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Dynamic Fractional Inequalities Amplified on Time Scale Calculus Revealing Coalition of Discreteness and Continuity

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Abstract: In this paper, we present a generalization of Radon's inequality on dynamic time scale calculus, which is widely studied by many authors and an intrinsic inequality. Further, we present the classical Bergström's inequality and refinement of Nesbitt's inequality unified on dynamic time scale calculus in extended form.

Keywords: Radon's inequality; Bergström's inequality; Nesbitt's inequality; time scales

MSC: 26D15; 26D20; 26D99; 34N05

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1. Introduction

The beauty of Radon's inequality is its utility in many practical applications. Radon's inequality and its generalized form are equivalent to Rogers–Hölder's inequality and Bernoulli's inequality as given in [1]. Recently, it has been proven that the dynamic generalized Radon's inequality is equivalent to Radon's inequality, the weighted power mean inequality, Schlömilch's inequality, Rogers–Hölder's inequality and Bernoulli's inequality on dynamic time scale calculus, as given in [2].

The following inequality is a generalization of Radon's inequality as given in [3].

Theorem 1. *Let $c_1, c_2, c_3, c_4, x_k, y_k \in (0, \infty)$, where $k \in \{1, 2, ..., n\}$, $X_n = \sum\limits_{k=1}^{n} x_k$ and $Y_n = \sum\limits_{k=1}^{n} y_k$. If $\beta \in [1, \infty)$ and $\gamma, \zeta, \eta, \lambda \in [0, \infty)$ are such that $c_3 Y_n^\lambda > c_4 \max\limits_{1 \le k \le n} y_k^\lambda$, then:*

$$\frac{(c_1 n^\eta X_n^{\beta+\zeta} + c_2 X_n^{\beta+\eta})^{\gamma+1}}{(c_3 n^\lambda - c_4)^\gamma Y_n^{\gamma(\lambda+1)}} \cdot \frac{1}{n^{(\gamma+1)(\beta+\eta-1)-\gamma\lambda}} \le \sum_{k=1}^{n} \frac{(c_1 X_n^\zeta + c_2 x_k^\eta)^{\gamma+1} x_k^{\beta(\gamma+1)}}{(c_3 Y_n^\lambda - c_4 y_k^\lambda)^\gamma y_k^\gamma}. \tag{1}$$

The following inequality is a generalization of Nesbitt's inequality as given in [3].

Theorem 2. *Let $c_1 \ge 0$, $c_2, c_3, c_4, x_k \in (0, \infty)$, where $k \in \{1, 2, ..., n\}$, $X_n = \sum\limits_{k=1}^{n} x_k$ and $c_3 X_n^\gamma > c_4 \max\limits_{1 \le k \le n} x_k^\gamma$. If $\gamma \ge 1$, then:*

$$\frac{(c_1 n + c_2) n^\gamma}{c_3 n^\gamma - c_4} X_n^{1-\gamma} \le \sum_{k=1}^{n} \frac{c_1 X_n + c_2 x_k}{c_3 X_n^\gamma - c_4 x_k^\gamma}. \tag{2}$$

We will prove these results on time scale calculus. Time scale calculus was initiated by Stefan Hilger as given in [4]. A time scale is an arbitrary nonempty closed subset of the real numbers. The theory of time scale calculus is applied to reveal the symmetry of being continuous and discrete and to combine them in one comprehensive form. In time scale calculus, results are unified and extended. Time scale calculus is studied as delta calculus, nabla calculus and diamond-α calculus. This hybrid theory is also widely applied on dynamic inequalities. Basic work on dynamic inequalities using time scales was done by Agarwal, Anastassiou, Bohner, Peterson, O'Regan, Saker and and many other authors.

In this paper, it is assumed that all considerable integrals exist and are finite and $\mathbb{T}$ is a time scale, $a, b \in \mathbb{T}$, with $a < b$ and an interval $[a, b]_{\mathbb{T}}$ means the intersection of a real interval with the given time scale.

2. Preliminaries

We need here basic concepts of delta calculus. The results of delta calculus are adapted from [5,6].

