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Dirac Field, van der Waals Gas, Weyssenhoff Fluid, and Newton Particle

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Abstract: This article considers the Dirac field in polar formulation and shows that when torsion is taken in effective approximation the theory has the thermodynamic properties of a van der Waals gas. It is then shown that in the limit of zero chiral angle the van der Waals gas reduces to a Weyssenhoff fluid, and in spinlessness regime the Weyssenhoff fluid further reduces to a Newton particle. This nesting of approximations allows us to interpret the various spinor quantities. We will see that torsion will provide a form of negative pressure, while the chiral angle will be related to a type of temperature.

Keywords: spinor field; effective approximation; non-relativistic regime; classical limit

1. Introduction

Both in geometric construction, and for its far-reaching applications, the Dirac field is among the most important fields in mathematics and physics. Still, when confronted with possible interpretations, there appears to be a spread consensus that no one really knows what a spinor actually is. Of course, this situation is not limited to the relativistic spinor field. The Pauli field is affected by the very same condition. Nor is this situation confined to relativistic and non-relativistic spinors. The Schrödinger field, namely the usual quantum mechanical wave function, carries the same burden, i.e., the lack of visualizability that renders the understanding of quantum mechanics so difficult, if not impossible. Whether constituted by two chiral states or only one, or whether characterized by two helicities or a single one, what seems to be at the root of the problem is the fact that all these wave functions are intrinsically built to be complex-valued fields.

On the other hand, all complex quantities may always be written in polar form, in which complex functions are re-expressed as a product of modules times unitary phases, with modules and phases being real. Pauli spinors, having two helicities, need extra care in undergoing polar decomposition since, under rotations, the two components would mix. And even more care is required for Dirac spinors since, having two helicities as well as two chiralities, under Lorentz transformations all four components would mix. Still, the relativistic polar formulation is doable just as well, as was first shown in [1,2].

The advantage of the polar decomposition of relativistic spinor fields is that it converts the entire Dirac theory into a form that is genuinely hydrodynamic [3]. This does not only mean that all variables are real. It also means that all variables are in themselves perfectly visualizable in terms of the concepts of fluid mechanics. Indeed, of the four sets of variables in terms of which a spinor field can be decomposed, two are the density and velocity, exactly the same as those we have in standard hydrodynamics. Another is the spin, which is also a very well-known concept nowadays. The final one is the chiral angle, which is instead not yet easy to understand, although we hope that such an object will be better clarified in the light of the investigation that we intend to carry out in this work. We will see, in fact, that, under some very general conditions, the chiral angle can be interpreted as a form of generalized temperature for the Dirac field. When the chiral angle and density, spin, and velocity are all accounted for, one can then see that the Dirac field theory has

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indeed been re-formulated as a type of fluid with a temperature and a pressure verifying the relationships that they would satisfy in the case of the van der Waals gas. We will also see that in the zero-temperature regime, such a gas behaves as a Weyssenhoff fluid with a completely antisymmetric spin. And naturally, in the zero-spin limit, the laws of Newton dynamics are recovered.

The idea of re-formulating quantum mechanics as a type of fluid dates back to the work of Madelung, who first considered writing the wave function as a product of module and phase related to density and velocity, respectively. In turn, this would split the Schrödinger equation into a Hamilton–Jacobi equation with a quantum potential written in terms of the density and a continuity equation for the velocity. This was the basis upon which Bohm started to build his interpretation of quantum mechanics [4]. The treatment was revised by Takabayasi in [5]. The relativistic extension was attempted first by Bohm in [6]. And then it was undertaken by Takabayasi in a series of works culminating with [7].

All these works share the idea of trying to write relativistic quantum mechanics as a type of classical mechanics with our present treatment. None of these works, however, could reach a fully general covariant description because they never considered the polar form that was first proposed in [1,2]. It is our objective to show that when the polar form of [1,2] is used, as performed in [3], all the results of Bohm and Takabayasi will finally find their most generally covariant expression.

2. Dirac Field in Polar Form

2.1. Dirac Spinors

We start with a brief summary of the Dirac spinors to set our convention. To begin, let γ^i be matrices belonging to the Clifford algebra $\{\gamma^i,\gamma^j\}=2\mathbb{I}\eta^{ij}$ with η^{ij} the Minkowski matrix and where $\boldsymbol{\sigma}_{ik}=[\gamma_i,\gamma_k]/4$ are the generators of the Lorentz group. In terms of the completely antisymmetric Levi-Civita pseudo-tensor ε_{abcd} we can also have the validity of the relation $2i\boldsymbol{\sigma}_{ab}=\varepsilon_{abcd}\boldsymbol{\pi}\boldsymbol{\sigma}^{cd}$ implicitly giving the definition of the parity-odd matrix $\boldsymbol{\pi}$ whose existence stipulates that the Lorentz group is reducible (this is the fifth gamma matrix, which we will not indicate as a gamma with an index five to avoid the confusion coming from the dummy index. The Greek letter $\boldsymbol{\pi}$ corresponds to the Latin letter p and it stands for *parity* in the same way that the Greek letter $\boldsymbol{\sigma}$ corresponds to the Latin letter s and it stands for *spin*). The exponentiation of the generators gives an element of the Lorentz group $\boldsymbol{\Lambda}$ and therefore $\boldsymbol{S}=\boldsymbol{\Lambda}e^{iq\alpha}$ is an element of the spinor group also accounting for gauge transformations. A spinor field is an object that transforms like $\psi\rightarrow\boldsymbol{S}\psi$ and $\overline{\psi}\rightarrow\overline{\psi}\boldsymbol{S}^{-1}$ where $\overline{\psi}=\psi^\dagger\gamma^0$ is the adjoint operation. With a pair of adjoint spinors we can form the spinorial bi-linears

