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Multi-Point and Anti-Periodic Conditions for Generalized Langevin Equation with Two Fractional Orders

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Abstract: With anti-periodic and a new class of multi-point boundary conditions, we investigate, in this paper, the existence and uniqueness of solutions for the Langevin equation that has Caputo fractional derivatives of two different orders. Existence of solutions is obtained by applying Krasnoselskii–Zabreiko's and the Leray–Schauder fixed point theorems. The Banach contraction mapping principle is used to investigate the uniqueness. Illustrative examples are provided to apply of the fundamental investigations.

Keywords: Generalized Langevin equation; Krasnoselskii-Zabreiko's and Leray-Schauder fixed point theorems; anti-periodic and multi-point conditions; Existence and uniqueness

MSC: 34B15; 34A08; 26A33; 34A12

1. Introduction

Fractional calculus has appeared as an important area in a sundry scientific fields for instance, but not exclusively, physics, mathematics, chemistry and engineering. The Langevin equations is vastly utilized to depict the progression of physical phenomena in oscillating mediums [1]. As an emphatic growth of generalized derivatives, the fractional generalized Langevin equation has been provided by Mainardi and Pironi [2]. They presented a fractional Langevin equation as a special case of a generalized Langevin equation, and for the first time represented the velocity and displacement correlation functions in terms of the Mittag–Leffler functions. Eab and Lim [3] studied the possibility of application of fractional Langevin equation of distributed order for modeling single file diffusion and ultraslow diffusion. Also, they used the fractional generalized Langevin equation to model anomalous diffusive processes including single file-type diffusion. Sandev et al. [4,5] provided expressions for variances and mean squared displacement for fractional generalized Langevin equations for a free particle represented in the presence of the cases of internal and external noise. They discussed its application to model anomalous diffusive processes in complex media including phenomena similar to single file diffusion or possible generalizations thereof.

Recently, several contributions mindful with the uniqueness and existence results for fractional generalized Langevin equations, have been published, see [6–15] and the references given therein.

Fixed point theorems contribute with a substantial and great role in the study of the uniqueness and existence of integral, differential and integro-differential equations. Although there are a large number of these theorems, but a limited number of them have been focused by the authors in this area such as Krasnoselskii's, nonlinear alternative Leray–Schauder, Banach contraction principle and Leray–Schauder degree. Krasnoselskii–Zabreiko's fixed point theorem for asymptotically linear mappings is one of the immutable point theorems that give important and accurate results in the existence of solutions for differential equations. However, it did not adequately draw the attention of many authors in their applications. Of contributions that used Krasnoselskii–Zabreiko's fixed point theorem [16–18], it is worth pointing out that this theorem was provided at the first time by [19].

Motivated by the studies above, we focus in this paper on discussing the existence and uniqueness of solutions for the problem:

$$D^{\beta}(D^{\alpha}+\lambda)x(t)=f(t,x(t)), \qquad t\in[0,1] \tag{1}$$

additional to the new boundary conditions

$$x(0)+x(1)=0, \qquad {}^{c}D^{\alpha}x(0)=0, \qquad {}^{c}D^{\alpha}x(1)=\sum_{i=1}^{m}\mu_i x(q_i) \tag{2}$$

where ${}^{c}D^{\alpha}$ and ${}^{c}D^{\beta}$ are the Caputo's derivative of generalized orders $\beta\in(1,2]$ and $\alpha\in(0,1]$, $\lambda\in\mathbb{R}$, $q_i\in(0,1), i=1,2,3,\ldots,m$ with $m\in\mathbb{N}$ and the function $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuously differentiable.

In the fractional Langevin Equation (1), $x(t)$ represents the displacement of particle depending on the time $t\in[0,T]$, $T>0$ (for the simplicity we using the transformation $t=t/T$ to take t in the unit interval $[0,1]$), ${}^{c}D^{\alpha}x(t)$, $0<\alpha\le 1$ represents the velocity instead of its defining as a first time derivative of the displacement and ${}^{c}D^{\beta}x(t)$, $1<\beta\le 2$ represents the acceleration instead of its definition as a second time derivative of the displacement. The product of two fractional derivatives gives the term ${}^{c}D^{\alpha+\beta}$, which represents the acceleration if $1<\alpha+\beta\le 2$ and the abberancy of the curve or the jerk term [20–22] if $2<\alpha+\beta\le 3$ instead of its definition as a third time derivative of the displacement.

The first condition in (2), called anti-periodic boundary condition, means the particle takes the same value on the opposite direction at the initial and terminal points. It is worth noting that this condition appeared in physics in a variety of situations (see [23] and the reference therein). The second boundary condition in (2) indicates that the particle starts its motion from stillness. The latest condition in (2), which looks as a linear mixture of the value of the fractional derivative of obscure function at the terminal point and summation of the values of this function at the middle points, can be explained as "the value of the fractional derivative of obscure function at the terminal point proportionates to the sum of values of the obscure function at midst nonlocal m-points that lay between the initial and terminal points". The studying of the generalized Langevin Equation (1) simultaneously with multi-point and anti-periodic conditions (2) makes our problem new especially when using Krasnoselskii–Zabreiko's immutable point theorem for asymptotically linear mappings.

Our systematic in this research is taken as follows. In Section 2, we render requisite definitions of the generalized integral and derivative and preparatory results that are necessity to accomplish this paper. In Section 3, we employ Krasnoselskii–Zabreiko's and the Leray–Schauder nonlinear alternative theorems to identify the constraints of the solutions existence for our problem. We do so by means of the Banach contraction principle. In Section 4, we will determine under which conditions the solution uniqueness is satisfied. Also, after the terminal of each theorem in the former two section, we will accord an example to explicate our fundamental conditions in this theorem.

2. Preliminaries and Relevant Lemmas

We submit important definitions, lemmas and basic results that we need in this paper, which are taking from the books [24,25].