For $t \in \mathbb{T}$, forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by:

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

The mapping $\mu : \mathbb{T} \to \mathbb{R}_0^+ = [0, \infty)$ such that $\mu(t) := \sigma(t) - t$ is called the forward graininess function. The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by:

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

The mapping $\nu : \mathbb{T} \to \mathbb{R}_0^+ = [0, \infty)$ such that $\nu(t) := t - \rho(t)$ is called the backward graininess function. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. Furthermore, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. If $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

For a function $f : \mathbb{T} \to \mathbb{R}$, the delta derivative f^{Δ} is defined as follows:

Let $t \in \mathbb{T}^k$; if there exists $f^{\Delta}(t) \in \mathbb{R}$ such that for all $\epsilon > 0$, there exists a neighborhood U of t, such that:

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|,$$

for all $s \in U$, then f is said to be delta differentiable at t and $f^{\Delta}(t)$ is called the delta derivative of f at t.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous (rd-continuous), if it is continuous at each right-dense point and there exists a finite left limit at every left-dense point. The set of all rd-continuous functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

The next definition is given in [5,6].

Definition 1. *A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \to \mathbb{R}$, provided that $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^k$, then the delta integral of f is defined by:*

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

The following results of nabla calculus are taken from [5–7].

If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. For $f : \mathbb{T} \to \mathbb{R}$, a function f is called nabla differentiable at $t \in \mathbb{T}_k$, with nabla derivative $f^{\nabla}(t)$, if there exists $f^{\nabla}(t) \in \mathbb{R}$ such that for any given $\epsilon > 0$, there exists a neighborhood V of t, such that:

$$|f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s)| \leq \epsilon|\rho(t) - s|,$$

for all $s \in V$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be left-dense continuous (ld-continuous), provided it is continuous at every left-dense point in $\mathbb{T}$ and its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$. The set of all ld-continuous functions is denoted by $C_{ld}(\mathbb{T}, \mathbb{R})$.

The next definition is given in [5–7].

Definition 2. *A function $G : \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $g : \mathbb{T} \to \mathbb{R}$, provided that $G^{\nabla}(t) = g(t)$ holds for all $t \in \mathbb{T}_k$, then the nabla integral of g is defined by:*

$$\int_a^b g(t)\nabla t = G(b) - G(a).$$

Now, we present a short introduction of the diamond-α derivative as given in [8,9].

Let $\mathbb{T}$ be a time scale and $f(t)$ be differentiable on $\mathbb{T}$ in the Δ and ∇ senses. For $t \in \mathbb{T}_k^k$, where $\mathbb{T}_k^k = \mathbb{T}^k \cap \mathbb{T}_k$, diamond-$\alpha$ dynamic derivative $f^{\diamond_\alpha}(t)$ is defined by:

$$f^{\diamond_\alpha}(t) = \alpha f^{\Delta}(t) + (1-\alpha) f^{\nabla}(t), \quad 0 \le \alpha \le 1.$$

Thus, f is diamond-α differentiable if and only if f is Δ and ∇ differentiable.

The diamond-α derivative reduces to the standard Δ-derivative for $\alpha = 1$, or the standard ∇-derivative for $\alpha = 0$. It represents a weighted dynamic derivative for $\alpha \in (0, 1)$.

Theorem 3. [9] *Let $f, g : \mathbb{T} \to \mathbb{R}$ be diamond-α differentiable at $t \in \mathbb{T}$. Then:*

(i) *$f \pm g : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$, with:*

$$(f \pm g)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t) \pm g^{\diamond_\alpha}(t).$$

(ii) *$fg : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$, with:*

$$(fg)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t)g(t) + \alpha f^{\sigma}(t)g^{\Delta}(t) + (1-\alpha)f^{\rho}(t)g^{\nabla}(t).$$

(iii) *For $g(t)g^{\sigma}(t)g^{\rho}(t) \neq 0$, $\frac{f}{g} : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$, with:*

$$\left(\frac{f}{g}\right)^{\diamond_\alpha}(t) = \frac{f^{\diamond_\alpha}(t)g^{\sigma}(t)g^{\rho}(t) - \alpha f^{\sigma}(t)g^{\rho}(t)g^{\Delta}(t) - (1-\alpha)f^{\rho}(t)g^{\sigma}(t)g^{\nabla}(t)}{g(t)g^{\sigma}(t)g^{\rho}(t)}.$$

Definition 3. [9] *Let $a, t \in \mathbb{T}$ and $h : \mathbb{T} \to \mathbb{R}$. Then, the diamond-α integral from a to t of h is defined by:*

$$\int_a^t h(s) \diamond_\alpha s = \alpha \int_a^t h(s)\Delta s + (1-\alpha)\int_a^t h(s)\nabla s, \quad 0 \le \alpha \le 1,$$

provided that there exist delta and nabla integrals of h on $\mathbb{T}$, as given in Definitions 1 and 2, respectively.

