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Fractional Definite Integral

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Received: 14 June 2017; Accepted: 30 June 2017; Published: 2 July 2017

Abstract: This paper proposes the definition of fractional definite integral and analyses the corresponding fundamental theorem of fractional calculus. In this context, we studied the relevant properties of the fractional derivatives that lead to such a definition. Finally, integrals on $\mathbb{R}^2$ and $\mathbb{R}^3$ are also proposed.

Keywords: fractional integral; fractional derivative; definite fractional integral

1. Introduction

Fractional Calculus has evolved considerably during the last 30 years and has become popular in many scientific and technical areas [1–4]. The progress in applications vis-a-vis theoretical developments motivated the re-evaluation of past formulations. The concepts of fractional derivative (FD) and fractional integral (FI) assume various forms not always equivalent and also not compatible with each other. Notwithstanding this development a singular situation exists, since there is no definition of "fractional definite integral" [5,6]. Even more strangely, it seems that no-one has the goal to propose such a adefinition [7,8]. In fact, the term FI is used for an indefinite integral, which, as we will see, stands for a primitivation operation.

A review of the literature reveals two indirect attempts to formulate the "fractional generalization of the fundamental theorem of calculus" [9,10]. However, in [9] several formulations of the theorem were described, one for each derivative; the approach followed in [10] was more directed into application but also did not address the definite integral definition. None of those formulations introduced the notion of a definite integral that will be the topic of this paper. Starting from a revision of classic results, an approach based on a generalisation of the Barrow formula is used to propose the "definite fractional integral" and, from it, to formulate the "fractional fundamental theorem of calculus". These developments allowed the definition of double and triple integrals on rectangular spaces.

The paper is organised as follows. In Section 2, we introduce the "definite fractional integral" and the corresponding "fractional fundamental theorem of calculus". For this purpose we study the admissibility of fractional derivatives and we find those that are suitable for achieving the proposed integral. In Section 3 the definite integrals on $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$ are presented. Finally, in Section 4 the conclusions are drawn.

2. The Definite Fractional Integrals

2.1. On the One-Sided Integer Order Derivatives and Their Inverses

Let f be a function defined on a closed interval $[c, d] \in \mathbb{R}$ where f is continuous. Then, f is said to have a lefthand derivative at any point of the closed interval if

$$f'_l(x) = D_l f(x) = \lim_{h \to 0^+} \frac{f(x) - f(x-h)}{h} \tag{1}$$

exists as a finite value or $\pm\infty$. The righthand derivative is defined similarly by the expression

$$f'_r(x) = D_r f(x) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}. \tag{2}$$

If both derivatives exist at $x = x_0$ and are equal, then we say that f is differentiable at that point. In the following we work preferably with the left derivative and will omit the subscript "l" unless it is needed to clarify the formulae.

Proceeding as in [11] it is possible to obtain an operator, J, that is the inverse of the derivative:

$$Jf(x) = \lim_{h \to 0^+} h \sum_{n=0}^{\lfloor (x-c)/h \rfloor} f(x - nh), \tag{3}$$

where $\lfloor (x-c)/h \rfloor$ is the integer part of $(x-c)/h$. The summation in (3) is the so-called Riemann sum. Using (1) in (3) and vice-versa it is straightforward to show that

$$JDf(x) = DJf(x). \tag{4}$$

This means that J is simultaneously the left and right inverse of D.

As the derivative of a constant is zero, there are many right inverses, P, of the derivative having the form $Pf(x) = D^{-1}f(x) + C,\ C \in \mathbb{R}$. In fact, $DPf(x) = f(x)$, but $PDf(x) = C$. These inverses are called primitives of $f(x)$. The particular primitive $P = J = D^{-1}$ can be called "proper primitive" [12]. Here, it will be named anti-derivative and when needed a subscript will be inserted to indicate that it is a left operator D_l.