Definition 1. *Let $x(t)\in C[0,a]$, the Riemann–Liouville fractional integral with order $\alpha\in\mathbb{R}_+$ is defined as follows*

$$I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds$$

provided that the integral exists, where $\Gamma(\alpha)$ is the well-known Euler gamma function.

Definition 2. *Let $x(t) \in C[0,a]$, the Caputo fractional integral with order $\alpha \in \mathbb{R}_+$ is given by*

$$^cD^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^n(s) ds$$

where $n \in \mathbb{N}$ and $n-1 < \alpha \le n$, provided that the integral exists.

Lemma 1. *Let $x(t) \in C[0,a]$, $n \in \mathbb{N}$ and $n-1 < \alpha \le n$, then we have*

$$I^\alpha\, {}^cD^\alpha x(t) = x(t) + a_0 + a_1 t + \cdots + a_{n-1} t^{n-1}.$$

Consider the linear fractional Langevin equation

$$D^\beta (D^\alpha + \lambda) x(t) = h(t), \qquad t \in [0,1] \tag{3}$$

where $h(t)$ is a continuous function on $[0,1]$.

Lemma 2. *Let $h \in C[0,1]$, the unique solution of the boundary value problem for Langevin Equation (3) subject to the boundary conditions (2) satisfying*

$$\begin{aligned} x(t) = &\frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} h(s) ds - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds \\ &+ \frac{A_1(t)}{\Omega} \left\{ \frac{1}{\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} h(s) ds - \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} x(s) ds \right\} \\ &+ \frac{A_2(t)}{\Omega} \sum_{i=1}^m \mu_i \left\{ \frac{1}{\Gamma(\alpha+\beta)} \int_0^{q_i} (q_i-s)^{\alpha+\beta-1} h(s) ds - \frac{\lambda}{\Gamma(\alpha)} \int_0^{q_i} (q_i-s)^{\alpha-1} x(s) ds \right\} \\ &+ \frac{A_3(t)}{\Omega} \left\{ \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} h(s) ds \right\} \end{aligned} \tag{4}$$

where

$$\begin{aligned} \Omega = &2\lambda\Gamma(\alpha+1) - \lambda \sum_{i=1}^m \mu_i {q_i}^\alpha - \Gamma(\alpha+1) \sum_{i=1}^m \mu_i + \lambda \sum_{i=1}^m \mu_i {q_i}^{\alpha+1} \\ &+ 2\Gamma(\alpha+1) \sum_{i=1}^m \mu_i {q_i}^{\alpha+1} - \lambda\Gamma(\alpha+2) - 2\Gamma(\alpha+1)\Gamma(\alpha+2) \neq 0 \end{aligned} \tag{5}$$

$$\begin{aligned} A_1(t) = &t^{\alpha+1}\Gamma(\alpha+1) \left(\sum_{i=1}^m \mu_i - 2\lambda \right) + \lambda t^\alpha \left(\sum_{i=1}^m \mu_i {q_i}^\alpha t + \Gamma(\alpha+2) - \sum_{i=1}^m \mu_i {q_i}^{\alpha+1} \right) \\ &+ \Gamma(\alpha+1) \left(\Gamma(\alpha+2) - \sum_{i=1}^m \mu_i {q_i}^{\alpha+1} \right) \end{aligned} \tag{6}$$

$$A_2(t) = t^\alpha \left(\lambda - \lambda t - 2\Gamma(\alpha+1) \right) + \Gamma(\alpha+1) \tag{7}$$

$$A_3(t) = t^\alpha \left(\lambda t + 2\Gamma(\alpha+1) t - \lambda \right) - \Gamma(\alpha+1) \tag{8}$$

Proof. By using Lemma 1, we obtain:

$$^cD^\alpha x(t) + \lambda x(t) = I^\beta h(t) + a_0 + a_1 t. \tag{9}$$

Using the condition ${}^cD^\alpha x(0) = 0$ gives $a_0 = \lambda x(0)$. From Lemma 1 again with using the Definition 1, we have:

$$x(t) = \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} h(s)ds - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)ds + \frac{t^\alpha}{\Gamma(\alpha+1)} a_0 + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} a_1 + a_2.$$

Substituting $t = 0$, we get $x(0) = a_2$ which yields that $a_0 = \lambda a_2$ and so we have

$$x(t) = \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} h(s)ds - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)ds + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} a_1 + \frac{a_2}{\Gamma(\alpha+1)} \left(\lambda t^\alpha + \Gamma(\alpha+1)\right) \tag{10}$$

Using the ani-periodic condition and $x(0) = a_2$, then we get

$$\frac{a_1}{\Gamma(\alpha+2)} + \frac{a_2}{\Gamma(\alpha+1)} \left(\lambda + 2\Gamma(\alpha+1)\right) = \lambda I^\alpha x(1) - I^{\alpha+\beta} h(1). \tag{11}$$

Also, multiplying the ani-periodic condition $x(0) + x(1) = 0$ by λ and adding ${}^cD^\alpha x(1)$, we get:

$$\lambda a_2 +^c D^\alpha x(1) + \lambda x(1) =^c D^\alpha x(1).$$

From the condition ${}^cD^\alpha x(1) = \sum_{i=1}^m \mu_i x(q_i)$ and using (9), we get:

$$\frac{a_1}{\Gamma(\alpha+2)} \left(\Gamma(\alpha+2) - \sum_{i=1}^m \mu_i {q_i}^{\alpha+1} \right) + \frac{a_2}{\Gamma(\alpha+1)} \left(2\lambda\Gamma(\alpha+1) - \lambda \sum_{i=1}^m \mu_i {q_i}^\alpha - \Gamma(\alpha+1) \sum_{i=1}^m \mu_i \right) = \sum_{i=1}^m \mu_i \left[I^{\alpha+\beta} h(q_i) - \lambda I^\alpha x(q_i) \right] - I^\beta h(1). \tag{12}$$