$$\Sigma^{ab}=2\overline{\psi}\boldsymbol{\sigma}^{ab}\boldsymbol{\pi}\psi \qquad M^{ab}=2i\overline{\psi}\boldsymbol{\sigma}^{ab}\psi \tag{1}$$

$$S^a=\overline{\psi}\gamma^a\boldsymbol{\pi}\psi \qquad U^a=\overline{\psi}\gamma^a\psi \tag{2}$$

$$\Theta=i\overline{\psi}\boldsymbol{\pi}\psi \qquad \Phi=\overline{\psi}\psi \tag{3}$$

which are all real tensors. They verify the Hodge duality

$$\Sigma^{ab}=-\tfrac{1}{2}\varepsilon^{abij}M_{ij} \qquad M^{ab}=\tfrac{1}{2}\varepsilon^{abij}\Sigma_{ij} \tag{4}$$

beside the constitutive relations

$$M_{ik}U^i=\Theta S_k \qquad \Sigma_{ik}U^i=\Phi S_k \tag{5}$$

$$M_{ik}S^i=\Theta U_k \qquad \Sigma_{ik}S^i=\Phi U_k \tag{6}$$

as well as

$$M_{ab}\Phi-\Sigma_{ab}\Theta=U^jS^k\varepsilon_{jkab} \qquad M_{ab}\Theta+\Sigma_{ab}\Phi=U_{[a}S_{b]} \tag{7}$$

together with

$$\tfrac{1}{2}M_{ab}M^{ab} = -\tfrac{1}{2}\Sigma_{ab}\Sigma^{ab} = \Phi^2 - \Theta^2 \tag{8}$$

$$\tfrac{1}{2}M_{ab}\Sigma^{ab} = -2\Theta\Phi \tag{9}$$

and

$$U_aU^a = -S_aS^a = \Theta^2 + \Phi^2 \tag{10}$$

$$U_aS^a = 0 \tag{11}$$

called Fierz re-arrangements. They show that not all the bi-linears are independent, and in fact if $\Phi^2+\Theta^2 \neq 0$ both antisymmetric tensors M_{ab} and Σ_{ab} can be dropped in favor of the two vectors and the two scalars. In turn, under the same condition, the axial-vector and the vector S_a and U_a are space-like and time-like, showing that they can be recognized as spin and velocity, respectively [3].

The spinorial covariant derivative is defined as

$$\boldsymbol{\nabla}_\mu\psi = \partial_\mu\psi + \tfrac{1}{2}C_{ab\mu}\boldsymbol{\sigma}^{ab}\psi + iqA_\mu\psi \tag{12}$$

in which A_μ is the gauge potential of charge q and $C^{ab}{}_{\mu}$ is the spacetime spin connection in torsionless case. As we will see, full generality will be recovered by introducing torsion as an axial-vector field in the dynamics.

As usual, the commutator

$$[\boldsymbol{\nabla}_\mu, \boldsymbol{\nabla}_\nu]\psi = \tfrac{1}{2}R_{ab\mu\nu}\boldsymbol{\sigma}^{ab}\psi + iqF_{\mu\nu}\psi \tag{13}$$

defines the Riemann curvature and the Maxwell strength.

The dynamics is assigned by the torsion field equations

$$\nabla_\rho(\partial W)^{\rho\mu} + M^2W^\mu = XS^\mu \tag{14}$$

together with the gravitational field equations

$$\begin{aligned} R^{\rho\sigma} - \tfrac{1}{2}Rg^{\rho\sigma} - \Lambda g^{\rho\sigma} = \tfrac{1}{2}[\tfrac{1}{4}F^2g^{\rho\sigma} - F^{\rho\alpha}F^{\sigma}{}_{\alpha} +\\ + \tfrac{1}{4}(\partial W)^2g^{\rho\sigma} - (\partial W)^{\sigma\alpha}(\partial W)^{\rho}{}_{\alpha} + M^2(W^\rho W^\sigma - \tfrac{1}{2}W^2g^{\rho\sigma}) +\\ + \tfrac{i}{4}(\overline{\psi}\gamma^\rho\boldsymbol{\nabla}^\sigma\psi - \boldsymbol{\nabla}^\sigma\overline{\psi}\gamma^\rho\psi + \overline{\psi}\gamma^\sigma\boldsymbol{\nabla}^\rho\psi - \boldsymbol{\nabla}^\rho\overline{\psi}\gamma^\sigma\psi) - \tfrac{1}{2}X(W^\sigma S^\rho + W^\rho S^\sigma)] \end{aligned} \tag{15}$$

