$, then for $\zeta > 0$ and $x \in [\varsigma_1, \varsigma_2]$ we have:*

$$\left|\Big(\frac{(x-\varsigma_1)^{\frac{\zeta}{k}} + (\varsigma_2 - x)^{\frac{\zeta}{k}}}{\varsigma_2 - \varsigma_1}\Big)\Pi(x) - \frac{\Gamma_k(\zeta+k)}{\varsigma_2 - \varsigma_1}\Big[I^{\zeta,k}_{x-}\Pi(\varsigma_1) + I^{\zeta,k}_{x+}\Pi(\varsigma_2)\Big]\right|$$
$$\leq \frac{M}{(1+\zeta)^{1-\frac{1}{q}}}\left(\frac{\Gamma_k(\zeta+\frac{k}{2})\Gamma_k(\frac{k}{2})}{2\Gamma_k(\zeta+k)}\right)^{\frac{1}{q}}\frac{(x-\varsigma_1)^{\frac{\zeta}{k}+1} + (\varsigma_2 - x)^{\frac{\zeta}{k}+1}}{\varsigma_2 - \varsigma_1}.$$

5. Ostrowski-Type Fractional Integral Inequalities for Preinvex Functions

Definition 26 ([55]). *A set $K \subseteq \mathbb{R}^n$ is invex w.r.t $\eta : K \times K \to \mathbb{R}^n$, if $\forall\ x, y \in K$, we have*

$$x + t\eta(y, x) \in K.$$

Definition 27 ([55]). *A function $\Pi : K \to \mathbb{R}$ is preinvex w.r.t. η if*

$$\Pi(x + t\eta(y, x)) \leq (1 - t)\Pi(x) + t\Pi(y),$$

$\forall\ x, y \in K$, and all $t \in [0, 1]$.

Definition 28 ([56]). *The nonnegative function Π on the invex set K is prequasi invex w.r.t. η, if*

$$\Pi(x + t\eta(y, x)) \leq \max\{\Pi(x), \Pi(y)\},$$

for all $x, y \in K$ and $t \in [0, 1]$.

Condition C. [57] *Suppose $A \subseteq \mathbb{R}^n$ be an invex subset w.r.t. $\eta : K \times K \to \mathbb{R}^n$. We say that the function η satisfies the condition C if for any $x, y \in K$ and $t \in [0, 1]$,*

$$\eta(y, y + t\eta(x, y)) = -t\eta(x, y), \quad \eta(x, y + t\eta(x, y)) = (1 - t)\eta(x, y).$$

Ostrowski-type inequalities for preinvex and prequasiinvex functions are given in the next theorems.

Theorem 108 ([58]). *Let $K \subset \mathbb{R}$ be an open invex subset with respect to $\eta(\cdot, \cdot) : K \times K \to \mathbb{R}^n$ and $\varsigma_1, \varsigma_2 \in K$ with $\varsigma_1 < \varsigma_1 + \eta(\varsigma_2, \varsigma_1)$. Suppose that $\Pi : K \to \mathbb{R}$ is a differentiable function. If Π is integrable on $[\varsigma_1, \varsigma_1 + \eta(\varsigma_2, \varsigma_1)]$ and $|\Pi'|$ is a preinvex function on K, then*

$$\begin{aligned}
&\Big|[(x-\varsigma_1)^{\zeta} + (\eta(\varsigma_2,\varsigma_1)+\varsigma_1-x)^{\zeta}]\Pi(x) + \Gamma(\zeta+1)[J^{\zeta}_{x-}\Pi(\varsigma_1) + J^{\zeta}_{x+}\Pi(\varsigma_1+\eta(\varsigma_2,\varsigma_1))]\Big| \\
\leq &\Big[\frac{(x-\varsigma_1)^{\zeta+1}}{\zeta+1} + \frac{[(\eta(\varsigma_2,\varsigma_1)+\varsigma_1-x)^{\zeta+2} - (x-\varsigma_1)^{\zeta+2}]}{(\zeta+2)\eta(\varsigma_2,\varsigma_1)}\Big]|\Pi'(\varsigma_1)| \\
&+\Big[\frac{[(x-\varsigma_1)^{\zeta+2} - (\eta(\varsigma_2,\varsigma_1)+\varsigma_1-x)^{\zeta+2}}{(\zeta+2)\eta(\varsigma_2,\varsigma_1)} + \frac{(\eta(\varsigma_2,\varsigma_1)+\varsigma_1-x)^{\zeta+1}}{\zeta+1}\Big]|\Pi'(\varsigma_2)|,
\end{aligned}$$

for all $x \in [\varsigma_1, \varsigma_1 + \eta(\varsigma_2, \varsigma_1)]$.

Theorem 109 ([58]). *Let Π be as in Theorem 108. If Π is integrable on $[\varsigma_1, \varsigma_1 + \eta(\varsigma_2, \varsigma_1)]$ and $|\Pi'|^q, q > 1$ is a preinvex function on K, and η satisfies condition C, then*

$$\begin{aligned}
&\Big|[(x-\varsigma_1)^{\zeta} + (\eta(\varsigma_2,\varsigma_1)+\varsigma_1-x)^{\zeta}]\Pi(x) + \Gamma(\zeta+1)[J^{\zeta}_{x-}\Pi(\varsigma_1) + J^{\zeta}_{x+}\Pi(\varsigma_1+\eta(\varsigma_2,\varsigma_1))]\Big| \\
\leq &\Big(\frac{1}{p\zeta+1}\Big)^{\frac{1}{p}}\Big[(x-\varsigma_1)^{\zeta+1}\Big(\frac{|\Pi'(\varsigma_1)|^q + |\Pi'(x)|^q}{2}\Big)^{\frac{1}{q}} \\
&+(\varsigma_1+\eta(\varsigma_2,\varsigma_1)-x)^{\zeta+1}\Big(\frac{|\Pi'(\varsigma_1+\eta(\varsigma_2,\varsigma_1))|^q + |\Pi'(x)|^q}{2}\Big)^{\frac{1}{q}},
\end{aligned}$$

for all $x \in [\varsigma_1, \varsigma_1 + \eta(\varsigma_2, \varsigma_1)]$ where $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Theorem 110 ([58]). *Let the assumptions of this theorem be as stated in Theorem 108. If Π' is integrable on $[\varsigma_1, \varsigma_1 + \eta(\varsigma_2, \varsigma_1)]$ and $|\Pi'|$, is a prequasiinvex function on K, then*

$$\left|[(x-\varsigma_1)^\zeta+(\eta(\varsigma_2,\varsigma_1)+\varsigma_1-x)^\zeta]\Pi(x)-\Gamma(\zeta+1)[J^\zeta_{x-}\Pi(\varsigma_1)+J^\zeta_{x+}\Pi(\varsigma_1+\eta(\varsigma_2,\varsigma_1))]\right|$$
$$\leq \eta^{\zeta+1}(\varsigma_2,\varsigma_1)\max\{|\Pi'(\varsigma_1)|,|\Pi'(\varsigma_2)|\}\left(\frac{(x-\varsigma_1)^{\zeta+1}+(\eta(\varsigma_2,\varsigma_1)+\varsigma_1-x)^{\zeta+1}}{\zeta+1}\right),$$

for all $x\in[\varsigma_1,\varsigma_1+\eta(\varsigma_2,\varsigma_1)]$.

Theorem 111 ([58]). *Let the assumptions of this theorem be stated in Theorem 108. If* Π' *is integrable on* $[\varsigma_1,\varsigma_1+\eta(\varsigma_2,\varsigma_1)]$ *and* $|\Pi'|^q, q>1$ *is a prequasiinvex function on K,* η *satisfies condition C, then*

$$\left|[(x-\varsigma_1)^\zeta+(\eta(\varsigma_2,\varsigma_1)+\varsigma_1-x)^\zeta]\Pi(x)-\Gamma(\zeta+1)[J^\zeta_{x-}\Pi(\varsigma_1)+J^\zeta_{x+}\Pi(\varsigma_1+\eta(\varsigma_2,\varsigma_1))]\right|$$
$$\leq \max\{|\Pi'(\varsigma_1)|,|\Pi'(\varsigma_2)|\}\left(\frac{1}{p\zeta+1}\right)^{\frac{1}{p}}\left[(x-\varsigma_1)^{\zeta+1}+(\eta(\varsigma_2,\varsigma_1)+\varsigma_1-x)^{\zeta+1}\right],$$

for all $x\in[\varsigma_1,\varsigma_1+\eta(\varsigma_2,\varsigma_1)]$ *where* $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 112 ([59]). *Let* $\Pi:[\varsigma_1,\varsigma_1+\eta(\varsigma_2,\varsigma_1)]\to\mathbb{R}$ *be a differentiable mapping such that* $\eta(\varsigma_2,\varsigma_1)>0$ *and* $\Pi'\in L[\varsigma_1,\varsigma_1+\eta(\varsigma_2,\varsigma_1)]$. *If* $|\Pi'|$ *is a prequasiinvex function, then*

$$\left|\left(\left(\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^\zeta+\left(1-\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^\zeta\right)\Pi(x)-\frac{\Gamma(\zeta+1)}{(\eta(\varsigma_2,\varsigma_1))^\zeta}\left[J^\zeta_{x-}\Pi(\varsigma_1)+J^\zeta_{x+}\Pi(\varsigma_1,\varsigma_1+\eta(\varsigma_2,\varsigma_1))\right]\right|$$
$$\leq \frac{\eta(\varsigma_2,\varsigma_1)}{2}\left[\left(\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{\zeta+1}\max\{\Pi'(\varsigma_1)|,\Pi'(x)|\}\right.$$
$$\left.+\left(1-\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{\zeta+1}\max\{\Pi'(\varsigma_1+\eta(\varsigma_2,\varsigma_1))|,\Pi'(x)|\}\right],$$

for all $x\in[\varsigma_1,\varsigma_1+\eta(\varsigma_2,\varsigma_1)]$.

Theorem 113 ([59]). *Let the assumptions of this theorem be as stated in Theorem 112. If* $|\Pi'|^q$ *is a prequasiinvex function, where* $q>1, \frac{1}{p}+\frac{1}{q}=1$, *then*

$$\left|\left(\left(\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^\zeta+\left(1-\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^\zeta\right)\Pi(x)-\frac{\Gamma(\zeta+1)}{(\eta(\varsigma_2,\varsigma_1))^\zeta}\left[J^\zeta_{x-}\Pi(\varsigma_1)+J^\zeta_{x+}\Pi(\varsigma_1,\varsigma_1+\eta(\varsigma_2,\varsigma_1))\right]\right|$$
$$\leq \frac{\eta(\varsigma_2,\varsigma_1)}{(\zeta p+1)^{\frac{1}{q}}}\left[\left(\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{\zeta+1}\max\{|\Pi'(\varsigma_1)|^q,|\Pi'(x)|^q\}\right.$$
$$\left.+\left(1-\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{\zeta+1}\max\{|\Pi'(\varsigma_1+\eta(\varsigma_2,\varsigma_1))|,|\Pi'(x)|^q\}\right],$$

for all $x\in[\varsigma_1,\varsigma_1+\eta(\varsigma_2,\varsigma_1)]$.

In the next, we develop some fractional Ostrowski-type inequalities for twice differentiable preinvex mappings.

Theorem 114 ([60]). *Suppose that $\Pi : [\varsigma_1, \varsigma_1 + \eta(\varsigma_2, \varsigma_1)] \to \mathbb{R}$ is a twice differentiable mapping with $\varsigma_1 < \varsigma_1 + \eta(\varsigma_2, \varsigma_1)$. If $\Pi'' \in L[\varsigma_1, \varsigma_1 + \eta(\varsigma_2, \varsigma_1)]$ and $|\Pi''|$ is preinvex in $[\varsigma_1, \varsigma_1 + \eta(\varsigma_2, \varsigma_1)]$, then for all $x \in [\varsigma_1, \varsigma_1 + \eta(\varsigma_2, \varsigma_1)]$,*

$$\begin{aligned}
&\left| \frac{\eta^{\zeta+1}(x,\varsigma_1) - \eta^{\zeta+1}(\varsigma_2,x)}{(\zeta+1)\eta(\varsigma_2,\varsigma_1)}\Pi'(x) - \frac{\eta^{\zeta}(x,\varsigma_1) + \eta^{\zeta}(\varsigma_2,x)}{(\zeta+1)\eta(\varsigma_2,\varsigma_1)}\Pi(x) \right. \\
&\left. + \frac{\Gamma(\zeta+1)}{\eta(\varsigma_2,\varsigma_1)}\left\{ J^{\zeta}_{[\varsigma_1+\eta(\varsigma_2,\varsigma_1)]-}\Pi(\varsigma_1) + J^{\zeta}_{[\varsigma_1+\eta(\varsigma_2,\varsigma_1)]+}\Pi(\varsigma_2) \right\} \right| \\
\leq\ & \frac{\eta^{\zeta+2}(x,\varsigma_1)}{(\zeta+1)(\zeta+3)\eta(\varsigma_2,\varsigma_1)}\left\{ |\Pi''(x)| + |\Pi''(\varsigma_1)|\frac{1}{\zeta+2} \right\} \\
&+ \frac{\eta^{\zeta+2}(\varsigma_2,x)}{(\zeta+1)(\zeta+3)\eta(\varsigma_2,\varsigma_1)}\left\{ |\Pi''(x)| + |\Pi''(\varsigma_2)|\frac{1}{\zeta+2} \right\}.
\end{aligned}$$

Theorem 115 ([60]). *Suppose that $\Pi : [\varsigma_1, \varsigma_1 + \eta(\varsigma_2, \varsigma_1)] \to \mathbb{R}$ is a twice differentiable mapping with $\varsigma_1 < \varsigma_1 + \eta(\varsigma_2, \varsigma_1)$. If $\Pi'' \in L[\varsigma_1, \varsigma_1 + \eta(\varsigma_2, \varsigma_1)]$ and $|\Pi''|^q, q > 1$ is preinvex in $[\varsigma_1, \varsigma_1 + \eta(\varsigma_2, \varsigma_1)]$, then, for all $x \in [\varsigma_1, \varsigma_1 + \eta(\varsigma_2, \varsigma_1)]$,*

$$\begin{aligned}
&\left| \frac{\eta^{\zeta+1}(x,\varsigma_1) - \eta^{\zeta+1}(\varsigma_2,x)}{(\zeta+1)\eta(\varsigma_2,\varsigma_1)}\Pi'(x) - \frac{\eta^{\zeta}(x,\varsigma_1) + \eta^{\zeta}(\varsigma_2,x)}{(\zeta+1)\eta(\varsigma_2,\varsigma_1)}\Pi(x) \right. \\
&\left. + \frac{\Gamma(\zeta+1)}{\eta(\varsigma_2,\varsigma_1)}\left\{ J^{\zeta}_{[\varsigma_1+\eta(\varsigma_2,\varsigma_1)]-}\Pi(\varsigma_1) + J^{\zeta}_{[\varsigma_1+\eta(\varsigma_2,\varsigma_1)]+}\Pi(\varsigma_2) \right\} \right| \\
\leq\ & \left(\frac{1}{(\zeta+1)p+1} \right)^{\frac{1}{p}} \left[\frac{\eta^{\zeta+2}(x,\varsigma_1)}{(\zeta+1)\eta(\varsigma_2,\varsigma_1)} \left(\frac{|\Pi''(x)|^q + |\Pi''(\varsigma_1)|^q}{2} \right)^{\frac{1}{q}} \right. \\
&\left. + \frac{\eta^{\zeta+2}(\varsigma_2,x)}{(\zeta+1)\eta(\varsigma_2,\varsigma_1)} \left(\frac{|\Pi''(x)|^q + |\Pi''(\varsigma_2)|^q}{2} \right)^{\frac{1}{q}} \right],
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 116 ([60]). *Suppose that $\Pi : [\varsigma_1, \varsigma_1 + \eta(\varsigma_2, \varsigma_1)] \to \mathbb{R}$ is a function which is twice differentiable with $\varsigma_1 < \varsigma_1 + \eta(\varsigma_2, \varsigma_1)$. If $\Pi'' \in L[\varsigma_1, \varsigma_1 + \eta(\varsigma_2, \varsigma_1)]$ and $|\Pi''|^q, q \geq 1$ is preinvex in $[\varsigma_1, \varsigma_1 + \eta(\varsigma_2, \varsigma_1)]$, then, for all $x \in [\varsigma_1, \varsigma_1 + \eta(\varsigma_2, \varsigma_1)]$,*

$$\begin{aligned}
&\left| \frac{\eta^{\zeta+1}(x,\varsigma_1) - \eta^{\zeta+1}(\varsigma_2,x)}{(\zeta+1)\eta(\varsigma_2,\varsigma_1)}\Pi'(x) - \frac{\eta^{\zeta}(x,\varsigma_1) + \eta^{\zeta}(\varsigma_2,x)}{(\zeta+1)\eta(\varsigma_2,\varsigma_1)}\Pi(x) \right. \\
&\left. + \frac{\Gamma(\zeta+1)}{\eta(\varsigma_2,\varsigma_1)}\left\{ J^{\zeta}_{[\varsigma_1+\eta(\varsigma_2,\varsigma_1)]-}\Pi(\varsigma_1) + J^{\zeta}_{[\varsigma_1+\eta(\varsigma_2,\varsigma_1)]+}\Pi(\varsigma_2) \right\} \right| \\
\leq\ & \left(\frac{1}{\zeta+2} \right)^{1-\frac{1}{q}} \left[\frac{\eta^{\zeta+2}(x,\varsigma_1)}{(\zeta+1)\eta(\varsigma_2,\varsigma_1)} \left(\frac{|\Pi''(x)|^q}{\zeta+3} + \frac{|\Pi''(\varsigma_1)|^q}{(\zeta+2)(\zeta+3)} \right)^{\frac{1}{q}} \right. \\
&\left. + \frac{\eta^{\zeta+2}(\varsigma_2,x)}{(\zeta+1)\eta(\varsigma_2,\varsigma_1)} \left(\frac{|\Pi''(x)|^q}{\zeta+3} + \frac{|\Pi''(\varsigma_2)|^q}{(\zeta+2)(\zeta+3)} \right)^{\frac{1}{q}} \right].
\end{aligned}$$

Definition 29 ([61]). *A function $\Pi : K \subset (0,\infty) \to \mathbb{R}$ is s-preinvex in the second aspect w.r.t. η for some $s \in (0,1]$, if*

$$\Pi(x + t\eta(y,x)) \leq (1-t)^s\Pi(x) + t^s\Pi(y),$$

for all $x, y \in K$, and all $t \in [0,1]$.

Some Ostrowski-type inequalities for s-preinvex in the second sense, are given in the next theorems.