Now, solving the two Equations (11) and (12) for a_1 and a_2, we find that

$$\frac{a_1}{\Gamma(\alpha+2)} = \frac{1}{\Omega} \Bigg[\left(I^{\alpha+\beta} h(1) - \lambda I^\alpha x(1) \right) \left(\Gamma(\alpha+1) \sum_{i=1}^m \mu_i + \lambda \sum_{i=1}^m \mu_i {q_i}^\alpha - 2\lambda\Gamma(\alpha+1) \right) - \sum_{i=1}^m \mu_i \left(I^{\alpha+\beta} h(q_i) - \lambda I^\alpha x(q_i) \right) \left(2\Gamma(\alpha+1) + \lambda\right) + I^\beta h(1) \left(\lambda + 2\Gamma(\alpha+1)\right) \Bigg]$$

and

$$\frac{a_2}{\Gamma(\alpha+1)} = \frac{1}{\Omega} \Bigg[\sum_{i=1}^m \mu_i \left(I^{\alpha+\beta} h(q_i) - \lambda I^\alpha x(q_i) \right) - I^\beta h(1) + \left(I^{\alpha+\beta} h(1) - \lambda I^\alpha x(1) \right) \left(\Gamma(\alpha+2) - \sum_{i=1}^m \mu_i {q_i}^{\alpha+1} \right) \Bigg]$$

Inserting the values of a_1 and a_2 in (10), we get the solution (4). □

3. Existence of Solution

Assume that we have the Banach space $\mathcal{C} = C([0,1])$ containing of all continuous functions from $[0,1] \to \mathbb{R}$ defined by the norm

$$\|x\| = \sup_{t \in [0,1]} |x(t)|.$$

According to Lemma 2, we convert an operator S as follows:

$$Sx = S_1x + S_2x \tag{13}$$

where the operators $S_1, S_2 : \mathcal{C} \to \mathcal{C}$ are defined by

$$(S_1x)(t) = \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} h(s)ds - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)ds \tag{14}$$

and

$$\begin{aligned}(S_2x)(t) = & \frac{A_1(t)}{\Omega} \left\{ \frac{1}{\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} h(s)ds - \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} x(s)ds \right\} \\ & + \frac{A_2(t)}{\Omega} \sum_{i=1}^{m} \mu_i \left\{ \frac{1}{\Gamma(\alpha+\beta)} \int_0^{q_i} (q_i-s)^{\alpha+\beta-1} h(s)ds - \frac{\lambda}{\Gamma(\alpha)} \int_0^{q_i} (q_i-s)^{\alpha-1} x(s)ds \right\} \\ & + \frac{A_3(t)}{\Omega} \left\{ \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} h(s)ds \right\}\end{aligned} \tag{15}$$

To simplify this task, we put

$$\begin{aligned}\rho_1 = & \frac{1}{\Gamma(\alpha+\beta+1)} + \frac{\bar{A}_1}{|\Omega|\Gamma(\alpha+\beta+1)} + \frac{\bar{A}_2}{|\Omega|\Gamma(\alpha+\beta+1)} \sum_{i=1}^{m} |\mu_i| q_i^{\alpha+\beta} \\ & + \frac{\bar{A}_3}{|\Omega|\Gamma(\beta+1)}\end{aligned} \tag{16}$$

$$\rho_2 = \frac{|\lambda|}{\Gamma(\alpha+1)} + \frac{\bar{A}_1|\lambda|}{|\Omega|\Gamma(\alpha+1)} + \frac{\bar{A}_2|\lambda|}{|\Omega|\Gamma(\alpha+1)} \sum_{i=1}^{m} |\mu_i| q_i^{\alpha} \tag{17}$$

where

$$\bar{A}_1 = \sup_{t\in[0,1]} |A_1(t)| \qquad \bar{A}_2 = \sup_{t\in[0,1]} |A_2(t)| \qquad \bar{A}_3 = \sup_{t\in[0,1]} |A_3(t)| \tag{18}$$

where Ω, $A_1(t)$, $A_2(t)$ and $A_3(t)$ are defined by (5)–(8), respectively.

3.1. Existence via Krasnoselskii-Zabreiko Theorem

Now, we discuss the existence of the solution for the problem (1) and (2), by using Krasnoselskii–Zabreiko's fixed point theorem states as:

Lemma 3 ([26])**.** *Let $\mathcal{C}$ be a Banach space. Assume that $S : \mathcal{C} \to \mathcal{C}$ is completely continuous mapping and $\mathcal{L} : \mathcal{C} \to \mathcal{C}$ is a bounded linear mapping such that 1 is not an eigenvalue of $\mathcal{L}$ and*

$$\lim_{\|x\|\to\infty} \frac{\|Sx - \mathcal{L}x\|}{\|x\|} = 0. \tag{19}$$

Then S has a fixed point in $\mathcal{C}$

Theorem 1. *Suppose that the following conditions are satisfied*

($\Re_1$) *$f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(t,0)$ does not vanish identically in $[0,1]$*
($\Re_2$) $\lim_{\|x\|\to\infty} \frac{f(t,x)}{x} = \phi(t)$ *uniformly in $[0,1]$ and $\|\phi\| = \sup_{t\in[0,1]} |\phi(t)|$.*

Then the problem (1) and (2) has at least one nontrivial solution if $\|\phi\|\rho_1 + \rho_2 < 1$ where ρ_1 and ρ_2 are given by (16) and (17), respectively.