Theorem 4. [9] *Let $a, b, t \in \mathbb{T}$, $c \in \mathbb{R}$. Assume that $f(s)$ and $g(s)$ are $\diamond_\alpha$-integrable functions on $[a, b]_{\mathbb{T}}$, then:*

(i) $\int_a^t [f(s) \pm g(s)] \diamond_\alpha s = \int_a^t f(s) \diamond_\alpha s \pm \int_a^t g(s) \diamond_\alpha s$;
(ii) $\int_a^t cf(s) \diamond_\alpha s = c\int_a^t f(s) \diamond_\alpha s$;
(iii) $\int_a^t f(s) \diamond_\alpha s = -\int_t^a f(s) \diamond_\alpha s$;
(iv) $\int_a^t f(s) \diamond_\alpha s = \int_a^b f(s) \diamond_\alpha s + \int_b^t f(s) \diamond_\alpha s$;
(v) $\int_a^a f(s) \diamond_\alpha s = 0$.

We need the following result.

Definition 4. *[10]: A function $f:\mathbb{T}\to\mathbb{R}$ is called convex on $I_{\mathbb{T}}=I\cap\mathbb{T}$, where I is an interval of $\mathbb{R}$ (open or closed), if:*

$$f(\lambda t+(1-\lambda)s)\leq \lambda f(t)+(1-\lambda)f(s), \tag{3}$$

for all $t,s\in I_{\mathbb{T}}$ and all $\lambda\in[0,1]$ such that $\lambda t+(1-\lambda)s\in I_{\mathbb{T}}$.

The function f is strictly convex on $I_{\mathbb{T}}$ if (3) is strict for distinct $t,s\in I_{\mathbb{T}}$ and $\lambda\in(0,1)$.

The function f is concave (respectively, strictly concave) on $I_{\mathbb{T}}$, if $-f$ is convex (respectively, strictly convex).

Theorem 5. *[8]: Let $a,b\in\mathbb{T}$ and $c,d\in\mathbb{R}$. Suppose that $g\in C([a,b]_{\mathbb{T}},(c,d))$ and $h\in C([a,b]_{\mathbb{T}},\mathbb{R})$ with $\int\limits_a^b|h(s)|\diamond_\alpha s>0$. If $\Phi\in C((c,d),\mathbb{R})$ is convex, then the generalized Jensen's inequality is:*

$$\Phi\left(\frac{\int\limits_a^b|h(s)|g(s)\diamond_\alpha s}{\int\limits_a^b|h(s)|\diamond_\alpha s}\right)\leq\frac{\int\limits_a^b|h(s)|\Phi\left(g(s)\right)\diamond_\alpha s}{\int\limits_a^b|h(s)|\diamond_\alpha s}. \tag{4}$$

If Φ is strictly convex, then the inequality $\leq$ can be replaced by $<$.

3. Main Results

In order to present our main results, first we present an extension of Radon's inequality by applying Jensen's inequality for a convex function via time scales.

Theorem 6. *Let $w,f,g\in C([a,b]_{\mathbb{T}},\mathbb{R}-\{0\})$ be $\diamond_\alpha$-integrable functions and $c_1,c_2,c_3,c_4\in(0,\infty)$. If $\beta\geq1$, $\gamma,\zeta,\eta,\lambda\in[0,\infty)$ and $c_3\left(\int_a^b|w(x)||g(x)|\diamond_\alpha x\right)^\lambda>c_4\sup\limits_{x\in[a,b]_{\mathbb{T}}}|g(x)|^\lambda$, then:*

