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Nonlinear Vibration of a Nonlocal Nanobeam Resting on Fractional-Order Viscoelastic Pasternak Foundations

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Abstract: In the present study, the nonlinear vibration of a nanobeam resting on the fractional order viscoelastic Winkler–Pasternak foundation is studied using nonlocal elasticity theory. The D'Alembert principle is used to derive the governing equation and the associated boundary conditions. The approximate analytical solution is obtained by applying the multiple scales method. A detailed parametric study is conducted, and the effects of the variation of different parameters belonging to the application problems on the system are calculated numerically and depicted. We remark that the order and the coefficient of the fractional derivative have a significant effect on the natural frequency and the amplitude of vibrations.

Keywords: nanobeam; fractional-order; nonlinearity; Winkler–Pasternak foundation

1. Introduction

Due to the recent and rapid advances in nanomechanics, nanobeams have become the most important structures used extensively in technology such as nano-electromechanical systems (NEMs), opto-mechanical or nanoresonator devices. The exclusive properties of nanoscale beams are due to their size, and this size plays an important role in static and in dynamic analysis. In front of the difficulties of classical continuum mechanics in considering the size effect in modeling the behavior of this kind of structure, various size-dependent continuum theories have been developed. These theories include nonlocal continuum theory, strain gradient theory or a combination of both (nonlocal strain gradient theory), modified couple stress theory, micropolar theory and the surface elasticity theory. Among these theories, Eringen's nonlocal elasticity theory [1,2] was utilized by a number of researchers to capture size-effects.

These kinds of structures can be modeled as a beam structure on a viscoelastic foundation. The beam can be modeled as a Timoshenko beam [3,4], or as a Rayleigh beam [5], or as a Euler–Bernouilli beam [6], and the foundation as a Winkler model [7–9], or as a Pasternak model, or a combination of both (Winkler–Pasternak model), or as a nonlinear elastic model and the fractional order viscoelastic model [10]. The Winkler model is a one-parameter model, namely the Winkler-type elastic foundation consists of a series of closely-spaced elastic springs. The Pasternak model is a two-parameter model, namely the Pasternak-type viscoelastic foundation consists of Winkler-type elastic springs and transverse shear deformation. The nonlinear model is a three parameter one in which the layer is indicated by the linear elastic spring, shear deformation and cubic nonlinearity elastic spring.The fractional order Winkler–Pasternak [10] has been well developed. This fractional

order is due to the long memory effects of some kinds of viscoelastic materials. In vibration analysis of nanostructures, it is very important to evaluate the impact of the surrounding medium on the dynamic of beams. Niknam and Aghdam [11] proposed an analytical approach to study dynamic of nonlocal functionally graded beam resting on nonlinear elastic support. A meshless approach for free transverse vibration of single-walled carbon nanotube (SWCNT) was proposed by Kiani [12]. Eringen's nonlocal theory and Timoshenko beam theory were used to make a buckling analysis of SWCNT on elastic medium [13,14]. Non-conservative dynamic of nonlocal cantilever carbon nanotubes (CNTs) on viscoelastic medium is proposed [15]. Mikhasev [16] researched localized modes of free vibrations of SWCNT. Mustapha and Zhong [17] studied the dynamics of non-prismatic SWCNT in a viscoelastic medium. Lee and Chang [18] studied the dynamics of a viscous fluid conveying SWCNT. Kiani [19,20] examined elastically-restrained double-walled carbon nanotube (DWCNT) and SWCNT for delivering nanoparticles. Instability analysis of CNT conveying fluid was conducted [21]. Yas and Samadi [22] examined CNT-reinforced composite on and elastic medium. A small scale effect in nonuniform CNT conveying fluid on the viscoelastic medium was examined [23]. Aydogdu [24] analyzed nanorods on an elastic medium. Dynamic analysis of nanotubes on elastic matrix was conducted by Wang [25]. Dynamics of curved SWCNT on a Pasternak elastic foundation was examined [26]. Aydogdu and Arda [27] researched the torsional dynamics of nonlocal DWCNTs. Necla [28] studied the nonlinear vibration of a nonlocal nanobeam resting on a Winkler-type foundation. The work of Anague [10] is based on the dynamics of Rayleigh beams resting on a fractional order viscoelastic Pasternak foundation subjected to moving loads.

Many time-space differential equations are very difficult to solve; sometimes, these equations are exactly impossible to solve. In front of these difficulties, a sophisticated analytical and numerical method to find approximated solutions is needed. Ozturk and Coskun [29] proposed the homotopy perturbation method. The multiple scale method was used to analyze the nonlinear vibration of CNT [30–33]. He's variational method exhibited more advantages [12,34–37]. The direct iterative method was used in dynamical analysis of DWCNT [38]. The finite element method [21,23] and the differential quadrature method [13,22] also exhibited more advantages.

The above investigations clearly show that most of the studies presented in the literature are related to nonlocal and nonlinear structures, but studies on the nonlocal and nonlinear fractional order vibration are very limited. When it is observed in the field, the linear frequency amplitude and nonlinear frequency amplitude of beams are major topics, but dynamic analysis of beams embedded in a fractional order viscoelastic medium is very rare. The nonlinear free vibration of the nanotube with the damping effect was studied by using nonlocal elasticity theory [31]. To our knowledge, there is no published work on a fractional order nonlocal nonlinear vibration of a nanobeam resting on a viscoelastic foundation. The nonlinearity of the problem is obtained by considering the von Karman geometric nonlinearity, which introduces a cubic nonlinearity into the equations. In the present paper, we analyze the nonlinear vibration of a nanobeam resting on a fractional order viscoelastic Winkler–Pasternak foundation using Eringen's nonlocal elasticity. Nonlinear fractional order frequency response and mode shapes are drawn for the nanobeam with different end conditions.