and the electrodynamic field equations

$$\nabla_\sigma F^{\sigma\mu} = qU^\mu \tag{16}$$

where $(\partial W)_{\alpha\nu} = \nabla_\alpha W_\nu - \nabla_\nu W_\alpha$ and M is the torsion mass, and where we define $R^{\alpha}{}_{\rho\alpha\sigma} = R^{\rho\sigma}$ and $R^{\rho\sigma}g_{\rho\sigma} = R$ as the Ricci tensor and scalar and Λ is the cosmological constant.

For matter, the dynamics is assigned by the Dirac spinor field equation

$$i\gamma^\mu\boldsymbol{\nabla}_\mu\psi - XW_\sigma\gamma^\sigma\boldsymbol{\pi}\psi - m\psi = 0 \tag{17}$$

where W_σ is the Hodge dual of the torsion tensor and X the torsion–spin coupling constant, added to recover full generality, as we have already anticipated.

The set of field Equations (14) and (15) with (16) is conceived in this way so to give rise to conservation laws that turn out to be automatically satisfied when the Dirac spinorial field Equation (17) is valid, and so it is consistent [8].

2.2. Polar Decomposition

In the aforementioned case in which $\Phi^2+\Theta^2\neq 0$ we can perform what is called polar decomposition of the spinor field. Specifically, it is possible to demonstrate that under the above condition any spinor field can always be written, in chiral representation, as

$$\psi=\phi\, e^{-\frac{i}{2}\beta\boldsymbol{\pi}}\,\boldsymbol{L}^{-1}\left(\begin{array}{c}1\\0\\1\\0\end{array}\right) \tag{18}$$

for a pair of functions ϕ and β and for some $\boldsymbol{L}$ which has the structure of a spinor transformation [1,2] (see also Appendix A for a detailed derivation). As anticipated, the two antisymmetric tensors are expressed by means of the two vectors and the two scalars, and these are given by

$$S^a=2\phi^2 s^a \qquad U^a=2\phi^2 u^a \tag{19}$$

and

$$\Theta=2\phi^2\sin\beta \qquad \Phi=2\phi^2\cos\beta \tag{20}$$

when the polar form is implemented. The last two show that ϕ and β are a scalar and a pseudo-scalar, known as module and chiral angle. Then, (10) and (11) reduce to

$$u_a u^a=-s_a s^a=1 \tag{21}$$
$$u_a s^a=0 \tag{22}$$

showing that the velocity has only three independent components, the three spatial rapidities, whereas the spin has only two independent components, the two angles that, in the rest-frame, its spatial part forms with the third axis. As for $\boldsymbol{L}$ we can read its meaning as that of the specific transformation that takes a given spinor to its rest-frame with spin aligned along the third axis. For the spinorial fields in polar form, the eight real components are re-configured in such a way that the two scalars ϕ and β are isolated from the six parameters of $\boldsymbol{L}$ that can always be transferred into the frame and which are thus the Goldstone fields.

Because in general one can prove that

$$\boldsymbol{L}^{-1}\partial_\mu\boldsymbol{L}=iq\partial_\mu\zeta\mathbb{I}+\tfrac{1}{2}\partial_\mu\zeta_{ij}\boldsymbol{\sigma}^{ij} \tag{23}$$

for some ζ and ζ_{ij} that are in fact the Goldstone fields, it follows that we can define

$$R_{ij\mu}:=\partial_\mu\zeta_{ij}-C_{ij\mu} \tag{24}$$
$$P_\mu:=q(\partial_\mu\zeta-A_\mu) \tag{25}$$

which are real tensors. By reading these expressions one can see that after the Goldstone fields are transferred into the frame, they combine with spin connection and gauge potential to become the longitudinal components of the P_μ and $R_{ij\mu}$ tensors, hence called gauge and spacetime tensorial connections. From (18) with (25) and (24) we obtain

$$\boldsymbol{\nabla}_\mu\psi=(\nabla_\mu\ln\phi\mathbb{I}-\tfrac{i}{2}\nabla_\mu\beta\boldsymbol{\pi}-\tfrac{1}{2}R_{\alpha\nu\mu}\boldsymbol{\sigma}^{\alpha\nu}-iP_\mu\mathbb{I})\psi \tag{26}$$

as the polar form of the covariant derivative. Notice that

$$\nabla_\mu s_\nu=s^\alpha R_{\alpha\nu\mu} \qquad \nabla_\mu u_\nu=u^\alpha R_{\alpha\nu\mu} \tag{27}$$

as general identities. The covariant derivative of the velocity is the object with which one builds the strain-rate tensor in continuum mechanics. Expressions (27) are the extension to both velocity and spin of what makes $R_{ab\mu}$ interpretable as the strain-rate tensor.