Theorem 117 ([62])**.** *Let $\Pi : [\varsigma_1, \varsigma_1 + \eta(\varsigma_2, \varsigma_1)] \to \mathbb{R}$ be a differentiable function such that $\eta(\varsigma_2, \varsigma_1) > 0$ and $\Pi' \in L[\varsigma_1, \varsigma_1 + \eta(\varsigma_2, \varsigma_1)]$. If $|\Pi'|$ is s-preinvex, for $s \in (0, 1]$, then*

$$\begin{aligned}
&\left|\left(\left(\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{\zeta} + \left(1-\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{\zeta}\right)\Pi(x) - \frac{\Gamma(\zeta+1)}{(\eta(\varsigma_2,\varsigma_1))^{\zeta}}\left[J^{\zeta}_{x-}\Pi(\varsigma_1) + J^{\zeta}_{x+}\Pi(\varsigma_1,\varsigma_1+\eta(\varsigma_2,\varsigma_1))\right]\right| \\
\leq\ & \eta(\varsigma_2,\varsigma_1)\left(\left(B_{\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}}(\zeta+1,s+1) + \frac{1}{\zeta+s+1}\left(1-\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{\zeta+s+1}\right)|\Pi'(\varsigma_1)| \right.\\
&\left. + \left(\frac{1}{\zeta+s+1}\left(\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{\zeta+s+1} + B(s+1,\zeta+1) - B_{\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}}(s+1,\zeta+1)\right)|\Pi'(\varsigma_2)|\right),
\end{aligned}$$

for all $x \in [\varsigma_1, \varsigma_1 + \eta(\varsigma_2, \varsigma_1)]$.

Theorem 118 ([62])**.** *Let Π be as in Theorem 117. If $|\Pi'|^q$ is s-preinvex, for some fixed $s \in (0, 1]$, $q > 1$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then*

$$\begin{aligned}
&\left|\left(\left(\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{\zeta} + \left(1-\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{\zeta}\right)\Pi(x) - \frac{\Gamma(\zeta+1)}{(\eta(\varsigma_2,\varsigma_1))^{\zeta}}\left[J^{\zeta}_{x-}\Pi(\varsigma_1) + J^{\zeta}_{x+}\Pi(\varsigma_1,\varsigma_1+\eta(\varsigma_2,\varsigma_1))\right]\right| \\
\leq\ & \frac{\eta(\varsigma_2,\varsigma_1)}{(s+1)^{\frac{1}{q}}(\zeta p+1)^{\frac{1}{p}}}\left(\left(\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{\zeta+\frac{1}{p}}\left(\left(\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{s+1}|\Pi'(\varsigma_1)|^q \right.\right.\\
&\left. + \left(1-\left(1-\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{s+1}\right)|\Pi'(\varsigma_2)|^q\right)^{\frac{1}{q}} \\
&\left. + \left(1-\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{\zeta+\frac{1}{p}}\left(\left(1-\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{s+1}|\Pi'(\varsigma_1)|^q + \left(1-\left(\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{s+1}|\Pi'(\varsigma_2)|^q\right)^{\frac{1}{q}}\right),
\end{aligned}$$

for all $x \in [\varsigma_1, \varsigma_1 + \eta(\varsigma_2, \varsigma_1)]$.

Theorem 119 ([62])**.** *Let Π be as in Theorem 117. If $|\Pi'|^q$ is s-preinvex for some fixed $s \in (0, 1]$, $q > 1$, Then*

$$\begin{aligned}
&\left|\left(\left(\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{\zeta} + \left(1-\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{\zeta}\right)\Pi(x) - \frac{\Gamma(\zeta+1)}{(\eta(\varsigma_2,\varsigma_1))^{\zeta}}\left[J^{\zeta}_{x-}\Pi(\varsigma_1) + J^{\zeta}_{x+}\Pi(\varsigma_1,\varsigma_1+\eta(\varsigma_2,\varsigma_1))\right]\right| \\
\leq\ & \frac{\eta(\varsigma_2,\varsigma_1)}{(\zeta+1)^{1-\frac{1}{q}}}\left(\left(\left(\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{(\zeta+1)(1-\frac{1}{q})}\right)B_{\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}}(\zeta+1,s+1)|\Pi'(\varsigma_1)|^q \right.\\
& + \frac{1}{\zeta+s+1}\left(\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{\zeta+s+1}|\Pi'(\varsigma_2)|^q\Bigg)^{\frac{1}{q}} \\
& + \left(1-\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{(\zeta+1)(1-\frac{1}{q})}\left(\frac{1}{\zeta+s+1}\left(1-\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{\zeta+s+1}|\Pi'(\varsigma_1)|^q \right.\\
&\left. + \left(B(s+1,\zeta+1) - B_{\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}}(s+1,\zeta+1)\right)|\Pi'(\varsigma_2)|^q\right)^{\frac{1}{q}}\Bigg),
\end{aligned}$$

for all $x \in [\varsigma_1, \varsigma_1 + \eta(\varsigma_2, \varsigma_1)]$.

Definition 30 ([63])**.** *A non-negative function $\Pi : K \subset (0, \infty) \to \mathbb{R}$ is MT-preinvex w.r.t. η, if*

$$\Pi(x + t\eta(y, x)) \leq \frac{\sqrt{1-t}}{2\sqrt{t}}\Pi(x) + \frac{\sqrt{t}}{2\sqrt{1-t}}\Pi(y),$$

for all $x, y \in K$, and all $t \in (0, 1)$.

Here, we add some Ostrowski-type inequalities involving MT-preinvex via Reimann-Liouville integral operators.

Theorem 120 ([64])**.** *Suppose $\Pi : [\varsigma_1, \varsigma_1 + \eta(\varsigma_2, \varsigma_1)] \to \mathbb{R}$ be a differentiable mapping such that $\eta(\varsigma_2, \varsigma_1) > 0$ and $\Pi' \in L[\varsigma_1, \varsigma_1 + \eta(\varsigma_2, \varsigma_1)]$. If $|\Pi'|$ is MT-preinvex, then*

$$\left|\left(\left(\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{\zeta}+\left(1-\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{\zeta}\right)\Pi(x)-\frac{\Gamma(\zeta+1)}{(\eta(\varsigma_2,\varsigma_1))^{\zeta}}\left[J_{x-}^{\zeta}\Pi(\varsigma_1)+J_{x+}^{\zeta}\Pi(\varsigma_1,\varsigma_1+\eta(\varsigma_2,\varsigma_1))\right]\right|$$
$$\leq \frac{\eta(\varsigma_2,\varsigma_1)}{2}\left(\left(B_{\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}}\left(\zeta+\frac{1}{2},\frac{3}{2}\right)+B_{1-\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}}\left(\zeta+\frac{3}{2},\frac{1}{2}\right)\right)|\Pi'(\varsigma_1)|\right.$$
$$\left.+\left(B_{\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}}\left(\zeta+\frac{3}{2},\frac{1}{2}\right)+B_{1-\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}}\left(\zeta+\frac{1}{2},\frac{3}{2}\right)\right)|\Pi'(\varsigma_2)|,\right.$$

for all $x\in[\varsigma_1,\varsigma_1+\eta(\varsigma_2,\varsigma_1)]$.

Theorem 121 ([64])**.** *Let* $\Pi:[\varsigma_1,\varsigma_1+\eta(\varsigma_2,\varsigma_1)]\to\mathbb{R}$ *be a differentiable function such that* $\eta(\varsigma_2,\varsigma_1)>0$ *and* $\Pi'\in L[\varsigma_1,\varsigma_1+\eta(\varsigma_2,\varsigma_1)]$. *If* $|\Pi'|^q$ *is MT-preinvex,* $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\left|\left(\left(\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{\zeta}+\left(1-\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{\zeta}\right)\Pi(x)-\frac{\Gamma(\zeta+1)}{(\eta(\varsigma_2,\varsigma_1))^{\zeta}}\left[J_{x-}^{\zeta}\Pi(\varsigma_1)+J_{x+}^{\zeta}\Pi(\varsigma_1,\varsigma_1+\eta(\varsigma_2,\varsigma_1))\right]\right|$$
$$\leq \frac{\eta(\varsigma_2,\varsigma_1)}{2^{\frac{1}{q}}(\zeta p+1)^{\frac{1}{q}}}\left(\left(\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{\zeta+\frac{1}{p}}\left(B_{\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}}\left(\frac{1}{2},\frac{3}{2}\right)|\Pi'(\varsigma_1)|^q+B_{\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}}\left(\frac{3}{2},\frac{1}{2}\right)|\Pi'(\varsigma_2)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(1-\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{\zeta+\frac{1}{p}}\left(B_{1-\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}}\left(\frac{1}{2},\frac{3}{2}\right)|\Pi'(\varsigma_1)|^q+B_{1-\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}}\left(\frac{3}{2},\frac{1}{2}\right)|\Pi'(\varsigma_2)|^q\right)^{\frac{1}{q}}\right),$$

for all $x\in[\varsigma_1,\varsigma_1+\eta(\varsigma_2,\varsigma_1)]$.

Theorem 122 ([64])**.** *Let* $\Pi:[\varsigma_1,\varsigma_1+\eta(\varsigma_2,\varsigma_1)]\to\mathbb{R}$ *be a differentiable mapping such that* $\eta(\varsigma_2,\varsigma_1)>0$ *and* $\Pi'\in L[\varsigma_1,\varsigma_1+\eta(\varsigma_2,\varsigma_1)]$. *If* $|\Pi'|^q$ *is MT-preinvex,* $q>1$, *then*

$$\left|\left(\left(\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{\zeta}+\left(1-\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{\zeta}\right)\Pi(x)-\frac{\Gamma(\zeta+1)}{(\eta(\varsigma_2,\varsigma_1))^{\zeta}}\left[J_{x-}^{\zeta}\Pi(\varsigma_1)+J_{x+}^{\zeta}\Pi(\varsigma_1,\varsigma_1+\eta(\varsigma_2,\varsigma_1))\right]\right|$$
$$\leq \frac{\eta(\varsigma_2,\varsigma_1)}{2^{\frac{1}{q}}(\zeta p+1)^{1-\frac{1}{q}}}\left(\left(\left(\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{(\zeta+1)(1-\frac{1}{q})}\right)\right.$$
$$\times\left(B_{\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}}\left(\zeta+\frac{1}{2},\frac{3}{2}\right)|\Pi'(\varsigma_1)|^q+B_{\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}}\left(\zeta+\frac{3}{2},\frac{1}{2}\right)|\Pi'(\varsigma_2)|^q\right)^{\frac{1}{q}}$$
$$+\left(1-\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}\right)^{(\zeta+1)(1-\frac{1}{q})}$$
$$\left.\times\left(B_{1-\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}}\left(\zeta+\frac{3}{2},\frac{1}{2}\right)|\Pi'(\varsigma_1)|^q+B_{1-\frac{x-\varsigma_1}{\eta(\varsigma_2,\varsigma_1)}}\left(\zeta+\frac{1}{2},\frac{3}{2}\right)|\Pi'(\varsigma_2)|^q\right)^{\frac{1}{q}}\right),$$

for all $x\in[\varsigma_1,\varsigma_1+\eta(\varsigma_2,\varsigma_1)]$.

6. Ostrowski-Type Fractional Integral Inequalities via Riemann–Liouville Fractional Integrals of a Function with Respect to Another Function

In this section, we add some fractional Ostrowski-type inequalities w.r.t. another function.

Definition 31 ([2,65])**.** *Let* $(\varsigma_1,\varsigma_2)$ $(-\infty\leq\varsigma_1<\varsigma_2\leq\infty)$ *be the interval of* $\mathbb{R}$ *and* $\zeta>0$. *Suppose* $\psi(x)$ *is a positive monotone and increasing function on* $(\varsigma_1,\varsigma_2]$, *having* $\psi'(x)$ *on* $(\varsigma_1,\varsigma_2)$. *The* ψ*-Riemann–Liouville fractional integrals of a function (left and right sided) g w.r.t. another function* ψ *on* $[\varsigma_1,\varsigma_2]$ *are defined by*

$$J_{\varsigma_1+}^{\zeta;\psi}g(x)=\frac{1}{\Gamma(\zeta)}\int_{\varsigma_1}^{x}\psi'(t)(\psi(x)-\psi(t))^{\zeta-1}g(t)dt,$$
$$J_{\varsigma_2-}^{\zeta;\psi}g(x)=\frac{1}{\Gamma(\zeta)}\int_{x}^{\varsigma_2}\psi'(t)(\psi(t)-\psi(x))^{\zeta-1}g(t)dt,$$

respectively.

We start with Ostrowski-type fractional inequalities involving fractional integrals with respect to another function and h-convex functions.

Theorem 123 ([66]). *Let $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be a function which is differentiable on $(\varsigma_1, \varsigma_2)$ with $\varsigma_1 < \varsigma_2$, $\Pi' : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ is integrable on $[\varsigma_1, \varsigma_2]$. Additionally, let $|\Pi'|$ be h-convex on $[\varsigma_1, \varsigma_2]$ and $|\Pi'(x)| \leq M$, $|\psi'(x)| \leq L$, $x \in [\varsigma_1, \varsigma_2]$. Then, for all $x \in [\varsigma_1, \varsigma_2]$,*

$$\Big|\Pi(x) - \Gamma(\zeta+1)\Big[\frac{1}{2(\psi(x)-\psi(\varsigma_1))^{\zeta}} J^{\zeta,\psi}_{x-}\Pi(\varsigma_1) + \frac{1}{2(\psi((\varsigma_2)-\psi(x))^{\zeta}} J^{\zeta,\psi}_{x+}\Pi(\varsigma_2)\Big]\Big|$$
$$\leq \Big[\frac{(x-\varsigma_1)^{\zeta+1}}{2(\psi(x)-\psi(\varsigma_1))^{\zeta}} + \frac{(\varsigma_2-x)^{\zeta+1}}{2(\psi(\varsigma_2)-\psi(x))^{\zeta}}\Big] ML^{\zeta}\int_0^1 t^{\zeta}[h(t)+h(1-t)]dt.$$

Theorem 124 ([66]). *Let Π be as in Theorem 123. Additionally, let $|\Pi'|^q, q > 1$ be h-convex on $[\varsigma_1, \varsigma_2]$ and $|\Pi'(x)| \leq M$, $|\psi'(x)| \leq L$, $x \in [\varsigma_1, \varsigma_2]$. Then, for all $x \in [\varsigma_1, \varsigma_2]$,*

$$\Big|\Pi(x) - \Gamma(\zeta+1)\Big[\frac{1}{2(\psi(x)-\psi(\varsigma_1))^{\zeta}} J^{\zeta,\psi}_{x-}\Pi(\varsigma_1) + \frac{1}{2(\psi((\varsigma_2)-\psi(x))^{\zeta}} J^{\zeta,\psi}_{x+}\Pi(\varsigma_2)\Big]\Big|$$
$$\leq \Big[\frac{(x-\varsigma_1)^{\zeta+1}}{2(\psi(x)-\psi(\varsigma_1))^{\zeta}} + \frac{(\varsigma_2-x)^{\zeta+1}}{2(\psi(\varsigma_2)-\psi(x))^{\zeta}}\Big] \frac{ML^{\zeta}}{(\zeta p+1)^{\frac{1}{p}}}\Big(2\int_0^1 h(t)dt\Big)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 125 ([66]). *Let Π be as in Theorem 123. Additionally, let $|\Pi'|^q, q \geq 1$ be h-convex on $[\varsigma_1, \varsigma_2]$ and $|\Pi'(x)| \leq M$, $|\psi'(x)| \leq L$, $x \in [\varsigma_1, \varsigma_2]$. Then, for all $x \in [\varsigma_1, \varsigma_2]$,*

$$\Big|\Pi(x) - \Gamma(\zeta+1)\Big[\frac{1}{2(\psi(x)-\psi(\varsigma_1))^{\zeta}} J^{\zeta,\psi}_{x-}\Pi(\varsigma_1) + \frac{1}{2(\psi((\varsigma_2)-\psi(x))^{\zeta}} J^{\zeta,\psi}_{x+}\Pi(\varsigma_2)\Big]\Big|$$
$$\leq \Big[\frac{(x-\varsigma_1)^{\zeta+1}}{2(\psi(x)-\psi(\varsigma_1))^{\zeta}} + \frac{(\varsigma_2-x)^{\zeta+1}}{2(\psi(\varsigma_2)-\psi(x))^{\zeta}}\Big] ML^{\zeta}\Big(\frac{1}{\zeta+1}\Big)^{1-\frac{1}{q}}\Big(\int_0^1 t^{\zeta}[h(t)+h(1-t)]dt\Big)^{\frac{1}{q}}.$$

Theorem 126 ([66]). *Let Π be as in Theorem 123. Additionally, let $|\Pi'|^q, q > 1$ be h-convex on $[\varsigma_1, \varsigma_2]$ and $|\Pi'(x)| \leq M$, $|\psi'(x)| \leq L$, $x \in [\varsigma_1, \varsigma_2]$. Then, for all $x \in [\varsigma_1, \varsigma_2]$,*

$$\Big|\Pi(x) - \Gamma(\zeta+1)\Big[\frac{1}{2(\psi(x)-\psi(\varsigma_1))^{\zeta}} J^{\zeta,\psi}_{x-}\Pi(\varsigma_1) + \frac{1}{2(\psi(\varsigma_2)-\psi(x))^{\zeta}} J^{\zeta,\psi}_{x+}\Pi(\varsigma_2)\Big]\Big|$$
$$\leq \Big[\frac{(x-\varsigma_1)^{\zeta+1}}{2(\psi(x)-\psi(\varsigma_1))^{\zeta}} + \frac{(\varsigma_2-x)^{\zeta+1}}{2(\psi(\varsigma_2)-\psi(x))^{\zeta}}\Big] 2M^{q}L^{\zeta}\Big(\frac{1}{\zeta p+1}\Big)^{\frac{1}{p}}\Big(\int_0^1 h(t)dt\Big)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Now, we add some Ostrowski-type inequalities via fractional integrals with respect to another function, i.e., $\bar{\zeta}, \beta, \gamma, \delta$-convex functions in mixed kind, according to the following definition.

Definition 32 ([67]). *Let $(\bar{\zeta}, \beta, \gamma, \delta) \in (0,1]^4$. The function $\Pi : I \subset [0, \infty) \to [0, \infty)$ is $\bar{\zeta}, \beta, \gamma, \delta$-convex function, if*

$$\Pi(tx + (1-t)y) \leq t^{\bar{\zeta}\gamma}\Pi(x) + (1-t^{\beta})^{\delta}\Pi(y),$$

for all $x, y \in I$ and $t \in [0,1]$.

Theorem 127 ([67]). *Let $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be a function which is differentiable on $(\varsigma_1, \varsigma_2)$ with $\varsigma_1 < \varsigma_2$ and $\Pi' : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ is integrable on $[\varsigma_1, \varsigma_2]$. Additionally, let $|\Pi'|$ be a $\bar{\zeta}, \beta, \gamma, \delta$-convex* *function on* $[\varsigma_1, \varsigma_2]$ *and* $|\Pi'(x)| \leq M$, $|\psi'(x)| \leq L$ *for all* $x \in (\varsigma_1, \varsigma_2)$, ψ *is a Lipschizian function. Then*

$$\begin{aligned} &\left|\Pi(x) - \Gamma(\zeta+1)\Big[\frac{J^{\zeta,\psi}_{x+}\Pi(\varsigma_2)}{2(\psi(\varsigma_2)-\psi(x))^{\zeta}} + \frac{J^{\zeta,\psi}_{x-}\Pi(\varsigma_1}{2(\psi(\psi(x)-\varsigma_1))^{\zeta}}\Big]\right. \\ \leq\;& ML^{\zeta}\Big(\frac{1}{\zeta+\bar{\zeta}\gamma+1} + \frac{B\Big(\frac{\zeta+1}{\beta}, \delta+1\Big)}{\beta}\Big)\Big[\frac{(x-\varsigma_1)^{\zeta+1}}{2(\psi(x)-\psi(\varsigma_1))^{\zeta}} + \frac{(\varsigma_2 - x)^{\zeta+1}}{2(\psi(\varsigma_2)-\psi(x))^{\zeta}}\Big], \end{aligned}$$

for all $x \in (\varsigma_1, \varsigma_2)$.