2. Preliminaries

2.1. Fractional Order Viscoelasticity

Fractional calculus is a part of mathematical analysis that has found many applications in nanomechanics. The role of fractional calculus is to study arbitrary real or complex order integrals and derivatives. There are many definitions of fractional order integrals and derivatives that have been given by different authors. However, in our study, we will consider only Riemann–Liouville's definition of a fractional derivative as follows: If $x(\cdot)$ is an absolutely continuous function in $[a, b]$ and $0 \prec a \prec 1$, then:

1. The left Riemann–Liouville fractional derivative of order α is of the form:

$$ {}_aD_t^{\alpha} x(\tau) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{x(\tau)}{(t-\tau)^{\alpha}} d\tau, t \in [a,b] \tag{1} $$

2. The right Riemann–Liouville fractional derivative of order α is of the form:

$$ {}_tD_b^{\alpha} x(\tau) = \frac{1}{\Gamma(1-\alpha)} \left(-\frac{d}{dt}\right) \int_t^b \frac{x(\tau)}{(\tau-t)^{\alpha}} d\tau, t \in [a,b] \tag{2} $$

Fractional derivatives are used in the accurate modeling in rheology, as well as structural mechanics to model internal damping. In [39], it was shown that classical viscoelastic models failed to describe the damping of a viscoelastic solid and that improved fractional derivative-based models need to be considered. Such models have few advances. First, they are based on molecular theories [40]. Second, such models satisfy thermodynamic laws. At least, they need a few parameters to describe the viscoelastic behavior.

In the following, we give a constitutive relation of the fractional order viscoelastic Winkler–Pasternak foundation beam interaction force (per unit length of the beam's axis), which is obtained including the fractional derivative term as [10]:

$$ q(x,t) = kw(x,t) + c\frac{\partial w(x,t)}{\partial t} - [\mu_e + \mu_v D_t^{\alpha}] \frac{\partial^2 w(x,t)}{\partial x^2}, \tag{3} $$

in which the deformed beam can be described by the transverse deflection $w(x,t)$, k and c are the foundation stiffness and damping coefficients and μ_e and μ_v are the foundation shear elastic and viscosity coefficients. D_t^{α} is the fractional derivative with order α.

2.2. Nonlocal Theory

In the nonlocal elasticity theory, the stress at a point x is a function of the strains at all other points of an elastic body. The integral form of the nonlocal constitutive relation for a three-dimensional structure is:

$$ \sigma_{ij}(x) = \int \chi(|x-x'|, \tau) t_{ij}(x') \, dV(x'), \forall x \in V, \tag{4} $$

where σ_{ij} is the nonlocal stress tensor, t_{ij} is the local or classical stress tensors at a point x', $\chi(|x-x'|, \tau)$ denotes the attenuation function, which incorporates nonlocal effects into the constitutive equation, $|x-x'|$ is the distance in the Euclidean norm and $\tau = {}^{e_0 a}/_{l}$ is a nonlocal parameter, where l is the external characteristic length (crack length or wave length), a is the internal characteristic length (lattice parameter, granular, etc.) and e_0 is a material constant that can be determined from molecular dynamics simulations or by using the dispersive curve of the Born–Karman model of lattice dynamics. Later, Eringen [2] proposed a differential form of the constitutive relation with an appropriate kernel function as:

$$ \left(1 - \tau^2 l^2 \nabla^2\right) \sigma_{ij} = t_{ij}. \tag{5} $$

For the one-dimensional case, the local stress t_{xx} at a point x' can be explained according to Hooke's law as:

$$ t_{xx}(x') = E\varepsilon_{xx}(x'), \tag{6} $$

where E denotes the elastic modulus and ε_{xx} the strain. That yields the following differential form of the nonlocal constitutive equation for a one-dimensional elastic body:

$$ \sigma_{xx} - \mu \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E\varepsilon_{xx}, \tag{7} $$

where $\mu = (e_0 a)^2$ is the nonlocal parameter and σ_{xx} is the nonlocal stress.

3. Governing Equation of the Nanobeam Resting on the Fractional Order Viscoelastic Foundation

This study is carried out on the basis of the nonlocal Euler–Bernouilli nanobeam of length L, cross-sectional area A, density ρ and transverse deflection $w(x,t)$ in the z direction. Two types of boundary conditions, which are simple-simple and clamped-clamped, are considered in this work and shown in Figure 1. We assume that the cross-sectional area is constant along the x coordinate and that the material of the nanobeam is homogeneous. The nanobeam is resting on a fractional order viscoelastic Winkler–Pasternak foundation in which k and c are the stiffness and damping coefficient and μ_e and μ_v are the foundation shear elastic and viscosity coefficients. We also consider that the nanobeam is under the influence of time varying axial load. According to Euler–Bernouilli beam theory, the displacement fields at any point of the beam can be expressed as:

$$u_x(x,z,t) = u(x,t) - z\frac{\partial w(x,t)}{\partial x}, u_y = 0, u_z = w(x,t), \tag{8}$$

where u and w are the axial and transverse displacements, respectively. By assuming the von Karman nonlinear strain displacement relation for the given displacement fields, we get:

$$\varepsilon_0 = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \varepsilon_1 = -z\bar{k}, \bar{k} = \frac{\partial^2 w}{\partial x^2}. \tag{9}$$

where ε_0 is the nonlinear extensional strain and $\bar{k}$ is the bending strain. The von Karman nonlinear normal strain can be expressed as:

$$\varepsilon = \varepsilon_0 + \varepsilon_1 = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\frac{\partial^2 w}{\partial x^2}. \tag{10}$$

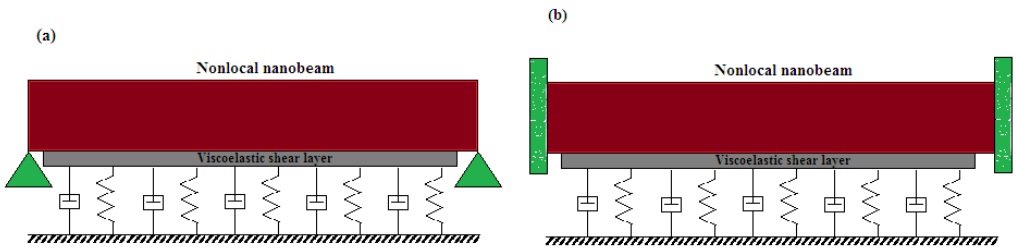

Figure 1. Boundary conditions for different beam supports. (**a**) Simple-simple case and (**b**) clamped-clamped case.