The tensorial connections are such that

$$-R^{i}{}_{j\mu\nu} = \nabla_{\mu}R^{i}{}_{j\nu} - \nabla_{\nu}R^{i}{}_{j\mu} + R^{i}{}_{k\mu}R^{k}{}_{j\nu} - R^{i}{}_{k\nu}R^{k}{}_{j\mu} \tag{28}$$

$$-qF_{\mu\nu} = \nabla_{\mu}P_{\nu} - \nabla_{\nu}P_{\mu} \tag{29}$$

therefore being the covariant potentials of the Riemann curvature and Maxwell strength.

In the gravitational field equations, the right-hand side aside for the factor $1/2$ is the energy density tensor, and it is expressed in polar variables according to

$$\begin{aligned} T^{\rho\sigma} = \tfrac{1}{4}F^{2}g^{\rho\sigma} - F^{\rho\alpha}F^{\sigma}{}_{\alpha} + \tfrac{1}{4}(\partial W)^{2}g^{\rho\sigma} - (\partial W)^{\sigma\alpha}(\partial W)^{\rho}{}_{\alpha} + M^{2}(W^{\rho}W^{\sigma} - \tfrac{1}{2}W^{2}g^{\rho\sigma}) + \\ +\phi^{2}[P^{\rho}u^{\sigma} + P^{\sigma}u^{\rho} + (\nabla^{\rho}\beta/2 - XW^{\rho})s^{\sigma} + (\nabla^{\sigma}\beta/2 - XW^{\sigma})s^{\rho} - \\ -\tfrac{1}{4}R_{\alpha\nu}{}^{\sigma}s_{\kappa}\varepsilon^{\rho\alpha\nu\kappa} - \tfrac{1}{4}R_{\alpha\nu}{}^{\rho}s_{\kappa}\varepsilon^{\sigma\alpha\nu\kappa}] \end{aligned} \tag{30}$$

where the explicit presence of the spacetime tensorial connection is seen.

The Dirac spinor field equations in polar form are

$$\nabla_{\mu}\beta - 2XW_{\mu} + B_{\mu} - 2P^{\iota}u_{[\iota}s_{\mu]} + 2ms_{\mu}\cos\beta = 0 \tag{31}$$

$$\nabla_{\mu}\ln\phi^{2} + R_{\mu} - 2P^{\rho}u^{\nu}s^{\alpha}\varepsilon_{\mu\rho\nu\alpha} + 2ms_{\mu}\sin\beta = 0 \tag{32}$$

in which $R_{\mu\nu}{}^{\nu} = R_{\mu}$ and $\frac{1}{2}\varepsilon_{\mu\alpha\nu\iota}R^{\alpha\nu\iota} = B_{\mu}$ were defined.

Upon the introduction of the potentials

$$2Y_{\mu} = \nabla_{\mu}\beta - 2XW_{\mu} + B_{\mu} \tag{33}$$

$$2Z_{\mu} = \nabla_{\mu}\ln\phi^{2} + R_{\mu} \tag{34}$$

it becomes easier to manipulate the polar spinor field Equations (31) and (32) in order to isolate the gauge tensorial connection

$$P^{\eta} = m\cos\beta u^{\eta} + Y_{\mu}u^{[\mu}s^{\eta]} + Z_{\mu}u_{\pi}s_{\tau}\varepsilon^{\mu\pi\tau\eta} \tag{35}$$

which is recognized to be the momentum of the field and with which the energy (30) acquires the form

$$\begin{aligned} T^{\rho\sigma} = \tfrac{1}{4}F^{2}g^{\rho\sigma} - F^{\rho\alpha}F^{\sigma}{}_{\alpha} + \tfrac{1}{4}(\partial W)^{2}g^{\rho\sigma} - (\partial W)^{\sigma\alpha}(\partial W)^{\rho}{}_{\alpha} + M^{2}(W^{\rho}W^{\sigma} - \tfrac{1}{2}W^{2}g^{\rho\sigma}) + \\ +\phi^{2}[2m\cos\beta u^{\rho}u^{\sigma} - 2Y_{\mu}s^{\mu}u^{\rho}u^{\sigma} + Y_{\mu}u^{\mu}(s^{\rho}u^{\sigma} + s^{\sigma}u^{\rho}) + Y^{\rho}s^{\sigma} + Y^{\sigma}s^{\rho} + \\ +Z_{\mu}u_{\pi}s_{\tau}(\varepsilon^{\mu\pi\tau\sigma}u^{\rho} + \varepsilon^{\mu\pi\tau\rho}u^{\sigma}) - \tfrac{1}{4}(R_{\alpha\nu\pi}\varepsilon^{\rho\alpha\nu\pi}g^{\sigma\kappa} + R_{\alpha\nu\pi}\varepsilon^{\sigma\alpha\nu\pi}g^{\rho\kappa} + \\ +R_{\alpha\nu}{}^{\sigma}\varepsilon^{\rho\alpha\nu\kappa} + R_{\alpha\nu}{}^{\rho}\varepsilon^{\sigma\alpha\nu\kappa})s_{\kappa}] \end{aligned} \tag{36}$$

in terms of the $R_{ab\mu}$ tensor and the Y_{μ} and Z_{μ} potentials.

