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Implementation and Convergence Analysis of Homotopy Perturbation Coupled With Sumudu Transform to Construct Solutions of Local-Fractional PDEs

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Abstract: In the present paper, the explicit solutions of some local fractional partial differential equations are constructed through the integration of local fractional Sumudu transform and homotopy perturbation such as local fractional dissipative and damped wave equations. The convergence aspect of this technique is also discussed and presented. The obtained results prove that the employed method is very simple and effective for treating analytically various kinds of problems comprising local fractional derivatives.

Keywords: homotopy perturbation coupled with Sumudu transform technique; local fractional derivative; converegnce analysis

MSC: 26A33; 65H20; 35A22

1. Introduction

In recent years, fractional calculus is considered to be a fascinating field of research, due to the wide applications of fractional integrals and derivatives in mathematical modeling of systems and processes in many fields of engineering and science [1–13]. Generally speaking, most of fractional differential equations are not solvable toward exact solutions. Therefore, numerous analytical and numerical methods were successfully utilized to treat this sort of problems. Among the aforementioned methods, we can refer to fractional Homotopy Perturbation [14], fractional Adomian decomposition [15], Yang-Laplace transform [16], Variational Iteration and function decomposition in local fractional sense [15,17].

Homotopy Perturbation coupled with Sumudu transform technique is an integration between two powerful methods: Sumudu transform, which was proposed by Watugala [18] in 1993, and Homotopy perturbation, which was introduced by He [19,20]. This combined method was applied to solve fractional nonlinear problems, arising in the field of nonlinear sciences such as engineering and mathematical physics. Several researchers used this simple tool to obtain solutions of nonlinear differential equations. For example, both Singh and Kumar [21] solved the nonlinear fractional gas dynamics equation. On the other hand, Sharma and Singh [22] derived the solutions for the fractional nonlinear partial differential equations. As far as Ait touchent and Belgacem [23] are concerned, they presented the solutions for the system of nonlinear fractional PDEs. For Eltayeb A. Youssif [24], he found the exact solution for nonlinear Schrodinger equation.

In recent years, not only was the local fractional analysis theory given much more consideration and concern, but also it was well employed for describing non-differentiable problems appearing in different sciences. For example, local fractional Laplace equation [15], diffusion equations on Cantor sets [25], Korteweg-ed Vries equation with local fraction operator [26] and fractal heat conduction equation [27] as well as fractal wave equation [28].

The aim of this paper is to implement the combination of local fractional Sumudu transform and homotopy perturbation in order to get analytical solutions of some local fractional problems in mathematical physics, for example dissipative and damped wave equations. Furthermore, we prove the convergence analysis and we show the advantages of this method for constructing solutions of local-fractional PDEs.

This article is divided into seven sections: In Section 2, we introduce some preliminaries about local fractional calculus. Section 3, is dedicated to present the local fractional Sumudu transform. In Section 4, the main steps of homotopy perturbation coupled with local fractional Sumudu transform are mentioned. Analysis on convergence with some examples are given in Sections 5 and 6, are followed by the conclusion in Section 7.

2. Local Fractional Calculus Preliminaries

The following section sheds light on some necessary definitions of local fractional calculus exploited in the present article.

Definition 1. *(see [29]) We say that the function g is continuous in local fractional sense at x_0, if*

$$\mid g(x) - g(x_0) \mid < \epsilon^r, \qquad 0 < r \leq 1, \tag{1}$$

with $\mid x - x_0 \mid < \delta$, *for* $\epsilon > 0$ *and* $\epsilon \in \mathbb{R}$.

Definition 2. *(see [29]) The derivative of g in local fractional sense at x_0 is presented as*

$$D_x^r g(x_0) = \frac{d^r g(x_0)}{dx^r} = \lim_{x \to x_0} \frac{\Delta^r(g(x) - g(x_0))}{(x - x_0)^r}, \tag{2}$$

where $\Delta^r(g(x) - g(x_0)) \cong \Gamma(r+1)\Delta(g(x) - g(x_0))$.

