Abstract: Computation of the solution of the nonlinear Caputo fractional differential equation is essential for using \( q \), which is the order of the derivative, as a parameter. The value of \( q \) can be determined to enhance the mathematical model in question using the data. The numerical methods available in the literature provide only the local existence of the solution. However, the interval of existence is known and guaranteed by the natural upper and lower solutions of the nonlinear differential equations. In this work, we develop monotone iterates, together with lower and upper solutions that converge uniformly, monotonically, and quadratically to the unique solution of the Caputo nonlinear fractional differential equation over its entire interval of existence. The nonlinear function is assumed to be the sum of convex and concave functions. The method is referred to as the generalized quasilinearization method. We provide a Caputo fractional logistic equation as an example whose interval of existence is \([0, \infty)\).

Keywords: Caputo fractional derivative; existence theorem; computational method; interval of existence; generalized quasilinearization method

MSC: 34A08; 34A12

1. Introduction

Qualitative study, such as investigating the existence, uniqueness, and stability of nonlinear differential equations with an initial value problem, plays an important role in the study of many branches of science and engineering. Furthermore, the computation of the solution to a nonlinear problem on its interval of existence is useful in enhancing mathematical models to align with the available data. However, computation of the explicit solution is rarely possible, even for integer-order nonlinear differential equations. This is even more challenging for Caputo fractional differential equations with initial conditions. One of the main reasons is that the product rule, the chain rule, separation of variable method, and the variation of parameter method, which are generally used in solving integer-order differential equations, are not available for Caputo fractional differential equations. The purpose of solving or finding the solution of the Caputo fractional differential equation is that the order of the derivative \( q \) can be used as a parameter to improve mathematical models, especially data-driven models. See [1,2], where the order of the derivative plays an important role. See also [1–20] for some applications of fractional differential equations. See also [3,8,15,21–27] for analysis and some more applications. Explicit computation of solutions of linear scalar and systems of Caputo fractional differential equations with constant coefficients and with initial conditions can be obtained. See [15,18,23,28] for the explicit solutions of the linear scalar and systems of Caputo fractional differential equations. The solutions of linear scalar and systems of Caputo fractional differential equations with initial conditions are in general in terms of Mittag–Leffler functions. See [15,18,23,26,29,30] for more details on the Mittag–Leffler functions. The majority of the numerical methods
available provide only the local existence of the solution and do not provide the procedure to compute over its interval of existence. See [31–33] for some of the numerical work on Caputo fractional differential equations. In [32], Picard’s iterative method for Caputo fractional differential equations was developed. It is to be noted that for the example 
\[ \mathcal{D}_0^q u = u^2, \quad u(0) = 1, \quad 0 < q < 1, \]
starting with \( u_0(t) = 1 \), each of the iterates exists for all time. It is easy to show (see [34]) that the solution of \( u' = u^2, \quad u(0) = 1 \), is a lower solution of the corresponding Caputo fractional differential equation. Consequently, the solution of the Caputo fractional differential equation does not exist for all time, even though the Picard iterates exist for all time. Fortunately, the natural upper and lower solutions guarantee the interval of existence. If the equilibrium solutions are the lower and upper solutions of the nonlinear Caputo fractional differential equation, then the solution of the nonlinear Caputo fractional differential equation exists for all time. Starting with the lower and upper solutions, we can construct increasing and decreasing sequences that converge to the maximal and minimal solutions by a monotone method when the nonlinear function is increasing or can be made increasing. To accommodate cases where the nonlinear function is the sum of increasing and decreasing sequences, the generalized monotone method was developed. All of these methods require comparison results relative to the lower and upper solutions. See [35–38] for some of these results. Although the generalized monotone method provides monotone iterates using natural lower and upper solutions with a restriction, the iterates are solutions of linear equations and they converge to the unique solution if the nonlinear problem has one. However, the restriction prevents the iterates from staying within the lower and upper solutions for all time on the interval of existence. See [39,40] for a numerical example that demonstrates the inadequacy of the generalized monotone method. In addition, the rate of convergence is linear. The method of quasilinearization yields monotone iterates that are solutions of linear differential equations, which converge quadratically to the unique solution of the nonlinear problem on its interval of existence. The method works if the nonlinear function is either a convex or concave function. Additionally, the interval of existence is not known a priori. See [41–43] for work on the method of quasilinearization. The method of generalization of quasilinearization is a more suitable method when the nonlinear function is the sum of an increasing and a decreasing function. In addition, the method uses both the lower and upper solutions of the nonlinear problem to develop the monotone iterates. See [44] for work on the generalized quasilinearization method for ordinary differential equations. In this work, we developed the generalized quasilinearization method for the scalar Caputo fractional differential equation with initial conditions. This was achieved by using upper and lower solutions and the fact that the nonlinear function is Lipschitzian on the sector defined by the upper and lower solutions. We used the scalar version of the existence in the large (which means existence of the solution on its interval of existence) and uniqueness has been proved using Picard’s method, which was developed for multi-order Caputo fractional differential systems in [30]. Even though the iterates are solutions of the linear equation, the order of iterates is not clear. Existence in the large is useful in proving that the iterates we develop in generalized quasilinearizations, which are solutions of the linear Caputo fractional differential equations with variable coefficients, exist and are unique on the entire interval of existence. We construct two sequences, which are monotonically increasing and decreasing sequences, starting with the initial approximation as the lower and upper solutions, respectively. These sequences converge uniformly, monotonically, and quadratically to the unique solution of the nonlinear Caputo differential equation with initial conditions. Each nth element of the sequences sandwiches the solution of the nonlinear Caputo differential equation with initial conditions on its entire interval of existence. Thus, the method developed in this work is both a theoretical and computational method to obtain the solution of the nonlinear Caputo fractional differential equation with initial conditions on its entire interval of existence. The novelty of this work is in computing the solution of the nonlinear Caputo fractional differential equation with initial conditions.
by a computational method on the entire interval of existence of the solution. The interval of existence is guaranteed by the upper and lower solutions.