By applying the D'Alembert principle to the infinitesimal element of the nanobeam, the equilibrium equation can be obtained as:

$$\rho A\frac{\partial^2 u}{\partial t^2} = \frac{\partial T}{\partial x}, \tag{11a}$$

$$\rho A\frac{\partial^2 w}{\partial t^2} = \frac{\partial Q}{\partial x} + T\frac{\partial^2 w}{\partial x^2} - q(x,t), \tag{11b}$$

$$\rho I\frac{\partial^3 w}{\partial x \partial t^2} = Q - \frac{\partial M}{\partial x}, \tag{11c}$$

in which the stress resultant is defined as:

$$(Q, T, M) = \int_0^A (\bar{\tau}_{xz}, \bar{\sigma}_{xx}, z\bar{\sigma}_{xx}) dA, \tag{12}$$

where Q, T and M are the transversal force, the axial force and the bending moment, respectively. $\bar{\tau}_{xz}$ and $\bar{\sigma}_{xx}$ are the shear and normal stress components. The longitudinal inertia $\frac{\partial^2 u}{\partial t^2}$ can be neglected based on the discussion about the nonlinear vibration of continuous systems [41,42], then the axial normal force T can be represented as:

$$T = F\cos\Omega t + \frac{EA}{2L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx. \tag{13}$$

Assuming that the axial force is periodic and time-dependent and combining Equations (7) and (11), the nonlinear vibration equation of motion for the nanobeam resting on the fractional order viscoelastic Pasternak-type foundation in terms of transversal displacements is obtained as follows:

$$\begin{gathered}\rho A\frac{\partial^2 w}{\partial t^2} - \rho I\frac{\partial^4 w}{\partial x^2\partial t^2} - \left(F\cos\Omega t + \frac{EA}{2L}\int\left(\frac{\partial w}{\partial x}\right)^2 dx\right)\frac{\partial^2 w}{\partial x^2} + kw + c\frac{\partial w}{\partial t} - \left(\mu_e + \mu_v D_t^\alpha\right)\frac{\partial^2 w}{\partial x^2} + EI\frac{\partial^4 w}{\partial x^4} \\ -\mu\frac{\partial^2}{\partial x^2}\left(\rho A\frac{\partial^2 w}{\partial t^2} - \rho I\frac{\partial^4 w}{\partial x^2\partial t^2}\right) + \mu\frac{\partial^2}{\partial x^2}\left(\left(F\cos\Omega t + \frac{EA}{2L}\int\left(\frac{\partial w}{\partial x}\right)^2 dx\right)\frac{\partial^2 w}{\partial x^2}\right) \\ -\mu\frac{\partial^2}{\partial x^2}\left(kw + c\frac{\partial w}{\partial t} - \left(\mu_e + \mu_v D_t^\alpha\right)\frac{\partial^2 w}{\partial x^2}\right) = 0,\end{gathered} \tag{14}$$

where F is the amplitude of axial load and Ω is the frequency of this load. The following non-dimensional quantities aim to study the problem in the general form as:

$$\bar{x} = \frac{x}{L}, \bar{w} = \frac{w}{L}, \bar{t} = \frac{t}{L^2}\sqrt{\frac{EI}{\rho A}}, \eta^2 = \frac{\mu}{L^2}, K = \frac{kL^4}{EI}, \varepsilon C = c\sqrt{\frac{L^4}{\rho A\,(EI)}}, K_P = \frac{\mu_e L^2}{EI}, \varepsilon\bar{F} = \frac{FL^2}{EI}$$

$$\varepsilon C_P = \frac{\mu_v L^{2(1-\alpha)}}{(\rho A)^{\frac{1}{2}\alpha}(EI)^{\frac{1}{2}(2-\alpha)}}, \delta = \frac{I}{AL^2}. \tag{15}$$

The non-dimensional Equations (14) and (15) can be expressed as:

$$\begin{gathered}\frac{\partial^2\bar{w}}{\partial\bar{t}^2} - \delta\frac{\partial^4\bar{w}}{\partial\bar{x}^2\partial\bar{t}^2} - \left(\epsilon\bar{F}\cos\bar{\Omega}\bar{t} + \tfrac{1}{2}\epsilon\int\left(\frac{\partial\bar{w}}{\partial\bar{x}}\right)^2 d\bar{x}\right)\frac{\partial^2\bar{w}}{\partial\bar{x}^2} + K\bar{w} + \epsilon C\frac{\partial\bar{w}}{\partial\bar{t}} + \left(K_P + \epsilon C_P D_{\bar{t}}^\alpha\right)\frac{\partial^2\bar{w}}{\partial\bar{x}^2} + \frac{\partial^4\bar{w}}{\partial\bar{x}^4} \\ -\eta^2\frac{\partial^2}{\partial\bar{x}^2}\left(\frac{\partial^2\bar{w}}{\partial\bar{t}^2} - \delta\frac{\partial^4\bar{w}}{\partial\bar{x}^2\partial\bar{t}^2}\right) + \eta^2\frac{\partial^2}{\partial\bar{x}^2}\left(\left(\epsilon\bar{F}\cos\bar{\Omega}\bar{t} + \tfrac{1}{2}\epsilon\int\left(\frac{\partial\bar{w}}{\partial\bar{x}}\right)^2 d\bar{x}\right)\frac{\partial^2\bar{w}}{\partial\bar{x}^2}\right) \\ -\eta^2\frac{\partial^2}{\partial\bar{x}^2}\left(K\bar{w} + \epsilon C\frac{\partial\bar{w}}{\partial\bar{t}} + \left(K_P + \epsilon C_P D_{\bar{t}}^\alpha\right)\frac{\partial^2\bar{w}}{\partial\bar{x}^2}\right) = 0,\end{gathered} \tag{16}$$

in which K and C denote dimensionless stiffness and viscosity medium, K_P and C_P denote the dimensionless shear elastic and viscosity coefficient, $\bar{F}$ represent the dimensionless amplitude of axial load and η, $\bar{w}$ and $\bar{t}$ denote the nonlocal parameter, transversal displacement and time, respectively, in dimensionless form. The small bookkeeping parameter ϵ is used to emphasize the transversal deformation, viscosity coefficients and tension fluctuation compared to the other terms.