On the other hand, if we represent J by D^{-1}, the Formulaes (2) and (3) can be joined in one:

$$f^{(\pm 1)}(x) = D^{\pm 1} f(x) = \lim_{h \to 0} \frac{\sum\limits_{n=0}^{\lfloor (x-c)/h \rfloor} (-1)^n \frac{(\mp)_n}{n!} f(x - nh)}{h^{\pm 1}}, \tag{5}$$

where $(a)_n = a(a+1)(a+2)\cdots(a+n-1), n \in \mathbb{N}$, with $(a)_0 = 1$, is the Pochhammer symbol for the rising factorial. In a similar way, the right hand derivative and inverse can be formulated as

$$f_r^{(\pm 1)}(x) = D_r^{\pm 1} f(x) = -\lim_{h \to 0} \frac{\sum\limits_{n=0}^{\lfloor (d-x)/h \rfloor} (-1)^n \frac{(\mp 1)_n}{n!} f(x + nh)}{h^{\pm 1}}. \tag{6}$$

This unified formulation opens a new perspective into the generalisation for fractional derivatives.

2.2. Order 1 Integral

There are several equivalent definitions of integral of a function in $[a, b],\ a \geq c,\ b \leq d$ (in particular c can be $-\infty$ and d can be $+\infty$), but there is no doubt about the meaning of $\int_a^b f(x)dx$ and its computation. Moreover, $\int_a^b f(x)dx$ enjoys relevant properties such as follows.

Let $f(x)$ as above and $F(x)$ be a function defined, for all $x \in [a, b]$, by

$$F(x) = \int_a^x f(t)dt. \tag{7}$$

Then, $F(x)$ is continuous on $[a, b]$, differentiable on the open interval (a, b), and

$$F'(x) = DF(x) = f(x), \tag{8}$$

for all $x \in (a, b)$. This constitutes the *fundamental theorem of integral calculus* and has as important consequence the well-known Barrow formula

$$\int_a^b f(x)dx = F(b) - F(a). \tag{9}$$

These results show that the function $F(x)$ is the anti-derivative and can be expressed by (4) giving

$$\int_a^b f(x)dx = f^{(-1)}(b) - f^{(-1)}(a). \tag{10}$$

If the upper limit is variable relation (10) gives

$$\int_a^x f(x)dx = f^{(-1)}(x) - f^{(-1)}(a), \tag{11}$$

establishing the connection and simultaneously the difference between the anti-derivative and the integral.

Also, for any $c \in [a, b]$, we have

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx = f^{(-1)}(c) - f^{(-1)}(a) + f^{(-1)}(b) - f^{(-1)}(c). \tag{12}$$

These expressions were presented in terms of the left derivative and anti-derivative. The corresponding right operators are readily obtained. However, in an integer order formulation, they give the same result in all the points where $f(x)$ is a continuous function.

2.3. Definite Fractional Integral

According to the above discussion, it is not clear how to obtain a generalisation of the definite integral. However, expression (10) suggests a shortcut for the solution: introduce it using a generalization of the Barrow formula.

Let $f(x)$, $x \in \mathbb{R}$, such that exist their left and right derivatives of any order. Denote the left and right anti-derivatives by $f_{lr}^{(-\alpha)}(x)$ with $\alpha > 0$.

Definition 1. *We define α-order fractional integral (FI) of $f(x)$ over the interval (a, b) through the fractional Barrow formula*

$$I_{lr}^{\alpha} f(a, b) = \int_a^b f(x)dx^{\alpha} = f_{lr}^{(-\alpha)}(b) - f_{lr}^{(-\alpha)}(a), \tag{13}$$

In coherence with the integer order result, this FI must lead to a *fractional formulation of the fundamental theorem of integral calculus*. Making variable the upper limit, $b = x \in \mathbb{R}$, and putting $f(x) = D_{lr}^{\alpha} g(x)$ in (13) it comes

$$I_{lr}^{\alpha} D_{lr}^{\alpha} g(a, x) = \int_a^x D_{lr}^{\alpha} g(t)dt^{\alpha} = D_{lr}^{-\alpha} D_{lr}^{\alpha} g(x) - D_{lr}^{-\alpha} D_{lr}^{\alpha} g(a),$$

but it is expected to obtain

$$I_{lr}^{\alpha} D_{lr}^{\alpha} g(a,x) = \int_a^x D_{lr}^{\alpha} g(t) dt^{\alpha} = g(x) - g(a). \tag{14}$$