We have structured our research article as follows: In the introduction, we have explained the reason, purpose, and methods adopted in our work. In the preliminary section, we have provided definitions, known comparison results, and known theoretical existence results that are needed for our main results. For the main result section, we developed a generalized quasilinearization method for the Caputo fractional differential equation with initial conditions when the nonlinear function is the sum of convex and concave functions. This method is both theoretical and computational. In our method, we have used natural lower and upper solutions of the nonlinear Caputo fractional differential equation of order $q$, where $0.5 < q < 1$. At the end of our main result section, we have presented an example of the Caputo fractional logistic equation for two different values of $q$ to demonstrate the application of our computational and theoretical result, namely the generalized quasilinearization method. Finally, at the end, we have included the concluding remarks of our result and open problems.

2. Preliminary Results

In this section, we recall some definitions and known results that play an important role in our main result.

**Definition 1.** The Riemann–Liouville fractional integral of order $q$ is defined by

$$D_{0+}^{-q}u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}u(s)ds,$$  \hfill (1)

where $0 < q \leq 1$ and $\Gamma(q)$ is the Gamma function.

**Definition 2.** The Riemann–Liouville (left-sided) fractional derivative of $u(t)$ of order $q$, when $0 < q < 1$, is defined as

$$D_{0+}^q u(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q}u(s)ds, \quad t > 0.$$  \hfill (2)

The Caputo integral of order $q$ for any function is the same as that of the Riemann–Liouville integral of order $q$.

**Definition 3.** The Caputo (left-sided) fractional derivative of $u(t)$ of order $nq$, $n-1 \leq nq < n$, is given by the following equation:

$$^{c}D_{0+}^{nq} u(t) = \frac{1}{\Gamma(n-nq)} \int_0^t (t-s)^{n-nq-1}u^{(n)}(s)ds, \quad t \in [0,\infty), \quad t > 0,$$  \hfill (3)

where $u^{(n)}(t) = \frac{d^n u}{dt^n}$.

In particular, if $q$ is an integer, then both the Caputo derivative and integer derivative are one and the same.

See [15,18,29] for more details on Caputo and Riemann–Liouville fractional derivatives.

**Definition 4.** The Caputo (left) fractional derivative of $u(t)$ of order $q$, when $0 < q < 1$, is defined as

$$^{c}D_{0+}^q u(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q}u'(s)ds.$$  \hfill (4)

We just replace $n$ by 1 in definition (3).

The next definition is useful in our basic Caputo fractional differential inequalities.
The two-parameter Mittag–Leffler function is defined as

\[ E_{q,r}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk+r)}, \]

(5)

where \( q, r > 0 \) and \( \lambda \) are constants. Furthermore, for \( r = q \), (5) reduces to

\[ E_{q,q}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk+q)}. \]

(6)

If \( r = 1 \) in (5), then

\[ E_{q,1}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk+1)}. \]

(7)

If \( q = 1 \) and \( r = 1 \) in (5), then we have

\[ E_{1,1}(\lambda t) = \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(k+1)} = e^{\lambda t}, \]

(8)

where \( e^{\lambda t} \) is the usual exponential function.

For more details on Mittag–Leffler functions and their applications, see [15,18,23,29,46]. Consider the \( q \)-th order Caputo fractional differential equation of the form

\[ ^cD^q u = F(t,u) = f(t,u) + g(t,u), \quad u(0) = u_0, \]

(9)

with \( t \in J; u \in C^q(J) \); and \( f,g \in C(J \times \mathbb{R}, \mathbb{R}) \), which is the space of continuous functions from \( J \times \mathbb{R} \) to \( \mathbb{R} \).

As seen in [15,18,23], if \( u \in C^q(J) \), then (9) is equivalent to the fractional integral equation

\[ u(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{(q-1)}F(s,u(s))ds. \]

(10)

In particular, if \( F(t,u) = \lambda u + h(t) \), then the solution \( u(t) \) is given by

\[ u(t) = u_0 E_{q,1}(\lambda t^q) + \int_0^t (t-s)^{(q-1)}E_{q,q}(\lambda (t-s)^q)h(s)ds. \]

(11)

Note that this solution is only a \( C^q \) solution on the closed interval \([0,T]\).

Furthermore, if we consider the Caputo fractional differential inequality

\[ ^cD^q u \leq \lambda u + h(t), \quad u(0) = u_0, \]

(12)
with \( t \in J \) and \( u \in C^q(J) \), then (11) yields the corresponding integral inequality,

\[
    u(t) \leq u_0E_{q,1}(\lambda t^q) + \int_0^t (t-s)^{(q-1)}E_{q,\eta}(\lambda (t-s)^q)h(s)ds. \tag{13}
\]

Note that in the above inequality, if \( h(t) \) is a constant, say \( k \) and \( \lambda \neq 0 \), then (13) simplifies to

\[
    u(t) \leq u_0E_{q,1}(\lambda t^q) + \frac{k}{\lambda}[E_{q,1}(\lambda t^q) - 1]. \tag{14}
\]

The inequality (14) is useful in our main result to prove the quadratic convergence of the linear iterates.

Next, we provide the definitions of the lower and upper solutions of (9).