The non-dimensional form of boundary conditions can be expressed as

Simple-simple case:

$$\bar{w}(0) = 0, \bar{w}(1) = 0, \bar{w}''(0) = 0, \bar{w}''(1) = 0; \tag{17}$$

Clamped-clamped case:

$$\bar{w}(0) = 0, \bar{w}(1) = 0, \bar{w}'(0) = 0, \bar{w}'(1) = 0. \tag{18}$$

3.1. Solution of the Governing Equation

The dimensionless fractional order nonlinear partial differential equation, Equation (16), describes the transversal vibration of the nanobeam resting on a fractional order viscoelastic foundation under the influence of periodic axial load. In order to obtain the asymptotic approximate solution in the first order for the problem, the perturbation method of multiple scales will be employed. By applying the Galerkin method, we assume the asymptotic approximate solution in the following form:

$$\bar{w}(\bar{x}, \bar{t}) = q(\bar{t})\,\phi(\bar{x}), \tag{19}$$

in which $q(\bar{t})$ is the unknown time function and $\phi(\bar{x})$ is the linear mode shape determined from the boundary conditions. The linear mode shapes of Equations (17) and (18) are given by:

$$\phi(\bar{x}) = c_1 \exp i\alpha_1\bar{x} + c_2 \exp i\alpha_2\bar{x} + c_3 \exp i\alpha_3\bar{x} + c_4 \exp i\alpha_4\bar{x}. \tag{20}$$

The boundary conditions are applied, and the constants c_i and α_i can be obtained. Mode shapes of the linear first frequency are plotted in Figures 2 and 3.

By introducing Equation (19) into Equation (16), multiplying the results by the linear mode shape function $\phi(\bar{x})$ and then integrating them over the length of the nanobeam, we obtain a fractional order nonlinear ordinary differential equation expressed as:

$$\frac{d^2q}{d\bar{t}^2} + \epsilon\tilde{C}\frac{dq}{d\bar{t}} + \left(w_0^2 + \epsilon\gamma\bar{F}\cos\bar{\Omega}\bar{t}\right)q + \frac{1}{4}\epsilon\chi q^3 + \epsilon\tilde{C}_P\frac{d^\alpha q}{d\bar{t}^\alpha} = 0, \tag{21}$$

where w_0 is the natural frequency for the linear system, $\tilde{C}$ and $\tilde{C}_P$ are normal damping ratio and shear damping ratio, χ is the reduced nonlinear stiffness and γ is the constant:

$$\begin{gathered} w_0^2 = \frac{K(a_1 - \eta^2 a_2) + K_P(-a_2 + \eta^2 a_3) + a_3}{(a_1 - \eta^2 a_2) + \delta(-a_2 + \eta^2 a_3)}, \tilde{C} = \frac{C(a_1 - \eta^2 a_2)}{(a_1 - \eta^2 a_2) + \delta(-a_2 + \eta^2 a_3)}, \\ \gamma = \frac{(a_2 - \eta^2 a_3)}{(a_1 - \eta^2 a_2) + \delta(-a_2 + \eta^2 a_3)}, \chi = \frac{2a_4(-a_2 + \eta^2 a_3)}{(a_1 - \eta^2 a_2) + \delta(-a_2 + \eta^2 a_3)}, \\ \tilde{C}_P = \frac{C_P(-a_2 + \eta^2 a_3)}{(a_1 - \eta^2 a_2) + \delta(-a_2 + \eta^2 a_3)}, \{a_1, a_2, a_3, a_4\} = \int_0^L \left\{\phi^2, \phi\phi', \phi\phi^{IV}, (\phi')^2\right\}. \end{gathered} \tag{22}$$

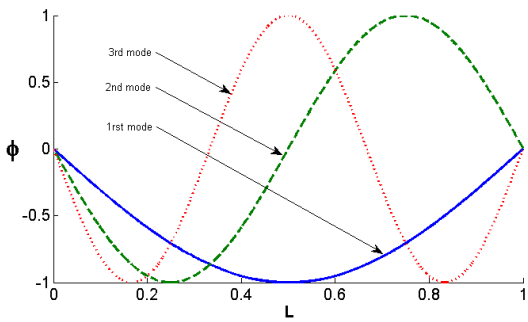

Figure 2. First three vibration mode shapes for the simple-simple case boundary condition.

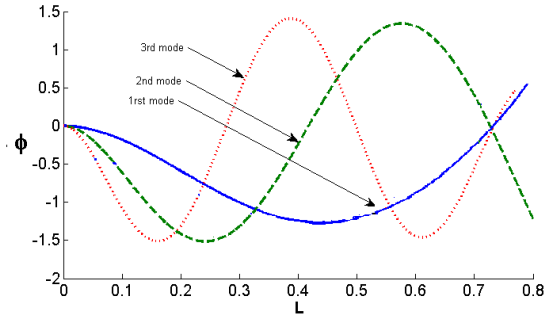

Figure 3. First three vibration modes shapes for the clamped-clamped case boundary condition.

Equation (21) is a new form of parametrical; 7-excited Duffing differential equation due to the presence of the fractional order term. In order to determine the asymptotic approximate solution with combined effects of nonlinearity, parametric excitation and fractional order damping, we will apply the method of multiple scales. A straightforward asymptotic expansion can be introduced:

$$q\left(\bar{t};\epsilon\right) = \epsilon^0 q_0\left(T_0, T_1\right) + \epsilon^1 q_1\left(T_0, T_1\right), \tag{23}$$

where $T_0 = \bar{t}$ and $T_1 = \epsilon\bar{t}$ represent the fast and low timescale. The fast timescale is associated with the linear unperturbed system, while the slow timescale is characterized by the modulation of the amplitude and phase in the presence of possible resonance. Denoting $D_0 = {}^{\partial}/_{\partial T_0}$, $D_1 = {}^{\partial}/_{\partial T_1}$, the ordinary time derivatives can be transformed into the partial derivative as:

$$\frac{d}{dt} = D_0 + \epsilon D_1, \frac{d^2}{dt^2} = D_0^2 + 2\epsilon D_0 D_1, \left(\frac{d}{dt}\right)^{\alpha} = D_0^{\alpha} + \epsilon\alpha D_0^{\alpha-1} D_1 + ..., \tag{24}$$

Inserting Equations (23) and (24) into Equation (21), we obtain the following relation:

$$\left(\epsilon^0\right) : D_0^2 q_0 + \omega_0^2 q_0 = 0, \tag{25a}$$

$$\left(\epsilon^1\right) : D_0^2 q_1 + \omega_0^2 q_1 = -2D_0 D_1 q_0 - \tilde{C}_P D_0^{\alpha} q_0 - \tilde{C} D_0 q_0 - \frac{1}{4}\chi q_0^3 + \gamma\bar{F}\cos\left(\bar{\Omega}T_0\right), \tag{25b}$$