This means that we should have

$$D_{lr}^{-\alpha} D_{lr}^{\alpha} g(x) = g(x). \tag{15}$$

On the other hand, according to (8), the right hand side in the equation,

$$D_{lr}^{\alpha} f \left[I_{lr}^{\alpha} f(a,x) \right] = D_{lr}^{\alpha} \left[f_{lr}^{(-\alpha)}(x) - f_{lr}^{(-\alpha)}(a) \right], \tag{16}$$

should be equal to $f(x)$. This implies that

$$D_{lr}^{\alpha} \left[f_{lr}^{(-\alpha)}(x) \right] = f(x) \tag{17}$$

and that the derivative of a constant is zero

$$D_{lr}^{\alpha} \left[f_{lr}^{(-\alpha)}(a) \right] = 0. \tag{18}$$

These results show that not all the FD can be used to generalise the notion of FI. For now, let us assume that the derivatives to be adopted verify (15), (17), and (18).

In this case, (12) can be readily generalised to

$$\int_a^b f(x) dx^{\alpha} = \int_a^c f(x) dx^{\alpha} + \int_c^b f(x) dx^{\alpha} = f^{(-\alpha)}(c) - f^{(-\alpha)}(a) + f^{(-\alpha)}(b) - f^{(-\alpha)}(c). \tag{19}$$

This definite integral makes sense if suitable FD are used. Therefore, the admissible derivatives have to agree with relations (15) to (19). In the next section, the most important derivatives are analysed to establish their admissibility.

2.4. Which Fractional Derivative?

In a previous paper we answered to the question "What is a fractional derivative?" [13] using two criteria as guidelines for deciding if a given operator is a FD. The *wide sense criterion* is as follows.

An operator is considered as a FD under ^{1}**P** criterion if it enjoys the properties:

- ^{1}**P1** Linearity
- ^{1}**P2** Identity
- ^{1}**P3** Backward compatibility
- ^{1}**P4** The index law holds for negative orders
- ^{1}**P5** Generalised Leibniz rule

The index law property can be modified to include positive orders. This led to the *strict sense criterion*. Therefore, criterion ^{2}**P** keeps four conditions and ^{1}**P4** is modified to:

^{2}P4 The index law

$$D^{\alpha} D^{\beta} f(x) = D^{\alpha+\beta} f(x) \tag{20}$$

holds for any real α and β. The difference between the two criteria lies in the validity of the index law for positive values only, or for any real order.

Let us consider relation (20) again and the case $\alpha > 0$ with $\beta = -\alpha$. We conclude that for each derivative operator, D^{α}, there is an anti-derivative, $D^{-\alpha}$. As in the classic framework we can say that, given a function, it can have an infinite number of primitives, but only one of them does not depend on arbitrary constants.

2.5. The Riemann-Liouville and Caputo Derivatives

For the RL and C derivatives the negative order operator is given by the *Riemann-Liouville fractional integral* (RL-FI) that can be introduced as follows. Let $[a,b]$ $(-\infty < a < b < \infty)$ be an interval on the real axis $\mathbb{R}$. The *left-sided RL-FI*, $I^{\alpha}_{a+}f$, of order $\alpha \in \mathbb{R}^{+}$ is given by [6]

$$ {}^{RL}\mathcal{I}^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} \frac{f(t)dt}{(x-t)^{1-\alpha}}, \quad x > a,\ \alpha > 0. \tag{21} $$

The left *Riemann-Liouville fractional derivative* (RL-FD), $D^{\alpha}_{a+}f$, of order $\alpha \in \mathbb{R}^{+}_{0}$ is defined as

$$ \begin{aligned} {}^{RL}\mathcal{D}^{\alpha}_{a+}f(x) &= \left(\frac{d}{dt}\right)^{n} I^{n-\alpha}_{a+}f(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^{n} \int_{a}^{x} \frac{f(t)dt}{(x-t)^{\alpha-n+1}}, \quad n = \lfloor \alpha \rfloor + 1,\ x > a. \end{aligned} \tag{22} $$