**Definition 7.** The functions \( v_0, w_0 \in C^q(J, \mathbb{R}) \) are called the natural upper and lower solutions of (9) if

\[
\begin{align*}
    &\mathcal{D}^q_{0+}v_0 \leq f(t, v_0) + g(t, v_0), \quad v_0(0) \leq u_0, \\
    &\mathcal{D}^q_{0+}w_0 \geq f(t, w_0) + g(t, w_0), \quad w_0(0) \geq u_0, \quad t \in J.
\end{align*}
\]

**Definition 8.** The functions \( v_0, w_0 \in C^q(J, \mathbb{R}) \) are called the coupled upper and lower solutions of (9) type I if

\[
\begin{align*}
    &\mathcal{D}^q_{0+}v_0 \leq f(t, v_0) + g(t, w_0), \quad v_0(0) \leq u_0, \\
    &\mathcal{D}^q_{0+}w_0 \geq f(t, w_0) + g(t, v_0), \quad w_0(0) \geq u_0, \quad t \in J.
\end{align*}
\]

In order to bring in a relation between the lower and upper solution of the Caputo fractional differential equations, we need to obtain a basic fractional differential inequality result related to the Riemann–Liouville derivative. For this purpose, we need the following definition of a \( C_\rho \) continuous function.

**Definition 9.** We say \( m(t) \in C_\rho([t_0, T], \mathbb{R}), 1 - q = p, \) is a \( C_\rho \) continuous function on \([t_0, T]\) if \( m(t) \) is not continuous on \([t_0, T]\), but \((t-t_0)^p m(t) = u(t) \in C([t_0, T], \mathbb{R})\).

**Lemma 1.** Let \( m \in C_\rho([0, T], \mathbb{R}) \). Then, for any \( t_1 \in (0, T] \), we have

\[
    m(t_1) = 0, \text{ and } m(t) \leq 0, \text{ for } 0 \leq t \leq t_1.
\]

Then, it follows that \( D^q m(t_1) \geq 0 \).

Note that the above lemma was proved as Lemma 2.3.1 in [23] with an extra assumption, that the function \( m(t) \) is Hölder continuous of order \( \lambda > q \). The proof without the Hölder continuity assumption is presented in [47] as Lemma 2.7. The Hölder continuity assumption is not applicable in iterative methods. Hence, our lemma without the Hölder continuity assumption is more useful.

Note that the above lemma can also be easily extended to the Caputo derivative by simply using the relationship between the Caputo derivative and the Riemann–Liouville derivative, namely,

\[
    \mathcal{D}^q_{0+}m(t) = D^q(m(t) - m(0)).
\]

The result can be stated as follows.

**Lemma 2.** Let \( m \in C^q([0, T], \mathbb{R}) \). Then, for any \( t_1 \in (0, T] \), we have

\[
    m(t_1) = 0, \text{ and } m(t) \leq 0, \text{ for } 0 \leq t \leq t_1.
\]
Then, it follows that,

\[ cD_t^\alpha m(t_1) \geq 0. \]

This is useful in proving the relationship between the lower and upper solutions of the nonlinear scalar Caputo fractional differential Equation (9) with an initial condition.

**Theorem 1.** Let \( v_0, w_0 \in C^1(J, R) \) be the natural lower and upper solutions of (9), respectively. Suppose that \( F(t, x) - F(t, y) \leq L(x - y) \) whenever \( x \geq y \), and \( L > 0 \) is a constant. Then, \( v_0(0) \leq w_0(0) \) implies that \( v_0(t) \leq w_0(t), t \in J \).

The proof follows along the same lines as in the integer case. Assuming one of the inequalities to be strict, the conclusion should be true with a strict inequality. If the conclusion with a strict inequality is not true, then by using Lemma 2, we can obtain a contradiction. Using the Mittag–Leffler function \( E_{\eta,1}(2\lambda^q t^q) \) in place of \( e^{2t^q} \), in the proof of the integer case, we can construct a new function \( v_{0,e}(t) = v_0 - \epsilon E_{\eta,1}(2\lambda^q t^q) \). Using the one-sided Lipschitz condition of \( v \), we can obtain the strict Caputo fractional differential inequality with respect to the newly constructed function \( v_{0,e}(t) \). The conclusion \( v_{0,e}(t) < w_0(t) \) on \( J \) can be obtained using the strict inequality result. The conclusion of the theorem follows by taking the limit as \( \epsilon \) tends to 0.

In the next result, we prove that if the solution \( u(t) \) of (9) exists, the solution is bounded under suitable conditions. For this purpose, we define a sector defined by the upper and lower solutions of (9).

Let \( v_0, w_0 \in C^1(J, R) \) with \( v_0(t) \leq w_0(t), t \in J \). We define the following:

\[ \Omega = \{(t, u) : v_0(t) \leq u \leq w_0(t), \quad t \in J = [0, T] \}. \]

In our main result, we make an assumption that \( f(t, u) \) is convex in \( u \) and \( g(t, u) \) is concave in \( u \), on the sector \( \Omega \). However, we need

\[ F(t, u) = f(t, u) + g(t, u) \]

to be Lipschitzian on the sector \( \Omega \) for our next result.

**Theorem 2.** Assume the following:

1. \( v_0, w_0 \in C^1(J, R) \) with \( v_0(t) \leq w_0(t) \) on \( J \), and

\[ cD_t^\alpha v_0 \leq f(t, v_0) + g(t, v_0), \quad v_0(0) \leq u_0, \]

\[ cD_t^\alpha w_0 \geq f(t, w_0) + g(t, w_0), \quad w_0(0) \geq u_0, \quad t \in J. \]

2. \( f, g \in C[\Omega, R], f_u, g_u \) exists and is continuous for \( (t, u) \) in \( \Omega \).

Then, there exists a solution \( u(t) \) of (9) such that \( v_0(t) \leq u(t) \leq w_0(t) \) on \( J \) provided that \( v_0(0) \leq u(0) = u_0 \leq w_0(0) \). Furthermore, the solution is unique if it exists.