Theorem 128 ([67])**.** *Let* Π *be as in Theorem 127. Additionally, let* $|\Pi'|^q, q \geq 1$ *be a* $\bar{\zeta}, \beta, \gamma, \delta$*-convex function on* $[\varsigma_1, \varsigma_2]$ *and* $|\Pi'(x)| \leq M$, $|\psi'(x)| \leq L$ *for all* $x \in (\varsigma_1, \varsigma_2)$, ψ *is a Lipschizian function. Then*

$$\begin{aligned} &\left|\Pi(x) - \Gamma(\zeta+1)\Big[\frac{J^{\zeta,\psi}_{x+}\Pi(\varsigma_2)}{2(\psi(\varsigma_2)-\psi(x))^{\zeta}} + \frac{J^{\zeta,\psi}_{x-}\Pi(\varsigma_1}{2(\psi(\psi(x)-\varsigma_1))^{\zeta}}\Big]\right. \\ \leq\;& \frac{ML^{\zeta}}{(\zeta+1)^{1-\frac{1}{q}}}\Big(\frac{1}{\zeta+\bar{\zeta}\gamma+1} + \frac{B\Big(\frac{\zeta+1}{\beta}, \delta+1\Big)}{\beta}\Big)^{\frac{1}{q}}\Big[\frac{(x-\varsigma_1)^{\zeta+1}}{2(\psi(x)-\psi(\varsigma_1))^{\zeta}} + \frac{(\varsigma_2 - x)^{\zeta+1}}{2(\psi(\varsigma_2)-\psi(x))^{\zeta}}\Big], \end{aligned}$$

for all $x \in (\varsigma_1, \varsigma_2)$.

Theorem 129 ([67])**.** *Let* Π *be as in Theorem 127. Additionally, let* $|\Pi'|^q$ *be a* $\bar{\zeta}, \beta, \gamma, \delta$*-convex function on* $[\varsigma_1, \varsigma_2]$, $q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $|\Pi'(x)| \leq M$, $|\psi'(x)| \leq L$ *for all* $x \in (\varsigma_1, \varsigma_2)$, ψ *is a Lipschizian function. Then*

$$\begin{aligned} &\left|\Pi(x) - \Gamma(\zeta+1)\Big[\frac{J^{\zeta,\psi}_{x+}\Pi(\varsigma_2)}{2(\psi(\varsigma_2)-\psi(x))^{\zeta}} + \frac{J^{\zeta,\psi}_{x-}\Pi(\varsigma_1}{2(\psi(\psi(x)-\varsigma_1))^{\zeta}}\Big]\right. \\ \leq\;& \frac{ML^{\zeta}}{(\zeta p+1)^{\frac{1}{p}}}\Big(\frac{1}{\bar{\zeta}\gamma+1} + \frac{B\Big(\frac{1}{\beta}, \delta+1\Big)}{\beta}\Big)^{\frac{1}{q}}\Big[\frac{(x-\varsigma_1)^{\zeta+1}}{2(\psi(x)-\psi(\varsigma_1))^{\zeta}} + \frac{(\varsigma_2 - x)^{\zeta+1}}{2(\psi(\varsigma_2)-\psi(x))^{\zeta}}\Big], \end{aligned}$$

for all $x \in (\varsigma_1, \varsigma_2)$.

Here, we add some fractional Ostrowski-type inequalities for functions with respect to another function.

Theorem 130 ([68])**.** *Let* $\Pi : I \to \mathbb{R}$ *be a mapping differentiable on* I° *and* $\varsigma_1, \varsigma_2 \in I^{\circ}$, $\varsigma_1 < \varsigma_2$ *and* $|\Pi'(x)| \leq M$, *for all* $x \in [\varsigma_1, \varsigma_2]$*. Suppose* $\psi \in C^1(I)$ *is positive monotone and increasing, and* $\psi'(x) \geq 1$ *for all* $x \in I$*. Let* $J^{\zeta,\psi}_{\varsigma_1+}$ *and* $J^{\beta,\psi}_{\varsigma_2-}$ *be the left- and right-Riemmansided fractional integrals. Then*

$$\begin{aligned} &\left|\Big((\psi(\varsigma_2)-\psi(x))^{\beta} + (\psi(x)-\psi(\varsigma_1))^{\zeta}\Big)\Pi(x) - \Big(\Gamma(\beta+1)J^{\beta,\psi}_{\varsigma_2-}\Pi(x) + \Gamma(\zeta+1)J^{\beta,\psi}_{\varsigma_1+}\Pi(x)\Big)\right| \\ \leq\;& M\Big(\frac{\beta}{\beta+1}(\psi(\varsigma_2)-\psi(x))^{\beta+1} + \frac{\zeta}{\zeta+1}(\psi(x)-\psi(\varsigma_1))^{\zeta+1}\Big), \end{aligned}$$

where $\zeta, \beta > 0$ *and* $x \in [\varsigma_1, \varsigma_2]$.

Theorem 131 ([68])**.** *Assume that* Π *and* ψ *are as in Theorem 130. If* $m \leq \Pi'(x)| \leq M$, *for all* $M \geq 0$, $m \leq 0$ *and all* $x \in [\varsigma_1, \varsigma_2]$, *then*

$$\begin{aligned}
&\Big((\psi(x)-\psi(\varsigma_1))^{\zeta}-(\psi(\varsigma_2)-\psi(x))^{\beta}\Big)\Pi(x)-\Big(\Gamma(\zeta+1)J^{\zeta,\psi}_{\varsigma_1+}\Pi(x)-\Gamma(\beta+1)J^{\beta,\psi}_{\varsigma_2-}\Pi(x)\Big)\Big|\\
\leq\ & M\Big(\frac{\zeta}{\zeta+1}(\psi(x)-\psi(\varsigma_1))^{\zeta+1}+\frac{\beta}{\beta+1}(\psi(\varsigma_2)-\psi(x))^{\beta+1}\Big),
\end{aligned}$$

and

$$\begin{aligned}
&\Big((\psi(\varsigma_2)-\psi(x))^{\beta}-(\psi(x)-\psi(\varsigma_1))^{\zeta}\Big)\Pi(x)+\Big(\Gamma(\zeta+1)J^{\zeta,\psi}_{\varsigma_1+}\Pi(x)-\Gamma(\beta+1)J^{\beta,\psi}_{\varsigma_2-}\Pi(x)\Big)\Big|\\
\leq\ & -m\Big(\frac{\beta}{\beta+1}(\psi(\varsigma_2)-\psi(x))^{\beta+1}+\frac{\zeta}{\zeta+1}(\psi(x)-\psi(\varsigma_1))^{\zeta+1}\Big),
\end{aligned}$$

where $\zeta,\beta>0$ *and* $x\in[\varsigma_1,\varsigma_2]$.

Theorem 132 ([68]). *Assume that* Π *and* ψ *are as in Theorem 130. Then*

$$\begin{aligned}
&\Big|\Big((\psi(\varsigma_2)-\psi(x))^{\beta}\Pi(\varsigma_2)+(\psi(x)-\psi(\varsigma_1))^{\zeta}\Pi(\varsigma_1)\Big)-\Big(\Gamma(\beta+1)J^{\beta,\psi}_{\varsigma_2-}+\Gamma(\zeta+1)J^{\beta,\psi}_{\varsigma_1+}\Pi(x)\Big)\Big|\\
\leq\ & M\Big(\frac{\beta}{\beta+1}(\psi(\varsigma_2)-\psi(x))^{\beta+1}+\frac{\zeta}{\zeta+1}(\psi(x)-\psi(\varsigma_1))^{\zeta+1}\Big),
\end{aligned}$$

where $\zeta,\beta>0$ *and* $x\in[\varsigma_1,\varsigma_2]$.

7. Mercer-Ostrowski-Type Fractional Integral Inequalities for Riemann–Liouville Fractional Integral Operator

In this section, we present Mercer-Ostrowski-type fractional integral inequalities for first order differentiable functions for the Riemann–Liouville integral operator.

Theorem 133 ([69]). *Let* $\Pi:[\varsigma_1,\varsigma_2]\to\mathbb{R}$ *be a differentiable function on* $(\varsigma_1,\varsigma_2)$ *with* $\varsigma_1<\varsigma_2$ *such that* $\Pi'\in L[\varsigma_1,\varsigma_2]$. *If* $|\Pi'|$ *is a convex function on* $[\varsigma_1,\varsigma_2]$, *then*

$$\begin{aligned}
&\Big|\Big\{(\ell-s_1)^{\zeta}\Pi(\ell+\varsigma_1-s_1)+(s_2-\ell)^{\zeta}\Pi(\ell+\varsigma_2-s_2)\Big\}\\
&-\Gamma(\zeta+1)\Big\{J^{\zeta}_{(\ell+\varsigma_1-s_1)-}\Pi(\varsigma_1)+J^{\zeta}_{(\ell+\varsigma_2-s_2)+}\Pi(\varsigma_2)\Big\}\Big|\\
\leq\ & (\ell-s_1)^{\zeta}\Big\{\frac{1}{\zeta+1}(|\Pi'(\ell)|+|\Pi'(\varsigma_1)|)-\Big[\frac{1}{\Gamma(\zeta+2)}|\Pi'(s_1)|+\frac{1}{(\zeta+1)(\zeta+2)}|\Pi'(\ell)|\Big]\Big\}\\
&+(s_2-\ell)^{\zeta}\Big\{\frac{1}{\zeta+1}(|\Pi'(\ell)|+|\Pi'(\varsigma_2)|)-\Big[\frac{1}{\Gamma(\zeta+2)}|\Pi'(s_2)|+\frac{1}{(\zeta+1)(\zeta+2)}|\Pi'(\ell)|\Big]\Big\}.
\end{aligned}$$

Theorem 134 ([69]). *Let* Π *be as in Theorem 133. If* $|\Pi'|^q$ *is a convex function on* $[\varsigma_1,\varsigma_2]$, $q>1$, *then*

$$\begin{aligned}
&\Big|\Big\{(\ell-s_1)^{\zeta}\Pi(\ell+\varsigma_1-s_1)+(s_2-\ell)^{\zeta}\Pi(\ell+\varsigma_2-s_2)\Big\}\\
&-\Gamma(\zeta+1)\Big\{J^{\zeta}_{(\ell+\varsigma_1-s_1)-}\Pi(\varsigma_1)+J^{\zeta}_{(\ell+\varsigma_2-s_2)+}\Pi(\varsigma_2)\Big\}\Big|\\
\leq\ & (\ell-s_1)^{\zeta}\Big(\frac{1}{\zeta p+1}\Big)^{\frac{1}{p}}\Big\{(|\Pi'(\ell)|^q+|\Pi'(\varsigma_1)|^q)-\frac{1}{2}\Big[|\Pi'(s_1)|^q+|\Pi'(\ell)|\Big]\Big\}^{\frac{1}{q}}\\
&+(s_2-\ell)^{\zeta}\Big(\frac{1}{\zeta p+1}\Big)^{\frac{1}{p}}\Big\{(|\Pi'(\ell)|^q+|\Pi'(\varsigma_2)|^q)-\Big[\frac{1}{2}|\Pi'(s_2)|^q+|\Pi'(\ell)^q|\Big]\Big\}^{\frac{1}{q}},
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 135 ([69])**.** *Let Π be as in Theorem 133. If $|\Pi'|^q$ is a convex function on $[\varsigma_1, \varsigma_2]$, $q \geq 1$, then*

$$\begin{aligned} & \Big|\Big\{(\ell - s_1)^{\zeta}\Pi(\ell + \varsigma_1 - s_1) + (s_2 - \ell)^{\zeta}\Pi(\ell + \varsigma_2 - s_2)\Big\} \\ & -\Gamma(\zeta+1)\Big\{J^{\zeta}_{(\ell+\varsigma_1-s_1)-}\Pi(\varsigma_1) + J^{\zeta}_{(\ell+\varsigma_2-s_2)+}\Pi(\varsigma_2)\Big\}\Big| \\ \leq\; & (\ell - s_1)^{\zeta}\Big(\frac{1}{\zeta+1}\Big)^{1-\frac{1}{q}}\Big\{\frac{1}{\zeta+1}(|\Pi'(\ell)|^q + |\Pi'(\varsigma_1)|^q) \\ & -\Big[\frac{1}{\Gamma(\zeta+2)}|\Pi'(s_1)|^q + \frac{1}{(\zeta+1)(\zeta+2)}|\Pi'(\ell)|^q\Big]\Big\}^{\frac{1}{q}} \\ & +(s_2 - \ell)^{\zeta}\Big(\frac{1}{\zeta+1}\Big)^{1-\frac{1}{q}}\Big\{\frac{1}{\zeta+1}(|\Pi'(\ell)|^q + |\Pi'(\varsigma_2)|^q) \\ & -\Big[\frac{1}{\Gamma(\zeta+2)}|\Pi'(s_2)|^q + \frac{1}{(\zeta+1)(\zeta+2)}|\Pi'(\ell)|^q\Big]\Big\}^{\frac{1}{q}}. \end{aligned}$$

Throughout this portion, Mercer-Ostrowski-type inequalities for differentiable functions via ψ-Riemann–Liouville fractional integral operators are obtained.

Theorem 136 ([70])**.** *Let $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be a function which is differentiable on $(\varsigma_1, \varsigma_2)$ with $\varsigma_1 < \varsigma_2$, $\Pi' : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be integrable on $[\varsigma_1, \varsigma_2]$. Let $\psi(x)$ be an increasing and positive monotone function on $(\varsigma_1, \varsigma_2]$, having a continuous derivative $\psi'(x)$ on $(\varsigma_1, \varsigma_2)$. If $|\Pi'|$ is a convex function on $[\varsigma_1, \varsigma_2]$, then for $\zeta > 0$ and $x, a, b, y \in [\varsigma_1, \varsigma_2]$, we have:*

$$\begin{aligned} & \Big|\frac{(y-a)^{\zeta}}{b-a}\Pi(x + \varsigma_1 - a) + \frac{(b-y)^{\zeta}}{b-a}\Pi(x + \varsigma_2 - b) - \frac{\Gamma(\zeta+1)}{b-a}\Big[J^{\zeta,\psi}_{\psi^{-1}(x+\varsigma_1-a)-}(\Pi \circ \psi)(\psi^{-1}(x + \varsigma_1 - y)) \\ & + J^{\zeta,\psi}_{\psi^{-1}(x+\varsigma_2-b)+}(\Pi \circ \psi)(\psi^{-1}(x + \varsigma_2 - y))\Big] \\ \leq\; & \frac{(y-a)^{\zeta+1}}{b-a}\Big\{\frac{1}{\zeta+1}\Big(|\Pi'(x)| + |\Pi'(\varsigma_1)|\Big) - \Big[\frac{1}{\zeta+2}|\Pi'(a)| + \frac{1}{(\zeta+1)(\zeta+2)}|\Pi'(y)|\Big]\Big\} \\ & + \frac{(b-y)^{\zeta+1}}{b-a}\Big\{\frac{1}{\zeta+1}\Big(|\Pi'(x)| + |\Pi'(\varsigma_2)|\Big) - \Big[\frac{1}{\zeta+2}|\Pi'(b)| + \frac{1}{(\zeta+1)(\zeta+2)}|\Pi'(y)|\Big]\Big\}. \end{aligned}$$

Theorem 137 ([70])**.** *Assume that Π and ψ are as in Theorem 136. If $|\Pi'|^q$ is a convex function on $[\varsigma_1, \varsigma_2]$, then for $\zeta > 0$ and $x, a, b, y \in [\varsigma_1, \varsigma_2]$, we have:*

$$\begin{aligned} & \Big|\frac{(y-a)^{\zeta}}{b-a}\Pi(x + \varsigma_1 - a) + \frac{(b-y)^{\zeta}}{b-a}\Pi(x + \varsigma_2 - b) - \frac{\Gamma(\zeta+1)}{b-a}\Big[J^{\zeta,\psi}_{\psi^{-1}(x+\varsigma_1-a)-}(\Pi \circ \psi)(\psi^{-1}(x + \varsigma_1 - y)) \\ & + J^{\zeta,\psi}_{\psi^{-1}(x+\varsigma_2-b)+}(\Pi \circ \psi)(\psi^{-1}(x + \varsigma_2 - y))\Big] \\ \leq\; & \frac{(y-a)^{\zeta+1}}{b-a}\Big(\frac{1}{\zeta p+1}\Big)^{\frac{1}{p}}\Big\{\Big(|\Pi'(x)|^q + |\Pi'(\varsigma_1)|^q\Big) - \frac{1}{2}\Big[|\Pi'(a)|^q + |\Pi'(y)|^q\Big]\Big\}^{\frac{1}{p}} \\ & + \frac{(b-y)^{\zeta+1}}{b-a}\Big(\frac{1}{\zeta p+1}\Big)^{\frac{1}{p}}\Big\{\Big(|\Pi'(x)|^q + |\Pi'(\varsigma_2)|^q\Big) - \frac{1}{2}\Big[|\Pi'(b)|^q + |\Pi'(y)|^q\Big]\Big\}^{\frac{1}{p}}, \end{aligned}$$

where $p, q > 1$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Theorem 138 ([70])**.** *Assume that Π and ψ are as in Theorem 136. If $|\Pi'|^q, q \geq 1$ is a convex function on $[\varsigma_1, \varsigma_2]$, then for $\zeta > 0$ and $x, a, b, y \in [\varsigma_1, \varsigma_2]$, we have:*

$$\left|\frac{(y-a)^{\zeta}}{b-a}\Pi(x+\varsigma_1-a)+\frac{(b-y)^{\zeta}}{b-a}\Pi(x+\varsigma_2-b)-\frac{\Gamma(\zeta+1)}{b-a}\left[J^{\zeta,\psi}_{\psi^{-1}(x+\varsigma_1-a)-}(\Pi\circ\psi)(\psi^{-1}(x+\varsigma_1-y))\right.\right.$$
$$\left.\left.+J^{\zeta,\psi}_{\psi^{-1}(x+\varsigma_2-b)+}(\Pi\circ\psi)(\psi^{-1}(x+\varsigma_2-y))\right]\right|$$
$$\leq \frac{(y-a)^{\zeta+1}}{b-a}\left(\frac{1}{\zeta+1}\right)^{1-\frac{1}{q}}\left\{\frac{1}{\zeta+1}\left(|\Pi'(x)|^q+|\Pi'(\varsigma_1)|^q\right)-\left[\frac{1}{\zeta+2}|\Pi'(a)|^q+\frac{1}{(\zeta+1)(\zeta+2)}|\Pi'(y)|^q\right]\right\}^{\frac{1}{q}}$$
$$+\frac{(b-y)^{\zeta+1}}{b-a}\left(\frac{1}{\zeta+1}\right)^{1-\frac{1}{q}}\left\{\frac{1}{\zeta+1}\left(|\Pi'(x)|^q+|\Pi'(\varsigma_2)|^q\right)-\left[\frac{1}{\zeta+2}|\Pi'(b)|^q+\frac{1}{(\zeta+1)(\zeta+2)}|\Pi'(y)|^q\right]\right\}^{\frac{1}{q}}.$$

8. Ostrowski-Type Fractional Integral Inequalities via Hadamard Fractional Integral

Definition 33 ([2]). *Hadamard fractional integrals (left and right) of order $\zeta\in\mathbb{R}^+$ of function Π are defined by*

$$({}_HJ^{\zeta}_{\varsigma_1+}\Pi)(x)=\frac{1}{\Gamma(\zeta)}\int_{\varsigma_1}^{x}\left(\ln\frac{x}{t}\right)^{\zeta-1}\Pi(t)\frac{dt}{t},\ \ 0<\varsigma_1<x\leq\varsigma_2,$$

and

$$({}_HJ^{\zeta}_{\varsigma_2-}\Pi)(x)=\frac{1}{\Gamma(\zeta)}\int_{x}^{\varsigma_2}\left(\ln\frac{t}{x}\right)^{\zeta-1}\Pi(t)\frac{dt}{t},\ \ 0<\varsigma_1\leq x<\varsigma_2.$$

Definition 34 ([71]). *The function $\Pi:I\subseteq(0,\infty)\to\mathbb{R}$ is called quasi-geometrically convex on I if*

$$\Pi(x^t y^{1-t})\leq\max\{\Pi(x),\Pi(y)\},$$

for all $x,y\in I$ and $t\in[0,1]$.