The formula of local fractional derivative of high order is given by

$$g^{(kr)}(x) = \overbrace{D_x^r D_x^r ... D_x^r}^{k \quad times} g(x). \tag{3}$$

The expression of local fractional partial differential operator of order $r \quad (0 < r < 1)$ has the form:

$$\frac{\partial^r}{\partial t^r} g(x_0, t) = \frac{\Delta^r(g(x_0, t) - g(x_0, t_0))}{(t - t_0)^r}, \tag{4}$$

where $\Delta^r(g(x_0, t) - g(x_0, t_0)) \cong \Gamma(r+1)\Delta(g(x_0, t) - g(x_0, t_0))$.

Moreover, the form of local fractional partial derivative of high order is presented below:

$$\frac{\partial^{kr}}{\partial x^{kr}} g(x, y) = \overbrace{\frac{\partial^r}{\partial x^r} \frac{\partial^r}{\partial x^r} \cdots \frac{\partial^r}{\partial x^r}}^{k \quad times} g(x, y). \tag{5}$$

The formula of local fractional derivative applied in this work, is in the following form [30]:

$$\frac{d^r}{dx^r} \frac{x^{nr}}{\Gamma(1 + nr)} = \frac{x^{(n-1)r}}{\Gamma(1 + (n-1)r)}, \quad n \in N. \tag{6}$$

Definition 3. *In fractal space, the Mittag–Leffler function is defined as*

$$E_r(x^r) = \sum_{m=0}^{\infty} \frac{x^{mr}}{\Gamma(1+mr)}, \qquad 0 < r \leq 1. \tag{7}$$

3. Local Fractional Sumudu Transform

The integral transform method named Sumudu transform was proposed in 1993 by Watugula, who applied it to obtain solutions for problems in mathematical physics. Some fundamental properties of this transform were introduced and investigated by Belgacem [31]. Katatbeh and Belgacem [32] put into practice this method to get solutions for fractional differential equations. Also Ait touchent and Belgacem used the combination of Sumudu transform and homotopy perturbation for handling nonlinear fractional PDEs [23]. In this part, we define the Sumudu transform in local fractional sense and we give some of its important properties.

The definition of this transform of function g is given by Srivastava et al. [29] as follows:

$$\begin{aligned} LFS_r[g(x)] &= G_r(z) \\ &= \frac{1}{\Gamma(1+r)} \int_0^{\infty} E_r(-z^{-r}x^r) \frac{g(x)}{z^r} (dx)^r, \qquad 0 < r \leq 1, \end{aligned} \tag{8}$$

where the integral of g in local fractional sense is given by [29]

$$_aI_b^r g(x) = \frac{1}{\Gamma(1+r)} \int_a^b g(x)(dx)^r = \frac{1}{\Gamma(1+r)} \lim_{\Delta x \to 0} \sum_{i=0}^{M-1} g(x_i)(\Delta x_i)^r, \tag{9}$$

where the partitions of the interval $[a, b]$ are $(x_i, x_{i+1}), i = 0, ..., M-1, \Delta x = x_{i+1} - x_i$.

The inverse of local fractional Sumudu transform is provided by

$$LFS_r^{-1}[G_r(z)] = g(x), \qquad 0 < r \leq 1. \tag{10}$$

The local fractional derivative is transformed by the local fractional Sumudu transform as is shown below [29]:

$$LFS_r\left[\frac{d^r g(x)}{dx^r}\right] = \frac{G_r(z) - g(0)}{z^r}, \tag{11}$$

where $G_r(z) = LFS_r[g(x)]$. Also, we have the following results [29]:

$$LFS_r\left[\frac{d^{nr} g(x)}{dx^{nr}}\right] = \frac{1}{z^{nr}}\left[G_r(z) - \sum_{k=0}^{n-1} z^{kr} g^{(kr)}(0)\right]. \tag{12}$$

When $n = 2$, from expression (12) we get

$$LFS_r\left[\frac{d^{2r} g(x)}{dx^{2r}}\right] = \frac{1}{z^{2r}}\left[G_r(z) - g(0) - z^r g^{(r)}(0)\right], \tag{13}$$

with $G_r(z) = LFS_r[g(x)]$.