**Proof.** From the hypotheses, it is clear that \( F(t, u) = f(t, u) + g(t, u) \) satisfies the Lipschitz condition:

\[ F(t, u_1) - F(t, u_2) \leq L(u_1 - u_2), \quad \text{for } u_1 \geq u_2, \quad L > 0. \]

Using Theorem 1 and letting the solution \( u(t) \) of (9) be the upper solution, we can prove that \( v_0(t) \leq u(t) \) on \( J \) since \( v_0(0) \leq u(0) \). Similarly, assuming the solution \( u(t) \) of (9) to be the lower solution and \( w_0(t) \) to be upper solution, we can prove that the solution of \( u(t) \leq w_0(t) \) on \( J \). Combining the two results, we obtain

\[ v_0(t) \leq u(t) \leq w_0(t) \]
on \( J \), provided that \( v_0(0) \leq u(0) = u_0 \leq w_0(0) \).
Since \(v_0(t)\) and \(w_0(t)\) are continuous functions on \([0, T]\), we obtain

\[
|u(t)| = |u(t) - v_0(t) + v_0(t)| \leq |u(t) - v_0(t)| + |v_0(t)| \leq |w_0(t) - v_0(t)| + |v_0(t)| \leq M,
\]

for some constant \(M\). This proves that the solution \(|u(t)| < M\). In order to prove the uniqueness of the solution of (9), assume that there are two solutions, say \(u_1\) and \(u_2\). Then, letting \(u_1\) be the lower solution and \(u_2\) be the upper solution, we obtain

\[
u_1(t) \leq u_2(t),
\]
on \(J\) since \(u_1(0) - u_2(0) = 0\). By a similar argument, by considering \(u_2(t)\) as the upper solution, we obtain

\[
u_2(t) \leq u_1(t),
\]
on \(J\). It follows that \(u_2(t) \equiv u_1(t) = u(t)\), which is the unique solution of (9) on \(J\).

**Remark 1.** Since \(f, g \in C(\Omega, \mathbb{R})\), this implies that \(F(t, u)\) is bounded. Using the local existence Theorem 2 of [32], we can prove that the solution of (9) exists and is unique on some interval \([0, t_1]\).

The next result is related to existence in the large for the unique solution of the scalar Caputo fractional differential equation. See [48]'s result for first-order ordinary differential equations. This result was extended to multi-order Caputo fractional differential equation. See Theorem 2 of [30]. Here, we recall the scalar version of this result.

**Theorem 3.** Let \(F(t, u)\) of Equation (9) be continuous and Lipschitzian with Lipschitz constant \(L\) on the strip defined by

\[
S = \{(t, u) : t \in J, \text{and } |u| < \infty\}.
\]

Then, the successive approximation defined by

\[
u_{n+1}(t) = \nu_0 + \frac{1}{\Gamma(q)} \int_0^t \frac{1}{(t-s)^{q-1}} F(s, \nu_n(s)) ds,
\]

exists for \(t \in J\) and converges uniformly to the unique solution of (9).

The above result can be stated in the following way.

**Theorem 4.** Let \(F(t, u)\) of equation (9) be continuous and Lipschitzian with Lipschitz constant \(L\) on the strip defined by

\[
S = \{(t, u) : t \in J, \text{and } |u| < \infty\}.
\]

Then, the solution of the Caputo nonlinear fractional differential Equation (9) exists and is unique on the strip \(S\).

Now, combining Theorems 2 and 4, we can obtain the next result, which plays a crucial role in our main result.

**Theorem 5.** Assume that

1. \(v_0, w_0 \in C^q(J, \mathbb{R})\) with \(v_0(t) \leq w_0(t)\) on \(J\), and
   \[
   ^cD_t^q v_0 \leq f(t, v_0) + g(t, v_0), \quad v_0(0) \leq u_0,
   \]
   \[
   ^cD_t^q w_0 \geq f(t, w_0) + g(t, w_0), \quad w_0(0) \geq u_0, \quad t \in J.
   \]
2. \(f, g \in C(\Omega, \mathbb{R})\), \(f_u\) and \(g_u\) exist and are continuous for \((t, u) \in \Omega\).

Then, there exists a solution \(u(t)\) of (9) such that \(v_0(t) \leq u(t) \leq w_0(t)\) on \(J\) provided that \(v_0(0) \leq u(0) = w_0(0) \leq w_0(0)\).

Furthermore, the solution of the Caputo nonlinear fractional differential Equation (9) exists and is unique on \(J\).
Proof. We can see from Theorem 2 that any solution of (9) is bounded by the upper and lower solutions $v_0(t)$ and $w_0(t)$. Using Theorem 4, there exists a unique solution of (9) on $J$ such that $v_0(t) \leq u(t) \leq w_0(t)$ on $J$ when $v_0(0) \leq u(0) = u_0 \leq w_0(0)$. □

Remark 2. Although Theorem 3 uses a computational method, namely, Picard’s method, the relationships between the successive iterates are not known. Furthermore, the rate of convergence of the iterates by Picard’s method is linear. In addition, the method does not take advantage of the lower and upper solutions except for the fact that the solution is bounded. This result is referred to as existence in the large, which proves that the solution exists on the interval when we know a priori the solution is bounded. This is the main motivation for our main result.