$$\begin{aligned}&\frac{\left(c_1\left(\int_a^b|w(x)|\diamond_\alpha x\right)^\eta\left(\int_a^b|w(x)||f(x)|\diamond_\alpha x\right)^{\beta+\zeta}+c_2\left(\int_a^b|w(x)||f(x)|\diamond_\alpha x\right)^{\beta+\eta}\right)^{\gamma+1}}{\left(c_3\left(\int_a^b|w(x)|\diamond_\alpha x\right)^\lambda-c_4\right)^\gamma\left(\int_a^b|w(x)||g(x)|\diamond_\alpha x\right)^{\gamma(\lambda+1)}}\\
&\quad\times\frac{1}{\left(\int_a^b|w(x)|\diamond_\alpha x\right)^{(\gamma+1)(\beta+\eta-1)-\gamma\lambda}}\\
&\leq\int_a^b|w(x)|\frac{\left(c_1\left(\int_a^b|w(x)||f(x)|\diamond_\alpha x\right)^\zeta+c_2|f(x)|^\eta\right)^{\gamma+1}|f(x)|^{\beta(\gamma+1)}}{\left(c_3\left(\int_a^b|w(x)||g(x)|\diamond_\alpha x\right)^\lambda-c_4|g(x)|^\lambda\right)^\gamma|g(x)|^\gamma}\diamond_\alpha x.\end{aligned} \tag{5}$$

Proof. Set $\Lambda=\int_a^b|w(x)|\diamond_\alpha x$, $\Upsilon=\int_a^b|w(x)||f(x)|\diamond_\alpha x$ and $\Omega=\int_a^b|w(x)||g(x)|\diamond_\alpha x$.

Let $j(x)=\left(c_1\Upsilon^\zeta+c_2|f(x)|^\eta\right)|f(x)|^\beta$ and $k(x)=\left(c_3\Omega^\lambda-c_4|g(x)|^\lambda\right)|g(x)|$.

The right-hand side of (5) can be written as:

$$\begin{aligned}&\int_a^b|w(x)|\frac{j^{\gamma+1}(x)}{k^\gamma(x)}\diamond_\alpha x\\
=&\int_a^b|w(x)|k(x)\left(\frac{j(x)}{k(x)}\right)^{\gamma+1}\diamond_\alpha x\\
=&\left(\int_a^b|w(x)|k(x)\diamond_\alpha x\right)\int_a^b\frac{|w(x)|k(x)}{\left(\int_a^b|w(x)|k(x)\diamond_\alpha x\right)}\left(\frac{j(x)}{k(x)}\right)^{\gamma+1}\diamond_\alpha x.\end{aligned}$$

Choosing $\Phi(x) = x^{\gamma+1}$, which for $\gamma > 0$ is a convex function on $x \in (0, \infty)$, then Jensen's inequality given in (4) takes the form:

$$\left(\int_a^b \frac{|w(x)|k(x)}{\int_a^b |w(x)|k(x) \diamond_\alpha x} \left(\frac{j(x)}{k(x)}\right) \diamond_\alpha x\right)^{\gamma+1} \leq \int_a^b \frac{|w(x)|k(x)}{\int_a^b |w(x)|k(x) \diamond_\alpha x} \left(\frac{j(x)}{k(x)}\right)^{\gamma+1} \diamond_\alpha x. \tag{6}$$

Now, (6) takes the simplified form:

$$\frac{\left(\int_a^b |w(x)|j(x) \diamond_\alpha x\right)^{\gamma+1}}{\left(\int_a^b |w(x)|k(x) \diamond_\alpha x\right)^{\gamma}} \leq \int_a^b |w(x)| \frac{j^{\gamma+1}(x)}{k^{\gamma}(x)} \diamond_\alpha x. \tag{7}$$