4. Local Fractional Sumudu Transform Coupled with Homotopy Perturbation (LHPSTM)

The following section is dedicated to the introduction of the main steps of the exploited method (LHPSTM) [33]. To illustrate this, we consider the next local fractional differential problem:

$$Lw(x,t) + Rw(x,t) = h(x,t), \quad 0 < x < 1, \quad 0 < t < 1, \tag{14}$$

where L and R represent respectively local fractional differential linear operator and nonlinear operator and the function h is a source term.

Applying the technique of local fractional Sumudu transform on both sides of Equation (14) we obtain

$$w_r(x,z) = w(x,0) - z^r LFS_r[Rw(x,t)] + z^r LFS_r[h(x,t)]. \tag{15}$$

The inverse local fractional Sumudu transform method implies that

$$w(x,t) = w(x,0) - LFS_r^{-1}\left[z^r LFS_r\left[Rw(x,t)\right]\right] + LFS_r^{-1}\left[z^r LFS_r\left[h(x,t)\right]\right]. \tag{16}$$

By the homotopy perturbation for (16), we get:

$$H(w,p) = w(x,t) - w(x,0) + p \times \left[LFS_r^{-1}\left[z^r LFS_r\left[Rw(x,t)\right]\right]\right] - LFS_r^{-1}\left[z^r LFS_r\left[h(x,t)\right]\right] = 0, \tag{17}$$

where $p \in [0,1]$ is a homotopy parameter, which gives

$$w(x,t) = w(x,0) - p \times \left[LFS_r^{-1}\left[z^r LFS_r\left[Rw(x,t)\right]\right]\right] + LFS_r^{-1}\left[z^r LFS_r\left[h(x,t)\right]\right], \tag{18}$$

and

$$w(x,t) = \sum_{n=0}^{\infty} p^n w_n(x,t), \tag{19}$$

and the nonlinear term is decomposed as:

$$Rw(x,t) = \sum_{n=0}^{\infty} p^n H_n(w), \tag{20}$$

for some He's polynomials $H_n(w)$ that are given by:

$$H_n(w_0, w_1, ..., w_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[R\sum_{i=0}^{n} p^i w_i\right]_{p=0}, \quad n = 0,1,2,... \tag{21}$$

Substituting Equations (19) and (20) in Equation (18), we obtain:

$$\begin{aligned} \textstyle\sum_{n=0}^{\infty} p^n w_n(x,t) \;=\;& w(x,0) - p \times \left[LFS_r^{-1}\left[z^r LFS_r\left[\textstyle\sum_{n=0}^{\infty} p^n H_n(w)\right]\right]\right] \\ &+ LFS_r^{-1}\left[z^r LFS_r\left[h(x,t)\right]\right], \end{aligned} \tag{22}$$

which is the combination of local fractional Sumudu transform and HPM with He's polynomials.

Making a comparison of the terms having same power of p, we achieve:

$$\begin{aligned} p^0 : w_0(x,t) &= w(x,0) + LFS_r^{-1}[z^r LFS_r[h(x,t)]], \\ p^1 : w_1(x,t) &= -LFS_r^{-1}[z^r LFS_r[H_0(w)]], \\ p^2 : w_2(x,t) &= -LFS_r^{-1}[z^r LFS_r[H_1(w)]], \\ &. \\ &. \\ &. \end{aligned} \tag{23}$$

Finally, the analytical solution of Equation (14) has this structure:

$$w(x,t) = \lim_{N\to\infty} \sum_{n=0}^{N} w_n(x,t). \tag{24}$$

5. Analysis on Convergence

The goal of this section is to demonstrate the convergence theorem which is used further in this work.

Lemma 1. *$C_r([0,1],\mathbb{R})$ is a Banach space of local fractional continuous functions on the interval $[0,1]$ with the norm*

$$\parallel w \parallel = \max_{t\in[0,1]} |w(x,t)|. \tag{25}$$

Proof. Let $(f_n)_{n\in\mathbb{N}}$ a Cauchy sequence in $C_r([0,1],\mathbb{R})$. For t fixed in $[0,1]$.

$$\forall p,q \in \mathbb{N}, \text{ we have } |f_p(t) - f_q(t)| \leq \|f_p - f_q\| < \epsilon^r, \tag{26}$$

which implies that $f_n(t)$ is a Cauchy sequence in $\mathbb{R}$. While $\mathbb{R}$ is complete, then $f_n(t)$ is convergent.