3. Main Results

Computing the solution of a nonlinear differential equation, whether it is of integer order or Caputo fractional order, is rarely possible. Most of the numerical methods available in the literature yield the local numerical solution. In addition, the Caputo fractional differential equation has an additional disadvantage due to the lack of a product rule and the separation of variables. However, natural lower and upper solutions guarantee the interval of existence of the solution. In particular, if the lower and upper solutions are equilibrium solutions, then it guarantees the existence of a global solution. In this work, we develop a theoretical and computational method to compute the solution over the entire interval of its existence. We develop the generalized quasilinearization method using natural lower and upper solutions for scalar nonlinear Caputo fractional differential equations. Specifically, we develop the generalized quasilinearization method for Caputo fractional differential equations with initial conditions when the nonlinear function is the sum of convex and concave functions. We develop iterates that are solutions of linear Caputo fractional differential equations with variable coefficients and initial conditions. These iterates start with lower and upper solutions, and each iterate depends on both the lower and upper solutions. The sequences $\{v_n\}$ and $\{w_n\}$ starting with $v_0$ and $w_0$, which are the lower and upper solutions, are monotonically increasing and decreasing sequences on the entire interval of existence. The sequences converge uniformly, monotonically, and quadratically to the unique solution of the nonlinear Caputo fractional nonlinear differential equation. It is to be noted that $v_n(t)$ and $w_n(t)$ are solutions of the linear Caputo differential equation with variable coefficients and the same initial condition of the nonlinear problem. The existence and uniqueness of these linear iterates on the entire interval of the existence of the nonlinear problem can be proved using Theorem 5. In addition, the computation of both $v_n(t)$ and $w_n(t)$ depends on the previous iterates $v_{n-1}(t)$ and $w_{n-1}(t)$.

This method is known as the generalized quasilinearization method. See [44]’s result for nonlinear ordinary differential equations.

For this purpose, consider the first-order differential equation of the form

$$\frac{d^q}{dt^q} u(t) = f(t, u), \quad u(0) = u_0, \text{ on } [0, T] = J,$$

(15)

where $f, g \in C(J \times \mathbb{R}, \mathbb{R})$, which is the space of continuous functions from $J \times \mathbb{R}$ to $\mathbb{R}$.

Let $v_0, w_0 \in C^q[J, \mathbb{R}]$, with $v_0(t) \leq w_0(t)$, on $J$ and define

$$\Omega = \{(t, u) : v_0(t) \leq u \leq w_0(t), t \in J = [0, T]\}.$$

Theorem 6. Assume that

1. $v_0, w_0 \in C^q[J, \mathbb{R}]$, $v_0(t) \leq w_0(t)$, on $J$, where $v_0(t)$ and $w_0(t)$ are the natural lower and upper solutions of (15). That is,

$$\frac{d^q}{dt^q} v_0 \leq f(t, v_0), \quad v_0(0) \leq u_0,$$

$$\frac{d^q}{dt^q} w_0 \geq f(t, w_0), \quad w_0(0) \geq u_0, \quad t \in J = [0, T].$$
2. $f, g \in C[\Omega, \mathbb{R}], f_{uu}, g_{uu}, f_{nu}, g_{nu}$, and $g_{uu}$ exist, are continuous, and satisfy

$$f_{uu}(t, u) \geq 0, \quad g_{uu}(t, u) \leq 0, \quad \text{for } (t, u) \in \Omega.$$ 

Then, there exist monotone sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ that converge uniformly to the unique solution of (15), and the convergence is quadratic.

**Proof.** Observe that the assumptions of Theorem 6 satisfy all the assumptions of Theorem 5 relative to Equation (15). As such, it is clear that the solution of the Caputo nonlinear fractional differential Equation (15) with initial conditions exists and is unique on $J$.

Now, using the assumptions $f_{uu}(t, u) \geq 0$ and $g_{uu}(t, u) \leq 0$ yields the following inequalities:

$$f(t, u) \geq f(t, v) + f_u(t, v)(u - v), \quad (16)$$

$$g(t, u) \geq g(t, v) + g_u(t, u)(u - v), \quad (17)$$

for $u \geq v$, respectively.

In addition, $f$ and $g$ satisfies

$$-L(u_1 - u_2) \leq f(t, u_1) - f(t, u_2) \leq L(u_1 - u_2),$$

and

$$-L(u_1 - u_2) \leq g(t, u_1) - g(t, u_2) \leq L(u_1 - u_2),$$

for any $v_0(t) \leq u_2 \leq u_1 \leq w_0(t)$ for some $L > 0$.

Observe that the inequalities (16) and (17) are used throughout our proof when constructing the monotone sequences $\{v_n(t)\}$ and $\{w_n(t)\}$.

Consider the following IVPs (initial value problems):

$$cD_0^q v(t) = F(t, v_0, w_0; v),$$

$$= f(t, v_0) + f_u(t, v_0)(v - v_0) + g(t, v_0) + g_u(t, w_0)(v - v_0), \quad (18)$$

$$v(0) = u_0, \quad \text{on } [0, T] = J,$$

and

$$cD_0^q w(t) = G(t, v_0, w_0; w),$$

$$= f(t, w_0) + f_u(t, w_0)(w - w_0) + g(t, w_0) + g_u(t, w_0)(w - w_0), \quad (19)$$

$$w(0) = u_0, \quad \text{on } [0, T] = J,$$

where $v_0 \leq u(0) = u_0 \leq w_0$.

We show that the solutions $v$ and $w$ of (18) and (19) exist and are unique on $J$. For this purpose, we show that $v_0(t)$ and $w_0(t)$ are the lower and upper solutions of the linear Caputo fractional differential Equations (18) and (19) on $J$ using the inequalities (16) and (17):

$$cD_0^q v_0(t) \leq f(t, v_0) + g(t, v_0) \equiv F(t, v_0, w_0; v_0),$$

$$cD_0^q w_0(t) \geq f(t, w_0) + g(t, w_0),$$

$$\geq f(t, v_0) + f_u(t, v_0)(v_0 - v_0)$$

$$+ g(t, v_0) + g_u(t, w_0)(w_0 - v_0),$$

$$\equiv F(t, v_0, w_0; v_0),$$

$$v_0(0) \leq u_0, \quad \text{on } [0, T] = J.$$
Since Equation (18) is linear, it satisfies all the hypotheses of Theorem 5. Hence, there exists a unique solution of the linear Caputo fractional differential equation (18), say \( v_1(t) \) with \( v_1(0) = u_0 \) such that \( v_0(t) \leq v_1(t) \leq \bar{w}_0(t) \) on \( J \).