Fundamental frequencies are obtained by solving the first order of expansion, and the solvability condition is obtained by solving the second order of expansion. The solution of the first order equation is given as:

$$q_0\left(T_0, T_1\right) = A\left(T_1\right)\exp i\omega_0 T_0 + \bar{A}\left(T_1\right)\exp -i\omega_0 T_0, \tag{26}$$

where $i = \sqrt{-1}$, A is a complex function of slow timescale and $\bar{A}$ is the complex conjugate. Excitation frequency is assumed close to one of the natural frequencies of the system; the dimensionless form of this excitation frequency can be written as:

$$\bar{\Omega} = \omega_0 + \epsilon\sigma, \tag{27}$$

where σ is a detuning parameter. Substituting Equation (26) into the second order of expansion and using the dimensionless form of excitation frequency yield:

$$\begin{aligned} D_0^2 q_1 + \omega_0^2 q_1 = {} & -2i\omega_0\left(D_1 A + \frac{1}{2}\tilde{C}A\right)\exp i\omega_0 T_0 \\ & - \left(\frac{3}{4}\chi A^2\bar{A} + \left(i\omega_0\right)^{\alpha}\tilde{C}_P A + \frac{1}{2}\gamma\bar{A}\bar{F}\exp\left(\sigma T_1\right)\right)\exp i\omega_0 T_0 + cc + NST, \end{aligned} \tag{28}$$

where cc and NST represent the complex conjugate and the non-secular term, respectively. The solvability condition for Equation (28) is obtained as follows:

$$2i\omega_0\left(D_1A+\frac{1}{2}\tilde{C}A\right)+\frac{3}{4}\chi A^2\bar{A}+(i\omega_0)^{\alpha}\tilde{C}_PA+\frac{1}{2}\gamma\bar{A}\bar{F}\exp\left(\sigma T_1\right)=0. \tag{29}$$

Taking into account the real amplitude a and phase β, the complex amplitude A can be written as:

$$A=a\left(T_1\right)\exp i\beta\left(T_1\right). \tag{30}$$

Then, the amplitude and phase modulation equations are:

$$D_1a+\frac{1}{2}\tilde{C}a+\frac{1}{2}\omega_0^{\alpha-1}\tilde{C}_Pa\sin\frac{\alpha\pi}{2}+\frac{1}{4}\frac{\gamma a\bar{F}}{\omega_0}\sin\psi=0, \tag{31a}$$

$$D_1\beta-\frac{3\chi}{8\omega_0}a^2-\frac{1}{2}\omega_0^{\alpha-1}\tilde{C}_P\cos\frac{\alpha\pi}{2}-\frac{1}{4}\frac{\gamma\bar{F}}{\omega_0}\cos\psi=0, \tag{31b}$$

in which $\psi=\sigma T_1-2\beta$ is the new phase angle. In the steady-case, Equation (31) will be solved in a further section.

4. Numerical Results

Numerical examples of frequencies are presented in this section. The linear fundamental frequencies for different kinds of boundary conditions will be evaluated, and the fractional order nonlinear frequencies for free vibrations will also be evaluated in the case of the steady-state. To show the correctness of the presented study, we compared the obtained results to the results proposed by Mustapha and Zhong [17], Yokoyama [43] and Togun et al. [28]. A detailed parametric study was conducted to investigate the effects of system parameters such as stiffness, damping, the nonlocal parameter and fractional parameter on the dimensional fractional order nonlinear natural frequencies of the nanobeam with simple-simple boundary conditions and the frequency response curve obtained by the perturbation method. For free vibration $\bar{F}=0$, in the case of the steady-state, we obtain:

$$D_1a=0\Rightarrow a=a_0. \tag{32}$$

By introducing Equation (32) into Equation (31b), we get:

$$\beta\left(T_1\right)=\left(\frac{3\chi}{8\omega_0}a_0^2+\frac{1}{2}\omega_0^{\alpha-1}\tilde{C}_P\cos\frac{\alpha\pi}{2}\right)T_1+\beta_0, \tag{33}$$

where a_0 and β_0 are the constants' steady-state real amplitude and phase, which are determined from the initial conditions. Introducing the obtained results into Equation (26) gives the first order vibration response:

$$q_0\left(T_0,T_1\right)=a_0\exp i\left(\frac{3\chi}{8\omega_0}a_0^2+\frac{1}{2}\omega_0^{\alpha-1}\tilde{C}_P\cos\frac{\alpha\pi}{2}\right)\epsilon\bar{t}\times\ \exp i\left(\omega_0\bar{t}+\beta_0\right)+cc, \tag{34}$$

and hence, the fractional order nonlinear frequency is:

$$\omega_{nl}^{(\alpha)}=\omega_0+D_1\beta=\omega_0+\epsilon\frac{3\chi}{8\omega_0}a_0^2+\epsilon\frac{1}{2}\omega_0^{\alpha-1}\tilde{C}_P\cos\frac{\alpha\pi}{2}, \tag{35}$$

where $\lambda = \frac{3\chi}{8\omega_0}$ is the nonlinear correction coefficient and the third term is a correction of the natural frequency due to the fractional order damping term. At the steady-state, $D_1 a = 0$ and $D_1 \psi = 0$. The detuning parameter or the amplitude-frequency response is as follows:

$$\sigma = \frac{3\chi}{4\omega_0} a_0^2 + \omega_0^{\alpha-1} \tilde{C}_P \cos\frac{\alpha\pi}{2} \pm \sqrt{\frac{1}{4}\frac{\gamma^2 \bar{F}^2}{\omega_0^2} - \left(\tilde{C} + \omega_0^{\alpha-1} \tilde{C}_P \sin\frac{\alpha\pi}{2}\right)^2}. \tag{36}$$