Let $f(t)$ its limit, where the function $f : [0,1] \to \mathbb{R}$ verify:

$$f(t) = \lim_{n\longrightarrow\infty} f_n(t) \qquad \forall t \in [0,1]. \tag{27}$$

Now, we prove that f is local fractional continuous. The sequence $(f_n)_{n\in\mathbb{N}}$ is a sequence of local fractional continuous functions. Therefore

$$|f_n(t) - f_n(t_0)| < \epsilon^r \quad \forall t \in [0,1], \text{ with } |t - t_0| < \delta. \tag{28}$$

Using limit we get

$$|f(t) - f(t_0)| < \epsilon^r \qquad t \in [0,1]. \tag{29}$$

Hence the function $f \in C_r([0,1],\mathbb{R})$.

Let's prove that $(f_n)_{n\in\mathbb{N}}$ converges to f in $C_r([0,1],\mathbb{R})$. There exist N such that:

$$\forall p,q \geq N \quad \|f_p - f_q\| < \epsilon^r. \tag{30}$$

Therefore, for fixed t in $[0,t]$, we have:

$$\forall p \geq N, \forall q \geq N \qquad |f_p(t) - f_q(t)| \leq \|f_p - f_q\| < \epsilon^r. \tag{31}$$

Now, we fixe p and we let q to infinity in the last expression, we get $|f_p(t) - f(t)| < \epsilon^r$ which is true for every $t \in [0,1]$, we have $\|f_p - f\| < \epsilon^r$ for every $p \geq N$, then f_p converges to f.

Finally, every Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ converges in $C_r([0,1],\mathbb{R})$, then $C_r([0,1],\mathbb{R})$ is complete. Therefore $C_r([0,1],\mathbb{R})$ is a Banach space. □

Theorem 1. *Let $(w_n)_{n\in\mathbb{N}}$ in Banach space $C_r([0,1],\mathbb{R})$. If $\parallel w_{k+1} \parallel \leq q \parallel w_k \parallel$ for $0 < q < 1$ then, the series $\sum_{n=0}^{\infty} w_n$ is convergent.*

Theorem 2. *If the series solution $\sum_{n=0}^{\infty} w_n$ given in (24) is convergent then, it is an exact solution of the local fractional problem (14).*

Proof. We determine the sequence

$$\mathcal{A}_n = w_1 + w_2 + ... + w_n, \tag{32}$$

by using the iterative scheme

$$\mathcal{A}_0 = 0, \tag{33}$$

$$\mathcal{A}_{n+1} = w(x,0) - LFS^{-1}\left[z^r LFS\left[R_n(\mathcal{A}_n + w_0)\right]\right] + LFS^{-1}\left[z^r LFS\left[h\right]\right], \tag{34}$$

where

$$R_n\left(\sum_{i=0}^{n} w_i\right) = \sum_{i=0}^{n} H_i, \quad n = 0,1,2,... \tag{35}$$

Suppose that the series solution (24) converges, and $\psi = \sum_{n=0}^{\infty} w_n$, then we have,

$$\lim_{n\to\infty} \mathcal{A}_{n+1} = w(x,0) - LFS^{-1}\left[z^r LFS\left[\lim_{n\to\infty} R_n(\mathcal{A}_n + w_0)\right]\right] + LFS^{-1}\left[z^r LFS\left[h\right]\right], \tag{36}$$

then we obtain

$$\begin{aligned}\psi =& w(x,0) - LFS^{-1}\left[z^r LFS\left[\lim_{n\to\infty} R_n \sum_{i=0}^{n} w_i\right]\right] + LFS^{-1}\left[z^r LFS\left[h\right]\right],\\ =& w(x,0) - LFS^{-1}\left[z^r LFS\left[\lim_{n\to\infty} \sum_{i=0}^{n} H_i\right]\right] + LFS^{-1}\left[z^r LFS\left[h\right]\right],\\ =& w(x,0) - LFS^{-1}\left[z^r LFS\left[\sum_{i=0}^{\infty} H_i\right]\right] + LFS^{-1}\left[z^r LFS\left[h\right]\right],\end{aligned}$$