Similarly, we obtain

\[
\begin{align*}
^cD^\alpha_0 v_0(t) & \leq f(t, v_0) + g(t, v_0), \\
& \leq f(t, w_0) + f_u(t)(v_0 - w_0) + g(t, w_0) + g_u(t)(v_0 - w_0), \\
& \equiv G(t, v_0, w_0; v_0), \\
^cD^\alpha_0 \bar{w}_0(t) & \geq f(t, w_0) + g(t, w_0), \\
& \equiv G(t, v_0, w_0; \bar{w}_0), \\
v_1(0) &= u_0, \text{ on } [0, T] = J.
\end{align*}
\] (21)

This proves that \( v_0(t) \) and \( \bar{w}_0(t) \) are the lower and upper solutions of the linear Caputo fractional differential equation (19). In addition, the Caputo fractional initial value problem (19) is linear in \( w(t) \). Using Theorem 5, the solution of the linear Caputo fractional differential equation (19) exists on \( J \) and it is unique. Labeling this solution as \( w_1(t) \), we obtain \( v_0(t) \leq w_1(t) \leq \bar{w}_0(t) \) on \( J \) such that \( w_1(0) = u_0 \).

Now, using the inequalities (16) and (17), we show that \( v_1(t) \) and \( w_1(t) \) are the lower and upper solutions of the nonlinear Caputo fractional differential equations (15) on \( J \). Using the fact that \( v_0(t) \leq v_1(t) \) and \( w_1(t) \leq \bar{w}_0(t) \), and using the inequalities (16) and (17), we obtain

\[
\begin{align*}
^cD^\alpha_0 v_1(t) &= F(t, v_0, \bar{w}_0; v_1), \\
& = f(t, v_0) + f_u(t)(v_1 - v_0) + g(t, v_0) + g_u(t)(v_1 - v_0), \\
& \leq f(t, v_1) + g(t, v_1), \\
v_1(0) &= u_0, \text{ on } [0, T] = J,
\end{align*}
\] (22)

and

\[
\begin{align*}
^cD^\alpha_0 w_1(t) &= G(t, v_0, \bar{w}_0; w_1), \\
& = f(t, w_0) + f_u(t)(w_1 - w_0) + g(t, w_0) + g_u(t)(w_1 - w_0), \\
& \geq f(t, w_1) + g(t, w_1), \\
w_1(0) &= u_0, \text{ on } [0, T] = J.
\end{align*}
\] (23)

This proves that \( v_1(t) \) and \( w_1(t) \) are natural lower and upper solutions of the nonlinear Caputo fractional differential equations (15). Using assumption 2 of Theorem 5, we obtain

\[
v_1(t) \leq u(t) \leq w_1(t),
\]
on \( J \).

This, together with \( v_0(t) \leq v_1(t) \) and \( w_1(t) \leq \bar{w}_0(t) \), allows us to obtain

\[
v_0(t) \leq v_1(t) \leq u(t) \leq w_1(t) \leq \bar{w}_0(t) \text{ on } J,
\] (24)

where \( u(t) \) is the unique solution of (15) on \( J \).

Now, we can continue the same process to compute \( v_{n+1}(t) \) and \( w_{n+1}(t) \) for \( n \geq 1 \). The next set of equations provide the set of Caputo fractional differential equations with initial conditions to compute \( v_{n+1}(t) \) and \( w_{n+1}(t) \) for any integer \( n = 1, 2, 3, \ldots \). For this purpose, consider the pair of following Caputo fractional differential equations with the initial conditions:
\[ cD^q_{0+} v_{n+1}(t) = F(t, v_n, w_n; v_{n+1}), \]
\[ = f(t, v_n) + f_u(t, v_n)(v_{n+1} - v_n) + g(t, v_n) + g_u(t, w_n)(v_{n+1} - v_n), \]
\[ v_{n+1}(0) = u_0, \text{ on } [0, T] = J, \tag{25} \]

and
\[ cD^q_{0+} w_{n+1}(t) = G(t, v_n, w_n; w_{n+1}), \]
\[ = f(t, w_n) + f_u(t, v_n)(w_{n+1} - w_n) + g(t, w_n) + g_u(t, w_n)(w_{n+1} - w_n), \]
\[ w_{n+1}(0) = u_0, \text{ on } [0, T] = J. \tag{26} \]

Consider the above Caputo fractional differential equations with initial conditions for \( n = 1 \). We proved that \( v_1(t) \) and \( w_1(t) \) are already the lower and upper solutions of the nonlinear Caputo fractional differential Equations (15). Using this and the inequalities (16) and (17), we obtain
\[ cD^q_{0+} v_1(t) \leq f(t, v_1) + g(t, v_1) \equiv F(t, v_1, w_1; v_2), \]
\[ cD^q_{0+} w_1(t) \geq f(t, w_1) + g(t, w_1), \]
\[ \geq f(t, v_1) + f_u(t, v_1)(v_1 - v_1) + g(t, v_1) + g_u(t, w_1)(v_1 - v_1), \]
\[ \equiv F(t, v_1, w_1; v_2), \]
\[ v_1(0) = u_0, \text{ on } [0, T] = J, \tag{27} \]

and
\[ cD^q_{0+} v_1(t) \leq f(t, v_1) + g(t, v_1), \]
\[ \leq f(t, w_1) + f_u(t, v_1)(v_1 - w_1) + g(t, v_1) + g_u(t, w_1)(v_1 - w_1), \]
\[ \equiv G(t, v_1, w_1; v_2), \]
\[ cD^q_{0+} w_1(t) \geq f(t, w_1) + g(t, w_1), \]
\[ \equiv G(t, v_1, w_1; w_2), \]
\[ w_1(0) = u_0, \text{ on } [0, T] = J. \tag{28} \]