The C-FD, ${}^{C}\mathcal{D}^{\alpha}_{a+}f(x)$, of order $\alpha \in \mathbb{R}^{+}$ on $[a,b]$ is introduced via the above RL-FD as [5]:

$$ {}^{C}\mathcal{D}^{\alpha}_{a+}f(x) := {}^{RL}\mathcal{D}^{\alpha}_{a+} \left[y(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^{k} \right]. \tag{23} $$

With the RL-FI formula above introduced, we verify [5,13] that:

- The RL-FD is an inverse operator of the left RL-FI

$$ {}^{RL}\mathcal{D}^{\alpha}_{a+}\ {}^{RL}\mathcal{I}^{\alpha}_{a+}f(x) = f(x) \tag{24} $$

- The C-FD is also an inverse operator of the left RL-FI

$$ {}^{C}\mathcal{D}^{\alpha}_{a+}\ {}^{RL}\mathcal{I}^{\alpha}_{a+}f(x) = f(x). \tag{25} $$

Therefore, the RL-FI is the right inverse of the RL and C derivatives. The RL-FI is not a left inverse, because we have [5]:

- If $f(x) \in L_{1}(a,b)$ and $f_{n-\alpha}(x) = I^{n-\alpha}_{a+}f(x) \in AC^{n}[a,b]$, then

$$ {}^{RL}\mathcal{I}^{\alpha}_{a+}\ {}^{RL}\mathcal{D}^{\alpha}_{a+}f(x) = f(x) - \sum_{j=1}^{n} \frac{f^{(\alpha-j)}(a)}{\Gamma(\alpha-j+1)}(x-a)^{\alpha-j} \tag{26} $$

- If $f(x) \in C^{n}[a,b]$ or $f(x) \in AC^{n}[a,b]$, then

$$ {}^{RL}\mathcal{I}^{\alpha}_{a+}\ {}^{C}\mathcal{D}^{\alpha}_{a+}f(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^{k} \tag{27} $$

In order to study the admissiblity of RL and C derivatives we compare their properties with those required in the previous sub-section, namely conditions (15) to (18).

The RL derivative is not acceptable because it does not give the result zero for the derivative of a constant. On the other hand, (26) is not suitable also since it introduces terms that do not verify (19). Concerning the C derivative, it is acceptable from the point of view of (25) and gives zero for the derivative of a constant, but (27) is only acceptable if $\alpha \leq 1$. Also, it creates difficulties when trying to use (19), since we must have

$$ {}^{RL}\mathcal{I}^{\alpha}_{a+}\ {}^{C}\mathcal{D}^{\alpha}_{a+}f(b) = {}^{RL}\mathcal{I}^{\alpha}_{a+}\ {}^{C}\mathcal{D}^{\alpha}_{a+}f(c) + {}^{RL}\mathcal{I}^{\alpha}_{c+}\ {}^{C}\mathcal{D}^{\alpha}_{c+}f(b), $$

that is true if, and only if the following equation is valid

$$f(b) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k =$$
$$f(c) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(c-a)^k + f(b) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(b-c)^k.$$

However, in general, this relation is not valid.

2.6. Grünwald-Letnikov Derivatives

In a recent paper [14], we considered the problem of selecting derivatives suitable for generalizing standard laws of physics and results while maintaining fully compatibility with them. It was shown that the Grünwald-Letnikov (GL) derivative verifies the above strict sense criterion and is fully compatible with the classic laws and tools of Physics. Here it is important to study the GL derivative from the point of view of the fractional integral introduced in Section 2.3.

Let $f(x)$ be a function defined on $\mathbb{R}$. We introduce the left (forward) FD by means of the fractional incremental ratio

$$D_l^\alpha f(x) = \lim_{h\to 0} \frac{\sum\limits_{n=0}^{\infty} (-1)^n \frac{(-\alpha)_n}{n!} f(x-nh)}{h^\alpha}, \tag{28}$$

where $h > 0$. Relation (28) expresses the called left GL-FD [6,15,16]. Without intending to explore existence problems (see [16]) we can say that $f(x)$ must decrease to zero as x goes to $-\infty$. It is interesting to remark that (28) generalises (5) and this corresponds to making $\alpha = \pm 1$.