using (20), for $p = 1$, we get

$$\psi = w(x,0) - LFS^{-1}\left[z^r LFS\left[R\sum_{i=0}^{\infty} w_i\right]\right] + LFS^{-1}\left[z^r LFS\left[h\right]\right], \tag{37}$$

then

$$\psi = w(x,0) - LFS^{-1}\left[z^r LFS\left[R\psi\right]\right] + LFS^{-1}\left[z^r LFS\left[h\right]\right], \tag{38}$$

applying the LFHST on both sides of Equation (37) then, we obtain,

$$\frac{LFS\left[\psi\right] - w(x,0)}{z^r} = LFS\left[R\psi\right] + LFS\left[h\right], \tag{39}$$

$$LFS\left[L\psi(x,t)\right] = LFS\left[R\psi(x,t)\right] + LFS\left[h(x,t)\right], \tag{40}$$

Now, by applying the inverse of LFHST we obtain,

$$L\psi(x,t) + R\psi(x,t) = h(x,t), \tag{41}$$

therefore, $\psi = \sum_{n=0}^{\infty} w_n$ is an exact solution of Equation (14). □

6. Application

6.1. Exemple 1

Considering the following diffusion equation involving local fractional derivative

$$D_t^r w(x,t) = \frac{1}{2}x^2 w_{xx}(x,t), \qquad 0 < x < 1, 0 < r \leq 1, \tag{42}$$

with the next conditions:

$$\begin{aligned} w(x,0) &= x^2, \\ w(0,t) &= 0, \\ w(1,t) &= E_r(t^r), \end{aligned} \tag{43}$$

where the expression of exact solution is $w(x,t) = x^2 E_r(t^r)$.

Employing the expression of local fractional Sumudu transform for Equation (42) we have

$$LFS_r[D_t^r w(x,t)] = LFS_r\left[\frac{1}{2}x^2 w_{xx}(x,t)\right], \tag{44}$$

then, we obtain

$$w_r(x,z) = w(x,0) + z^r LFS_r\left[\frac{1}{2}x^2 w_{xx}(x,t)\right], \tag{45}$$

which gives

$$w_r(x,z) = x^2 + z^r LFS_r\left[\frac{1}{2}x^2 w_{xx}(x,t)\right]. \tag{46}$$

The inverse of local fractional Sumudu transform method implies that

$$w(x,t) = x^2 + LFS_r^{-1}\left[z^r LFS_r\left[\frac{1}{2}x^2 w_{xx}(x,t)\right]\right]. \tag{47}$$

The homotopy perturbation gives

$$\sum_{n=0}^{\infty} p^n w_n(x,t) = x^2 + p \times LFS_r^{-1}\left[z^r LFS_r\left(\frac{1}{2}x^2 \sum_{n=0}^{\infty} p^n (w_n)_{xx}(x,t)\right)\right]. \tag{48}$$

By comparison of the terms owning similar power of p, we get:

$$\begin{aligned} p^0 : w_0(x,t) &= x^2, \\ p^1 : w_1(x,t) &= LFS_r^{-1}\left[z^r LFS_r(\tfrac{1}{2}x^2(w_0)_{xx}\right] = \tfrac{x^2 t^r}{\Gamma(r+1)}, \\ p^2 : w_2(x,t) &= LFS_r^{-1}\left[z^r LFS_r(\tfrac{1}{2}x^2(w_1)_{xx}\right] = \tfrac{x^2 t^{2r}}{\Gamma(2r+1)}, \\ &\cdot \\ &\cdot \\ &\cdot \\ p^n : w_n(x,t) &= LFS_r^{-1}\left[z^r LFS_r(\tfrac{1}{2}x^2(w_{n-1})_{xx}\right] = \tfrac{x^2 t^{nr}}{\Gamma(nr+1)}. \end{aligned} \tag{49}$$

Therefore, the solution is formed as shown below:

$$w(x,t) = x^2\left[1 + \frac{t^r}{\Gamma(r+1)} + \frac{t^{2r}}{\Gamma(2r+1)} + \frac{t^{3r}}{\Gamma(3r+1)} + ... + \frac{t^{nr}}{\Gamma(nr+1)} + ...\right]. \tag{50}$$

Hence

$$w(x,t) = x^2 \sum_{m=0}^{\infty} \frac{(t^r)^m}{\Gamma(mr+1)}. \tag{51}$$

Finally, we come up with the following:

$$w(x,t) = x^2 E_r(t^r), \tag{52}$$

which is the exact solution of Equation (42).