Proof. Define two bounded linear operators $\mathcal{L}_1, \mathcal{L}_2 : \mathcal{C} \to \mathcal{C}$ with setting $f(t, x(t)) = \phi(t)x(t)$ as follow

$$\mathcal{L}_1 x(t) = \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} \phi(s)x(s)ds - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)ds$$

and

$$\begin{aligned}\mathcal{L}_2 x(t) = & \frac{A_1(t)}{\Omega} \left\{ \frac{1}{\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} \phi(s)x(s)ds - \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} x(s)ds \right\} \\ & + \frac{A_2(t)}{\Omega} \sum_{i=1}^m \mu_i \left\{ \frac{1}{\Gamma(\alpha+\beta)} \int_0^{q_i} (q_i-s)^{\alpha+\beta-1} \phi(s)x(s)ds - \frac{\lambda}{\Gamma(\alpha)} \int_0^{q_i} (q_i-s)^{\alpha-1} x(s)ds \right\} \\ & + \frac{A_3(t)}{\Omega} \left\{ \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} \phi(s)x(s)ds \right\}.\end{aligned}$$

Consider $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$, we claim that 1 is not an eigenvalue of $\mathcal{L}$. Suppose that 1 is an eigenvalue of $\mathcal{L}$, then we find,

$$\begin{aligned}\|\mathcal{L}_1 x\| = & \sup_{t\in[0,1]} |\mathcal{L}_1 x(t)| \\ = & \sup_{t\in[0,1]} \left| \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} \phi(s)x(s)ds - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)ds \right| \\ \leq & \sup_{t\in[0,1]} \left\{ \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} |\phi(s)x(s)|ds + \frac{|\lambda|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |x(s)|ds \right\} \\ \leq & \sup_{t\in[0,1]} \left\{ \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} |\phi(s)|ds + \frac{|\lambda|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \right\} \|x\| \\ \leq & \|x\| \left\{ \frac{\|\phi\|}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda|}{\Gamma(\alpha+1)} \right\}\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{L}_2 x\| = & \sup_{t\in[0,1]} |(\mathcal{L}_2 x)(t)| \\ = & \sup_{t\in[0,1]} \left| \frac{A_1(t)}{\Omega} \left\{ \frac{1}{\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} \phi(s)x(s)ds - \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} x(s)ds \right\} \right. \\ & + \frac{A_2(t)}{\Omega} \sum_{i=1}^m \mu_i \left\{ \frac{1}{\Gamma(\alpha+\beta)} \int_0^{q_i} (q_i-s)^{\alpha+\beta-1} \phi(s)x(s)ds - \frac{\lambda}{\Gamma(\alpha)} \int_0^{q_i} (q_i-s)^{\alpha-1} x(s)ds \right\} \\ & \left. + \frac{A_3(t)}{\Omega} \left\{ \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} \phi(s)x(s)ds \right\} \right| \\ \leq & \sup_{t\in[0,1]} \left[\frac{A_1(t)}{|\Omega|} \left\{ \frac{1}{\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} |\phi(s)x(s)|ds + \frac{|\lambda|}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |x(s)|ds \right\} \right. \\ & + \frac{A_2(t)}{|\Omega|} \sum_{i=1}^m |\mu_i| \left\{ \frac{1}{\Gamma(\alpha+\beta)} \int_0^{q_i} (q_i-s)^{\alpha+\beta-1} |\phi(s)x(s)|ds + \frac{|\lambda|}{\Gamma(\alpha)} \int_0^{q_i} (q_i-s)^{\alpha-1} |x(s)|ds \right\} \\ & \left. + \frac{A_3(t)}{|\Omega|} \left\{ \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} |\phi(s)x(s)|ds \right\} \right] \\ \leq & \frac{\|x\|}{|\Omega|} \left\{ \frac{\|\phi\|}{\Gamma(\alpha+\beta+1)} \left(\bar{A}_1 + \bar{A}_2 \sum_{i=1}^m |\mu_i| {q_i}^{\alpha+\beta} \right) + \frac{|\lambda|}{\Gamma(\alpha+1)} \left(\bar{A}_1 + \bar{A}_2 \sum_{i=1}^m |\mu_i| {q_i}^{\alpha} \right) \right. \\ & \left. + \frac{\|\phi\| \bar{A}_3}{\Gamma(\beta+1)|} \right\}\end{aligned}$$

These conclude that

$$\|\mathcal{L}x\| \leq \|x\|(\|\phi\|\rho_1 + \rho_2) < \|x\|.$$

The source of the contradiction is our supposition that 1 is an eigenvalue of the operator $\mathcal{L}$. Therefore, 1 is not an eigenvalue of $\mathcal{L}$.

Let $r > 0$, $\|f\| = \sup_{t\in[0,1]} |f(t, x(t))|$ and $\mathcal{U}_r = \{x \in \mathcal{C} : \|x\| \leq r\}$ be a closed ball with fix radius $r \geq \|f\|\rho_1|1 - \rho_2|^{-1}$. These imply that

$$\begin{aligned}\|S_1x\| \leq &\sup_{t\in[0,1]} \left\{\frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}|f(s,x(s))|ds + \frac{|\lambda|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|x(s)|ds\right\}\\ \leq &\frac{\|f\|}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda|r}{\Gamma(\alpha+1)}\end{aligned}$$

and

$$\begin{aligned}\|S_2x\| \leq &\sup_{t\in[0,1]} \left[\frac{A_1(t)}{|\Omega|}\left\{\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}|f(s,x(s))|ds + \frac{|\lambda|}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}|x(s)|ds\right\}\right.\\ &+\frac{A_2(t)}{|\Omega|}\sum_{i=1}^m |\mu_i|\left\{\frac{1}{\Gamma(\alpha+\beta)}\int_0^{q_i} (q_i-s)^{\alpha+\beta-1}|f(s,x(s))|ds + \frac{|\lambda|}{\Gamma(\alpha)}\int_0^{q_i} (q_i-s)^{\alpha-1}|x(s)|ds\right\}\\ &\left.+\frac{A_3(t)}{|\Omega|}\left\{\frac{1}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}|f(s,x(s))|ds\right\}\right]\\ \leq &\frac{1}{|\Omega|}\left\{\frac{\|f\|}{\Gamma(\alpha+\beta+1)}\left(\bar{A}_1 + \bar{A}_2\sum_{i=1}^m |\mu_i|{q_i}^{\alpha+\beta}\right) + \frac{|\lambda|r}{\Gamma(\alpha+1)}\left(\bar{A}_1 + \bar{A}_2\sum_{i=1}^m |\mu_i|{q_i}^{\alpha}\right)\right.\\ &\left.+\frac{\|f\|\bar{A}_3}{\Gamma(\beta+1)|}\right\}\end{aligned}$$

This means that $\|Sx\| \leq \|f\|\rho_1 + r\rho_2 \leq r$. Therefore, S is uniformly bounded on $\mathcal{U}_r$.