Fractional Ostrowski-type fractional integral inequalities for functions which are differentiable and quasi-geometrically convex, are given now.

Theorem 139 ([72]). *Let $\Pi:[\varsigma_1,\varsigma_2]\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable mapping on $(\varsigma_1,\varsigma_2)$ with $\varsigma_1<\varsigma_2$. Let $g:[\varsigma_1,\varsigma_2]\to(0,\infty)$ be a continuous, positive and geometrically symmetric to $\sqrt{\varsigma_1\varsigma_2}$ and $\Pi'\in L[\varsigma_1,\varsigma_2]$. If $|\Pi'|$ is quasi-geometrically convex, then, for all $x\in[\varsigma_1,\varsigma_2]$,*

$$\left|{}_HJ^{\zeta}_{x-}g\Pi(\varsigma_1)+{}_HJ^{\zeta}_{x+}g\Pi(\varsigma_2)-\left[{}_HJ^{\zeta}_{x-}\Pi(\varsigma_1)+{}_HJ^{\zeta}_{x+}\Pi(\varsigma_2)\right]\right|$$
$$\leq\frac{\left(\ln\left(\frac{\varsigma_2}{x}\right)\right)^{\zeta+1}}{\Gamma(\zeta+2)}\max\{|\Pi'(x)|,|\Pi'(\varsigma_2)|\}\|g\|_{[x,\varsigma_2],\infty}$$
$$+\frac{\left(\ln\left(\frac{\varsigma_1}{x}\right)\right)^{\zeta+1}}{\Gamma(\zeta+2)}\max\{|\Pi'(x)|,|\Pi'(\varsigma_1)|\}\|g\|_{[\varsigma_1,x],\infty}.$$

Theorem 140 ([72]). *Let $\Pi:[\varsigma_1,\varsigma_2]\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable mapping on $(\varsigma_1,\varsigma_2)$ with $\varsigma_1<\varsigma_2$. Let $g:[\varsigma_1,\varsigma_2]\to(0,\infty)$ be a continuous, positive and geometrically symmetric to $\sqrt{\varsigma_1\varsigma_2}$ and $\Pi'\in L[\varsigma_1,\varsigma_2]$. If $|\Pi'|^q$ is quasi-geometrically convex, $q>1$ and $\dfrac{1}{p}+\dfrac{1}{q}=1$, then*

$$\left|{}_HJ^{\zeta}_{x-}g\Pi(\varsigma_1)+{}_HJ^{\zeta}_{x+}g\Pi(\varsigma_2)-\left[{}_HJ^{\zeta}_{x-}\Pi(\varsigma_1)+{}_HJ^{\zeta}_{x+}\Pi(\varsigma_2)\right]\right|$$
$$\leq\frac{\left(\ln\left(\frac{\varsigma_2}{x}\right)\right)^{\zeta+1}}{(\zeta p+1)^{\frac{1}{p}}\Gamma(\zeta+1)}\left(\max\{|\Pi'(x)|,|\Pi'(\varsigma_2)|\}\right)^{\frac{1}{q}}\|g\|_{[x,\varsigma_2],\infty}$$
$$+\frac{\left(\ln\left(\frac{\varsigma_1}{x}\right)\right)^{\zeta+1}}{(\zeta p+1)^{\frac{1}{p}}\Gamma(\zeta+1)}\left(\max\{|\Pi'(x)|,|\Pi'(\varsigma_1)|\}\right)^{\frac{1}{q}}\|g\|_{[\varsigma_1,x],\infty},$$

for all $x \in [\varsigma_1, \varsigma_2]$.

Theorem 141 ([72]). *Let* $\Pi : [\varsigma_1, \varsigma_2] \subseteq (0, \infty) \to \mathbb{R}$ *be a differentiable mapping on* $(\varsigma_1, \varsigma_2)$ *with* $\varsigma_1 < \varsigma_2$. *Let* $g : [\varsigma_1, \varsigma_2] \to (0, \infty)$ *be a continuous, positive and geometrically symmetric to* $\sqrt{\varsigma_1\varsigma_2}$ *and* $\Pi' \in L[\varsigma_1, \varsigma_2]$. *If* $|\Pi'|^q, q \geq 1$ *is quasi-geometrically convex, then, for all* $x \in [\varsigma_1, \varsigma_2]$,

$$\begin{aligned}
&\left| {}_HJ^{\zeta}_{x-}g\Pi(\varsigma_1) + {}_HJ^{\zeta}_{x+}g\Pi(\varsigma_2) - \left[{}_HJ^{\zeta}_{x-}\Pi(\varsigma_1) + {}_HJ^{\zeta}_{x+}\Pi(\varsigma_2)\right]\right| \\
\leq\; & \frac{\left(\ln\left(\frac{\varsigma_2}{x}\right)\right)^{\zeta+1}}{\Gamma(\zeta+2)}\Big(\max\{|\Pi'(x)|, |\Pi'(\varsigma_2)|\}\Big)^{\frac{1}{q}}\|g\|_{[x,\varsigma_2],\infty} \\
& + \frac{\left(\ln\left(\frac{\varsigma_1}{x}\right)\right)^{\zeta+1}}{\Gamma(\zeta+2)}\Big(\max\{|\Pi'(x)|, |\Pi'(\varsigma_1)|\}\Big)^{\frac{1}{q}}\|g\|_{[\varsigma_1,x],\infty}.
\end{aligned}$$

Definition 35 ([73]). *A function* $\Pi : I \subset (0, \infty) \to \mathbb{R}$ *is said to satisfy the* $s - e$*-condition if*

$$\Pi(e^{tx+(1-t)y}) \leq t^s\Pi(e^x) + (1-t)^s\Pi(e^y),$$

for all $x, y \in I, t \in [0, 1]$ *and for some fixed* $s \in (0, 1]$.

Here, we add some fractional Ostrowski inequalities for $s - e$-condition.

Theorem 142 ([73]). *Let* $\Pi : [\varsigma_1, \varsigma_2] \subset (0, \infty) \to \mathbb{R}$ *be a function which is differentiable on* $(\varsigma_1, \varsigma_2)$ *with* $0 < \varsigma_1 < \varsigma_2$ *such that* $\Pi' \in L[\varsigma_1, \varsigma_2]$. *If* $|\Pi'|$ *satisfies the* $s - e$*-condition on* $[\varsigma_1, \varsigma_2]$ *for* $s \in (0, 1]$ *and* $|\Pi'(x)| \leq M, x \in [\varsigma_1, \varsigma_2]$, *then*

$$\begin{aligned}
&\left|\frac{(\ln x - \ln\varsigma_1)^{\zeta} + (\ln\varsigma_2 - \ln x)^{\zeta}}{\ln\varsigma_2 - \ln\varsigma_1}\Pi(x) - \frac{\Gamma(\zeta+1)}{\ln\varsigma_2 - \ln\varsigma_1}\left[{}_HJ^{\zeta}_{x-}\Pi(\varsigma_1) + {}_HJ^{\zeta}_{x+}\Pi(\varsigma_2)\right]\right| \\
\leq\; & \frac{M\varsigma_2}{\ln\varsigma_2 - \ln\varsigma_1}\left(1 + \frac{\Gamma(\zeta+1)\Gamma(s+1)}{\Gamma(\zeta+s+1)}\right)\frac{(\ln x - \ln\varsigma_1)^{\zeta+1} + (\ln\varsigma_2 - \ln x)^{\zeta+1}}{\zeta+s+1},
\end{aligned}$$

for all $x \in (\varsigma_1, \varsigma_2)$.

Theorem 143 ([73]). *Let* Π *be as in Theorem 142. If* $|\Pi'|^q, q > 1$ *satisfies the* $s - e$*-condition on* $[\varsigma_1, \varsigma_2]$ *for* $s \in (0, 1]$ *and* $|\Pi'(x)| \leq M, x \in [\varsigma_1, \varsigma_2]$, *then*

$$\begin{aligned}
&\left|\frac{(\ln x - \ln\varsigma_1)^{\zeta} + (\ln\varsigma_2 - \ln x)^{\zeta}}{\ln\varsigma_2 - \ln\varsigma_1}\Pi(x) - \frac{\Gamma(\zeta+1)}{\ln\varsigma_2 - \ln\varsigma_1}\left[{}_HJ^{\zeta}_{x-}\Pi(\varsigma_1) + {}_HJ^{\zeta}_{x+}\Pi(\varsigma_2)\right]\right| \\
\leq\; & \frac{M\varsigma_2}{(1+p\zeta)^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\frac{(\ln x - \ln\varsigma_1)^{\zeta+1} + (\ln\varsigma_2 - \ln x)^{\zeta+1}}{\ln\varsigma_2 - \ln\varsigma_1},
\end{aligned}$$

for all $x \in (\varsigma_1, \varsigma_2)$, *where* $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 144 ([73]). *Let the assumptions of this theorem be stated in Theorem 143. Then*

$$\begin{aligned}
&\left|\frac{(\ln x - \ln\varsigma_1)^{\zeta} + (\ln\varsigma_2 - \ln x)^{\zeta}}{\ln\varsigma_2 - \ln\varsigma_1}\Pi(x) - \frac{\Gamma(\zeta+1)}{\ln\varsigma_2 - \ln\varsigma_1}\left[{}_HJ^{\zeta}_{x-}\Pi(\varsigma_1) + {}_HJ^{\zeta}_{x+}\Pi(\varsigma_2)\right]\right| \\
\leq\; & M\varsigma_2\left(\frac{1}{\zeta q+s+1}\right)^{\frac{1}{q}}\left(1 + \frac{\Gamma(\zeta q+1)\Gamma(s+1)}{\Gamma(\zeta q+s+1)}\right)^{\frac{1}{q}}\frac{(\ln x - \ln\varsigma_1)^{\zeta+1} + (\ln\varsigma_2 - \ln x)^{\zeta+1}}{\ln\varsigma_2 - \ln\varsigma_1},
\end{aligned}$$

for all $x \in (\varsigma_1, \varsigma_2)$, *where* $\frac{1}{p} + \frac{1}{q} = 1$.

9. Ostrowski-Type Fractional Integral Inequalities for Exponential Kernel

Here, in this section, we add some Ostrowski-type inequalities for exponential kernel.

Definition 36 ([74]). *Let $\Pi \in L(\varsigma_1, \varsigma_2)$. The left and right fractional integrals of order $\zeta \in (0,1)$ are defined by*

$$I^{\zeta}_{\varsigma_1}\Pi(x) = \frac{1}{\zeta}\int_{\varsigma_1}^{x} \exp\Big(-\frac{1-\zeta}{\zeta}(x-s)\Big)\Pi(s)ds, \;\; x > \varsigma_1,$$

and

$$I^{\zeta}_{\varsigma_2}\Pi(x) = \frac{1}{\zeta}\int_{x}^{\varsigma_2} \exp\Big(-\frac{1-\zeta}{\zeta}(s-x)\Big)\Pi(s)ds, \;\; x < \varsigma_2,$$

respectively.

Theorem 145 ([75]). *Let $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be a function which is differentiable with $0 \le \varsigma_1 < \varsigma_2$ and $\Pi \in L[\varsigma_1, \varsigma_2]$. If $|\Pi'|^q, q > 1$, is a convex function on $[\varsigma_1, \varsigma_2]$, then, for all $x \in [\varsigma_1, \varsigma_2]$,*

$$\begin{aligned}
&\left|\Pi(x) - \frac{1-\zeta}{2-\exp\left\{-\frac{1-\zeta}{\zeta}(x-\varsigma_1)\right\} - \exp\left\{-\frac{1-\zeta}{\zeta}(\varsigma_2-x)\right\}}\left[I^{\zeta}_{x-}\Pi(\varsigma_1) + ^{\zeta}_{x+}\Pi(\varsigma_2)\right]\right| \\
\le\;& \frac{1-\zeta}{2-\exp\{-\theta_{\varsigma_1}\} - \exp\{-\theta_{\varsigma_2}\}}\left[(x-\varsigma_1)A_1(\zeta,p)\Big(\frac{|\Pi'(x)|^q + |\Pi'(\varsigma_1)|^q}{2}\Big)^{\frac{1}{q}}\right. \\
&\left.+(\varsigma_2 - x)A_2(\zeta,p)\Big(\frac{|\Pi'(x)|^q + |\Pi'(\varsigma_2)|^q}{2}\Big)^{\frac{1}{q}}\right],
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q} = 1$, $\theta_{\varsigma_1} = \frac{1-\zeta}{\zeta}(x-\varsigma_1)$, $\theta_{\varsigma_2} = \frac{1-\zeta}{\zeta}(\varsigma_2 - x)$ and

$$\begin{aligned}
A_1(\zeta,p) &= \left(\int_0^1 [1-\exp\Big\{-\frac{1-\zeta}{\zeta}(x-\varsigma_1)t]^p dt\Big\}\right)^{\frac{1}{p}}, \\
A_2(\zeta,p) &= \left(\int_0^1 [1-\exp\Big\{-\frac{1-\zeta}{\zeta}(\varsigma_2-x)t]^p dt\Big\}\right)^{\frac{1}{p}}.
\end{aligned}$$

Theorem 146 ([75]). *Let $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be a function which is differentiable with $0 \le \varsigma_1 < \varsigma_2$ and $\Pi \in L[\varsigma_1, \varsigma_2]$. If $|\Pi'|^q, q \ge 1$, is a convex function on $[\varsigma_1, \varsigma_2]$, then*

$$\begin{aligned}
&\left|\Pi(x) - \frac{1-\zeta}{2-\exp\left\{-\frac{1-\zeta}{\zeta}(x-\varsigma_1)\right\} - \exp\left\{-\frac{1-\zeta}{\zeta}(\varsigma_2-x)\right\}}\left[I^{\zeta}_{x-}\Pi(\varsigma_1) + ^{\zeta}_{x+}\Pi(\varsigma_2)\right]\right| \\
\le\;& \frac{1}{2-\exp\{-\theta_{\varsigma_1}\} - \exp\{-\theta_{\varsigma_2}\}}\left\{\frac{x-\varsigma_1}{\theta_{\varsigma_1}^{1-\frac{1}{q}}}[1+\theta_{\varsigma_1} - \exp\{-\theta_{\varsigma_1}\}]^{1-\frac{1}{q}}\right. \\
&\times\Big(|\Pi'(x)|^q\Big[\frac{1}{2} - \frac{1}{\theta_{\varsigma_1}^2}[1-\exp\{-\theta_{\varsigma_1}\}(\theta_{\varsigma_1}+1)]\Big] \\
&+|\Pi'(\varsigma_1)|^q\Big[\frac{1}{2} - \frac{1}{\theta_{\varsigma_1}^2}[\theta_{\varsigma_1} + \exp\{-\theta_{\varsigma_1}\} - 1]\Big]\Big)^{\frac{1}{q}} \\
&+\frac{\varsigma_2 - x}{\theta_{\varsigma_2}^{1-\frac{1}{q}}}[1+\theta_{\varsigma_2} - \exp\{-\theta_{\varsigma_2}\}]^{1-\frac{1}{q}}\Big(|\Pi'(x)|^q\Big[\frac{1}{2} - \frac{1}{\theta_{\varsigma_2}^2}[1-\exp\{-\theta_{\varsigma_2}\}(\theta_{\varsigma_2}+1)]\Big] \\
&\left.+|\Pi'(\varsigma_2)|^q\Big[\frac{1}{2} - \frac{1}{\theta_{\varsigma_2}^2}[\theta_{\varsigma_2} + \exp\{-\theta_{\varsigma_2}\} - 1]\Big]\Big)^{\frac{1}{q}}\right\},
\end{aligned}$$

for all $x \in [\varsigma_1, \varsigma_2]$, where $\theta_{\varsigma_1}, \theta_{\varsigma_2}$ defined in the previous theorem.

10. Ostrowski-Type Fractional Integral Inequalities via Atangana-Baleanu Fractional Integrals Operator

In this section, we give Ostrowski-type fractional integral inequalities for Atangana-Baleanu fractional integral operator for twice differentiable functions.

Definition 37 ([76]). *The left and right Atangana-Baleanu fractional integrals operators with nonlocal kernel of a function $\Pi \in H^1(\varsigma_1, \varsigma_2) = \{\Pi \in L^2(\varsigma_1, \varsigma_2) : \Pi' \in L^2(\varsigma_1, \varsigma_2)\}$, are defined as*

$$
{}^{AB}{}_{t}I^{\delta}_{\varsigma_1}\Pi(t) = \frac{1-\delta}{M(\delta)}\Pi(t) + \frac{\delta}{M(\delta)\Gamma(\delta)}\int_{\varsigma_1}^{t}\Pi(y)(t-y)^{\delta-1}dy,
$$
$$
{}^{AB}{}_{t}I^{\delta}_{\varsigma_2}\Pi(t) = \frac{1-\delta}{M(\delta)}\Pi(t) + \frac{\delta}{M(\delta)\Gamma(\delta)}\int_{t}^{\varsigma_2}\Pi(y)(y-t)^{\delta-1}dy,
$$

for $\delta \in [0,1]$ and $M(\delta)$ a normalization function satisfying $M(0) = M(1) = 1$.