4.1. Validation Study

Studies related to the nonlinear nonlocal nanobeam resting on a Winkler–Pasternak viscoelastic foundation in the literature are very limited. In order to validate the present analytical results for the amplitude-frequency response of the dynamical fractional order nonlinear nonlocal nano-beam with the simple-simple boundary condition, we compared the obtained results proposed by Mustapha and Zhong [17], Yokoyama [43] and Togun et al. [28]. Let us consider the case of free vibration and only the classical damping influence ($\alpha = 1$); thus, the fractional order correction to the natural frequency is absent in Equations (35) and (36), and then, we recognize the common form of nonlinear frequency and the detuning parameter:

$$\omega_{nl} = \omega_0 \left(1 + \epsilon \frac{3\chi}{8\omega_0^2} a_0^2\right), \tag{37}$$

$$\sigma = \frac{3\chi}{4\omega_0} a_0^2 \pm \sqrt{\frac{1}{4}\frac{\gamma^2 \bar{F}^2}{\omega_0^2} - \mu^2}, \tag{38}$$

where $\lambda = \frac{3\chi}{8\omega_0}$ is the nonlinear correction coefficient and $\mu = \tilde{C} + \tilde{C}_P$ the damping coefficient. The work of Mustapha and Zhong [17] studied the non-uniform SWCNT depending on a nonlocal Rayleigh beam resting on a Pasternak-type foundation. Yokoyama [43] studied the free transverse vibration of the classical Euler–Bernouilli beam resting on a Winkler–Pasternak foundation, and Togun et al. [28] studied the nonlinear vibration of a nonlocal nanobeam on a Winkler–Pasternak foundation using Euler–Bernouilli beam theory. A comparison study was performed to check the correctness of the present study. To this aim, the linear frequency of the local case of our nanobeam resting on a Winkler–Pasternak foundation for the simple-simple boundary condition was compared to those of the work of Mustapha and Zhong [17], Yokoyama [43] and Togun et al. [28]. It can be seen from Tables 1 and 2 that there is good harmony among the four results.

Figure 4 shows the nonlocal parameter effect on the fractional nonlinear frequency; we observe that the natural frequency decreases when the nonlocal parameter increases. The variation of the fractional nonlinear frequency with the amplitude for the first three modes of vibration is shown in Figure 5; this figure shows that the fractional nonlinear frequencies increase with an increase in the mode number.

Table 1. The first five non-dimensional natural frequencies of a local Euler–Bernouilli beam resting on a Winkler–Pasternak foundation for the simple-simple boundary condition ($\eta = 0$, $\delta = 0$, $K = 25$, $K_p = 25$).

Mode	Present	Ref. [28]	Ref. [17]	Ref. [43]
1	19.2133	19.2133	19.2178	19.21
2	50.7002	50.7002	50.7804	50.71
3	100.6767	100.677	-	-
4	170.0281	170.028	-	-
5	258.9868	258.987	-	-

Table 2. The first five non-dimensional natural frequencies of a local Euler–Bernouilli beam resting on a Winkler–Pasternak foundation for the simple-simple boundary condition ($\eta = 0$, $\delta = 0$, $K = 36$, $K_p = 36$).

Mode	Present	Ref. [28]	Ref. [17]	Ref. [43]
1	22.1069	22.1069	22.1112	-
2	54.9160	54.916	55.1873	-
3	105.4698	105.47	-	-
4	175.0932	175.093	-	-
5	264.1956	264.196	-	-

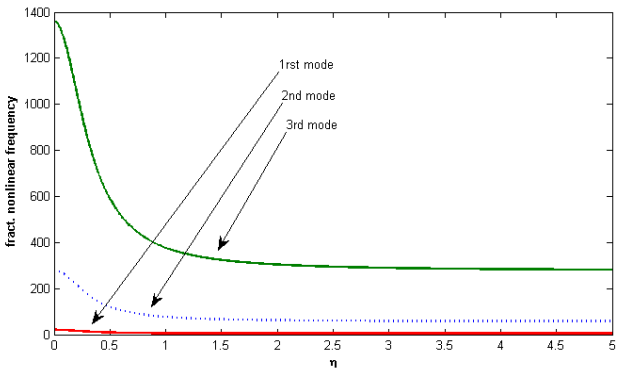

Figure 4. First three modes of the fractional nonlinear frequency versus nonlocality η ($\alpha = 0.5$, $K = 5$, $Kp = 2$, $Cp = 0.001$).

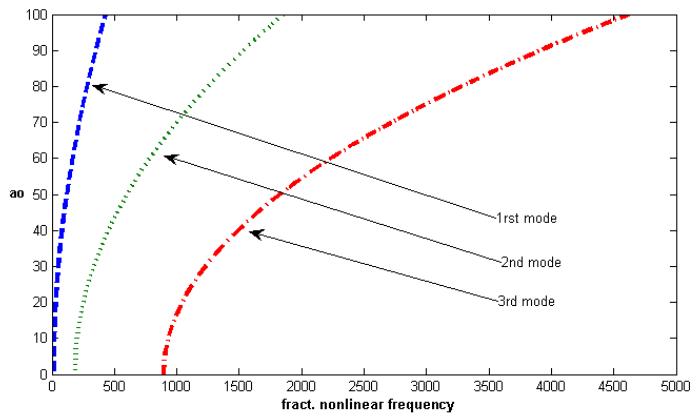

Figure 5. First three modes of the fractional nonlinear frequency versus amplitude ($\alpha = 0.5$, $K = 5$, $Kp = 2$, $Cp = 0.001$, $\eta = 0.5$).

In Figures 6–8, the fractional nonlinear frequency versus amplitude for different values of the system parameter are shown for the first mode of vibration. Figure 6 shows the effect of Pasternak parameter Kp on the fractional nonlinear frequency versus amplitude curves, and we can observe in this figure that the fractional nonlinear frequency increases with an increase of Kp. In Figure 7, the fractional nonlinear frequency also increases with an increase of Winkler stiffness parameter K. In Figure 8, the fractional nonlinear frequency versus amplitude for different values of fractional damping coefficient Cp is drawn, and we can notice that the fractional nonlinear frequency increases slowly when the fractional damping coefficient increases. This is normal because the fractional nonlinear frequency has a direct relation to Cp. Furthermore a hardening behavior can be observed in Figures 6–8 because the fractional nonlinear frequency increases as the amplitude increases.