3. Torsion Effective Approximation and van der Waals Gas

3.1. General Thermodynamic Variables

With this general presentation, we are now ready to begin the study of the various approximations, starting from the torsion effective approximation. Before this, however, we recall a very general construction that takes place in thermodynamics. In a thermodynamical context, when combining the two principles of thermodynamics into one, we obtain the fundamental relation $dU = TdS - pdV$ as is well known. Considering that dS is an exact differential form, one can extract the expression

$$\left(\frac{\partial U}{\partial V}\right)_{T} = \left(\frac{\partial p}{\partial T}\right)_{V} T - p \tag{37}$$

in case V and T are the independent variables. This equation is called internal energy equation, and with it one can deduce the internal energy from the equation of state.

For example, take the simplest non-perfect gas, that is the van der Waals gas, whose equation of state is

$$\left(p+\frac{a}{V^2}\right)(V-b)=RT \tag{38}$$

in which a is a constant related to the effective pressure due to forces between the molecules, positive in the case of attraction, and b is the volume that is occupied by the molecules. By means of (37) one can deduce that

$$U=\omega+C_V T-\frac{a}{V} \tag{39}$$

where ω is a generic constant.

It is important to notice that with the equation of the internal energy we are giving an axiomatic definition of thermodynamical variables, in the sense that we are assigning a meaning to the different terms entering (37) according to the role they play in such an equation. For example, if we knew that U had a given dependence on V then the right-hand side of (37) would be known, and any pair of variables satisfying the right-hand side of (37) in exactly the way p and T are would, respectively, be interpreted as pressure and temperature. With this in mind, we are now going to investigate the thermodynamic structure of the Dirac spinor field theory.

3.2. Massive Propagating Torsion

We will consider the Dirac theory with torsion taken to be massive enough to allow the effective approximation. In effective approximation, the torsion field loses all its propagating properties, with field equations reducing to

$$M^2 W^\mu = XS^\mu \tag{40}$$

so that now torsion can be replaced in terms of the spin.

When this is performed in the expression of the energy density tensor (36) remarkable simplifications occur. Taking in particular the purely spinorial contribution, it reads

$$\begin{aligned}E^{\rho\sigma}=\phi^2[2(m\cos\beta-\phi^2X^2/M^2-s^\mu\nabla_\mu\beta/2-\tfrac{1}{4}R^{\pi\tau\eta}s^\kappa\varepsilon_{\kappa\pi\tau\eta})u^\rho u^\sigma+\\+2\phi^2X^2/M^2(g^{\rho\sigma}-u^\rho u^\sigma)+(s^\rho u^\sigma+s^\sigma u^\rho)u^\mu\nabla_\mu\beta/2+s^\rho\nabla^\sigma\beta/2+s^\sigma\nabla^\rho\beta/2+\\+Z_\mu u_\pi s_\tau(\varepsilon^{\mu\pi\tau\sigma}u^\rho+\varepsilon^{\mu\pi\tau\rho}u^\sigma)-\tfrac{1}{4}R_{\pi\tau\eta}s_\kappa(\varepsilon^{\rho\pi\tau\kappa}g^{\sigma\eta}+\varepsilon^{\sigma\pi\tau\kappa}g^{\rho\eta}+\\+\varepsilon^{\pi\tau\eta\mu}u_\mu u^\sigma g^{\rho\kappa}+\varepsilon^{\pi\tau\eta\mu}u_\mu u^\rho g^{\sigma\kappa})]\end{aligned} \tag{41}$$

which can be worked out in detail in the following way.

Defining the quantities given by

$$\mu=E_{\rho\sigma}u^\rho u^\sigma \tag{42}$$
$$p=-\tfrac{1}{3}E_{\rho\sigma}(g^{\rho\sigma}-u^\rho u^\sigma) \tag{43}$$

it is easy to prove that

$$E^{\rho\sigma}=\mu u^\rho u^\sigma-p(g^{\rho\sigma}-u^\rho u^\sigma) \tag{44}$$

as a general identity. Re-writing the energy density with this structure helps identifying the quantities μ and p as the internal energy density and the pressure of the field.

As a consequence, in our case we have that

$$\mu = 2\phi^2(m\cos\beta - \phi^2 X^2/M^2) - [2\phi^2(s^\mu \nabla_\mu \beta/2 - \tfrac{1}{2}\varepsilon^{\kappa\alpha\mu\nu} s_\kappa u_\alpha \nabla_\mu u_\nu)] \tag{45}$$

$$p = -2\phi^4 X^2/M^2 - \tfrac{1}{3}[2\phi^2(s^\mu \nabla_\mu \beta/2 - \tfrac{1}{2}\varepsilon^{\kappa\alpha\mu\nu} s_\kappa u_\alpha \nabla_\mu u_\nu)] \tag{46}$$

are the internal energy density and pressure of Dirac spinors.