The GL derivative enjoys relevant characteristics, namely the index law (20) [15,16]. This means that given an FD of order $\alpha > 0$, there is an FD, of negative order, that we will call "anti-derivative" and verifying

$$D^\alpha D^{-\alpha} f(x) = D^{-\alpha} D^\alpha f(x) = f(x). \tag{29}$$

It can be shown [14] that the FD of the constant function is identically null. The corresponding righthand (backward) GL derivative is defined by

$$D_r^\alpha f(x) = e^{-i\pi\alpha} \lim_{h\to 0} \frac{\sum\limits_{n=0}^{\infty} (-1)^n \frac{(-\alpha)_n}{n!} f(x+nh)}{h^\alpha}. \tag{30}$$

Contrarily to the above situation, here a suitable function, $f(x)$, must decrease to zero when $x \to +\infty$. When dealing with space derivative we do not need to impose causality, because we can move in all the directions in the 3D space. Therefore, the factor $e^{-i\pi\alpha}$ can be removed. This allows us to write a unified formula joining the lefthand and righthand derivatives

$$D_{lr}^\alpha f(x) := \lim_{h\to 0} \frac{\sum\limits_{n=0}^{\infty} (-1)^n \frac{(-\alpha)_n}{n!} f(x \pm nh)}{h^\alpha}. \tag{31}$$

Attending to (29) and to the fact that the GL derivative of a constant is zero, we arrive to the conclusion that the GL derivative is suitable for computing the definite integral introduced in Section 2.3.

2.7. Liouville Derivatives

The name "Liouville derivative" is frequently attached to the RL derivative when the integration domain is $\mathbb{R}$ [6]. This derivative has the inconvenience of being divergent when the function is constant.

Another alternative is given by the so called Liouville-Caputo derivative [1,17,18] that is also similar to the C derivative but is also defined on $\mathbb{R}$. This derivative poses strong requirements to the integrand function, since this must have the nth order derivative with $n \geq \alpha$. A third alternative was proposed by Liouville in the 19th century and can be obtained from the system interpretation of the GL derivative. Let $h(x),\ x \in \mathbb{R}$ be the impulse response of the GL derivative [15]

$$h(x) = \delta^{(\alpha)}(x) = \frac{x^{-\alpha-1}}{\Gamma(-\alpha)} u(x), \tag{32}$$

where $u(x)$ is the Heaviside unit step. In a linear system perspective, when the input is a given function, $f(x)$, the output is a α-order derivative

$$D_l^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty f(x-\tau)\tau^{-\alpha-1} d\tau, \tag{33}$$

that generalises the well known result $f'(x) = \int_{-\infty}^{\infty} f(x-y)\delta'(y)dy$,and represents an integral formulation of the left side FD. For $\alpha > 0$, the above integral is singular. However, it can be regularised, leading to what we will call the left Liouville (L) derivative, defined as follows:

Definition 2. *Liouville derivative*

$$D_l^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \left[f(x-\tau) - u(\alpha) \sum_{n=0}^{N} \frac{(-1)^n f^{(n)}(x)}{n!} \tau^m \right] \tau^{-\alpha-1} d\tau, \tag{34}$$

where N is the integer part of α, so that $\alpha - 1 < N \leq \alpha$, $N \in \mathbb{N}_0$.

This derivative verifies also relations (29). The equivalence of GL and L derivatives is valid for at least functions with Laplace transform, although we can assume it for a broader class of functions and use them interchangeably.

Joining the left and right Liouville derivatives, it becomes

$$D_{lr}^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \left[f(x \pm \tau) - u(\alpha) \sum_{n=0}^{N} \frac{(-1)^n f^{(n)}(x)}{n!} \tau^m \right] \tau^{-\alpha-1} d\tau. \tag{35}$$

3. Definite Fractional Integrals

Using the Liouville anti-derivative expression we can write

$$I_{lr}^\alpha f(a,b) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \left[f(b \pm x) - f(a \pm x) \right] dx^\alpha. \tag{36}$$

From the standard (integer order) Barrow formula $\int_a^b f'(x)dx = f(b) - f(a)$ we obtain the expression

$$I_{lr}^\alpha f(a,b) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \int_a^b f'(y \pm x) dy dx^\alpha. \tag{37}$$

Using (37) it comes

$$I_{lr}^\alpha f(a,b) = \int_a^b f_{lr}^{(-\alpha+1)}(x) dx. \tag{38}$$

because the GL and Liouville derivatives yield the result 0 for the derivative of the constant function. This formula reduces the α-order integral to one integer order.