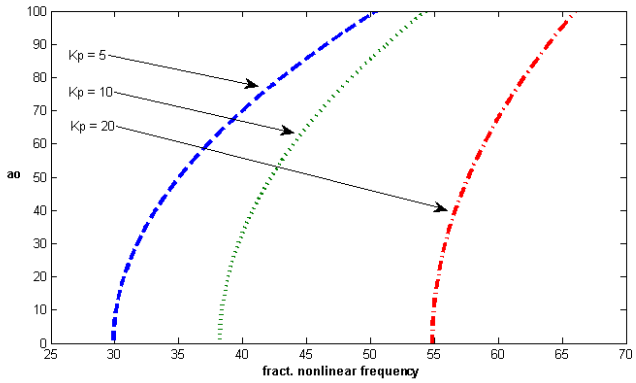

Figure 6. Fractional nonlinear frequency versus amplitude for different values of Kp ($\alpha = 0.5$, $K = 100$, $Cp = 0.001$, $\eta = 0.5$).

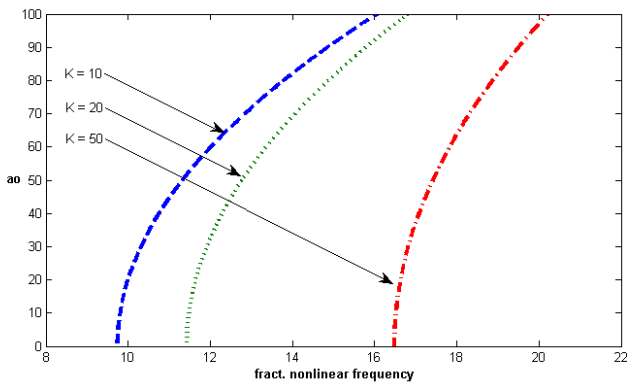

Figure 7. Fractional nonlinear frequency versus amplitude for different values of K ($\alpha = 0.5$, $Kp = 5$, $Cp = 0.001$, $\eta = 0.5$).

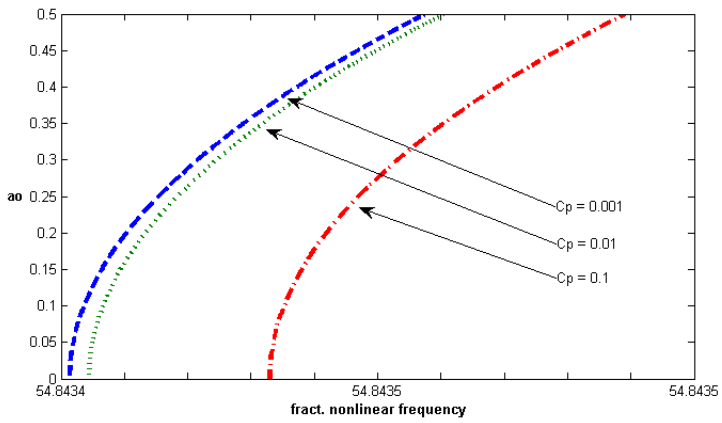

Figure 8. Fractional nonlinear frequency versus amplitude for different values of Cp ($\alpha = 0.5$, $Kp = 5$, $K = 100$, $\eta = 0.5$).

Frequency response curves are presented in Figure 9 for different values of the nondimensional nonlinear coefficient. In this figure, nonlinearity is actually observed. In Figures 10–12, the fractional contribution frequency versus Winkler parameter K and nonlocal parameter η for different values of fractional parameter α are shown. It can be seen from Figures 10–12 that the fractional contribution frequency increases and reaches the constant maximum value when the nonlocal parameter increases.

For the small values of the nonlocal parameter, the fractional contribution increases quickly, but for the high values, this contribution is constant.

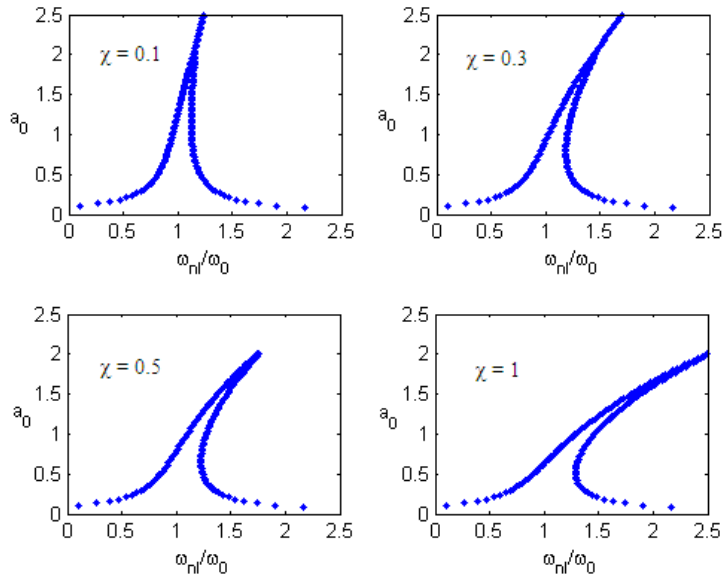

Figure 9. Frequency-response curves versus amplitude for different values of χ ($\alpha = 1$, $C = 0.025$, $Cp = 0.025$, $wo = 1$, $\bar{F} = 0.2$).

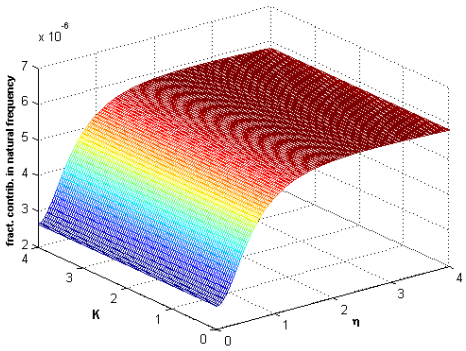

Figure 10. Fractional contribution frequency versus stiffness K and nonlocality η ($\alpha = 0.2$).

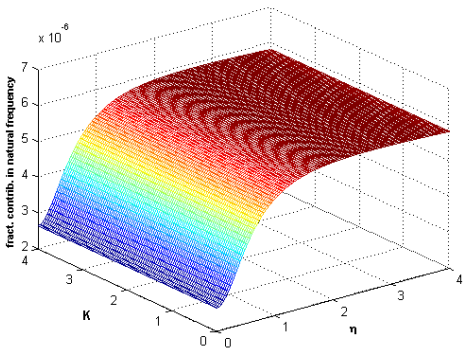

Figure 11. Fractional contribution frequency versus stiffness K and nonlocality η ($\alpha = 0.5$).