6.2. Example 2: Local Fractional Dissipative Wave Equation

Considering the next dissipative wave equation including local fractional derivative [34]:

$$D_{tt}^{2r}w(x,t) - D_t^r w(x,t) - D_{xx}^{2r}w(x,t) - D_x^r w(x,t) - \frac{t^r}{\Gamma(r+1)} = 0 \qquad 0 \le x \le l, t > 0, \tag{53}$$

with the initial conditions

$$\begin{aligned} w(x,0) &= \frac{x^r}{\Gamma(1+r)} \\ D_t^r w(x,0) &= 0. \end{aligned} \tag{54}$$

Using the Sumudu transform in local fractional sense for Equation (53) we obtain

$$LFS_r\left[D_{tt}^{2r}w(x,t)\right] = LFS_r\left[D_t^r w(x,t) + D_{xx}^{2r}w(x,t) + D_x^r w(x,t) + \frac{t^r}{\Gamma(r+1)}\right]. \tag{55}$$

Then, we get

$$w_r(x,z) = w(x,0) + z^{2r} LFS_r\left[D_t^r w(x,t) + D_{xx}^{2r}w(x,t) + D_x^r w(x,t) + \frac{t^r}{\Gamma(r+1)}\right], \tag{56}$$

which implies

$$w_r(x,z) = \frac{x^r}{\Gamma(1+r)} + z^{2r} LFS_r\left[D_t^r w(x,t) + D_{xx}^{2r}w(x,t) + D_x^r w(x,t) + \frac{t^r}{\Gamma(r+1)}\right]. \tag{57}$$

By the inverse local fractional Sumudu transform, it yields

$$\begin{aligned} w(x,t) &= \tfrac{x^r}{\Gamma(1+r)} + LFS_r^{-1}\left[z^{2r} LFS_r\left[D_t^r w(x,t) + D_{xx}^{2r}w(x,t)\right.\right. \\ &+ \left.\left. D_x^r w(x,t) + \tfrac{t^r}{\Gamma(r+1)}\right]\right]. \end{aligned} \tag{58}$$

Now, the use of homotopy perturbation gives

$$\begin{aligned} \textstyle\sum_{n=0}^{\infty} p^n w_n(x,t) &= \tfrac{x^r}{\Gamma(1+r)} + p \times LFS_r^{-1}\left[z^{2r} LFS_r\left[D_t^r \textstyle\sum_{n=0}^{\infty} p^n w_n(x,t)\right.\right. \\ &+ \left.\left. D_{xx}^{2r} \textstyle\sum_{n=0}^{\infty} p^n w_n(x,t) \;\; D_x^r \textstyle\sum_{n=0}^{\infty} p^n w_n(x,t) + \tfrac{t^r}{\Gamma(r+1)}\right]\right]. \end{aligned} \tag{59}$$

Taking the terms of alike power of p, we obtain

$$\begin{aligned} p^0 : w_0(x,t) &= \tfrac{x^r}{\Gamma(1+r)} \\ p^1 : w_1(x,t) &= LFS_r^{-1}\left[z^{2r} LFS_r\left[D_t^r w_0(x,t) + D_{xx}^{2r} w_0(x,t) + D_x^r w_0(x,t) + \tfrac{t^r}{\Gamma(r+1)}\right]\right] \\ &. \\ &. \\ &. \\ p^n : w_n(x,t) &= LFS_r^{-1}\left[z^{2r} LFS_r\left[D_t^r w_{n-1}(x,t) + D_{xx}^{2r} w_{n-1}(x,t) + D_x^r w_{n-1}(x,t)\right]\right]. \end{aligned} \tag{60}$$