Theorem 147 ([77]). *Let $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be a differentiable mapping on $(\varsigma_1, \varsigma_2)$ such that $\Pi' \in L[\varsigma_1, \varsigma_2]$. If $|\Pi'|$ is a convex on $[\varsigma_1, \varsigma_2]$, then $\forall\ x \in [\varsigma_1, \varsigma_2]$, $\delta \in (0,1]$ the inequality is given as*

$$
\begin{aligned}
&\Bigg|\frac{(x-\varsigma_1)^{\delta} + (\varsigma_2 - x)^{\delta}}{\varsigma_2 - \varsigma_1}\Pi(x) + \frac{1-\delta}{\varsigma_2-\varsigma_1}[\Pi(\varsigma_1) + \Pi(\varsigma_2)] \\
&\quad - \frac{M(\delta)\Gamma(\delta)}{\varsigma_2-\varsigma_1}\left\{{}^{AB}{}_{x}I^{\delta}_{\varsigma_1}\Pi(\varsigma_1) + {}^{AB}{}_{x}I^{\delta}_{\varsigma_2}\Pi(\varsigma_2)\right\}\Bigg| \\
\leq\ & \frac{(x-\varsigma_1)^{\delta+1}}{\varsigma_2-\varsigma_1}\left\{\frac{|\Pi'(x)|}{\delta+2} + \frac{|\Pi'(\varsigma_1)|}{(\delta+1)(\delta+2)}\right\} \\
&\quad + \frac{(\varsigma_2-x)^{\delta+1}}{\varsigma_2-\varsigma_1}\left\{\frac{|\Pi'(x)|}{\delta+2} + \frac{|\Pi'(\varsigma_2)|}{(\delta+1)(\delta+2)}\right\}.
\end{aligned}
$$

Theorem 148 ([77]). *Let $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be a differentiable mapping on $(\varsigma_1, \varsigma_2)$ such that $\Pi' \in L[\varsigma_1, \varsigma_2]$. If $|\Pi'|^q$ is a convex on $[\varsigma_1, \varsigma_2]$, then $\forall\ x \in [\varsigma_1, \varsigma_2]$, $\delta \in (0,1]$ the inequality is given as:*

$$
\begin{aligned}
&\Bigg|\frac{(x-\varsigma_1)^{\delta} + (\varsigma_2 - x)^{\delta}}{\varsigma_2 - \varsigma_1}\Pi(x) + \frac{1-\delta}{\varsigma_2-\varsigma_1}[\Pi(\varsigma_1) + \Pi(\varsigma_2)] \\
&\quad - \frac{M(\delta)\Gamma(\delta)}{\varsigma_2-\varsigma_1}\left\{{}^{AB}{}_{x}I^{\delta}_{\varsigma_1}\Pi(\varsigma_1) + {}^{AB}{}_{x}I^{\delta}_{\varsigma_2}\Pi(\varsigma_2)\right\}\Bigg| \\
\leq\ & \frac{(x-\varsigma_1)^{\delta+1}}{\varsigma_2-\varsigma_1}\left(\frac{1}{\delta p+1}\right)^{\frac{1}{p}}\left(\frac{|\Pi'(x)|^q + |\Pi'(\varsigma_1)|^q}{2}\right)^{\frac{1}{q}} \\
&\quad + \frac{(\varsigma_2-x)^{\delta+1}}{\varsigma_2-\varsigma_1}\left(\frac{1}{\delta p+1}\right)^{\frac{1}{p}}\left(\frac{|\Pi'(x)|^q + |\Pi'(\varsigma_2)|^q}{2}\right)^{\frac{1}{q}},
\end{aligned}
$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $q > 1$.

Theorem 149 ([78]). *Let $\Pi : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on $(\varsigma_1, \varsigma_2)$ with $\varsigma_1, \varsigma_2 \in I$, $\varsigma_1 < \varsigma_2$ such that $\Pi'' \in L[\varsigma_1, \varsigma_2]$. If $|\Pi''|$ is a convex function on $[\varsigma_1, \varsigma_2]$, then for all $\delta \in (0,1]$ the inequality is given as:*

$$\begin{aligned}
&\Bigg|\frac{(x-\varsigma_1)^{\delta+1}+(\varsigma_2-x)^{\delta+1}}{(\delta+1)(\varsigma_2-\varsigma_1)}\Pi'(x)-\frac{(x-\varsigma_1)^{\delta}+(\varsigma_2-x)^{\delta}}{\varsigma_2-\varsigma_1}\Pi(x)\\
&-\frac{M(\delta)\Gamma(\delta)}{\varsigma_2-\varsigma_1}\Big\{{}^{AB}{}_{x}I^{\delta}_{\varsigma_1}\Pi(\varsigma_1)+{}^{AB}{}_{x}I^{\delta}_{\varsigma_2}\Pi(\varsigma_2)\Big\}-\frac{2(1-\delta)\Gamma(\delta)}{\varsigma_2-\varsigma_1}\Pi(x)\Bigg|\\
\leq\;&\frac{(x-\varsigma_1)^{\delta+2}}{(\delta+1)(\delta+3)(\varsigma_2-\varsigma_1)}\Big\{|\Pi''(x)|+|\Pi''(\varsigma_1)|\frac{1}{\delta+2}\Big\}\\
&+\frac{(\varsigma_2-x)^{\delta+2}}{(\delta+1)(\delta+3)(\varsigma_2-\varsigma_1)}\Big\{|\Pi''(x)|+|\Pi''(\varsigma_2)|\frac{1}{\delta+2}\Big\},
\end{aligned}$$

for all $x \in [\varsigma_1, \varsigma_2]$.

Theorem 150 ([78]). *Let $\Pi : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on $(\varsigma_1, \varsigma_2)$ with $\varsigma_1, \varsigma_2 \in I$, $\varsigma_1 < \varsigma_2$ such that $\Pi'' \in L[\varsigma_1, \varsigma_2]$. If $|\Pi''|^q$ is a convex function on $[\varsigma_1, \varsigma_2]$, $q > 1$ then for all $\delta \in (0,1]$ the inequality is given as:*

$$\begin{aligned}
&\Bigg|\frac{(x-\varsigma_1)^{\delta+1}+(\varsigma_2-x)^{\delta+1}}{(\delta+1)(\varsigma_2-\varsigma_1)}\Pi'(x)-\frac{(x-\varsigma_1)^{\delta}+(\varsigma_2-x)^{\delta}}{\varsigma_2-\varsigma_1}\Pi(x)\\
&-\frac{M(\delta)\Gamma(\delta)}{\varsigma_2-\varsigma_1}\Big\{{}^{AB}{}_{x}I^{\delta}_{\varsigma_1}+{}^{AB}{}_{x}I^{\delta}_{\varsigma_2}\Big\}-\frac{2(1-\delta)\Gamma(\delta)}{\varsigma_2-\varsigma_1}\Pi(x)\Bigg|\\
\leq\;&\Big(\frac{1}{(\delta+1)p+1}\Big)^{\frac{1}{p}}\Big[\frac{(x-\varsigma_1)^{\delta+2}}{(\delta+1)(\varsigma_2-\varsigma_1)}\Big(\frac{|\Pi''(x)|^q+|\Pi''(\varsigma_1)|^q}{2}\Big)^{\frac{1}{q}}\\
&+\frac{(\varsigma_2-x)^{\delta+2}}{(\delta+1)(\varsigma_2-\varsigma_1)}\Big(\frac{|\Pi''(x)|^q+|\Pi''(\varsigma_2)|^q}{2}\Big)^{\frac{1}{q}}\Big],
\end{aligned}$$

for all $x \in [\varsigma_1, \varsigma_2]$, where $\dfrac{1}{p}+\dfrac{1}{q}=1$.

Theorem 151 ([78]). *Let $\Pi : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on $(\varsigma_1, \varsigma_2)$ with $\varsigma_1, \varsigma_2 \in I$, $\varsigma_1 < \varsigma_2$ such that $\Pi'' \in L[\varsigma_1, \varsigma_2]$. If $|\Pi''|^q$ is a convex on $[\varsigma_1, \varsigma_2]$, $q \geq 1$ then $\forall$ $\delta \in (0,1]$ the inequality is given as:*

$$\begin{aligned}
&\Bigg|\frac{(x-\varsigma_1)^{\delta+1}+(\varsigma_2-x)^{\delta+1}}{(\delta+1)(\varsigma_2-\varsigma_1)}\Pi'(x)-\frac{(x-\varsigma_1)^{\delta}+(\varsigma_2-x)^{\delta}}{\varsigma_2-\varsigma_1}\Pi(x)\\
&-\frac{M(\delta)\Gamma(\delta)}{\varsigma_2-\varsigma_1}\Big\{{}^{AB}{}_{x}I^{\delta}_{\varsigma_1}+{}^{AB}{}_{x}I^{\delta}_{\varsigma_2}\Big\}-\frac{2(1-\delta)\Gamma(\delta)}{\varsigma_2-\varsigma_1}\Pi(x)\Bigg|\\
\leq\;&\Big(\frac{1}{\delta+2}\Big)^{1-\frac{1}{q}}\Big[\frac{(x-\varsigma_1)^{\delta+2}}{(\delta+1)(\varsigma_2-\varsigma_1)}\Big(\frac{|\Pi''(x)|^q}{\delta+3}+\frac{|\Pi''(\varsigma_1)|^q}{(\delta+2)(\delta+3)}\Big)^{\frac{1}{q}}\\
&+\frac{(\varsigma_2-x)^{\delta+2}}{(\delta+1)(\varsigma_2-\varsigma_1)}\Big(\frac{|\Pi''(x)|^q}{\delta+3}+\frac{|\Pi''(\varsigma_2)|^q}{(\delta+2)(\delta+3)}\Big)^{\frac{1}{q}}\Big],
\end{aligned}$$

for all $x \in [\varsigma_1, \varsigma_2]$.

Now, we give Ostrowski-type fractional integral inequalities for Atangana-Baleanu fractional integral operators for twice differentiable s-convex functions.

Theorem 152 ([79]). *Let $\Pi : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on $(\varsigma_1,\varsigma_2)$ with $\varsigma_1,\varsigma_2 \in I$, $\varsigma_1 < \varsigma_2$ such that $\Pi'' \in L[\varsigma_1,\varsigma_2]$. If $|\Pi''|$ is s-convex in the second sense on $[\varsigma_1,\varsigma_2]$ for $s \in (0,1]$, then, for all $x \in [\varsigma_1,\varsigma_2]$,*

$$\begin{aligned}
&\left|\frac{(x-\varsigma_1)^{\delta+1}+(\varsigma_2-x)^{\delta+1}}{(\delta+1)(\varsigma_2-\varsigma_1)}\Pi'(x)-\frac{(x-\varsigma_1)^{\delta}+(\varsigma_2-x)^{\delta}}{\varsigma_2-\varsigma_1}\Pi(x)\right.\\
&\left.-\frac{M(\delta)\Gamma(\delta)}{\varsigma_2-\varsigma_1}\left\{{}^{AB}{}_{x}I^{\delta}_{\varsigma_1}\Pi(\varsigma_1)+{}^{AB}{}_{x}I^{\delta}_{\varsigma_2}\Pi(\varsigma_2)\right\}-\frac{2(1-\delta)\Gamma(\delta)}{\varsigma_2-\varsigma_1}\Pi(x)\right|\\
\le\;&\frac{(x-\varsigma_1)^{\delta+2}}{(\delta+1)(\varsigma_2-\varsigma_1)}\left\{\frac{|\Pi''(x)|}{\delta+s+2}+|\Pi''(\varsigma_1)|B(\delta+2,s+1)\right\}\\
&+\frac{(\varsigma_2-x)^{\delta+2}}{(\delta+1)(\varsigma_2-\varsigma_1)}\left\{\frac{|\Pi''(x)|}{\delta+s+2}+|\Pi''(\varsigma_2)|B(\delta+2,s+1)\right\}.
\end{aligned}$$

Theorem 153 ([79]). *Let Π be as in Theorem 152. If $|\Pi''|^q, q > 1$ is s-convex in the second sense on $[\varsigma_1,\varsigma_2]$ for $s \in (0,1]$, then, for all $x \in [\varsigma_1,\varsigma_2]$,*

$$\begin{aligned}
&\left|\frac{(x-\varsigma_1)^{\delta+1}+(\varsigma_2-x)^{\delta+1}}{(\delta+1)(\varsigma_2-\varsigma_1)}\Pi'(x)-\frac{(x-\varsigma_1)^{\delta}+(\varsigma_2-x)^{\delta}}{\varsigma_2-\varsigma_1}\Pi(x)\right.\\
&\left.-\frac{M(\delta)\Gamma(\delta)}{\varsigma_2-\varsigma_1}\left\{{}^{AB}{}_{x}I^{\delta}_{\varsigma_1}\Pi(\varsigma_1)+{}^{AB}{}_{x}I^{\delta}_{\varsigma_2}\Pi(\varsigma_2)\right\}-\frac{2(1-\delta)\Gamma(\delta)}{\varsigma_2-\varsigma_1}\Pi(x)\right|\\
\le\;&\left(\frac{1}{(\delta+1)p+1}\right)^{\frac{1}{p}}\left[\frac{(x-\varsigma_1)^{\delta+2}}{(\delta+1)(\varsigma_2-\varsigma_1)}\left(\frac{|\Pi''(x)|^q+|\Pi''(\varsigma_1)|^q}{s+1}\right)^{\frac{1}{q}}\right.\\
&\left.+\frac{(\varsigma_2-x)^{\delta+2}}{(\delta+1)(\varsigma_2-\varsigma_1)}\left(\frac{|\Pi''(x)|^q+|\Pi''(\varsigma_2)|^q}{s+1}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where $\dfrac{1}{p}+\dfrac{1}{q}=1$.

Theorem 154 ([79]). *Let Π be as in Theorem 152. If $|\Pi''|^q, q \ge 1$ is s-convex in the second sense on $[\varsigma_1,\varsigma_2]$ for $s \in (0,1]$, then, for all $x \in [\varsigma_1,\varsigma_2]$,*

$$\begin{aligned}
&\left|\frac{(x-\varsigma_1)^{\delta+1}+(\varsigma_2-x)^{\delta+1}}{(\delta+1)(\varsigma_2-\varsigma_1)}\Pi'(x)-\frac{(x-\varsigma_1)^{\delta}+(\varsigma_2-x)^{\delta}}{\varsigma_2-\varsigma_1}\Pi(x)\right.\\
&\left.-\frac{M(\delta)\Gamma(\delta)}{\varsigma_2-\varsigma_1}\left\{{}^{AB}{}_{x}I^{\delta}_{\varsigma_1}\Pi(\varsigma_1)+{}^{AB}{}_{x}I^{\delta}_{\varsigma_2}\Pi(\varsigma_2)\right\}-\frac{2(1-\delta)\Gamma(\delta)}{\varsigma_2-\varsigma_1}\Pi(x)\right|\\
\le\;&\left(\frac{1}{\delta+2}\right)^{1-\frac{1}{q}}\left[\frac{(x-\varsigma_1)^{\delta+2}}{(\delta+1)(\varsigma_2-\varsigma_1)}\left(\frac{|\Pi''(x)|^q}{\delta+s+2}+B(\delta+2,s+1)|\Pi''(\varsigma_1)|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\frac{(\varsigma_2-x)^{\delta+2}}{(\delta+1)(\varsigma_2-\varsigma_1)}\left(\frac{|\Pi''(x)|^q}{\delta+s+2}+B(\delta+2,s+1)|\Pi''(\varsigma_2)|^q s+1\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Theorem 155 ([80])**.** *Let $\Pi : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on $(\varsigma_1, \varsigma_2)$ with $\varsigma_1, \varsigma_2 \in I$, $\varsigma_1 < \varsigma_2$ such that $\Pi' \in L[\varsigma_1, \varsigma_2]$. If $|\Pi''|$ is an s-convex function in the second sense on $[\varsigma_1, \varsigma_2]$ and $|\Pi'(x)| \leq K$, for all $x \in [\varsigma_1, \varsigma_2]$, for some fixed $s \in (0,1]$, then, for all $x \in [\varsigma_1, \varsigma_2]$,*

$$\begin{aligned}
&\Bigg| \frac{(x-\varsigma_1)^\delta + (\varsigma_2 - x)^\delta}{M(\delta)\Gamma(\delta)(\varsigma_2-\varsigma_1)}\Pi(x) - \frac{1}{\varsigma_2-\varsigma_1}\Big\{ {}^{AB}{}_{x}I^{\delta}_{\varsigma_1}\Pi(\varsigma_1) + {}^{AB}{}_{x}I^{\delta}_{\varsigma_2}\Pi(\varsigma_2)\Big\} \\
&-\frac{1-\delta}{(\varsigma_2-\varsigma_1)M(\delta)}\Big[\Pi(\varsigma_1)+\Pi(\varsigma_2)\Big]\Bigg| \\
\leq\ & \frac{K}{M(\delta)\Gamma(\delta)}\Big(\frac{(x-\varsigma_1)^{\delta+1} + (\varsigma_2-x)^{\delta+1}}{\varsigma_2-\varsigma_1}\Big)\Big(\frac{1}{\delta+s+1} + B(\delta+1, s+2)\Big).
\end{aligned}$$

Theorem 156 ([80])**.** *Let Π be as in Theorem 155. If $|\Pi''|^q$ is an s-convex function in the second sense on $[\varsigma_1, \varsigma_2]$ and $|\Pi'(x)| \leq K$, for all $x \in [\varsigma_1, \varsigma_2]$, for some fixed $s \in (0,1]$, then*

$$\begin{aligned}
&\Bigg| \frac{(x-\varsigma_1)^\delta + (\varsigma_2 - x)^\delta}{M(\delta)\Gamma(\delta)(\varsigma_2-\varsigma_1)}\Pi(x) - \frac{1}{\varsigma_2-\varsigma_1}\Big\{ {}^{AB}{}_{x}I^{\delta}_{\varsigma_1}\Pi(\varsigma_1) + {}^{AB}{}_{x}I^{\delta}_{\varsigma_2}\Pi(\varsigma_2)\Big\} \\
&-\frac{1-\delta}{(\varsigma_2-\varsigma_1)M(\delta)}\Big[\Pi(\varsigma_1)+\Pi(\varsigma_2)\Big]\Bigg| \\
\leq\ & \frac{K}{M(\delta)\Gamma(\delta)}\Big(\frac{1}{\delta p+1}\Big)^{\frac{1}{p}}\Big(\frac{2}{s+1}\Big)^{\frac{1}{q}}\Big(\frac{(x-\varsigma_1)^{\delta+1} + (\varsigma_2-x)^{\delta+1}}{\varsigma_2-\varsigma_1}\Big),
\end{aligned}$$

for all $x \in [\varsigma_1, \varsigma_2]$, where $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 157 ([80])**.** *Let the assumptions of this theorem be as stated in Theorem 156. Then*

$$\begin{aligned}
&\Bigg| \frac{(x-\varsigma_1)^\delta + (\varsigma_2 - x)^\delta}{M(\delta)\Gamma(\delta)(\varsigma_2-\varsigma_1)}\Pi(x) - \frac{1}{\varsigma_2-\varsigma_1}\Big\{ {}^{AB}{}_{x}I^{\delta}_{\varsigma_1}\Pi(\varsigma_1) + {}^{AB}{}_{x}I^{\delta}_{\varsigma_2}\Pi(\varsigma_2)\Big\} \\
&-\frac{1-\delta}{(\varsigma_2-\varsigma_1)M(\delta)}\Big[\Pi(\varsigma_1)+\Pi(\varsigma_2)\Big]\Bigg| \\
\leq\ & \frac{K}{M(\delta)\Gamma(\delta)}\Big(\frac{1}{\delta+1}\Big)^{\frac{1}{p}}\Big(\frac{2}{\delta+s+1} + B(\delta+1, s+1)\Big)^{\frac{1}{q}}\Big(\frac{(x-\varsigma_1)^{\delta+1} + (\varsigma_2-x)^{\delta+1}}{\varsigma_2-\varsigma_1}\Big),
\end{aligned}$$

for all $x \in [\varsigma_1, \varsigma_2]$.