Next, we prove that $S : \mathcal{C} \to \mathcal{C}$ is equicontinuous. Assume that $t_1, t_2 \in [0,1]$ provided that $t_1 < t_2$ then

$$\begin{aligned}\|(S_1x)(t_2) - (S_1x)(t_1)\| \leq &\frac{\|f\|}{\Gamma(\alpha+\beta)}\left(\int_0^{t_1}\left[(t_2-s)^{\alpha+\beta-1} - (t_1-s)^{\alpha+\beta-1}\right]ds\right.\\ &\left.+\int_{t_1}^{t_2}(t_2-s)^{\alpha+\beta-1}ds\right) + \frac{|\lambda|r}{\Gamma(\alpha)}\left(\int_0^{t_1}\left[(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\right]ds\right.\\ &\left.+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\right)\\ \leq &\frac{\|f\|}{\Gamma(\alpha+\beta+1)}(t_2^{\alpha+\beta} - t_1^{\alpha+\beta}) + \frac{|\lambda|r}{\Gamma(\alpha+1)}(2(t_2-t_1)^\alpha + t_1^\alpha - t_2^\alpha)\\ \leq &\frac{\|f\|}{\Gamma(\alpha+\beta+1)}(t_2^{\alpha+\beta} - t_1^{\alpha+\beta}) + \frac{2|\lambda|r}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha\end{aligned}$$

and

$$
\begin{aligned}
\|(S_2x)(t_2)-(S_2x)(t_1)\| \le & \frac{A_1(t_2)-A_1(t_1)}{|\Omega|}\left(\frac{1}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}|f(s,x(s))|ds\right.\\
&\left.+\frac{|\lambda|}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}|x(s)|ds\right)\\
&+\frac{A_2(t_2)-A_2(t_1)}{|\Omega|}\sum_{i=1}^{m}|\mu_i|\left(\frac{1}{\Gamma(\alpha+\beta)}\int_0^{q_i}(q_i-s)^{\alpha+\beta-1}|f(s,x(s))|ds\right.\\
&\left.+\frac{|\lambda|}{\Gamma(\alpha)}\int_0^{q_i}(q_i-s)^{\alpha-1}|x(s)|ds\right)\\
&+\frac{A_3(t_2)-A_3(t_1)}{|\Omega|}\left(\frac{1}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}|f(s,x(s))|ds\right)\\
\le & \frac{A_1(t_2)-A_1(t_1)}{|\Omega|}\left(\frac{\|f\|}{\Gamma(\alpha+\beta+1)}+\frac{|\lambda|r}{\Gamma(\alpha+1)}\right)\\
&+\frac{A_2(t_2)-A_2(t_1)}{|\Omega|}\sum_{i=1}^{m}|\mu_i|\left(\frac{\|f\|{q_i}^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{|\lambda|{q_i}^{\alpha}r}{\Gamma(\alpha+1)}\right)\\
&+\frac{A_3(t_2)-A_3(t_1)}{|\Omega|}\frac{\|f\|}{\Gamma(\beta+1)}
\end{aligned}
$$

It is clear that $|S_1x(t_2)-S_1x(t_1)|\to 0$ uniformly as $t_2\to t_1$ and $|S_2x(t_2)-S_2x(t_1)|\to 0$ uniformly as $t_2\to t_1$. Then, by applying the Arzela–Ascoli theorem, we conclude that the operator S is completely continuous.

To prove $\frac{\|Sx-\mathcal{L}x\|}{\|x\|}\to 0$ as $\|x\|\to\infty$ for $t\in[0,1]$, we have

$$
\begin{aligned}
\|S_1x-\mathcal{L}_1x\| = & \sup_{t\in[0,1]}|S_1x(t)-\mathcal{L}_1x(t)|\\
\le & \sup_{t\in[0,1]}\left\{\frac{1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}|f(s,x(s))-x(s)\phi(s)|ds\right.\\
&\left.+\frac{|\lambda|}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|x(s)-x(s)|ds\right\}\\
\le & \sup_{t\in[0,1]}\left\{\frac{1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}\left|\frac{f(s,x(s))}{x(s)}-\phi(s)\right||x(s)|ds\right\}\\
\le & \left(\frac{1}{\Gamma(\alpha+\beta+1)}\right)\left|\frac{f(s,x(s))}{x(s)}-\phi(s)\right|\|x\|
\end{aligned}
$$

