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Article

The Proof of a Conjecture Relating Catalan Numbers to an Averaged Mandelbrot-Möbius Iterated Function

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Abstract: In 2021, Mork and Ulness studied the Mandelbrot and Julia sets for a generalization of the well-explored function $\eta_\lambda(z) = z^2 + \lambda$. Their generalization was based on the composition of η_λ with the Möbius transformation $\mu(z) = \frac{1}{z}$ at each iteration step. Furthermore, they posed a conjecture providing a relation between the coefficients of (each order) iterated series of $\mu(\eta_\lambda(z))$ (at $z = 0$) and the Catalan numbers. In this paper, in particular, we prove this conjecture in a more precise (quantitative) formulation.

Keywords: fractal; Mandelbrot set; Julia set; Möbius transformation; iterated function; Catalan numbers

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1. Introduction

Let $\eta : \mathbb{C} \to \mathbb{C}$ be a monic complex polynomial of degree $d \geq 2$. We denote by η^j the j-th iterate of η, that is,

$$\eta^j(z) = \overbrace{\eta(\eta(\eta(\cdots\eta}^{j-\text{times}}(z)\cdots))).$$

The *filled-in Julia set* of η is defined as

$$K(\eta) = \{z \in \mathbb{C} : \eta^j(z) \text{ does not diverge}\}$$

and the *Julia set* $J(\eta)$ of the function η is defined to be the boundary of the set $K(\eta)$, i.e., $J(\eta) = \partial K(\eta)$ (see, e.g., [1]).

In this work, we are interested in a modified version of the "classical" filled-in Julia set $K(\eta_\lambda)$ and the Julia set $J(\eta_\lambda)$ of functions in the *quadratic family* $(\eta_\lambda(z))_{\lambda\in\mathbb{C}} = (z^2 + \lambda)_{\lambda\in\mathbb{C}}$. We observe that the Mandelbrot set $M(\eta_\lambda)$ is the fractal defined as

$$M(\eta_\lambda) = \{\lambda \in \mathbb{C} : J(\eta_\lambda) \text{ is connected}\}.$$

We point out that there is a more "workable" way of considering the Mandelbrot set (we refer to [2], Theorem 14.14) for a proof of the usually referred fundamental theorem of the Mandelbrot set):

$$\lambda \in M(\eta_\lambda) \iff \eta_\lambda^j(0) \text{ does not diverge}.$$

Some other recent results related to the Mandelbrot set can be found for example in [3–10].

In 2019, Mork et al. [11] constructed filled-in Julia sets for a *lacunary function* $\eta_{N,k}(z) = \sum_{n=1}^{N} z^{P_k(n)}$, where $(P_n(k))_n = (\frac{1}{2}(k\,n^2 - k\,n - 2))_n$ is the sequence of *centered k-gonal numbers* and k is any positive integer (for more facts and history of lacunary functions see, e.g., [12–14]).

In 2021, Mork et al. [15] followed up on the aforementioned article and considered a generalization of the filled-in Julia sets and their corresponding Mandelbrot sets by composing the lacunary function $\eta(z) = \sum_{n=1}^{N} z^{P_k(n)}$ with a fixed Möbius transformation $\mathcal{M}(z) = e^{i\theta}\frac{z-a}{\bar{a}z-1}$ (with $(\theta, a) \in \mathbb{R} \times \mathbb{D}$, where $\mathbb{D}$ denotes the the unit disc) at each iteration step. More precisely

$$h^j(z; a, k, N, \theta) = \overbrace{\mathcal{M}(\eta_{N,k}(\mathcal{M}(\eta_{N,k}(\cdots \mathcal{M}(\eta_{N,k}}^{j-\text{times}}(z)\cdots))))).$$

Very recently, Mork and Ulness [16] continued the previous line of research by dealing with the so-called *j-averaged Mandelbrot set* which is a set generated by iterating a function obtained by composing the function η_λ and the Möbius transformation $\mu_A(z) = \frac{az+b}{cz+d}$, where $A = (a, b, c, d) \in \mathbb{C}^4$. Thus,

$$h^j(z; A) = \overbrace{\mu_A(\eta_\lambda(\mu_A(\eta_\lambda(\cdots \mu_A(\eta_\lambda}^{j-\text{times}}(z)\cdots))))).$$