Putting values of $j(x)$ and $k(x)$ in the right-hand side of (7), we get:

$$\begin{aligned}
&\int_a^b |w(x)| \frac{\left[\left(c_1 \Upsilon^\zeta + c_2|f(x)|^\eta\right)|f(x)|^\beta\right]^{\gamma+1}}{\left[\left(c_3\Omega^\lambda - c_4|g(x)|^\lambda\right)|g(x)|\right]^\gamma} \diamond_\alpha x \\
\geq\ & \frac{\left[\int_a^b |w(x)|\left(c_1 \Upsilon^\zeta + c_2|f(x)|^\eta\right)|f(x)|^\beta \diamond_\alpha x\right]^{\gamma+1}}{\left[\int_a^b |w(x)|\left(c_3\Omega^\lambda - c_4|g(x)|^\lambda\right)|g(x)| \diamond_\alpha x\right]^\gamma} \\
=\ & \frac{\left[c_1 \Upsilon^\zeta \int_a^b |w(x)||f(x)|^\beta \diamond_\alpha x + c_2 \int_a^b |w(x)||f(x)|^{\beta+\eta} \diamond_\alpha x\right]^{\gamma+1}}{\left[c_3\Omega^{\lambda+1} - c_4 \int_a^b |w(x)||g(x)|^{\lambda+1} \diamond_\alpha x\right]^\gamma} \\
\geq\ & \frac{\left[c_1 \Upsilon^{\beta+\zeta}\left(\frac{1}{\Lambda}\right)^{\beta-1} + c_2 \Upsilon^{\beta+\eta}\left(\frac{1}{\Lambda}\right)^{\beta+\eta-1}\right]^{\gamma+1}}{\left[c_3\Omega^{\lambda+1} - c_4\Omega^{\lambda+1}\left(\frac{1}{\Lambda}\right)^\lambda\right]^\gamma} \\
=\ & \frac{\left(c_1 \Upsilon^{\beta+\zeta}\Lambda^\eta + c_2\Upsilon^{\beta+\eta}\right)^{\gamma+1}}{\left(c_3\Lambda^\lambda - c_4\right)^\gamma \Omega^{\gamma(\lambda+1)}\Lambda^{(\gamma+1)(\beta+\eta-1)-\gamma\lambda}},
\end{aligned}$$

completing the proof of our claim. □

Remark 1. *Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $w(x) = 1$, $f(x) = x_k$ and $g(x) = y_k$ for $k \in \{1, 2, ..., n\}$ be sets of positive values, $X_n = \sum_{k=1}^{n} x_k$, $Y_n = \sum_{k=1}^{n} y_k$ and $c_1, c_2, c_3, c_4 \in (0, \infty)$. If $\beta \geq 1$, $\gamma, \zeta, \eta, \lambda \in [0, \infty)$ and $c_3 Y_n^\lambda > c_4 \max_{1 \leq k \leq n} y_k^\lambda$, then (5) reduces to (1).*

Remark 2. *Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $w(x) = 1$, $f(x) = x_k \in [0, \infty)$ and $g(x) = y_k \in (0, \infty)$ for $k \in \{1, 2, ..., n\}$, $X_n = \sum_{k=1}^{n} x_k$ and $Y_n = \sum_{k=1}^{n} y_k$. If $\beta = 1$ and $\zeta = \eta = \lambda = 0$, then (5) reduces to:*

$$\frac{\left(\sum_{k=1}^{n} x_k\right)^{\gamma+1}}{\left(\sum_{k=1}^{n} y_k\right)^{\gamma}} \leq \sum_{k=1}^{n} \frac{x_k^{\gamma+1}}{y_k^{\gamma}}, \tag{8}$$

which is Radon's inequality, as given in [11].

Remark 3. *Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $w(x) = 1$, $f(x) = x_k \in \mathbb{R}$ and $g(x) = y_k \in (0, \infty)$ for $k \in \{1, 2, ..., n\}$, $X_n = \sum\limits_{k=1}^{n} x_k$ and $Y_n = \sum\limits_{k=1}^{n} y_k$. If $\beta = 1$, $\gamma = 1$, $\zeta = \eta = \lambda = 0$, then (5) reduces to:*

$$\frac{\left(\sum\limits_{k=1}^{n} x_k\right)^2}{\sum\limits_{k=1}^{n} y_k} \leq \sum_{k=1}^{n} \frac{x_k^2}{y_k}, \tag{9}$$

which is called Bergström's inequality in the literature, as given in [12–15].

The inequality given in upcoming corollary is called Schlömilch's inequality. Its other versions are also given in [16,17].