and

$$
\begin{aligned}
\|S_2x - \mathcal{L}_2x\| &= \sup_{t\in[0,1]} |S_2x(t) - (\mathcal{L}_2x)(t)| \\
&\leq \sup_{t\in[0,1]} \Bigg[\frac{A_1(t)}{|\Omega|} \left\{ \frac{1}{\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} |f(s,x(s)) - x(s)\phi(s)| ds \right. \\
&\left. + \frac{|\lambda|}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |x(s) - x(s)| ds \right\} \\
&+ \frac{A_2(t)}{|\Omega|} \sum_{i=1}^m |\mu_i| \left\{ \frac{1}{\Gamma(\alpha+\beta)} \int_0^{q_i} (q_i-s)^{\alpha+\beta-1} |f(s,x(s)) - x(s)\phi(s)| ds \right. \\
&\left. + \frac{|\lambda|}{\Gamma(\alpha)} \int_0^{q_i} (q_i-s)^{\alpha-1} |x(s) - x(s)| ds \right\} \\
&+ \frac{A_3(t)}{|\Omega|} \left\{ \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} |f(s,x(s)) - x(s)\phi(s)| ds \right\} \Bigg] \\
&\leq \sup_{t\in[0,1]} \Bigg[\frac{A_1(t)}{|\Omega|} \left\{ \frac{1}{\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} \left| \frac{f(s,x(s))}{x(s)} - \phi(s) \right| |x(s)| ds \right. \\
&+ \frac{A_2(t)}{|\Omega|} \sum_{i=1}^m |\mu_i| \left\{ \frac{1}{\Gamma(\alpha+\beta)} \int_0^{q_i} (q_i-s)^{\alpha+\beta-1} \left| \frac{f(s,x(s))}{x(s)} - \phi(s) \right| |x(s)| ds \right. \\
&+ \frac{A_3(t)}{|\Omega|} \left\{ \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} \left| \frac{f(s,x(s))}{x(s)} - \phi(s) \right| |x(s)| ds \right\} \Bigg] \\
&\leq \left(\frac{\bar{A}_1}{\Gamma(\alpha+\beta+1)|\Omega|} + \frac{\bar{A}_2 \sum_{i=1}^m |\mu_i| {q_i}^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)|\Omega|} + \frac{\bar{A}_3}{\Gamma(\beta+1)|\Omega|} \right) \left| \frac{f(s,x(s))}{x(s)} - \phi(s) \right| \|x\|
\end{aligned}
$$

which means that

$$
\begin{aligned}
\frac{\|Sx - \mathcal{L}x\|}{\|x\|} \leq & \left(\frac{1}{\Gamma(\alpha+\beta+1)} + \frac{\bar{A}_1}{\Gamma(\alpha+\beta+1)|\Omega|} + \frac{\bar{A}_2 \sum_{i=1}^m |\mu_i| {q_i}^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)|\Omega|} + \frac{\bar{A}_3}{\Gamma(\beta+1)|\Omega|} \right) \\
& \times \left| \frac{f(s,x(s))}{x(s)} - \phi(s) \right|
\end{aligned}
$$

According to assumption $(\Re_2)$, we get

$$
\lim_{\|x\|\to\infty} \frac{\|Sx - Lx\|}{\|x\|} = 0.
$$

In consequence, by Theorem 1, the boundary value problem (1) and (2) has at least one nontrivial solution. □

Example 1. *Consider the following fractional Langevin equations with multi-point and anti-periodic conditions:*

$$
\begin{cases}
{}^cD^{\frac{3}{2}}({}^cD^{\frac{1}{2}} + \lambda)x(t) = f(t,x(t)), & 0 < t < 1 \\
x(0) + x(1) = 0, \quad {}^cD^{\frac{1}{2}}x(0) = 0, & {}^cD^{\frac{1}{2}}x(1) = \frac{4}{5}x(\frac{3}{4}) - \frac{9}{5}x(\frac{4}{5})
\end{cases} \tag{20}
$$

Here $\alpha = \frac{1}{2}$, $\beta = \frac{3}{2}$, $\lambda = 1/8$, $m = 2, \mu_1 = \frac{4}{5}, \mu_2 = \frac{-9}{5}, q_1 = \frac{3}{4}, q_2 = \frac{4}{5}$.

Define the function $f(t,x(t))$ *by*

$$
f(t,x(t)) = \frac{t^3 x}{5(t+1)^2} + \frac{1}{3}, \qquad t \in [0,1].
$$

Observe that $f(t,0) = \frac{1}{3}$ *and*

$$\lim_{\|x\|\to\infty} \frac{f(t,x(t))}{x} = \frac{t^3}{5(t+1)^2}.$$

Choosing $\phi(t) = \frac{t^3}{5(t+1)^2}$. *It is easy to show that* $\phi(t)$ *is increasing on* $[0,1]$ *which means that* $\|\phi\| \le 1/20$. *By the given values and carrying out the software of Mathematica 11, we get* $\|\phi\|\rho_1 + \rho_2 \simeq 0.410122 < 1$. *Therefore, the conditions of the theorem (1) are satisfied. Hence the BVP has at least one nontrivial solution.*

3.2. Existence via Nonlinear Alternative Leray-Schauder Fixed Point Theorem

Next, we study the existence of the solution for the problem (1) and (2) by using nonlinear alternative Leray–Schauder theorem which states.

Lemma 4 ([27,28])**.** *Let* $\mathbb{E}$ *be a Banach space,* M *a closed, convex subset of* $\mathbb{E}$ *and* U *an open subset of* M *with* $0 \in U$. *Suppose that the operator* $T : \overline{U} \to M$ *is a continuous and compact map (that is,* $T(\overline{U})$ *is a relatively compact subset of* M*). Then either*

(i) T *has a fixed point* $x^* \in \overline{U}$, *or*
(ii) *There is* $x \in \partial U$ *(the boundary of* U *in* M*) and* $\rho \in (0,1)$ *such that* $\rho T(x) = x$.

Theorem 2. *Let* $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ *be a jointly continuous function satisfying the following conditions*

$(\Re_3)$ *There exists a nonnegative function* $v \in C([0,1], \mathbb{R}^+)$ *and a nondecreasing function* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$|f(t,u)| \le v(t)\psi(\|x\|), \qquad \forall (t,x) \in [0,1] \times \mathbb{R}.$$

$(\Re_4)$ *There exists a constant* $N > 0$ *such that*

$$\frac{N}{\rho_1 \|v\| \psi(N) + \rho_2 N} > 1$$

where ρ_1 *and* ρ_2 *are defined as in(16) and (17), respectively.*

Then the boundary value problem (1) and (2) has at least one solution on $[0,1]$.