3.1. Integrals in $\mathbb{R}^2$ and $\mathbb{R}^3$

The above result can be enlarged for double and triple integrals. This will be done next for rectangular domains, meaning that in $\mathbb{R}^2$ the domain is obtained by joining rectangles so that the limiting line is made of segments parallel to one of the coordinated axes. Similarly in $\mathbb{R}^3$ the domain is made of parallelepipeds with a boundary constituted by rectangles parallel to one of the coordinated planes.

Let us assume that the function f is dependent of a variable, x_1, and a parameter, x_2, that is kept fixed, $f(x_1, x_2)$. Consider the parametric integral

$$I^{\alpha}_{lr} f(a_1, b_1, x_2) = \fint_{a_1}^{b_1} f(x_1, x_2) dx_1^{\alpha_1} = f_{lr}^{(-\alpha_1)}(b_1, x_2) - f_{lr}^{(-\alpha_1)}(a_1, x_2), \tag{39}$$

and similarly, fixing x_1, another one

$$I^{\alpha}_{lr} f(x_1, a_2, b_2) = \fint_{a_2}^{b_2} f(x_1, x_2) dx_2^{\alpha_2} = f_{lr}^{(-\alpha_2)}(x_1, b_2) - f_{lr}^{(-\alpha_2)}(x_1, a_2). \tag{40}$$

This leads to the following definition

Definition 3. *The FI on a rectangular region $(a_1, b_1) \times (a_2, b_2)$ is given by*

$$I^{\alpha}_{lr} f(a_1, b_1, a_2, b_2) = \fint_{a_1}^{b_1} \fint_{a_2}^{b_2} f(x_1, x_2) dx_1^{\alpha_1} dx_2^{\alpha_2}, \tag{41}$$

where each integration is performed by the fractional Barrow formula.

If the orders of the FD are equal (i.e., $\alpha_1 = \alpha_2 = \alpha$) we put $x_1^{\alpha} x_2^{\alpha} = S^{\alpha}$ to get then

$$I^{\alpha}_{lr} f(a_1, b_1, a_2, b_2) = \fint \fint_S f(x_1, x_2) dS^{\alpha}, \tag{42}$$

The integral in $\mathbb{R}^3$ is obtained in a similar way.

Definition 4. *The FI in $\mathbb{R}^3$ is defined by*

$$I^{\alpha}_{lr} f(a_1, b_1, a_2, b_2, a_3, b_3) = \fint_{a_1}^{b_1} \fint_{a_2}^{b_2} \fint_{a_3}^{b_3} f(x_1, x_2, x_3) dx_1^{\alpha_1} dx_2^{\alpha_2} dx_3^{\alpha_3}, \tag{43}$$

where each integration is performed by means of the fractional Barrow formula.

If the derivative orders are equal (i.e., $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$), then we get

$$I^{\alpha}_{lr} f(a_1, b_1, a_2, b_2, a_3, b_3) = \fint_V f(x_1, x_2, x_3) dV^{\alpha}, \tag{44}$$

where we adopt the notation $V^{\alpha} = x_1^{\alpha} x_2^{\alpha} x_3^{\alpha}$.

4. Conclusions

In this paper, a definition of fractional definite integral was introduced and the corresponding fundamental theorem of fractional calculus formulated. For this purpose, the mostly used fractional derivatives were studied and a compatibility with the definite fractional integral was saught. It was shown that the Grünwald-Letnikov and Liouville derivatives were suitable for such a definition. Finally, integrals on $\mathbb{R}^2$ and $\mathbb{R}^3$ were also proposed.

Acknowledgments: This work was funded by Portuguese National Funds through the FCT—Foundation for Science and Technology—under the project PEst- UID/EEA/00066/2013.

Author Contributions: These two authors contribute equally to this paper.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

FD	Fractional derivative
FI	Fractional integral
RL	Riemann-Liouville
L	Liouville
C	Caputo
GL	Grünwald-Letnikov

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