We find

$$\begin{aligned} w_0(x,t) &= \tfrac{x^r}{\Gamma(1+r)} \\ w_1(x,t) &= \tfrac{t^{2r}}{\Gamma(1+2r)} + \tfrac{t^{3r}}{\Gamma(1+3r)} \\ &. \\ &. \\ &. \\ w_n(x,t) &= \tfrac{t^{(n+1)r}}{\Gamma(1+(n+1)r)} + \tfrac{t^{(n+2)r}}{\Gamma(1+(n+2)r)}. \end{aligned} \tag{61}$$

Therefore, the solution is given as

$$w(x,t) = \frac{x^r}{\Gamma(1+r)} - \sum_{j=0}^{1} \frac{t^{jr}}{\Gamma(1+jr)} - \sum_{j=0}^{2} \frac{t^{jr}}{\Gamma(1+jr)} + 2E_r(t^r). \tag{62}$$

For the convergence of the above series, we have to prove that

$$\lim_{n\to\infty} \frac{\|w_n\|}{\|w_{n-1}\|} < 1. \tag{63}$$

Proof. For $0 < t < 1, \quad 0 < r < 1$, we have

$$\begin{aligned} \tfrac{w_n}{w_{n-1}} &= \left(\tfrac{t^{(n+1)r}}{\Gamma(1+(n+1)r)} + \tfrac{t^{(n+2)r}}{\Gamma(1+(n+2)r)}\right)\left(\tfrac{t^{nr}}{\Gamma(nr+1)} + \tfrac{t^{(n+1)r}}{\Gamma(1+(n+1)r)}\right)^{-1} \\ &= \tfrac{t^{(n+1)r}}{t^{nr}}\left(\tfrac{1}{\Gamma(1+(n+1)r)} + \tfrac{t^r}{\Gamma(1+(n+2)r)}\right)\left(\tfrac{1}{\Gamma(nr+1)} + \tfrac{t^r}{\Gamma(1+(n+1)r)}\right)^{-1} \\ &= t^r\left(\tfrac{1}{\Gamma(1+(n+1)r)} + \tfrac{t^r}{\Gamma(1+(n+2)r)}\right)\left(\tfrac{1}{\Gamma(nr+1)} + \tfrac{t^r}{\Gamma(1+(n+1)r)}\right)^{-1}. \end{aligned} \tag{64}$$

While the function Gamma is increasing, we find

$$\tfrac{1}{\Gamma(1+(n+1)r)} + \tfrac{t^r}{\Gamma(1+(n+2)r)} < \tfrac{1}{\Gamma(nr+1)} + \tfrac{t^r}{\Gamma(1+(n+1)r)}. \tag{65}$$

Then, we get

$$\left(\tfrac{1}{\Gamma(1+(n+1)r)} + \tfrac{t^r}{\Gamma(1+(n+2)r)}\right)\left(\tfrac{1}{\Gamma(nr+1)} + \tfrac{t^r}{\Gamma(1+(n+1)r)}\right)^{-1} < 1. \tag{66}$$

Hence, we have this inequality

$$t^r\left(\tfrac{1}{\Gamma(1+(n+1)r)} + \tfrac{t^r}{\Gamma(1+(n+2)r)}\right)\left(\tfrac{1}{\Gamma(nr+1)} + \tfrac{t^r}{\Gamma(1+(n+1)r)}\right)^{-1} < t^r < 1. \tag{67}$$

Then,

$$\frac{\|w_n\|}{\|w_{n-1}\|} < t^r < 1. \tag{68}$$

Therefore,

$$\forall n \in \mathbb{N} \qquad \frac{\|w_n\|}{\|w_{n-1}\|} < t^r < 1. \tag{69}$$

Finally,

$$\lim_{n\to\infty} \frac{\|w_n\|}{\|w_{n-1}\|} \le t^r < 1. \tag{70}$$

□

Consequently, Theorem 1 ensures the convergence of the corresponding series for all values of t satisfying $0 < t < 1$.