Proof. Firstly, we will show that the operator S maps bounded sets into bounded sets in $C([0,T],\mathbb{R})$. Consider the closed ball $\mathcal{U}_r = \{x \in \mathcal{C} : \|x\| \le r\}$ with the radius $r \ge \|v\|\psi(r)\rho_1|1-\rho_2|^{-1}$. By the condition $(\mathcal{H}_3)$, we have

$$|S_1x(t)| \le \sup_{t\in[0,1]} \left\{ \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} v(t)\psi(\|x\|)ds + \frac{|\lambda|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|x(s)|ds \right\}$$

and

$$\begin{aligned} |S_2x(t)| \le \sup_{t\in[0,1]} \Bigg[& \frac{A_1(t)}{|\Omega|} \left\{ \frac{1}{\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} v(t)\psi(\|x\|)ds + \frac{|\lambda|}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}|x(s)|ds \right\} \\ & + \frac{A_2(t)}{|\Omega|} \sum_{i=1}^m |\mu_i| \left\{ \frac{1}{\Gamma(\alpha+\beta)} \int_0^{q_i} (q_i-s)^{\alpha+\beta-1} v(t)\psi(\|x\|)ds + \frac{|\lambda|}{\Gamma(\alpha)} \int_0^{q_i} (q_i-s)^{\alpha-1}|x(s)|ds \right\} \\ & + \frac{A_3(t)}{|\Omega|} \left\{ \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1}|f(s,x(s))|ds \right\} \Bigg] \end{aligned}$$

Taking the norm on $[0,1]$, we have

$$\|S_1x\| \le \frac{\|v\|\psi(r)}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda|r}{\Gamma(\alpha+1)}$$

and

$$\|S_2x\| \leq \frac{1}{|\Omega|}\left\{\frac{\|v\|\psi(r)}{\Gamma(\alpha+\beta+1)}\left(\bar{A}_1+\bar{A}_2\sum_{i=1}^{m}|\mu_i|{q_i}^{\alpha+\beta}\right)+\frac{|\lambda|r}{\Gamma(\alpha+1)}\left(\bar{A}_1+\bar{A}_2\sum_{i=1}^{m}|\mu_i|{q_i}^{\alpha}\right)\right.$$
$$\left.+\frac{\|v\|\psi(r)\bar{A}_3}{\Gamma(\beta+1)|}\right\}$$

Consequently, $\|Sx\| \leq \|v\|\psi(r)\rho_1 + r\rho_2 \leq r$. As in Theorem 1, it is clear that the operator $S : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is completely continuous. Finally, we will show that there exist an open subset U_N of $C([0,1],\mathbb{R})$ such that $x \notin \delta S(x)$ for $\delta \in (0,1)$ and $x \in \partial U_N$.

Using the same computations above we obtain

$$\|x\| = \sup_{t\in[0,1]}|x(t)| = \sup_{t\in[0,1]}|\delta S(x)(t)| \leq \rho_1\|v\|\psi(\|x\|)+\rho_2\|x\|$$

which yield to

$$\frac{\|x\|}{\rho_1\|v\|\psi(\|x\|)+\rho_2\|x\|} \leq 1$$

But by $(\Re_4)$ there exist a constant N such that $\|x\| \neq N$. Letting $U_N = \{x \in C([0,1],\mathbb{R}) : \|x\| < N\}$. Notice that the operator $S : \overline{U_N} \to C([0,1],\mathbb{R})$ is completely continuous. From the choice of U_N, there is no $x \in \partial U_N$ such that $x \in \delta S(x)$ and $\delta \in (0,1)$. Therefore, Lemma 4 concludes that S has a fixed point $x \in \overline{U_N}$. This is a solution for the problem. □

Example 2. *Consider the following fractional Langevin equations with multi-point and anti-periodic conditions:*

$$\begin{cases} {}^cD^{\frac{3}{2}}({}^cD^{\frac{1}{4}}+\lambda)x(t) = f(t,x(t)), & 0<t<1 \\ x(0)+x(1)=0, \quad {}^cD^{\frac{1}{4}}x(0)=0, & {}^cD^{\frac{1}{4}}x(1)=\frac{1}{3}x(\frac{3}{4})-\frac{2}{3}x(\frac{3}{7}) \end{cases} \tag{21}$$

Here $\alpha = \frac{1}{4}$, $\beta = \frac{3}{2}$, $\lambda = \frac{1}{4}$, $m = 2$, $\mu_1 = \frac{1}{3}$, $\mu_2 = \frac{-2}{3}$, $q_1 = \frac{3}{4}$, $q_2 = \frac{3}{7}$.
Let $f(t,x(t)) = \frac{e^t}{10(3+t)}(x+2cosx), t \in [0,1]$. *Obviously*

$$|f(t,x(t))| \leq \frac{e^t}{10(3+t)}(\|x\|+2).$$

with a choice $v(t) = e^t/[10(t+3)]$ *and* $\psi(\|x\|) = \|x\|+2$. *It is not difficult to see that* $v(t)$ *is increasing on* $[0,1]$ *and so* $v(t) \leq v(1) = e/40$. *From assumption* $(\Re_4)$, *we have*

$$N > \frac{2\rho_1\|v\|}{1-\rho_1\|v\|-\rho_2} \sim 0.857376.$$

Hence, the BVP has at least one solution on $[0,1]$.

4. Uniqueness of Solution

We will show the uniqueness of solution of problem by applying Banach contraction mapping principle.