Introducing the volume $2\phi^2 = 1/V$ as well as the internal energy $U = \mu V$ we can write them, respectively, as

$$U = m\cos\beta + 3RT - \frac{X^2}{2M^2}\frac{1}{V} \tag{47}$$

$$\left(p + \frac{X^2}{2M^2}\frac{1}{V^2}\right)V = RT \tag{48}$$

in which

$$3RT = -s^\mu \nabla_\mu \beta/2 + \tfrac{1}{2}\varepsilon^{\kappa\alpha\mu\nu} s_\kappa u_\alpha \nabla_\mu u_\nu \tag{49}$$

has also been defined.

Notice that (48) is exactly the van der Waals equation of state in the case in which $b = 0$ and $2a = X^2/M^2$ showing that the torsional effective force is indeed attractive. Also notice that (47) can be recognized as the van der Waals gas internal energy if $C_V = 3R$ and $m = \omega$ for small values of the chiral angle. It is essential to remark that in order for (48) and (47) to be structurally similar to those of a van der Waals gas, condition (49) must hold. The validity of (49) can be interpreted as the definition of temperature for the Dirac field, and it can be read as the fact that the internal dynamics of the Dirac field obtains contributions from its chiral angle and its vorticity. It is not surprising that the chiral angle, the phase difference between the chiral parts, be tied to the internal dynamics and, thus, thermodynamically associated with the concept of temperature. Such an association is also clear in the fact that $m\cos\beta$ is another contribution of the chiral angle to the internal energy, which is just the relativistic mass. We recall to the reader that the association of the chiral angle to temperature, while justified by an interpretation employing the concept of internal dynamics, is only the axiomatic type of connection in the sense explained above. The definition of temperature assigned by means of the internal energy and its Equation (37) is formal and not functional: we have defined T according to (49) with the aim of rendering (37) satisfied but T does not represent a chaotic motion of particles in the kinetic theory of gases.

The definition of temperature as given by (49) seems to us the only way to define something conceptually close to the idea of temperature even for systems that are not constituted by randomly distributed particles.

4. Zero Chiral Angle and Weyssenhoff Fluid

4.1. Non-Relativistic Regime

In the previous section, we have identified the chiral angle as what gives rise to a type of internal dynamics thus related to a generalized form of temperature for the Dirac spinor field. In [3], and references therein, we have discussed the idea of non-relativistic limit as the regime for which

$$\vec{u} \to 0 \tag{50}$$

$$\beta \to 0 \tag{51}$$

characterizing the difference between the two conditions in the fact that, while the first represents the lost motion, the second represents the loss of the dynamical properties that would remain even in rest-frame, thus the intrinsic, internal dynamics. This fits well in the discussion above, where it is even more reasonably justified the fact that, in non-relativistic regime, the temperature (49) would lose all contributions coming from the material distribution. In other words, we may say that our generalized definition of temperature for the Dirac field is such that it would tend to zero in non-relativistic cases.

While the pair of conditions (50) and (51) are the non-relativistic limit, the single condition (51) can be defined as internal triviality. Or equivalently, when the chiral angle vanishes we lose the internal dynamics. This is also reasonable if we consider that $\beta = 0$ means no difference between the two chiral parts. Or that zitterbewegung effects vanish [9,10].

The condition of internal triviality has also the advantage of being covariant, so it makes sense to see what happens when it is assumed.

4.2. Hydrodynamics with Spin

Assuming $\beta = 0$ from the start implies that the bi-linear pseudo-scalar $\Theta = 0$ identically, and, therefore, we have that

$$M_{ik}u^i = 0 \qquad \Sigma_{ik}u^i = 2\phi^2 s_k \tag{52}$$

$$M_{ik}s^i = 0 \qquad \Sigma_{ik}s^i = 2\phi^2 u_k \tag{53}$$

alongside

$$M_{ab} = 2\phi^2 u^j s^k \varepsilon_{jkab} \tag{54}$$

$$\Sigma_{ab} = 2\phi^2 u_{[a} s_{b]} \tag{55}$$

and

$$M_{ab}M^{ab} = -\Sigma_{ab}\Sigma^{ab} = 8\phi^4 \tag{56}$$

$$M_{ab}\Sigma^{ab} = 0 \tag{57}$$

as Fierz identities. By employing (4) into (53) one has

$$M_{ik}u^i = 0 \qquad \tfrac{1}{2}\varepsilon_{kiab}M^{ab}u^i = 2\phi^2 s_k \tag{58}$$

$$M_{ik}s^i = 0 \qquad \tfrac{1}{2}\varepsilon_{kiab}M^{ab}s^i = 2\phi^2 u_k \tag{59}$$

so that focusing in particular on the first, we can re-write the two expressions according to

$$M^{ki}u_i = 0 \tag{60}$$

$$M^{[ab}u^{c]} = \varepsilon^{abck} S_k \tag{61}$$

telling that the momentum is orthogonal to the velocity and that the completely antisymmetric part of $M^{ij}u^k$ is the Hodge dual of the spin density axial-vector. It follows that the momentum M^{ki} is the fundamental spin tensor of Weyssenhoff fluids [11,12], where (60) is the constitutive condition and (61) the condition saying that it is only the completely antisymmetric part of the spin density tensor that is excited. This is expected as the Dirac spinor has a completely antisymmetric spin.