6.3. Example 3: Local Fractional Damped Wave Equation

Consider the damped wave equation with local fractional derivative as follows [34]

$$D_{tt}^{2r}w(x,t) - D_t^r w(x,t) - D_{xx}^{2r}w(x,t) - \frac{x^r}{\Gamma(r+1)} = 0 \qquad 0 \le x \le l, t > 0, \tag{71}$$

subject to the following conditions

$$\begin{aligned} w(x,0) &= 0, \\ D_t^r w(x,0) &= -\tfrac{x^r}{\Gamma(1+r)}. \end{aligned} \tag{72}$$

Using the expression of local fractional Sumudu transform for Equation (71) we find

$$LFS_r\left[D_{tt}^{2r}w(x,t)\right] = LFS_r\left[D_t^r w(x,t) + D_{xx}^{2r}w(x,t) + \frac{x^r}{\Gamma(r+1)}\right], \tag{73}$$

then, we get

$$w_r(x,z) = z^r w^r(x,0) + z^{2r}LFS_r\left[D_t^r w(x,t) + D_{xx}^{2r}w(x,t) + \frac{x^r}{\Gamma(r+1)}\right], \tag{74}$$

it yields

$$w_r(x,z) = -z^r\frac{x^r}{\Gamma(1+r)} + z^{2r}LFS_r\left[D_t^r w(x,t) + D_{xx}^{2r}w(x,t) + \frac{x^r}{\Gamma(r+1)}\right]. \tag{75}$$

By applying the inverse local fractional Sumudu transform, we find

$$\begin{aligned} w(x,t) &= -\tfrac{t^r}{\Gamma(1+r)}\tfrac{x^r}{\Gamma(1+r)} + LFS_r^{-1}\left[z^{2r}LFS_r\left[D_t^r w(x,t)\right.\right. \\ &+ \left.\left. D_{xx}^{2r}w(x,t) + \tfrac{x^r}{\Gamma(r+1)}\right]\right]. \end{aligned} \tag{76}$$

Using the homotopy perturbation, we obtain

$$\begin{aligned} \textstyle\sum_{n=0}^{\infty} p^n w_n(x,t) &= -\tfrac{t^r}{\Gamma(1+r)}\tfrac{x^r}{\Gamma(1+r)} + p \times LFS_r^{-1}\left[z^{2r}LFS_r\left[D_t^r \textstyle\sum_{n=0}^{\infty} p^n w_n(x,t)\right.\right. \\ &+ \left.\left. D_{xx}^{2r}\textstyle\sum_{n=0}^{\infty} p^n w_n(x,t) + \tfrac{x^r}{\Gamma(r+1)}\right]\right]. \end{aligned} \tag{77}$$

Collecting the terms of similar powers of p, we get:

$$\begin{aligned} p^0 : w_0(x,t) &= -\tfrac{t^r}{\Gamma(1+r)}\tfrac{x^r}{\Gamma(1+r)}, \\ p^1 : w_1(x,t) &= LFS_r^{-1}\left[z^{2r}LFS_r\left[D_t^r w_0(x,t) + D_{xx}^{2r}w_0(x,t) + \tfrac{x^r}{\Gamma(r+1)}\right]\right], \\ &. \\ &. \\ &. \\ p^n : w_n(x,t) &= LFS_r^{-1}\left[z^{2r}LFS_r\left[D_t^r w_{n-1}(x,t) + D_{xx}^{2r}w_{n-1}(x,t)\right]\right]. \end{aligned} \tag{78}$$

we get

$$\begin{aligned} w_0(x,t) &= -\tfrac{t^r}{\Gamma(1+r)}\tfrac{x^r}{\Gamma(1+r)}, \\ w_1(x,t) &= 0, \\ &\cdot \\ &\cdot \\ &\cdot \\ w_n(x,t) &= 0. \end{aligned} \tag{79}$$

Hence the solution is

$$w(x,t) = -\frac{t^r}{\Gamma(1+r)}\frac{x^r}{\Gamma(1+r)}, \tag{80}$$

which is an exact solution of Equation (71).

7. Conclusions

In this work, the technique that combines homotopy perturbation and Sumudu transform in local fractional sense was applied, to obtain solutions of local fractional PDEs. Further, we presented the sufficient condition for the convergence of this method. The obtained solutions show that this technique is a powerful tool to solve different types of local fractional PDEs arising in mathematical physics.

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