We will give now results on Ostrowski-type fractional integral inequalities containing second order derivatives for s-convex functions in the second sense.

Theorem 158 ([80]). *Let $\Pi : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on $(\varsigma_1, \varsigma_2)$ with $\varsigma_1, \varsigma_2 \in I$, $\varsigma_1 < \varsigma_2$ such that $\Pi'' \in L[\varsigma_1, \varsigma_2]$. If $|\Pi''|$ is s-convex in the second sense on $[\varsigma_1, \varsigma_2]$ for $s \in (0,1]$, then, for all $x \in [\varsigma_1, \varsigma_2]$,*

$$\begin{aligned}&\Bigg|\frac{1}{\varsigma_2-\varsigma_1}\Big\{{}^{AB}{}_{x}I^{\delta}_{\varsigma_1}\Pi(\varsigma_1)+{}^{AB}{}_{x}I^{\delta}_{\varsigma_2}\Pi(\varsigma_2)\Big\}-\frac{1-\delta}{(\varsigma_2-\varsigma_1)M(\delta)}\Big[\Pi(\varsigma_1)+\Pi(\varsigma_2)\Big]\\&-\frac{(x-\varsigma_1)^{\delta}+(\varsigma_2-x)^{\delta}}{M(\delta)\Gamma(\delta)(\varsigma_2-\varsigma_1)}\Pi(x)+\frac{(x-\varsigma_1)^{\delta+1}+(\varsigma_2-x)^{\delta+1}}{M(\delta)\Gamma(\delta)(\varsigma_2-\varsigma_1)(\delta+1)}\Pi'(x)\Bigg|\\\leq\;&\frac{|\Pi''(x)|}{\delta+s+1}\left[\frac{(x-\varsigma_1)^{\delta+2}+(\varsigma_2-x)^{\delta+2}}{M(\delta)\Gamma(\delta)(\varsigma_2-\varsigma_1)(\delta+1)}\right]\\&+\frac{B(\delta+3,s+1)}{M(\delta)\Gamma(\delta)(\delta+1)}\left[\frac{(x-\varsigma_1)^{\delta+2}|\Pi''(\varsigma_1)|+(\varsigma_2-x)^{\delta+2}|\Pi''(\varsigma_2)|}{\varsigma_2-\varsigma_1}\right].\end{aligned}$$

Theorem 159 ([80]). *Let Π be as in Theorem 158. If $|\Pi''|^q$ is s-convex in the second sense on $[\varsigma_1, \varsigma_2]$ for $s \in (0,1]$, then*

$$\begin{aligned}&\Bigg|\frac{1}{\varsigma_2-\varsigma_1}\Big\{{}^{AB}{}_{x}I^{\delta}_{\varsigma_1}\Pi(\varsigma_1)+{}^{AB}{}_{x}I^{\delta}_{\varsigma_2}\Pi(\varsigma_2)\Big\}-\frac{1-\delta}{(\varsigma_2-\varsigma_1)M(\delta)}\Big[\Pi(\varsigma_1)+\Pi(\varsigma_2)\Big]\\&-\frac{(x-\varsigma_1)^{\delta}+(\varsigma_2-x)^{\delta}}{M(\delta)\Gamma(\delta)(\varsigma_2-\varsigma_1)}\Pi(x)+\frac{(x-\varsigma_1)^{\delta+1}+(\varsigma_2-x)^{\delta+1}}{M(\delta)\Gamma(\delta)(\varsigma_2-\varsigma_1)(\delta+1)}\Pi'(x)\Bigg|\\\leq\;&\frac{(x-\varsigma_1)^{\delta+2}}{M(\delta)\Gamma(\delta)(\varsigma_2-\varsigma_1)(\delta+1)}\left(\frac{1}{(\delta+1)p+1}\right)^{\frac{1}{p}}\left(\frac{|\Pi''(x)|^q+|\Pi''(\varsigma_1)|^q}{s+1}\right)^{\frac{1}{q}}\\&+\frac{(\varsigma_2-x)^{\delta+2}}{M(\delta)\Gamma(\delta)(\varsigma_2-\varsigma_1)(\delta+1)}\left(\frac{1}{(\delta+1)p+1}\right)^{\frac{1}{p}}\left(\frac{|\Pi''(x)|^q+|\Pi''(\varsigma_2)|^q}{s+1}\right)^{\frac{1}{q}},\end{aligned}$$

for all $x \in [\varsigma_1, \varsigma_2]$, where $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

11. Ostrowski-Type Fractional Integral Inequalities via Generalized Fractional Integrals

We define the left and right sided generalized fractional integrals as:

Definition 38 ([81]). *The left and right-sided generalized fractional integrals given as follows:*

$$_{\varsigma_1+}I_{\varphi}\Pi(\varkappa)=\int_{\varsigma_1}^{\varkappa}\frac{\varphi(\varkappa-\sigma)}{\varkappa-\sigma}\Pi(\sigma)d\sigma,\quad \varkappa>\varsigma_1, \tag{6}$$

$$_{\varsigma_2-}I_{\varphi}\Pi(\varkappa)=\int_{\varkappa}^{\varsigma_2}\frac{\varphi(\sigma-\varkappa)}{\sigma-\varkappa}\Pi(\sigma)d\sigma,\quad \varkappa<\varsigma_2, \tag{7}$$

where the function $\varphi : [0,\infty) \to [0,\infty)$ satisfies $\int_0^1 \frac{\varphi(\sigma)}{\sigma} d\sigma < \infty$.

Some inequalities connected with the Ostrowski-type inequality using of generalized fractional integral operators are presented now.

Theorem 160 ([82]). *Let* $\Pi : [\varsigma_1, \varsigma_2] \subseteq (0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* $(\varsigma_1, \varsigma_2)$ *and* $|\Pi'(x)| \leq M$ *for all* $x \in [\varsigma_1, \varsigma_2]$*. Then, for all* $x \in [\varsigma_1, \varsigma_2]$*,*

$$\begin{aligned} & \left| \frac{\varphi(\varsigma_2 - x)}{\varsigma_2 - x}\Pi(x) - {}_{\varsigma_1+}I_\varphi(P_1(x,\varsigma_2)\Pi(\varsigma_2)) - \frac{1}{\varsigma_2 - \varsigma_1}\, {}_{\varsigma_1+}I_\varphi \Pi(\varsigma_2) \right| \\ \leq\; & \frac{M}{\varsigma_2 - \varsigma_1}\Big[\int_{\varsigma_1}^{x} \left|\frac{\varphi(\varsigma_2 - t)}{\varsigma_2 - t}\right| (t-\varsigma_1)dt + \int_{x}^{\varsigma_2} \left|\frac{\varphi(\varsigma_2 - t)}{\varsigma_2 - t}\right| (\varsigma_2 - t)dt \Big], \end{aligned}$$

where $P_1(x,t) = \begin{cases} \dfrac{t-\varsigma_1}{\varsigma_2-\varsigma_1}, & \varsigma_1 \leq t < x, \\ \dfrac{t-\varsigma_2}{\varsigma_2-\varsigma_1}, & x \leq t \leq \varsigma_2 \end{cases}$ *is the Peano kernel function.*

Theorem 161 ([82]). *Suppose* $\Pi : [\varsigma_1, \varsigma_2] \subseteq (0,\infty) \to \mathbb{R}$ *be a function which is differentiable on* $(\varsigma_1, \varsigma_2)$ *and* $|\Pi'(x)|^q \leq M_1$ *for all* $x \in [\varsigma_1, \varsigma_2]$ *and* $p, q \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$*. Then*

$$\begin{aligned} & \left| \frac{\varphi(\varsigma_2 - x)}{\varsigma_2 - x}\Pi(x) - {}_{\varsigma_1+}I_\varphi(P_1(x,\varsigma_2)\Pi(\varsigma_2)) - \frac{1}{\varsigma_2 - \varsigma_1}\, {}_{\varsigma_1+}I_\varphi \Pi(\varsigma_2) \right| \\ \leq\; & M_1\Big[\Big(\int_{\varsigma_1}^{\varsigma_2} \left|\frac{\varphi(\varsigma_2 - t)}{\varsigma_2 - t}\right|^p |P_1(x,t)|^p dt \Big)^{\frac{1}{p}} Big], \end{aligned}$$

for all $x \in [\varsigma_1, \varsigma_2]$*, where* $P_1(x,t)$ *is the Peano kernel function defined in previous theorem.*

In the following, we present some Ostrowski-type inequalities for differentiable harmonically convex functions via the generalized fractional integrals.

Theorem 162 ([83]). *Let* $\Pi : [\varsigma_1, \varsigma_2] \subseteq (0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* $(\varsigma_1, \varsigma_2)$ *such that* $\Pi' \in L[\varsigma_1, \varsigma_2]$*. If* $|\Pi'|^q$ *is harmonically convex on* $[\varsigma_1, \varsigma_2]$ *for some* $q \geq 1$*, then for all* $x \in [\varsigma_1, \varsigma_2]$*,*

$$\begin{aligned} & \left| (\Delta(1) + \Lambda(1))\Pi(\varkappa) - \left[{}_{\frac{1}{\varkappa}+}I_\varphi(\Pi \circ g)\left(\frac{1}{\varsigma_1}\right) + {}_{\frac{1}{\varkappa}-}I_\varphi(\Pi \circ g)\left(\frac{1}{\varsigma_2}\right) \right] \right| \\ \leq\; & \varsigma_1\varkappa(\varkappa - \varsigma_1)\Theta_1^{1-\frac{1}{q}}\left(\Theta_2|\Pi'(\varkappa)|^q + \Theta_3|\Pi'(\varsigma_1)|^q\right)^{\frac{1}{q}} \\ & + \varsigma_2\varkappa(\varsigma_2 - \varkappa)\Theta_4^{1-\frac{1}{q}}\left(\Theta_5|\Pi'(\varkappa)|^q + \Theta_6|\Pi'(\varsigma_2)|^q\right)^{\frac{1}{q}}, \end{aligned}$$

where the mappings Δ *and* Λ *are defined as:*

$$\Delta(\sigma) = \int_0^\sigma \frac{\varphi\left(\frac{\varkappa - \varsigma_1}{\varsigma_1\varkappa}s\right)}{s}ds < +\infty, \qquad \Lambda(\sigma) = \int_0^\sigma \frac{\varphi\left(\frac{\varsigma_2 - \varkappa}{\varsigma_2\varkappa}s\right)}{s}ds < +\infty,$$

and

$$\begin{aligned} \Theta_1 &= \int_0^1 \frac{\Delta(\sigma)}{(\sigma\varsigma_1 + (1-\sigma)\varkappa)^2}d\sigma, & \Theta_2 &= \int_0^1 \frac{\sigma\Delta(\sigma)}{(\sigma\varsigma_1 + (1-\sigma)\varkappa)^2}d\sigma, \\ \Theta_3 &= \int_0^1 \frac{(1-\sigma)\Delta(\sigma)}{(\sigma\varsigma_1 + (1-\sigma)\varkappa)^2}d\sigma, & \Theta_4 &= \int_0^1 \frac{\Lambda(\sigma)}{(\sigma\varsigma_2 + (1-\sigma)\varkappa)^2}d\sigma, \\ \Theta_5 &= \int_0^1 \frac{\sigma\Lambda(\sigma)}{(\sigma\varsigma_2 + (1-\sigma)\varkappa)^2}d\sigma, & \Theta_6 &= \int_0^1 \frac{(1-\sigma)\Lambda(\sigma)}{(\sigma\varsigma_2 + (1-\sigma)\varkappa)^2}d\sigma. \end{aligned}$$

Theorem 163 ([83]). *Let $\Pi : [\varsigma_1, \varsigma_2] \subseteq (0, \infty) \to \mathbb{R}$ be a differentiable mapping on $(\varsigma_1, \varsigma_2)$ such that $\Pi' \in L[\varsigma_1, \varsigma_2]$. If $|\Pi'|^q$ is harmonically convex on $[\varsigma_1, \varsigma_2]$ for some $q > 1$, then, for all $x \in [\varsigma_1, \varsigma_2]$,*

$$\left|(\Delta(1) + \Lambda(1))\Pi(\varkappa) - \left[{}_{\frac{1}{\varkappa}+}I_{\varphi}(\Pi \circ g)\left(\frac{1}{\varsigma_1}\right) + {}_{\frac{1}{\varkappa}-}I_{\varphi}(\Pi \circ g)\left(\frac{1}{\varsigma_2}\right)\right]\right|$$
$$\leq \varsigma_1\varkappa(\varkappa - \varsigma_1)\Theta_7^{\frac{1}{p}}\left(\frac{|\Pi'(\varkappa)|^q + |\Pi'(\varsigma_1)|^q}{2}\right)^{\frac{1}{q}} + \varsigma_2\varkappa(\varsigma_2 - \varkappa)\Theta_8^{\frac{1}{p}}\left(\frac{|\Pi'(\varkappa)|^q + |\Pi'(\varsigma_2)|^q}{2}\right)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$ *and*

$$\Theta_7 = \int_0^1 \left(\frac{\Delta(\sigma)}{(\sigma\varsigma_1 + (1-\sigma)\varkappa)^2}\right)^p d\sigma, \qquad \Theta_8 = \int_0^1 \left(\frac{\Lambda(\sigma)}{(\sigma\varsigma_2 + (1-\sigma)\varkappa)^2}\right)^p d\sigma.$$

12. Ostrowski-Type Fractional Integral Inequalities via Quantum Calculus

Definition 39 ([84]). *Let function $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be continuous. Then*

$${}_{\varsigma_1}D_q\Pi(t) = \frac{\Pi(t) - \Pi(qt + (1-q)\varsigma_1)}{(1-q)(t - \varsigma_1)}, t \neq \varsigma_1, \quad {}_{\varsigma_1}D_q\Pi(\varsigma_1) = \lim_{t \to \varsigma_1} {}_{\varsigma_1}D_q\Pi(t),$$

is called q_{ς_1}-derivative of Π at $t \in [\varsigma_1, \varsigma_2]$.

Definition 40 ([84]). *Let function $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be continuous. Then*

$$\int_{\varsigma_1}^t \Pi(x)\ {}_{\varsigma_1}d_qx = (1-q)(t - \varsigma_1)\sum_{n=0}^{\infty} q^n\Pi(q^nt + (1-q^n)\varsigma_1),$$

is called q_{ς_1}-integral of Π for $x \in [\varsigma_1, \varsigma_2]$.

Definition 41 ([85]). *Let function $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be continuous. Then*

$${}^{\varsigma_2}D_q\Pi(t) = \frac{\Pi(qt + (1-q)\varsigma_2) - \Pi(t)}{(1-q)(\varsigma_2 - t)}, t \neq \varsigma_2, \quad {}^{\varsigma_2}D_q\Pi(\varsigma_2) = \lim_{t \to \varsigma_2} {}^{\varsigma_2}D_q\Pi(t),$$

is called q^{ς_2}-derivative of Π at $t \in [\varsigma_1, \varsigma_2]$.

Definition 42 ([85]). *Let function $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be continuous. Then*

$$\int_t^{\varsigma_2} \Pi(x)^{\varsigma_2}d_qx = (1-q)(\varsigma_2 - t)\sum_{n=0}^{\infty} q^n\Pi(q^nt + (1-q^n)\varsigma_2),$$

is called q^{ς_2}-integral of Π for $x \in [\varsigma_1, \varsigma_2]$.

We give now some Ostrowski-type inequalities for q-differentiable convex functions.

Theorem 164 ([86]). *Let $\Pi : I = [\varsigma_1, \varsigma_2] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on I° with ${}_{\varsigma_1}D_q$ continuous and integrable on I where $0 < q < 1$. If $|{}_{\varsigma_1}D_q\Pi|^r$ is a convex function and $|{}_{\varsigma_1}D_q\Pi(x)| \leq M$, then, for all $x \in [\varsigma_1, \varsigma_2]$,*

$$\left|\Pi(x) - \frac{1}{\varsigma_2 - \varsigma_1}\int_{\varsigma_1}^{\varsigma_2} \Pi(u)\ {}_{\varsigma_1}d_qu\right| \leq \frac{qM[(x-\varsigma_1)^2 + (\varsigma_2 - x)^2]}{(\varsigma_2 - \varsigma_1)(1+q)}.$$

Theorem 165 ([86]). *Let $\Pi : I = [\varsigma_1, \varsigma_2] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on I° with ${}_{\varsigma_1}D_q$ continuous and integrable on I where $0 < q < 1$. If $|{}_{\varsigma_1}D_q\Pi|$ is a convex function and $|{}_{\varsigma_1}D_q\Pi(x)| \leq M$, then for $p, r > 1$, $\frac{1}{p} + \frac{1}{r} = 1$, we have for all $x \in [\varsigma_1, \varsigma_2]$,*

$$\left|\Pi(x) - \frac{1}{\varsigma_2 - \varsigma_1}\int_{\varsigma_1}^{\varsigma_2} \Pi(u)\ {}_{\varsigma_1}d_q u\right| \leq \frac{qM[(x-\varsigma_1)^2 + (\varsigma_2 - x)^2]}{(\varsigma_2 - \varsigma_1)}\left(\frac{1-q}{1-q^{p+1}}\right)^{\frac{1}{p}}.$$

For q-differentiable bounded functions, we present some Ostrowski-type inequalities.