5. Spinlessness and Newton Mechanics

5.1. Classical Limit

At last, we discuss spinlessness. Such a case is obtained in the approximation

$$s^a \to 0 \tag{62}$$

and it means that we are losing quantum effects. Indeed, if we were not to choose natural units, the spin would be seen to be proportional to $\hbar$ and the limit $\hbar \to 0$ is what would give rise to the condition encoding the classical approximation.

Notice also that the validity of the Dirac equation gives

$$\nabla_i S^i = 2m\Theta \tag{63}$$

showing that $\beta \to 0$ is implied by $S^i \to 0$ and stating that there can be no chirality if there is no helicity. The present limit is therefore compatible with the limit that we discussed in the previous section.

5.2. Point Particle

Let us then re-consider the momentum (35) as well as the energy density tensor (36) in effective approximation and in this limit. We have

$$P^\eta = (m - 2\phi^2 X^2/M^2)u^\eta \tag{64}$$

and

$$T^{\rho\sigma} = \frac{1}{4}F^2 g^{\rho\sigma} - F^{\rho\alpha}F^\sigma{}_\alpha + 2\phi^2(m - 2\phi^2 X^2/M^2)u^\rho u^\sigma + 2\phi^4 X^2/M^2 g^{\rho\sigma} \tag{65}$$

which next we discuss in view of their conservation laws.

To this purpose, set $2\phi^2 = \rho$ with ρ being the density distribution of the material field. The last two expressions now become

$$P^\eta = (m - \rho X^2/M^2)u^\eta \tag{66}$$

and

$$T^{\rho\sigma} = \frac{1}{4}F^2 g^{\rho\sigma} - F^{\rho\alpha}F^\sigma{}_\alpha + \rho(m - \rho X^2/M^2)u^\rho u^\sigma + \frac{1}{2}\rho^2 X^2/M^2 g^{\rho\sigma} \tag{67}$$

and for them we know that

$$\nabla_\alpha T^{\alpha\nu} = 0 \tag{68}$$

and

$$\nabla_\alpha(\rho u^\alpha) = 0 \tag{69}$$

must be valid as a consequence of the Dirac spinorial field equations. Taking (66) into (67) and the result into (68) and then employing (69) we arrive at

$$\frac{1}{2}F_{\alpha\pi}\nabla^\sigma F^{\alpha\pi} + F_{\alpha\pi}\nabla^\pi F^{\sigma\alpha} - \nabla^\eta F_{\eta\alpha}F^{\sigma\alpha} + \rho u^\nu \nabla_\nu P^\sigma - \nabla^\sigma p = 0 \tag{70}$$

where the pressure $p = -\frac{1}{2}\rho^2 X^2/M^2$ was used.

By employing now the Maxwell Equation (16) we obtain

$$\rho u^\nu \nabla_\nu P^\sigma = \nabla^\sigma p + q\rho F^{\sigma\alpha}u_\alpha \tag{71}$$

which is the Newton equation of hydrodynamic motion.

In total absence of torsion, no pressure remains so that it becomes possible to simplify the density on both sides and we reduce to the final

$$u^\nu \nabla_\nu P^\sigma = qF^{\sigma\alpha}u_\alpha \tag{72}$$

as the Newton equation for the motion of material points.

It is important to remark that the Newton law has been obtained without any assumption on localization for the matter distribution. With this we do not mean to imply that matter distributions cannot be localized, but only that there is no need for this assumption.

6. Conclusions

In this work, we have considered the Dirac spinor field theory re-formulated in terms of the polar variables given by the ϕ and β scalars with the u_a and s_a vectors. After conversion, the full relativistic quantum mechanics turn into a type of hydrodynamics in which $2\phi^2$ is the density distribution and β is the chiral angle while u_a is the velocity and s_a is the spin. This hydrodynamics is, therefore, an extension of the usual one since not only are the density and velocity present, but the chiral angle and spin are also present.

However, the general construction can be restricted to the standard hydrodynamics by removing these two extra variables. The general theory, with torsion in its effective approximation, has the same thermodynamic features of a van der Waals gas, with van der Waals pressure due to torsion being always negative since torsion is always attractive, and with the temperature and internal energy being tied to the chiral angle. In the limit $\beta \to 0$ (corresponding to the requirement of losing the phase difference between chiral parts) the general theory reduces to that of a Weyssenhoff fluid with completely antisymmetric spin. And, for $s_a \to 0$ (corresponding to the condition of non-quantum limit) it reduces to a Newton fluid in the presence of pressure due to torsion. By vanishing torsion, the usual Newton equation for the motion of material points is eventually recovered.