The above inequalities prove that \( v_1(t) \) and \( w_1(t) \) are the lower and upper solutions for the linear Caputo fractional differential Equations (25) and (26) for \( n = 0 \), respectively. Now, using Theorem 5, the solutions \( v_2(t) \) and \( w_2(t) \) of (25) and (26) for \( n = 1 \) exist and are unique such that \( v_0(t) \leq v_1(t) \leq v_2(t) \leq w_1(t) \leq w_0(t) \) and \( v_0(t) \leq v_1(t) \leq w_2(t) \leq w_1(t) \leq w_0(t) \) on \( J \).

Using the inequalities (24), (16), and (17), we can obtain the following inequalities:
\[ cD^q_{0+} v_2(t) = F(t, v_1, w_1; v_2), \]
\[ = f(t, v_1) + f_u(t, v_1)(v_2 - v_1) + g(t, v_1) + g_u(t, v_1)(v_2 - v_1), \]
\[ v_2(0) = u_0, \text{ on } [0, T] = J, \tag{29} \]

and
\[ cD^q_{0+} w_2(t) = G(t, v_1, w_1; w_2), \]
\[ = f(t, v_1) + f_u(t, v_1)(w_2 - w_1) + g(t, v_1) + g_u(t, v_1)(w_2 - w_1), \]
\[ \geq f(t, w_2) + g(t, w_2), \]
\[ w_2(0) = u_0, \text{ on } [0, T] = J. \tag{30} \]
This proves that $v_2(t)$ and $w_2(t)$ are the lower and upper solutions of the nonlinear Caputo fractional differential Equation (15) with an initial condition on $J$.

From this, it follows that

$$v_0(t) \leq v_1(t) \leq v_2(t) \leq u(t) \leq w_2(t) \leq w_1(t) \leq w_0(t) \quad \text{on } J,$$

where $u(t)$ is the unique solution of (15) on $J$.

Continuing this process, we can prove that for any $n$

$$v_0(t) \leq v_1(t) \leq v_2(t) \cdots \leq v_n(t) \leq u(t) \leq w_2(t) \leq \cdots \leq w_1(t) \leq w_0(t),$$

on $J$, where $u(t)$ is the unique solution of (15) on $J$.

It is easy to show that the sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ are equi-continuous and uniformly bounded. Then, using standard arguments, one can show that the increasing and decreasing sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ converge uniformly and monotonically to the unique solution of the nonlinear Caputo fractional differential Equation (15) with initial conditions on $J$.

Finally, we show that the rate of convergence of the sequences is quadratic. For this purpose, we consider the function

$$p_{n+1}(t) = w_{n+1}(t) - v_{n+1}(t).$$

We can combine the mean value theorem on $f(t,u)$, $g(t,u)$, $f_u(t,u)$, and $g_u(t,u)$, and the fact that

1. $f_u(t,u)$ and $g_u(t,u)$ are increasing and decreasing functions of $u$, respectively.
2. $f_u, f_{uu}, g_u$, and $g_{uu}$ are bounded on the sector $\Omega$.

This allows us to obtain the following Caputo fractional differential inequality:

$$\begin{align*}
\mathcal{D}_0^\alpha p_{n+1}(t) &= \mathcal{D}_0^\alpha w_{n+1}(t) - \mathcal{D}_0^\alpha v_{n+1}(t), \\
&= \mathcal{G}(t, v_n, w_n; w_{n+1}) - \mathcal{G}(t, v_n, w_n; v_{n+1}), \\
&= f(t, w_n) + f_u(t, w_n)(w_{n+1} - w_n) + g(t, w_n) + g_u(t, w_n)(w_{n+1} - w_n) - f(t, v_n) - f_u(t, v_n)(v_{n+1} - v_n) - g(t, v_n) - g_u(t, v_n)(v_{n+1} - v_n), \\
&= f_u(t, \xi) p_n + g_u(t, \eta) p_n - f_u(t, v_n) p_n - g_u(t, v_n) p_n + [f_u(t, v_n) + g_u(t, v_n)] p_{n+1}, \\
&= [f_{uu}(t, \sigma) - g_{uu}(t, \delta)] p_n^2 + [f_u(t, v_n) + g_u(t, v_n)] p_{n+1}, \\
&\leq K p_n^2 + M p_{n+1}, \\
p_{n+1}(0) &= w_{n+1}(0) - v_{n+1}(0) = u_0 - u_0 = 0, \quad \text{on } [0,T] = J.
\end{align*}$$

We can rewrite the final form of the differential inequality as

$$\begin{align*}
\mathcal{D}_0^\alpha (p_{n+1}(t)) &\leq K p_n^2 + M p_{n+1}, \quad p_{n+1}(0) = 0, \quad \text{on } J.
\end{align*}$$

Using the differential inequality (34), we can obtain the following integral inequality:

$$\begin{align*}
p_{n+1}(t) &\leq p_{n+1}(0) E_{q,1}(M t^\alpha) + \int_0^t (t-s)^{(q-1)} E_{q,1}(M(t-s)^\alpha) K p_n^2(s) ds, \quad \text{on } J.
\end{align*}$$

From this, we can obtain

$$\max_{t \in J} p_{n+1}(t) \leq \frac{\max_{t \in J} p_n^2(t)}{K} [E_{q,1}(K T)^\alpha - 1].$$
This completes the proof. □

Next, we present an example of a Caputo fractional logistic equation.