Theorem 166 ([87]). *Let $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be a continuous and q-differentiable function on $[\varsigma_1, \varsigma_2]$. If $\left|{}_{\varsigma_1}D_q\Pi(t)\right|, \left|{}^{\varsigma_2}D_q\Pi(t)\right| \leq M$, then, for all $x \in [\varsigma_1, \varsigma_2]$,*

$$\begin{aligned}
&\left|(\varsigma_2 - \varsigma_1)h\frac{\Pi(\varsigma_1) + \Pi(\varsigma_2)}{2} + (\varsigma_2 - \varsigma_1)(1-h)\Pi(x)\right.\\
&\left.-\left[\int_{\varsigma_1}^{x} \Pi(qt + (1-q)\varsigma_1)\ {}_{\varsigma_1}d_q t + \int_{x}^{\varsigma_2} \Pi(qt + (1-q)\varsigma_2)\ {}^{\varsigma_2}d_q t\right]\right|\\
\leq\ & M(P(\varsigma_1, \varsigma_2, h, x; q) + Q(\varsigma_1, \varsigma_2, h, x; q)),
\end{aligned}$$

for $h \in [0,1]$ where

$$P(\varsigma_1, \varsigma_2, h, x; q) = \int_{\varsigma_1}^{x}\left|\left(t - \left(\varsigma_1 + h\frac{\varsigma_2 - \varsigma_1}{2}\right)\right)\right|\ {}_{\varsigma_1}d_q t$$

and

$$Q(\varsigma_1, \varsigma_2, h, x; q) = \int_{x}^{\varsigma_2}\left|\left(t - \left(\varsigma_2 - h\frac{\varsigma_2 - \varsigma_1}{2}\right)\right)\right|\ {}^{\varsigma_2}d_q t.$$

Theorem 167 ([87]). *Let $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be a continuous and q-differentiable function on $[\varsigma_1, \varsigma_2]$. If for $p > 1$, $\left|{}_{\varsigma_1}D_q\Pi(t)\right|^p, \left|{}^{\varsigma_2}D_q\Pi(t)\right|^p \leq M$, then, for all $x \in [\varsigma_1, \varsigma_2]$,*

$$\begin{aligned}
&\left|(\varsigma_2 - \varsigma_1)h\frac{\Pi(\varsigma_1) + \Pi(\varsigma_2)}{2} + (\varsigma_2 - \varsigma_1)(1-h)\Pi(x)\right.\\
&\left.-\left[\int_{\varsigma_1}^{x} \Pi(qt + (1-q)\varsigma_1)\ {}_{\varsigma_1}d_q t + \int_{x}^{\varsigma_2} \Pi(qt + (1-q)\varsigma_2)\ {}^{\varsigma_2}d_q t\right]\right|\\
\leq\ & M((x - \varsigma_1)A_1(\varsigma_1, \varsigma_2, h, x; q) + (\varsigma_2 - x)A_2(\varsigma_1, \varsigma_2, h, x; q))
\end{aligned}$$

where

$$\begin{aligned}
A_1(\varsigma_1, \varsigma_2, h, x; q) &= \left(\int_{\varsigma_1}^{x}\left|\left(t - \left(\varsigma_1 + h\frac{\varsigma_2 - \varsigma_1}{2}\right)\right)\right|^s\ {}_{\varsigma_1}d_q t\right)^{\frac{1}{s}},\\
A_2(\varsigma_1, \varsigma_2, h, x; q) &= \left(\int_{x}^{\varsigma_2}\left|\left(t - \left(\varsigma_2 - h\frac{\varsigma_2 - \varsigma_1}{2}\right)\right)\right|^s\ {}_{\varsigma_1}d_q t\right)^{\frac{1}{s}}
\end{aligned}$$

and $\frac{1}{p} + \frac{1}{s} = 1$.

In the next, we give Ostrowski-type inequalities for s-convex functions in the second sense.

Theorem 168 ([88])**.** *Let $\Pi : I = [\varsigma_1, \varsigma_2] \subset \mathbb{R}^+ \to \mathbb{R}$ be a q-differentiable function on I° with ${}_{\varsigma_1}D_q$ integrable on I. If ${}_{\varsigma_1}D_q$ is s-convex in the second sense on $[\varsigma_1, \varsigma_2]$ for unique $s \in (0,1]$ and $|{}_{\varsigma_1}D_q\Pi(x)| \le M$ for all $x \in [\varsigma_1, \varsigma_2]$, then, for all $x \in [\varsigma_1, \varsigma_2]$,*

$$\left|\Pi(x) - \frac{1}{\varsigma_2 - \varsigma_1}\int_{\varsigma_1}^{\varsigma_2} \Pi(u)\ {}_{\varsigma_1}d_q u\right| \le \frac{qM[(x-\varsigma_1)^2 + (\varsigma_2 - x)^2]}{(\varsigma_2 - \varsigma_1)}$$
$$\times\Big[-\frac{q}{[s+1]}\Big((1-q^{-1})_q^{s+1} + \frac{q(1-q^{-1})_q^{s+1}}{[s+2]} - \frac{q}{[s+2]}\Big) + \frac{1}{[s+2]}\Big].$$

Theorem 169 ([88])**.** *Let Π be as in Theorem 168. If $|{}_{\varsigma_1}D_q|^m$ is s-convex in the second sense on $[\varsigma_1, \varsigma_2]$ for unique $s \in (0,1]$, $m > 1, m^{-1} + n^{-1} = 1$ and $|{}_{\varsigma_1}D_q\Pi(x)| \le M$ for all $x \in [\varsigma_1, \varsigma_2]$, then, for all $x \in [\varsigma_1, \varsigma_2]$,*

$$\left|\Pi(x) - \frac{1}{\varsigma_2 - \varsigma_1}\int_{\varsigma_1}^{\varsigma_2} \Pi(u)\ {}_{\varsigma_1}d_q u\right|$$
$$\le \frac{qM}{[n+1]^{1/n}}\frac{[(x-\varsigma_1)^2 + (\varsigma_2 - x)^2]}{(\varsigma_2 - \varsigma_1)}\left[\frac{1 + q(1-(1-q^{-1})^{s+1}}{[s+1]}\right]^{1/m}.$$

Theorem 170 ([88])**.** *Let Π be as in Theorem 168. If $|{}_{\varsigma_1}D_q|^m$ is s-convex in the second sense on $[\varsigma_1, \varsigma_2]$ for unique $s \in (0,1]$, $m \ge 1$, and $|{}_{\varsigma_1}D_q\Pi(x)| \le M$ for all $x \in [\varsigma_1, \varsigma_2]$, then, for all $x \in [\varsigma_1, \varsigma_2]$,*

$$\left|\Pi(x) - \frac{1}{\varsigma_2 - \varsigma_1}\int_{\varsigma_1}^{\varsigma_2} \Pi(u)\ {}_{\varsigma_1}d_q u\right| \le qM\frac{[(x-\varsigma_1)^2 + (\varsigma_2 - x)^2]}{(\varsigma_2 - \varsigma_1)}$$
$$+\Big(\frac{1}{[2]}\Big)^{1-\frac{1}{m}}\Big[-\frac{q}{[s+1]}\Big((1-q^{-1})_q^{s+1} + \frac{q(1-q^{-1})_q^{s+2}}{[s+2]} - \frac{q}{[s+2]}\Big)\Big]^{\frac{1}{m}}.$$

We present Ostrowski-type inequalities for twice quantum differentiable functions involving the quantum integrals.

Theorem 171 ([89])**.** *Let $\Pi : [\varsigma_1, \varsigma_2] \subset \mathbb{R} \to \mathbb{R}$ be a twice q-differentiable function on $(\varsigma_1, \varsigma_2)$, such that ${}^{\varsigma_1}D_q^2\Pi$ and ${}_{\varsigma_1}D_q^2\Pi$ are continuous and integrable on $[\varsigma_1, \varsigma_2]$. Then, we have for all $x \in [\varsigma_1, \varsigma_2]$,*

$$\Bigg|\frac{(x-\varsigma_1)(\varsigma_2 - x)}{(1-q)q^3}[(x-\varsigma_1)q\Pi(qx + (1-x)\varsigma_2) + (\varsigma_2 - x)q\Pi(qx + (1-x)\varsigma_1)$$
$$-(q^2 + q - 1)(\varsigma_2 - \varsigma_1)\Pi(x) - \frac{[2]_q}{q^3}\Big[(x-\varsigma_1)^2\int_x^{\varsigma_2}\Pi(t)\ {}^{\varsigma_2}d_q t$$
$$+(\varsigma_2 - x)^2\int_{\varsigma_1}^x \Pi(t)\ {}_{\varsigma_1}d_q t\Big]\Bigg|$$
$$\le (x-\varsigma_1)^2(\varsigma_2 - x)^2\Big[(x-\varsigma_1)\Big(\frac{1}{[4]_q}|{}_{\varsigma_1}D_q^2\Pi(x)| + \frac{q^3}{[3]_q[4]_q}|{}_{\varsigma_1}D_q^2\Pi(x)|\Big)$$
$$+(\varsigma_2 - x)\Big(\frac{1}{[4]_q}|{}^{\varsigma_2}D_q^2\Pi(x)| + \frac{q^3}{[3]_q[4]_q}|{}^{\varsigma_2}D_q^2\Pi(x)|\Big)\Big].$$

Theorem 172 ([89])**.** *Let $\Pi : [\varsigma_1, \varsigma_2] \subset \mathbb{R} \to \mathbb{R}$ be a twice q-differentiable function on $(\varsigma_1, \varsigma_2)$, such that $|^{\varsigma_1}D_q^2\Pi|^{p_1}$ and $|_{\varsigma_1}D_q^2\Pi|^{p_1}, p_1 \geq 1$ are convex on $[\varsigma_1, \varsigma_2]$. Then, we have for all $x \in [\varsigma_1, \varsigma_2]$,*

$$\begin{aligned}
&\Bigg|\frac{(x-\varsigma_1)(\varsigma_2-x)}{(1-q)q^3}[(x-\varsigma_1)q\Pi(qx+(1-x)\varsigma_2)+(\varsigma_2-x)q\Pi(qx+(1-x)\varsigma_1)\\
&-(q^2+q-1)(\varsigma_2-\varsigma_1)\Pi(x)-\frac{[2]_q}{q^3}\Big[(x-\varsigma_1)^2\int_x^{\varsigma_2}\Pi(t)\ {}^{\varsigma_2}d_qt\\
&+(\varsigma_2-x)^2\int_{\varsigma_1}^{x}\Pi(t)\ {}_{\varsigma_1}d_qt\Big]\Bigg|\\
\leq\ &(x-\varsigma_1)^2(\varsigma_2-x)^2\Big(\frac{1}{[3]_q}\Big)^{1-\frac{1}{p_1}}\Big[(x-\varsigma_1)\Big(\frac{1}{[4]_q}|_{\varsigma_1}D_q^2\Pi(x)|^{p_1}+\frac{q^3}{[3]_q[4]_q}|_{\varsigma_1}D_q^2\Pi(x)|^{p_1}\Big)^{\frac{1}{p_1}}\\
&+(\varsigma_2-x)\Big(\frac{1}{[4]_q}|^{\varsigma_2}D_q^2\Pi(x)|^{p_1}+\frac{q^3}{[3]_q[4]_q}|^{\varsigma_2}D_q^2\Pi(x)|^{p_1}\Big)^{\frac{1}{p_1}}\Big].
\end{aligned}$$

Theorem 173 ([89])**.** *Let $\Pi : [\varsigma_1, \varsigma_2] \subset \mathbb{R} \to \mathbb{R}$ be a twice q-differentiable function on $(\varsigma_1, \varsigma_2)$, such that $|^{\varsigma_1}D_q^2\Pi|^{p_1}$ and $|_{\varsigma_1}D_q^2\Pi|^{p_1}$, for some $p_1 > 1$ and $\frac{1}{r_1} + \frac{1}{p_1} = 1$, are convex on $[\varsigma_1, \varsigma_2]$. Then, we have for all $x \in [\varsigma_1, \varsigma_2]$,*

$$\begin{aligned}
&\Bigg|\frac{(x-\varsigma_1)(\varsigma_2-x)}{(1-q)q^3}[(x-\varsigma_1)q\Pi(qx+(1-x)\varsigma_2)+(\varsigma_2-x)q\Pi(qx+(1-x)\varsigma_1)\\
&-(q^2+q-1)(\varsigma_2-\varsigma_1)\Pi(x)-\frac{[2]_q}{q^3}\Big[(x-\varsigma_1)^2\int_x^{\varsigma_2}\Pi(t)\ {}^{\varsigma_2}d_qt\\
&+(\varsigma_2-x)^2\int_{\varsigma_1}^{x}\Pi(t)\ {}_{\varsigma_1}d_qt\Big]\Bigg|\\
\leq\ &(x-\varsigma_1)^2(\varsigma_2-x)^2\Big(\frac{1}{[2r_1+1]_q}\Big)^{\frac{1}{r_1}}\Big[(x-\varsigma_1)\Big(\frac{{}_{\varsigma_1}D_q^2\Pi(x)|^{p_1}+q\ {}_{\varsigma_1}D_q^2\Pi(\varsigma_1)|^{p_1}}{[2]_q}\Big)^{\frac{1}{p_1}}\\
&+(\varsigma_2-x)\Big(\frac{{}_{\varsigma_1}D_q^2\Pi(x)|^{p_1}+q\ {}_{\varsigma_1}D_q^2\Pi(\varsigma_2)|^{p_1}}{[2]_q}\Big)^{\frac{1}{p_1}}\Big].
\end{aligned}$$

Definition 43 ([90])**.** *Let $\Pi \in L[\varsigma_1, \varsigma_2]$. The Riemann–Liouville q-integrals of order $\zeta > 0$ are defined by*

$$J^{\beta}_{q,\varsigma_1+}\Pi(x) = \frac{1}{\Gamma_q(\beta)}\int_{\varsigma_1}^{x}(x-qu)^{(\beta-1)}\Pi(u)d_qu, \quad \varsigma_1 < x,$$
$$J^{\beta}_{q,\varsigma_2-}\Pi(x) = \frac{1}{\Gamma_q(\beta)}\int_{qx}^{\varsigma_2}(u-qx)^{(\beta-1)}\Pi(u)d_qu, \quad \varsigma_2 > x.$$

In this section, q-fractional integral operators are used to construct a quantum analogue of Ostrowski-type fractional integral inequalities for the class of s-convex functions.

Theorem 174 ([91]). *Let $\Pi : [\varsigma_1, \varsigma_2] \subset (0, \infty) \to \mathbb{R}$ be a q-differentiable mapping in such a way $D_q\Pi \in L[\varsigma_1, \varsigma_2]$. If $|D_q\Pi|$ is s-convex in the second sense on $[\varsigma_1, \varsigma_2]$ for $s, q \in (0, 1]$ and $|D_q\Pi(x)| \leq M$, for all $x \in [\varsigma_1, \varsigma_2]$, then for $\beta > 0$ and for all $x \in [\varsigma_1, \varsigma_2]$,*

$$\begin{aligned} &\left| \left(\frac{(x-\varsigma_1)^\beta + (\varsigma_2 - x)^\beta}{\varsigma_2 - \varsigma_1} \right) \left(\frac{q^\beta + [\beta](1-q)}{q^\beta} \right) \Pi(x) \right. \\ &\left. - \frac{\Gamma_q(\beta+1)}{q^\beta(\varsigma_2 - \varsigma_1)} \left[J^\beta_{q,x-}\Pi(\varsigma_1) + J^\beta_{q,x+}\Pi(\varsigma_2) \right] \right| \\ \leq\ & \frac{M}{\varsigma_2 - \varsigma_1} \left(1 + \frac{\Gamma_q(\beta+1)\Gamma_q(s+1)}{\Gamma_q(\beta+s+1)} \right) \left[\frac{(x-\varsigma_1)^{\beta+1} + (\varsigma_2 - x)^{\beta+1}}{[\beta+s+1]} \right]. \end{aligned}$$

Theorem 175 ([91]). *Let Π be as in Theorem 174. If $|D_q\Pi|^m$ is s-convex in the second sense on $[\varsigma_1, \varsigma_2]$ for $s, q \in (0, 1]$, $n, m > 1$, $\frac{1}{n} + \frac{1}{m} = 1$ and $|D_q\Pi(x)| \leq M$, for all $x \in [\varsigma_1, \varsigma_2]$, then we have for $\beta > 0$ and for all $x \in [\varsigma_1, \varsigma_2]$,*

$$\begin{aligned} &\left| \left(\frac{(x-\varsigma_1)^\beta + (\varsigma_2 - x)^\beta}{\varsigma_2 - \varsigma_1} \right) \left(\frac{q^\beta + [\beta](1-q)}{q^\beta} \right) \Pi(x) \right. \\ &\left. - \frac{\Gamma_q(\beta+1)}{q^\beta(\varsigma_2 - \varsigma_1)} \left[J^\beta_{q,x-}\Pi(\varsigma_1) + J^\beta_{q,x+}\Pi(\varsigma_2) \right] \right| \\ \leq\ & \frac{M}{(1+n\beta)^{\frac{1}{n}}} \left[\frac{1 + q\{1 - (1-q^{-1})^{s+1}\}}{[s+1]} \right)^{\frac{1}{m}} \left[\frac{(x-\varsigma_1)^{\beta+1} + (\varsigma_2 - x)^{\beta+1}}{\varsigma_2 - \varsigma_1} \right]. \end{aligned}$$

Theorem 176 ([91]). *Let Π be as in Theorem 174. If $|D_q\Pi|^m$ is s-convex in the second sense on $[\varsigma_1, \varsigma_2]$ for $s, q \in (0, 1]$, $m \geq 1$ and $|D_q\Pi(x)| \leq M$, for all $x \in [\varsigma_1, \varsigma_2]$, then, we have for $\beta > 0$ and for all $x \in [\varsigma_1, \varsigma_2]$,*

$$\begin{aligned} &\left| \left(\frac{(x-\varsigma_1)^\beta + (\varsigma_2 - x)^\beta}{\varsigma_2 - \varsigma_1} \right) \left(\frac{q^\beta + [\beta](1-q)}{q^\beta} \right) \Pi(x) \right. \\ &\left. - \frac{\Gamma_q(\beta+1)}{q^\beta(\varsigma_2 - \varsigma_1)} \left[J^\beta_{q,x-}\Pi(\varsigma_1) + J^\beta_{q,x+}\Pi(\varsigma_2) \right] \right| \\ \leq\ & M \left(\frac{1}{[\beta+1]} \right)^{1-\frac{1}{m}} \left(\frac{1}{[\beta+s+1]} \right)^{\frac{1}{m}} \left(1 + \frac{\Gamma_q(\beta+1)\Gamma_q(s+1)}{\Gamma_q(\beta+s+1)} \right)^{\frac{1}{q}} \\ &\times \left[\frac{(x-\varsigma_1)^{\beta+1} + (\varsigma_2 - x)^{\beta+1}}{\varsigma_2 - \varsigma_1} \right]. \end{aligned}$$

In the following theorems, we present some post-quantum estimates of the Ostrowski-type inequality for n-polynomial convex functions.

Definition 44 ([92]). *Let $n \in \mathbb{N}$. A nonnegative function $\Pi : I \to \mathbb{R}$ is said to be an n-polynomial convex function if for every $x, y \in I$ and $t \in [0, 1]$, we have*

$$\Pi(tx + (1-t)y) \leq \frac{1}{n} \sum_{s=1}^{n} [1 - (1-t)^s] \Pi(x) + \frac{1}{n} \sum_{s=1}^{n} [1 - t^s] \Pi(y).$$

Definition 45 ([93]). *If function $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ is continuous, then the left (p,q)-derivative of Π at x is stated by*

$$ {}_{\varsigma_1}D_{p,q}\Pi(x) = \frac{\Pi(px + (1-p)\varsigma_1) - \Pi(qx + (1-q)\varsigma_1)}{(p-q)(x-\varsigma_1)}, \quad x \neq \varsigma_1, \tag{8} $$
$$ {}_{\varsigma_1}D_{p,q}\Pi(a) = \lim_{x \to x_1} {}_{\varsigma_1}D_{p,q}\Pi(x). $$

If ${}_{\varsigma_1}D_{p,q}\Pi(x)$ exists for all $x \in [\varsigma_1, \varsigma_2]$, then the function Π is called (p,q)-differentiable on $[\varsigma_1, \varsigma_2]$.