Aside from allowing us to see that torsion is a form of pressure and that the chiral angle can be interpreted like a type of temperature, the polar re-formulation of spinors allows for the relativistic quantum mechanics to convert into a specific hydrodynamics, whose variables may perfectly be visualized and, because of this, better understood.

The challenges of relativistic quantum mechanics have no resolution in a reformulation of the theory alone, and many questions still remain. Nonetheless, questions that can be answered more easily when made clearer will receive a boost by a Dirac spinor field theory formulated in terms of variables that are visualizable.

In its polar form, the Dirac theory is precisely this.

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Appendix A

In this appendix, we give a derivation of the polar form (18).

To begin, let us recall that we have assumed that $\Phi^2+\Theta^2\neq 0$ in general. In this case, the Fierz identity (10) tells us that $U_aU^a>0$ and thus U_a is time-like. It is a known result of relativity that in such a case it is always possible to perform up to three boosts to bring all three spatial components of such a vector to be equal to zero (conversely, one could always generate such components by boosting, and this tells us that these components are the spatial velocities, and so U_a is the velocity 4-vector).

Once these boosts have been performed, we are in the rest-frame. Because of the Fierz identity (11) we also have that $S^0=0$ in such a frame. Consequently, S_a has only spatial components. By employing only rotations around the first and second axes, we can always align the space part of S_a along, for example, the third axis. Namely, we are choosing the system of reference with respect to which $S^1=S^2=0$ too.

What this means is that in our case it is always possible to perform a series of Lorentz boosts and rotations for which the final result is that $U^1=U^2=U^3=S^1=S^2=0$ identically.

Let us now consider, in chiral representation, a general spinor field of the form

$$\psi = \begin{pmatrix} ce^{i\gamma} \\ de^{i\delta} \\ ae^{i\alpha} \\ be^{i\beta} \end{pmatrix} \tag{A1}$$

and impose $U^1 = U^2 = U^3 = 0$ on it. After some very straightforward manipulation we obtain

$$ab \cos(\beta - \alpha) = cd \cos(\delta - \gamma) \tag{A2}$$
$$ab \sin(\beta - \alpha) = cd \sin(\delta - \gamma) \tag{A3}$$
$$a^2 + d^2 = b^2 + c^2 \tag{A4}$$

which now have to be solved. The general solution of the above gives the rest-frame spinor field according to

$$\psi_{\mathrm{r}} = \begin{pmatrix} \pm \cos\omega e^{i\gamma} \\ \pm \sin\omega e^{i\delta} \\ \cos\omega e^{i\alpha} \\ \sin\omega e^{i\alpha} e^{i\delta} e^{-i\gamma} \end{pmatrix} \phi \tag{A5}$$

up to the sign of the ω function. We can now impose $S^1 = S^2 = 0$ and proceed as before. It is a matter of algebra to see that

$$\omega = n\frac{\pi}{2} \tag{A6}$$

so that after some re-naming of variables we obtain the rest-frame spin-eigenstate spinor as

$$\psi_{\mathrm{rs}} = \begin{pmatrix} \pm e^{\frac{i}{2}\zeta} \\ 0 \\ e^{-\frac{i}{2}\zeta} \\ 0 \end{pmatrix} e^{i\varphi}\phi \qquad \text{or} \qquad \psi_{\mathrm{rs}} = \begin{pmatrix} 0 \\ \pm e^{\frac{i}{2}\zeta} \\ 0 \\ e^{-\frac{i}{2}\zeta} \end{pmatrix} e^{i\varphi}\phi \tag{A7}$$

according to whether the axial-vector is aligned or anti-aligned with the third axis. Apart from the direction of the third axis, it is enough to consider only one of the above. And the sign can be re-absorbed through a shift of the ζ function. Therefore, it is enough to take

$$\psi_{\mathrm{rs}} = \begin{pmatrix} e^{\frac{i}{2}\zeta} \\ 0 \\ e^{-\frac{i}{2}\zeta} \\ 0 \end{pmatrix} e^{i\varphi}\phi \tag{A8}$$

as the final form. We can use now the rotation around the third axis to remove a global phase obtaining

$$\psi_{\mathrm{rs}} = \begin{pmatrix} e^{\frac{i}{2}\zeta} \\ 0 \\ e^{-\frac{i}{2}\zeta} \\ 0 \end{pmatrix} \phi \tag{A9}$$

or equivalently

$$\psi_{\text{rs}} = \phi e^{-\frac{i}{2}\zeta\pi} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{A10}$$

as rest-frame spin-eigenstate spinor. This form has been obtained by acting only with a series of Lorentz transformations, which we can collect into a single one called $\boldsymbol{L}$ and hence $\psi_{\text{rs}} = \boldsymbol{L}\psi$ where ψ_{rs} is the rest-frame spin-eigenstate spinor. By *undoing* the transformations that takes the general spinor to its rest-frame spin-eigenstate form we can write

$$\psi = \boldsymbol{L}^{-1}\psi_{\text{rs}} \tag{A11}$$

in general. Recalling that $[\boldsymbol{L}, \boldsymbol{\pi}] = 0$ we obtain (18).

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