Corollary 1. *Let $w, f \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ be $\diamond_\alpha$-integrable functions, where $\int_a^b |w(x)| \diamond_\alpha x = 1$. If $\eta_2 \geq \eta_1 > 0$, then:*

$$\left(\int_a^b |w(x)||f(x)|^{\eta_1} \diamond_\alpha x\right)^{\frac{1}{\eta_1}} \leq \left(\int_a^b |w(x)||f(x)|^{\eta_2} \diamond_\alpha x\right)^{\frac{1}{\eta_2}}. \tag{10}$$

Proof. If $\beta = 1$ and $\zeta = \eta = \lambda = 0$, then (5) reduces to:

$$\frac{\left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\gamma+1}}{\left(\int_a^b |w(x)||g(x)| \diamond_\alpha x\right)^{\gamma}} \leq \int_a^b \frac{|w(x)||f(x)|^{\gamma+1}}{|g(x)|^{\gamma}} \diamond_\alpha x. \tag{11}$$

Let $\gamma + 1 = \frac{\eta_2}{\eta_1} \geq 1$ for $\gamma \geq 0$ and $g(x) = 1$, then (11) becomes:

$$\left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\frac{\eta_2}{\eta_1}} \leq \int_a^b |w(x)||f(x)|^{\frac{\eta_2}{\eta_1}} \diamond_\alpha x. \tag{12}$$

Replacing $|f(x)|$ by $|f(x)|^{\eta_1}$ and taking power $\frac{1}{\eta_2}$, then (12) gives our required result. □

The upcoming result is the generalized Nesbitt's inequality on dynamic time scale calculus.

Theorem 7. *Let $w, f \in C([a,b]_{\mathbb{T}}, \mathbb{R} - \{0\})$ be $\diamond_\alpha$-integrable functions. If $c_1 \in [0, \infty)$, $c_2, c_3, c_4 \in (0, \infty)$, $\gamma, \zeta, \eta, \lambda \in [1, \infty)$ and $c_3 \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\gamma} > c_4 \sup\limits_{x \in [a,b]_{\mathbb{T}}} |f(x)|^{\gamma}$, then:*

$$\begin{aligned} &\frac{\left(c_1 \left(\int_a^b |w(x)| \diamond_\alpha x\right)^{\eta} + c_2\right)^{\lambda}}{\left(c_3 \left(\int_a^b |w(x)| \diamond_\alpha x\right)^{\gamma} - c_4\right)^{\zeta}} \left(\int_a^b |w(x)| \diamond_\alpha x\right)^{\gamma\zeta - \eta\lambda + 1} \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\eta\lambda - \gamma\zeta} \\ &\qquad \leq \int_a^b |w(x)| \frac{\left(c_1 \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\eta} + c_2 |f(x)|^{\eta}\right)^{\lambda}}{\left(c_3 \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\gamma} - c_4 |f(x)|^{\gamma}\right)^{\zeta}} \diamond_\alpha x. \end{aligned} \tag{13}$$

Proof. Let $|f(x)| = g(x) \int_a^b |w(x)||f(x)| \diamond_\alpha x$. Then, $\int_a^b |w(x)| g(x) \diamond_\alpha x = 1$.

The right-hand side of (13) is:

$$\int_a^b |w(x)| \frac{\left(c_1 \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^\eta + c_2|f(x)|^\eta\right)^\lambda}{\left(c_3 \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^\gamma - c_4|f(x)|^\gamma\right)^\zeta} \diamond_\alpha x$$
$$= \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\eta\lambda-\gamma\zeta} \left[\int_a^b |w(x)| \frac{\{c_1 + c_2 g^\eta(x)\}^\lambda}{\{c_3 - c_4 g^\gamma(x)\}^\zeta} \diamond_\alpha x\right]. \quad (14)$$

Let $\Phi(g(x)) = \frac{\{c_1+c_2 g^\eta(x)\}^\lambda}{\{c_3-c_4 g^\gamma(x)\}^\zeta} = \{c_1 + c_2 g^\eta(x)\}^\lambda \{c_3 - c_4 g^\gamma(x)\}^{-\zeta}$.

Clearly, $\Phi(g(x))$ is a convex function on $\left(0, \left(\frac{c_3}{c_4}\right)^{\frac{1}{\gamma}}\right)$, as it is the product of two convex functions.