Theorem 3. *Let* $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ *be a jointly continuous function satisfying the Lipschitz condition:*
$(\Re_5)$ *There exists a positive number L such that*

$$|f(t,x)-f(t,y)| \leq L|x-y|, \forall t \in [0,1], x,y \in \mathbb{R}.$$

Then the boundary value problem (1) and (2) has a unique solution on $[0,1]$ *if* $\sigma < 1$ *where*

$$\sigma = L\rho_1 + \rho_2$$

and ρ_1 *and* ρ_2 *are defined as in (16) and (17), respectively.*

Proof. we choose a bounded set as $\mathcal{U}_r = \{x \in \mathcal{C} : \|x\| \le r\}$ and also we choose a fix

$$r > \frac{\delta\rho_1}{1-\rho_2-L\rho_1} \qquad \text{where} \qquad \delta = \sup_{t\in[0,1]} |f(t,0)|.$$

First, we prove that $S\mathcal{U}_r \subseteq \mathcal{U}_r$. Then, for $x \in \mathcal{U}_r$, we have

$$\begin{aligned}\|S_1x\| \le& \sup_{t\in[0,1]} \left\{\frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}(|f(s,x(s))-f(s,0)|+|f(s,0)|)ds\right.\\ &\left.+\frac{|\lambda|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|x(s)|ds\right\}\\ \le&\frac{Lr+\delta}{\Gamma(\alpha+\beta+1)}+\frac{|\lambda|r}{\Gamma(\alpha+1)}\end{aligned}$$

and similarly

$$\begin{aligned}\|S_2x\| \le& \frac{1}{|\Omega|}\left\{\frac{Lr+\delta}{\Gamma(\alpha+\beta+1)}\left(\bar{A}_1+\bar{A}_2\sum_{i=1}^m|\mu_i|{q_i}^{\alpha+\beta}\right)+\frac{|\lambda|r}{\Gamma(\alpha+1)}\left(\bar{A}_1+\bar{A}_2\sum_{i=1}^m|\mu_i|{q_i}^{\alpha}\right)\right.\\ &\left.+\frac{(Lr+\delta)\bar{A}_3}{\Gamma(\beta+1)|}\right\}\end{aligned}$$

These imply that

$$\|Sx\| \le r(\rho_1L+\rho_2)+\delta\rho_1 = \sigma r+\delta\rho_1 \le r.$$

Next, we show that the operator S is a contraction mapping. Suppose that $x, y \in \mathcal{U}_r$, then

$$\begin{aligned}\|S_1x-S_1y\| =& \sup_{t\in[0,1]} |(S_1x)(t)-(S_1y)(t)|\\ \le& \sup_{t\in[0,1]}\left[\frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}|f(s,x(s))-f(s,y(s))|ds\right.\\ &\left.+\frac{|\lambda|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|x(s)-y(s)|ds\right]\\ \le& \|x-y\|\sup_{t\in[0,1]}\left[\frac{L}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}ds+\frac{|\lambda|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds\right]\\ \le& \|x-y\|\left[\frac{L}{\Gamma(\alpha+\beta+1)}+\frac{|\lambda|}{\Gamma(\alpha+1)}\right]\end{aligned}$$

and similarly

$$\begin{aligned}\|S_2x-S_2y\| \le& \frac{\|x-y\|}{|\Omega|}\left\{\frac{L}{\Gamma(\alpha+\beta+1)}\left(\bar{A}_1+\bar{A}_2\sum_{i=1}^m|\mu_i|{q_i}^{\alpha+\beta}\right)\right.\\ &\left.+\frac{|\lambda|r}{\Gamma(\alpha+1)}\left(\bar{A}_1+\bar{A}_2\sum_{i=1}^m|\mu_i|{q_i}^{\alpha}\right)+\frac{L\bar{A}_3}{\Gamma(\beta+1)|}\right\}\end{aligned}$$

which implies that

$$\|Sx-Sy\| \le (L\rho_1+\rho_2)\|x-y\| \le \sigma\|x-y\|.$$

Consequently, by the given assumption $\sigma < 1$, the operator S is a contraction. Therefore, from theorem (3), we conclude that the operator S has a fixed point, hence the BVP has a unique solution on $[0,1]$. □

Example 3. *Consider the following fractional Langevin equations with multi-point and anti-periodic conditions:*

$$\begin{cases} {}^cD^{\frac{8}{7}}({}^cD^{\frac{1}{7}} + \lambda)x(t) = f(t, x(t)), & 0 < t < 1 \\ x(0) + x(1) = 0, \quad {}^cD^{\frac{1}{7}}x(0) = 0, & {}^cD^{\frac{1}{7}}x(1) = \frac{3}{4}x(\frac{1}{3}) - \frac{1}{4}x(\frac{2}{3}) \end{cases} \tag{22}$$

Here $\alpha = \frac{1}{7}$, $\beta = \frac{8}{7}$, $\lambda = \frac{1}{4}$, $m = 2, \mu_1 = \frac{3}{4}, \mu_2 = \frac{-1}{4}, q_1 = \frac{1}{3}, q_2 = \frac{2}{3}$

Let the function $f(t, x(t)) = L(t + \tan^{-1} x(t))$. *Notice that* $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ *is continuous and satisfies Lipschitz condition:*

$$|f(t, x(t)) - f(t, y(t))| = L|\tan^{-1} x(t) - \tan^{-1} y(t)| \le L\|x - y\|.$$

By carry out Mathematica 11 and the previous values and quantities, we get $\rho_1 = 1.8725$ *and* $\rho_2 = 0.519117$ *which yields that for all* $L < (1 - \rho_2)/\rho_1 \sim 0.256814$ *we have* $\sigma = (L\rho_1 + \rho_2) < 1$. *Then, it follows by Theorem 3, this problem has a unique solution on* $[0,1]$.

5. Conclusions

We have discussed some results that concern the uniqueness and existence of solutions for generalized Langevin equation with two generalized orders including anti-periodic and multi-point boundary conditions. We establish an equivalence of problem by means of utilizing the tools of fractional calculus and immutable point theorems. To study our problem, we utilize Krasnoselskii–Zabreiko's immutable point type, the nonlinear alternative Leray–Schauder type and the Banach contraction principle. Our tactic is simple and serviceable to an assortment of real-world problems.

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