The left ${}_{\varsigma_1}(p,q)$-integral $\int_{\varsigma_1}^{x} \Pi(t)\ {}_{\varsigma_1}d_{p,q}t$ is defined by

$$ \int_{\varsigma_1}^{x} \Pi(t)\ {}_{\varsigma_1}d_{p,q}t = (p-q)(x-\varsigma_1)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\Pi\left(\frac{q^n}{p^{n+1}}x + \left(1-\frac{q^n}{p^{n+1}}\right)\varsigma_1\right). $$

Definition 46 ([94]). *If function $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ is continuous, then the right (p,q)-derivative of Π at x is stated by*

$$ {}^{\varsigma_2}D_{p,q}\Pi(x) = \frac{\Pi(px + (1-p)\varsigma_2) - \Pi(qx + (1-q)\varsigma_2)}{(p-q)(\varsigma_2 - x)}, \quad x \neq \varsigma_2, \tag{9} $$
$$ {}^{\varsigma_2}D_{p,q}\Pi(\varsigma_2) = \lim_{x \to \varsigma_2} {}^{\varsigma_2}D_{p,q}\Pi(x). $$

If ${}^{\varsigma_2}D_{p,q}\Pi(x)$ exists for all $x \in [\varsigma_1, \varsigma_2]$, then the function Π is called (p,q)-differentiable on $[\varsigma_1, \varsigma_2]$.

The right ${}_{\varsigma_2}(p,q)$-integral $\int_{x}^{\varsigma_2} \Pi(t)\ {}_{\varsigma_2}d_{p,q}t$ is defined by

$$ \int_{x}^{\varsigma_2} \Pi(t)\ {}_{\varsigma_2}d_{p,q}t = (p-q)(\varsigma_2 - x)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\Pi\left(\frac{q^n}{p^{n+1}}x + \left(1-\frac{q^n}{p^{n+1}}\right)\varsigma_2\right). $$

Theorem 177 ([94]). *Let $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be continuous and (p,q)- differentiable function on $(\varsigma_1, \varsigma_2)$ with $\varsigma_1 < \varsigma_2$ and ${}_{\varsigma_1}D_{p,q}\Pi, {}^{\varsigma_2}D_{p,q}\Pi$ be (p,q)-integrable. If $|{}_{\varsigma_1}D_{p,q}\Pi|, |{}^{\varsigma_2}D_{p,q}\Pi|$ are n-polynomial convex functions and $|{}_{\varsigma_1}D_{p,q}\Pi(x)|, |{}^{\varsigma_2}D_{p,q}\Pi(x)| \leq M$, for all $x \in [\varsigma_1, \varsigma_2]$, then*

$$ \begin{aligned} &\left|\Pi(x) - \frac{1}{p(\varsigma_2-\varsigma_1)}\int_{\varsigma_1}^{px+(1-p)\varsigma_1}\Pi(x)\ {}_{\varsigma_1+}d_{p,q}x - \frac{1}{p(\varsigma_2-\varsigma_1)}\int_{px+(1-p)\varsigma_2}^{\varsigma_2}\Pi(x)\ {}_{\varsigma_2-}d_{p,q}x\right| \\ \leq\ & \frac{qM(x-\varsigma_1)^2}{n(\varsigma_2-\varsigma_1)}\sum_{s=1}^{\infty}\left(\frac{2}{p+q} - \frac{p-q}{p^{s+2}-q^{s+2}} - (p-q)\sum_{n=0}^{\infty}\frac{q^{2n}}{p^{2n+2}}\left(1-\frac{q^n}{p^{n+1}}\right)^s\right) \\ &+ \frac{qM(\varsigma_2-x)^2}{n(\varsigma_2-\varsigma_1)}\sum_{s=1}^{\infty}\left(\frac{2(p+q-1)}{p+q} - (p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^{n+1}}\right)^s - (p-q)\sum_{n=0}^{\infty}\frac{q^{n(s+2)}}{p^{(n+1)(s+2)}}\right), \end{aligned} $$

for all $x \in [\varsigma_1, \varsigma_2]$.

Theorem 178 ([94]). *Let Π be as in Theorem 177. If $|{}_{\varsigma_1}D_{p,q}\Pi|, |{}^{\varsigma_2}D_{p,q}\Pi|$ are n-polynomial convex functions, $r, s > 1, \dfrac{1}{r} + \dfrac{1}{s} = 1$, and $|{}_{\varsigma_1}D_{p,q}\Pi(x)|, |{}^{\varsigma_2}D_{p,q}\Pi(x)| \leq M$, for all $x \in [\varsigma_1, \varsigma_2]$, then:*

$$ \begin{aligned} &\left|\Pi(x) - \frac{1}{p(\varsigma_2-\varsigma_1)}\int_{\varsigma_1}^{px+(1-p)\varsigma_1}\Pi(x)\ {}_{\varsigma_1+}d_{p,q}x - \frac{1}{p(\varsigma_2-\varsigma_1)}\int_{px+(1-p)\varsigma_2}^{\varsigma_2}\Pi(x)\ {}_{\varsigma_2-}d_{p,q}x\right| \\ \leq\ & \frac{qM[(\varsigma_2-x)^2+(x-\varsigma_1)^2]}{\varsigma_2-\varsigma_1}\left((p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(1-\frac{q^n}{p^{n+1}}\right)^r\right)^{\frac{1}{r}}\left(\frac{2}{n}\right)^{\frac{1}{s}}\sum_{s=1}^{n}\left((p-q)\sum_{n=0}^{\infty}\frac{q^n s+1}{p^{(s+1)(n+1)}}\right), \end{aligned} $$

for all $x \in [\varsigma_1, \varsigma_2]$.

Theorem 179 ([94]). *Let Π be as in Theorem 177. If $|{}_{\varsigma_1}D_{p,q}\Pi|, |{}^{\varsigma_2}D_{p,q}\Pi|$ are n-polynomial convex functions, $s > 1$, and $|{}_{\varsigma_1}D_{p,q}\Pi(x)|, |{}^{\varsigma_2}D_{p,q}\Pi(x)| \leq M$, for all $x \in [\varsigma_1, \varsigma_2]$, then:*

$$\left|\Pi(x) - \frac{1}{p(\varsigma_2-\varsigma_1)}\int_{\varsigma_1}^{px+(1-p)\varsigma_1}\Pi(x)\ _{\varsigma_1+}d_{p,q}x - \frac{1}{p(\varsigma_2-\varsigma_1)}\int_{px+(1-p)\varsigma_2}^{\varsigma_2}\Pi(x)\ _{\varsigma_2-}d_{p,q}x\right|$$
$$\le \frac{qM(x-\varsigma_1)^2}{n(\varsigma_2-\varsigma_1)}\left(\frac{1}{p+q}\right)^{1-\frac{1}{s}}\left[\sum_{s=1}^{\infty}\left(\frac{2}{p+q} - \frac{p-q}{p^{s+2}-q^{s+2}} - (p-q)\sum_{n=0}^{\infty}\frac{q^{2n}}{p^{2n+2}}\left(1-\frac{q^n}{p^{n+1}}\right)^s\right)\right]^{\frac{1}{s}}$$
$$+\frac{qM(\varsigma_2-x)^2}{n(\varsigma_2-\varsigma_1)}\left(\frac{p+q-1}{p+q}\right)^{1-\frac{1}{s}}$$
$$\times\left[\sum_{s=1}^{\infty}\left(\frac{2(p+q-1)}{p+q} - (p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^{n+1}}\right)^s - (p-q)\sum_{n=0}^{\infty}\frac{q^{n(s+2)}}{p^{(n+1)(s+2)}}\right)\right]^{\frac{1}{s}},$$

for all $x \in [\varsigma_1, \varsigma_2]$.

Now, we give some estimates of post quantum Ostrowski-type inequalities for twice (p,q)-differentiable functions involving $(p,q)_{\varsigma_1}$- and $(p,q)^{\varsigma_2}$-integrals. Let $J_1 = [\varsigma_2 - p(\varsigma_2 - x), \varsigma_2]$ and $J_2 = [\varsigma_1, \varsigma_1 + p(x - \varsigma_1)]$.

Theorem 180 ([95]). *If* $\Pi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ *is a twice* (p,q)*-differentiable function such that* ${}^{\varsigma_2}D^2_{p,q}\Pi$ *and* ${}_{\varsigma_1}D^2_{p,q}\Pi$ *are continuous and integrable functions on* J_1 *and* J_2*, respectively. Then*

$$\left|\frac{(x-\varsigma_1)(\varsigma_2-x)}{pq^3(p-q)}\Big[(x-\varsigma_1)pq\Pi(qx+(1-q)\varsigma_2) + (\varsigma_2-x)pq\Pi(qx+(1-q)\varsigma_1)\right.$$
$$-(x-\varsigma_1)(q^2+pq-p^2)\Pi(px+(1-p)\varsigma_2) - (\varsigma_2-x)(q^2+pq-p^2)\Pi(px+(1-p)\varsigma_1)\Big]$$
$$\left.-\frac{[2]_{p,q}}{p^3q^3}\left[(x-\varsigma_1)^2\int_{p^2x+(1-p^2)\varsigma_2}^{\varsigma_2}\Pi(t)\ ^{\varsigma_2}d_{p,q}t + (\varsigma_2-x)^2\int_{\varsigma_1}^{p^2x+(1-p^2)\varsigma_1}\Pi(t)\ _{\varsigma_1}d_{p,q}t\right]\right|$$
$$\le (x-\varsigma_1)^2(\varsigma_2-x)^2\left[(x-\varsigma_1)\left(\frac{1}{[4]_{p,q}}\left|{}_{\varsigma_1}D^2_{p,q}\Pi(x)\right| + \frac{[4]_{p,q}-[3]_{p,q}}{[3]_{p,q}[4]_{p,q}}\left|{}_{\varsigma_1}D^2_{p,q}\Pi(\varsigma_1)\right|\right)\right.$$
$$\left.+(\varsigma_2-x)\left(\frac{1}{[4]_{p,q}}\left|{}^{\varsigma_2}D^2_{p,q}\Pi(x)\right| + \frac{[4]_{p,q}-[3]_{p,q}}{[3]_{p,q}[4]_{p,q}}\left|{}^{\varsigma_2}D^2_{p,q}\Pi(\varsigma_2)\right|\right)\right].$$

Theorem 181 ([95]). *Let* Π *be as in Theorem 180. If* $|{}^{\varsigma_2}D^2_{p,q}\Pi|^r$ *and* $|{}_{\varsigma_1}D^2_{p,q}\Pi|^r$ *are convex functions for* $r > 1$*, then*

$$\left|\frac{(x-\varsigma_1)(\varsigma_2-x)}{pq^3(p-q)}\Big[(x-\varsigma_1)pq\Pi(qx+(1-q)\varsigma_2) + (\varsigma_2-x)pq\Pi(qx+(1-q)\varsigma_1)\right.$$
$$-(x-\varsigma_1)(q^2+pq-p^2)\Pi(px+(1-p)\varsigma_2) - (\varsigma_2-x)(q^2+pq-p^2)\Pi(px+(1-p)\varsigma_1)\Big]$$
$$\left.-\frac{[2]_{p,q}}{p^3q^3}\left[(x-\varsigma_1)^2\int_{p^2x+(1-p^2)\varsigma_2}^{\varsigma_2}\Pi(t)\ ^{\varsigma_2}d_{p,q}t + (\varsigma_2-x)^2\int_{\varsigma_1}^{p^2x+(1-p^2)a}\Pi(t)\ _{\varsigma_1}d_{p,q}t\right]\right|$$
$$\le (x-\varsigma_1)^2(\varsigma_2-x)^2\left(\frac{1}{[3]_{p,q}}\right)^{1-1/r}\left[(x-\varsigma_1)\left(\frac{1}{[4]_{p,q}}\left|{}_{\varsigma_1}D^2_{p,q}\Pi(x)\right|^r + \frac{[4]_{p,q}-[3]_{p,q}}{[3]_{p,q}[4]_{p,q}}\left|{}_{\varsigma_1}D^2_{p,q}\Pi(\varsigma_1)\right|^r\right)^{1/r}\right.$$
$$\left.+(\varsigma_2-x)\left(\frac{1}{[4]_{p,q}}\left|{}^{\varsigma_2}D^2_{p,q}\Pi(x)\right|^r + \frac{[4]_{p,q}-[3]_{p,q}}{[3]_{p,q}[4]_{p,q}}\left|{}^{\varsigma_2}D^2_{p,q}\Pi(\varsigma_2)\right|^r\right)^{1/r}\right].$$

Theorem 182 ([95]). *Let* Π *be as in Theorem 180. If* $|{}^{\varsigma_2}D^2_{p,q}\Pi|^r$ *and* $|{}_{\varsigma_1}D^2_{p,q}\Pi|^r$ *are convex functions for* $r > 1$ *and* $1/s + 1/r = 1$*, then*

$$
\begin{aligned}
&\Bigg|\frac{(x-\varsigma_1)(\varsigma_2-x)}{pq^3(p-q)}\Big[(x-\varsigma_1)pq\Pi(qx+(1-q)\varsigma_2)+(\varsigma_2-x)pq\Pi(qx+(1-q)\varsigma_1)\\
&-(x-\varsigma_1)(q^2+pq-p^2)\Pi(px+(1-p)\varsigma_2)-(\varsigma_2-x)(q^2+pq-p^2)\Pi(px+(1-p)\varsigma_1)\Big]\\
&-\frac{[2]_{p,q}}{p^3q^3}\left[(x-\varsigma_1)^2\int_{p^2x+(1-p^2)\varsigma_2}^{\varsigma_2}\Pi(t)\ {}^{\varsigma_2}d_{p,q}t+(\varsigma_2-x)^2\int_{\varsigma_1}^{p^2x+(1-p^2)\varsigma_1}\Pi(t)\ {}_{\varsigma_1}d_{p,q}t\right]\Bigg|\\
\leq\ &(x-\varsigma_1)^2(\varsigma_2-x)^2\left(\frac{1}{[2s+1]_{p,q}}\right)^{1/s}\left[(x-\varsigma_1)\left(\frac{\left|{}_{\varsigma_1}D_q^2\Pi(x)\right|^r+(p+q-1)\left|{}_{\varsigma_1}D_q^2\Pi(\varsigma_1)\right|^r}{[2]_{p,q}}\right)^{1/r}\right.\\
&\left.+(\varsigma_2-x)\left(\frac{\left|{}^{\varsigma_2}D_q^2\Pi(x)\right|^r+(p+q-1)\left|{}^{\varsigma_2}D_q^2\Pi(\varsigma_2)\right|^r}{[2]_{p,q}}\right)^{1/r}\right].
\end{aligned}
$$

13. Ostrowski-Type Tensorial Inequalities in Hilbert Space

In this section we present Ostrowski-type inequalities for twice differentiable functions in the Hilbert space of tensorial type. Some preliminary concepts are necessary [96]. Let $I_1,\dots,I_k$ be intervals from $\mathbb{R}$ and let $f: I_1\times\dots\times I_k\to\mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A=(A_1,\dots,A_k)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1,\dots,H_k$ such that the spectrum of A_i is contained in I_i for $i=1,\dots,k$. We say that such a k-tuple is in the domain of f. If

$$A_i=\int_{I_i}\lambda_i\,dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i=1,\dots,k$ we define

$$f(A_1,\dots,A_k):=\int_{I_1}\dots\int_{I_k}f(\lambda_1,\dots,\lambda_k)\,dE_1(\lambda_1)\otimes\dots\otimes dE_k(\lambda_k)$$

as bounded selfadjoint operator on the tensorial product $H_1\otimes\dots\otimes H_k$.

Now, we present Ostrowski-type inequalities for twice differentiable functions in the Hilbert space of tensorial type for fractional differential equations of order $\zeta>0$, for convex and quasi-convex functions.

Theorem 183 ([96]). *Suppose that f is continuously differentiable on I and $|f''|$ is convex and A,B are self-adjoint operators with $Sp(A),Sp(B)\subset I$, then*

$$
\begin{aligned}
&\Bigg\|\frac{1}{6}\Big(f(A)\otimes 1+4f\Big(\frac{A\otimes 1+1\otimes B}{2}\Big)+1\otimes f(B)\Big)\\
&-\frac{1}{2}\zeta\Big(\int_0^1 f\Big(\Big(\frac{1-k}{2}\Big)A\otimes 1+\Big(\frac{1+k}{2}\Big)1\otimes B\Big)k^{\zeta-1}dk\\
&+\int_0^1 f\Big(\Big(1-\frac{k}{2}\Big)A\otimes 1+\frac{k}{2}1\otimes B\Big)(1-k)^{\zeta-1}\Big)\Bigg\|\\
\leq\ &\|1\otimes B-A\otimes 1\|^2\frac{(\|f''(A)\|+\|f''(B)\|)(3\zeta^2+8\zeta+7)}{(\zeta+2)(24\zeta+24)}.
\end{aligned}
$$

Theorem 184 ([96]). *Suppose that f is continuously differentiable on I and $|f''|$ is quasi-convex and A, B are self-adjoint operators with $Sp(A), Sp(B) \subset I$, then*

$$\begin{aligned} & \Big\| \frac{1}{6}\Big(f(A)\otimes 1 + 4f\Big(\frac{A\otimes 1 + 1\otimes B}{2}\Big) + 1\otimes f(B)\Big) \\ & -\frac{1}{2}\zeta\Big(\int_0^1 f\Big(\Big(\frac{1-k}{2}\Big)A\otimes 1 + \Big(\frac{1+k}{2}\Big)1\otimes B\Big)k^{\zeta-1}dk \\ & + \int_0^1 f\Big(\Big(1-\frac{k}{2}\Big)A\otimes 1 + \frac{k}{2}1\otimes B\Big)(1-k)^{\zeta-1}\Big)\Big\| \\ \leq\ & \|1\otimes B - A\otimes 1\|^2 \frac{3\zeta^2 + 8\alpha + 7}{(\zeta+2)(24\zeta+24)} \\ & \times(\||f''(A)|\otimes 1 + 1\otimes |f''(B)|\| + \||f''(A)|\otimes 1 - 1\otimes |f''(B)|\|). \end{aligned}$$

14. Conclusions

In this survey, we have presented a variety of results on Ostrowski-type inequalities involving fractional integral operators and convex functions. This comprehensive review will inspire the researchers to acquire useful information about Ostrowski-type integral inequalities before pursuing their new research on the topic to develop it further. Moreover, it is expected that the present work will provide a guideline for developing numerous new results for Ostrowski-type inequalities involving the new fractional integral operators combined with different types of convex functions.

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