Now, we apply Jensen's inequality given in (4) and get:

$$\frac{\left[c_1 + c_2 \left\{\frac{\int_a^b |w(x)|g(x)\diamond_\alpha x}{\int_a^b |w(x)|\diamond_\alpha x}\right\}^\eta\right]^\lambda}{\left[c_3 - c_4 \left\{\frac{\int_a^b |w(x)|g(x)\diamond_\alpha x}{\int_a^b |w(x)|\diamond_\alpha x}\right\}^\gamma\right]^\zeta} \leq \frac{\int\limits_a^b |w(x)| \left[\frac{\{c_1+c_2 g^\eta(x)\}^\lambda}{\{c_3-c_4 g^\gamma(x)\}^\zeta}\right] \diamond_\alpha x}{\int\limits_a^b |w(x)| \diamond_\alpha x}; \quad (15)$$

hence, Inequality (13) is clear from (15).

Therefore, the proof of Theorem 7 is completed. □

Remark 4. *Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $w(x) = 1$, $f(x) = x_k$ for $k \in \{1, 2, ..., n\}$ be a set of positive values and $X_n = \sum_{k=1}^n x_k$. If $n \in \mathbb{N} - \{1\}$, $c_1 \in [0, \infty)$, $c_2, c_3, c_4 \in (0, \infty)$, $\gamma, \zeta, \eta, \lambda \in [1, \infty)$ and $c_3 X_n^\gamma > c_4 \max_{1\leq k\leq n} x_k^\gamma$, then (13) reduces to:*

$$\frac{(c_1 n^\eta + c_2)^\lambda}{(c_3 n^\gamma - c_4)^\zeta} n^{\gamma\zeta-\eta\lambda+1} X_n^{\eta\lambda-\gamma\zeta} \leq \sum_{k=1}^n \frac{(c_1 X_n^\eta + c_2 x_k^\eta)^\lambda}{(c_3 X_n^\gamma - c_4 x_k^\gamma)^\zeta}, \quad (16)$$

as given in [18].

Remark 5. *Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $w(x) = 1$, $f(x) = x_k$ for $k \in \{1, 2, ..., n\}$ be a set of positive values and $X_n = \sum_{k=1}^n x_k$. If $c_1 \in [0, \infty)$, $c_2, c_3, c_4 \in (0, \infty)$, $\gamma \geq 1$, $\zeta = \eta = \lambda = 1$ and $c_3 X_n^\gamma > c_4 \max_{1\leq k\leq n} x_k^\gamma$, then (13) reduces to (2).*

Corollary 2. *Let $w, f \in C([a, b]_\mathbb{T}, \mathbb{R} - \{0\})$ be $\diamond_\alpha$-integrable functions. If $c_3, c_4 \in (0, \infty)$, $\gamma, \zeta, \eta \in [1, \infty)$ and $c_3 \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^\gamma > c_4 \sup_{x\in[a,b]_\mathbb{T}} |f(x)|^\gamma$, then:*

$$\frac{\left(\int_a^b |w(x)| \diamond_\alpha x\right)^{\gamma\zeta-\eta+1}}{\left(c_3 \left(\int_a^b |w(x)| \diamond_\alpha x\right)^\gamma - c_4\right)^\zeta} \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\eta-\gamma\zeta}$$
$$\leq \int_a^b |w(x)| \frac{|f(x)|^\eta}{\left(c_3 \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^\gamma - c_4|f(x)|^\gamma\right)^\zeta} \diamond_\alpha x. \quad (17)$$

Proof. Put $c_1 = 0, c_2 = 1$ and $\lambda = 1$ in Theorem 7; we get our claim. □

Remark 6. *Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $w(x) = 1$, $f(x) = x_k$ for $k \in \{1,2,...,n\}$ be a set of positive values and $X_n = \sum_{k=1}^{n} x_k$. If $n \in \mathbb{N} - \{1\}$, $c_3, c_4 \in (0,\infty)$, $\gamma, \zeta, \eta \in [1,\infty)$ and $c_3 X_n^{\gamma} > c_4 \max_{1\le k\le n} x_k^{\gamma}$, then the discrete version of (17) reduces to:*

$$\frac{n^{\gamma\zeta-\eta+1}}{(c_3 n^{\gamma} - c_4)^{\zeta}} X_n^{\eta-\gamma\zeta} \le \sum_{k=1}^{n} \frac{x_k^{\eta}}{(c_3 X_n^{\gamma} - c_4 x_k^{\gamma})^{\zeta}}, \tag{18}$$

as given in [19].