**Example 1.** Consider the following nonlinear Caputo fractional differential equation

\[ cD^q_0u(t) = u - u^2, \quad u(0) = \frac{1}{2}, \quad (37) \]

for \( t \in [0, \infty) \).

Then, comparing with (15), we have \( f(t, u) = u \) and \( g(t, u) = -u^2 \).

It is easy to check that \( v_0(t) = \frac{1}{2} \) and \( w_0(t) = 1 \) are natural lower and upper solutions of (37). Then, using the iterative scheme (25) with \( n = 0 \), we obtain

\[ cD^q_0v_1(t) = \frac{3}{4} - v_1, \quad v_1(0) = \frac{1}{2} \]

and

\[ cD^q_0w_1(t) = 1 - w_1, \quad w_1(0) = \frac{1}{2} \]

for \( 0 < q < 1 \).

We compute \( v_1(t) \) and \( w_1(t) \) using the Laplace transform method. Taking the Laplace transform on both sides of the above equation, we obtain

\[ v_1(s) = \frac{s^{q-1}}{s^q + 1} u_0 + \frac{3}{4} \frac{1}{s(s^q + 1)} \]

and

\[ w_1(s) = \frac{s^{q-1}}{s^q + 1} u_0 + \frac{1}{s(s^q + 1)}. \]

Using \( u_0 = \frac{1}{2} \) and taking the inverse Laplace transform on both sides, we obtain

\[ v_1(t) = \frac{3}{4} - \frac{1}{4} E_{q,1}(-t^q), \]

and

\[ w_1(t) = 1 - \frac{1}{2} E_{q,1}(-t^q). \]

It is easy to see that

\[ v_1 - v_0 = \frac{1}{4} (1 - E_{q,1}(-t^q)) > 0 \] on \((0, T]\) for any \( T > 0 \).

Similarly, we obtain

\[ w_0 - w_1 = \frac{1}{2} E_{q,1}(-t^q) > 0 \] on \((0, T]\) for any \( T > 0 \).

This proves that

\[ v_0 \leq v_1 \leq w_1 \leq w_0 \] on \([0, T]\).

Since we cannot compute the solution of (37) explicitly, using our Theorem 6, we have

\[ v_0 \leq v_1 \leq u \leq w_1 \leq w_0 \] on \([0, T]\).

The Figures 1–3 illustrate that \( v_0 \leq v_1 \leq u \leq w_1 \leq w_0 \) for \( q = 0.8, 0.9, \) and 1. As \( q = 1 \), the solutions exactly match with integer order result as a special case. Note that the computation of \( v_2(t) \) and \( w_2(t) \) involves linear Caputo fractional differential equations.
with variable coefficients. There is no representation form for the solution of linear Caputo fractional differential equations with variable coefficients. Therefore, we plan to develop this result in our future work.

Here, we plot the figure of $v_0(t)$, $w_0(t)$, $v_1(t)$, and $w_1(t)$ using MATLAB for values of $q = 0.8, 0.9$, and $1$.

**Figure 1.** Graph of $v_0(t)$, $w_0(t)$, $v_1(t)$, and $w_1(t)$ for $q = 0.8$.

**Figure 2.** Graph of $v_0(t)$, $w_0(t)$, $v_1(t)$, and $w_1(t)$ for $q = 0.9$. 
Figure 3. Graph of $v_0(t)$, $w_0(t)$, $v_1(t)$, and $w_1(t)$ for $q = 1$.

More generally, we can see that the unique solution $u(t)$ is sandwiched between $v_n(t)$ and $w_n(t)$ on $[0, T]$ for any $T > 0$. In short, we can obtain the numerical solution of the nonlinear Caputo differential equation on its entire interval of existence to the accuracy we need.

4. Concluding Remarks

In this work, we develop the generalized quasilinearization method for the nonlinear Caputo fractional differential equation with initial conditions. The nonlinear function is assumed to be the sum of a convex function and a concave function. For this, in the preliminary phase, we include some results that are needed to develop our main result. One such result is that there exists a unique solution to the nonlinear problem if the initial value problem has upper and lower solutions and the nonlinear function is Lipschitzian in $u$. Additionally, the interval of existence is guaranteed on the entire interval of the common interval of the existence of the lower and upper solutions. In particular, if the lower and upper solutions are equilibrium solutions, then the global solution of the nonlinear problems exists. We develop monotone iterates that are solutions of linear Caputo fractional differential equations with variable coefficients. The solution of these iterates also exists and is unique for the entire interval of existence of the corresponding nonlinear problem. Although the explicit form of the solution of the iterates is not as simple as the corresponding integer problem, it can be computed numerically. Additionally, the iterates starting from the lower solution form an increasing sequence, and the iterates starting from the upper solution form a decreasing sequence. Although the lower and upper iterates depend on each other, they are independently scalar and linear equations. The sequences converge quadratically to the unique solution of the nonlinear problem. Since the solution of the nonlinear problem is sandwiched between the lower and upper iterates, we can compute sufficient iterates to minimize the error with the actual solution. The main advantage of this method is that the solution of the nonlinear problem is computed on its interval of existence rather than a small interval, as in most numerical methods. In fact, if we have the lower and upper solutions as equilibrium solutions, then the solution can be computed for any time $T > 0$ needed. If the data are available, we can choose the value of $q$ as a parameter to suit the model. Our result is useful in solving Ricatti-type nonlinear
equations. The open problems are whether our result can be extended to nonlinear systems of differential equations. This will be useful in obtaining solutions of nonlinear biological models, such as SIR and SIRS models.

We have developed numerical results for two different values of \( q \). However, we are yet to develop a numerical scheme to solve the linear Caputo fractional differential equation with variable coefficients, which we plan to do in our future work.

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