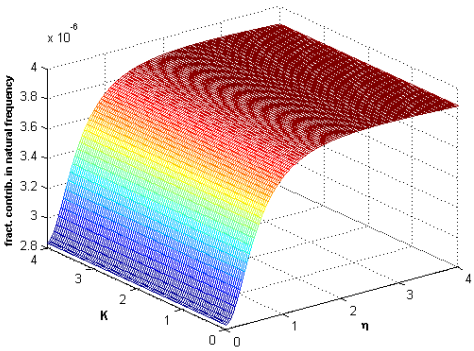

Figure 12. Fractional contribution frequency versus stiffness K and nonlocality η ($\alpha = 0.8$).

In Figures 13 and 14, the fractional contribution frequency versus Pasternak parameter Kp and nonlocal parameter η curves for different values of fractional parameter α are drawn. It is observed that the variation of the fractional contribution depends on the interval of variation of the nonlocal parameter η. For a small value of η, the fractional contribution increases, but for a high value of η, this fractional contribution decreases.

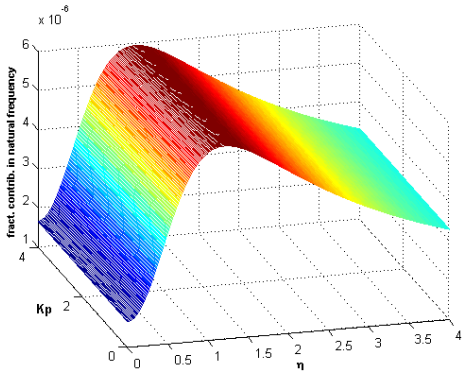

Figure 13. Fractional contribution frequency versus stiffness Kp and nonlocality η ($\alpha = 0.2$).

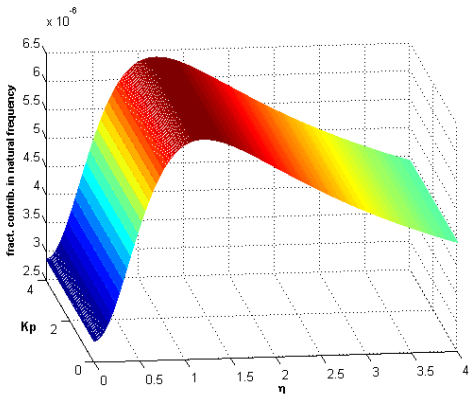

Figure 14. Fractional contribution frequency versus stiffness Kp and nonlocality η ($\alpha = 0.5$).

In Figures 15–17, the fractional contribution frequency versus fractional damping coefficient Cp and nonlocal parameter η curves are shown for different values of fractional parameter α. It can be

seen that the fractional contribution increases when Cp increases. In front of all these observations, it is normally easy to say that every system parameter has a significant effect on the natural frequency of the nanobeam, especially the fractional parameter and fractional damping coefficient.

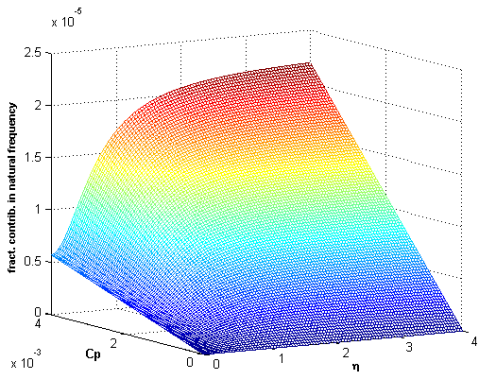

Figure 15. Fractional contribution frequency versus fractional damping coefficient Cp and nonlocality η ($\alpha = 0.2$).

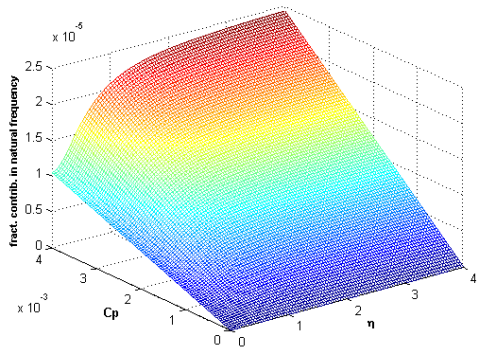

Figure 16. Fractional contribution frequency versus fractional damping coefficient Cp and nonlocality η ($\alpha = 0.5$).

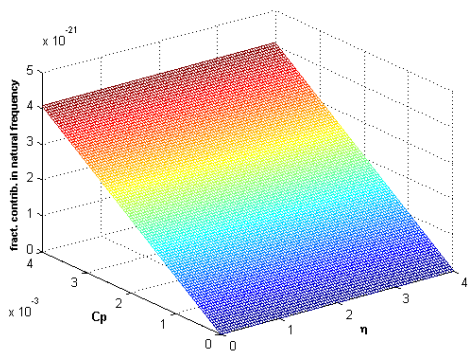

Figure 17. Fractional contribution frequency versus fractional damping coefficient Cp and nonlocality η ($\alpha = 1$).

5. Conclusions

In this study, using the fractional derivative, the nonlinear vibration of a nanobeam resting on a fractional order viscoelastic Winkler–Pasternak foundation is studied. For this purpose, Eringen's nonlocal elasticity theory, the von Karman geometric nonlinearity and the Euler–Bernouilli beam theory are employed. The D'Alembert principle is used to derive the governing equation. In the solution procedure, using the Galerkin scheme, the fractional integro-partial differential governing equation is first simplified into the time-dependent fractional ordinary differential equation. This new equation is known as the fractional order nonlinear Duffing equation which is then solved by the multiple scales method. A detailed parametric study is conducted to get the effects of the system parameter such as the Winkler stiffness parameter, Pasternak stiffness parameter, nonlocal parameter, nonlinear coefficient, fractional damping coefficient and fractional parameter on the fractional nonlinear frequency of the nanobeam. It is found that fractional nonlinear frequency decreases when the nonlocal parameter increases. Furthermore, this fractional nonlinear frequency increases when the Winkler parameter, Pasternak parameter, mode, fractional damping coefficient and amplitude increase. It is further found that every parameter of the system has a significant effect on the fractional contribution frequency.

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Abbreviations

The following abbreviations are used in this manuscript:

CNT	Carbon nanotube
SDCNT	Single-walled carbon nanotube
DWCNT	Double-walled carbon nanotube

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