Remark 7. *Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $w(x) = 1$, $f(x) = x_k$ for $k \in \{1,2,...,n\}$ be a set of positive values and $X_n = \sum_{k=1}^{n} x_k$. If $n \in \mathbb{N} - \{1\}$, $c_3, c_4 \in (0,\infty)$, $\gamma, \zeta, \eta \in [1,\infty)$ and $c_3 M^{\gamma} > c_4 \max_{1\le k\le n} x_k^{\gamma}$ for $M > 0$, then the discrete version of (17) takes the form:*

$$\frac{n^{\gamma\zeta-\eta+1}}{(c_3 n^{\gamma} M^{\gamma} - c_4 X_n^{\gamma})^{\zeta}} X_n^{\eta} \le \sum_{k=1}^{n} \frac{x_k^{\eta}}{(c_3 M_n^{\gamma} - c_4 x_k^{\gamma})^{\zeta}}, \tag{19}$$

as given in [19].

Corollary 3. *Let $w, f \in C([a,b]_{\mathbb{T}}, \mathbb{R} - \{0\})$ be $\diamond_\alpha$-integrable functions, $c_3, c_4 \in (0,\infty)$ and $c_3 \int_a^b |w(x)||f(x)| \diamond_\alpha x > c_4 \sup_{x\in[a,b]_{\mathbb{T}}} |f(x)|$, then:*

$$\int_a^b \frac{|w(x)|}{c_3 \int_a^b |w(x)| \diamond_\alpha x - c_4} \diamond_\alpha x \le \int_a^b \frac{|w(x)||f(x)|}{c_3 \int_a^b |w(x)||f(x)| \diamond_\alpha x - c_4 |f(x)|} \diamond_\alpha x. \tag{20}$$

Proof. Put $c_1 = 0$, $c_2 = 1$ and $\gamma = \zeta = \eta = \lambda = 1$ in Theorem 7, then Inequality (20) is clear. □

Remark 8. *Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $w(x) = 1$, $f(x) = x_k$ for $k \in \{1,2,...,n\}$ be a set of positive values and $X_n = \sum_{k=1}^{n} x_k$. If $n \in \mathbb{N} - \{1\}$, $c_3, c_4 \in (0,\infty)$ and $c_3 X_n > c_4 \max_{1\le k\le n} x_k$, then the discrete version of (20) takes the form:*

$$\frac{n}{c_3 n - c_4} \le \sum_{k=1}^{n} \frac{x_k}{c_3 X_n - c_4 x_k}, \tag{21}$$

as given in [20].

Further, if we set $n = 3$ and $c_3 = c_4 = 1$, then (21) takes the form:

$$\frac{3}{2} \le \frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_1} + \frac{x_3}{x_1 + x_2}, \tag{22}$$

where $x_1, x_2, x_3 > 0$. Inequality (22) is called Nesbitt's inequality, as given in [21].

Remark 9. *If we set $\alpha = 1$, then we get delta versions, and if we set $\alpha = 0$, then we get the nabla version of dynamic inequalities presented in this article.*

Furthermore, we get discrete versions, if we put $\mathbb{T} = \mathbb{Z}$, and we get continuous versions, if we put $\mathbb{T} = \mathbb{R}$, of dynamic inequalities presented in this article.

4. Conclusions and Future Work

In this research article, we have presented some fractional dynamic inequalities on diamond-α calculus. Recently, some dynamic inequalities on diamond-α calculus have been developed (see [17,22]). Some researchers developed various results concerning fractional calculus on time scales to produce related dynamic inequalities using the fractional Riemann–Liouville integral (see [23,24]). Similarly, we will continue to find further generalizations and applications of Radon's inequality, Bergström's inequality, Nesbitt's inequality and some other inequalities on dynamic time scale calculus.

In the future, we can generalize dynamic inequalities using functional generalization, the n-tuple diamond-α integral, the fractional Riemann–Liouville integral, quantum calculus and α,β-symmetric quantum calculus.

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