The name "*j*-averaged" is used here since the points of the resulting fractal are colored according to the total number of members of the following sequence of iterations $(\mathcal{H}_n)_{0 \le n \le j}$, that escaped from the circle with radius 2 (the concrete algorithm for coloring of points of this fractal you can find in Appendix 1 of [16]), see Figure 1,

$$\begin{aligned}(\mathcal{H}_n)_{0 \le n \le j} &= \{h^0(0; A), h^1(0; A), \ldots, h^j(0; A)\} \\ &= \{0, \mu_A(\eta_\lambda(0)), \ldots, (\overbrace{\mu_A(\eta_\lambda(\mu_A(\eta_\lambda(\cdots \mu_A(\eta_\lambda}^{j-\text{times}}(0)\cdots))))))\}.\end{aligned}$$

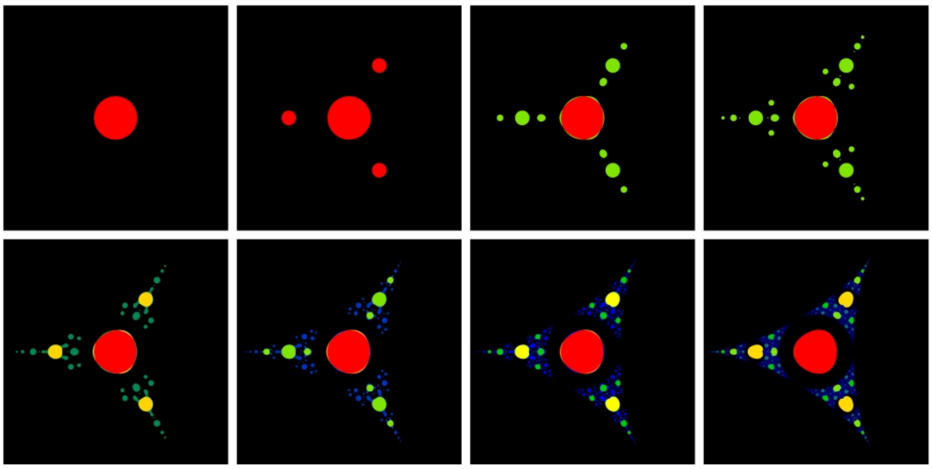

Figure 1. The *j*-averaged Mandelbrot sets for $A = (0, 0.5, 1, 0)$, $\lambda = x + i\,y$, with $x \in [-1.2, 0.8]$, $y \in [-1, 1]$, $j = 1, 2, 3, 4$ (the first row from the left to the right) and for and $j = 5, 7, 10, 100$ (the second row from the left to the right). We used functions in the software *Mathematica*® (see [17]) that are defined in Appendix 1 of [16].

Mork and Ulness ([16] Theorem 1) proved that the *j*-averaged Mandelbrot set for the Möbius transformation μ_A with $A = (0, 1, 1, 0)$ has threefold rotational symmetry and dihedral mirror symmetry. Additionally, they raised a conjecture (see [16], Conjecture 2)) concerning the coefficients of these iterations. Before stating their conjecture, we introduce some basic notations.

Let $\lambda \in \mathbb{D}$ be a non-zero complex number. Define the function $H(z, \lambda)$ by $H(z, \lambda) := \mu_A(\eta_\lambda(z))$, with $A = (0, 1, 1, 0)$. Therefore,

$$H(z, \lambda) = \frac{1}{z^2 + \lambda}.$$

Observe that the n-th iteration of H at $z = 0$ is a function of λ, say $h_n(\lambda)$, which satisfies the relations:

$$h_0(\lambda) = 0,\ h_1(\lambda) = H(0,\lambda) = \frac{1}{\lambda} \quad \text{and} \quad h_{n+1}(\lambda) = H(h_n(\lambda),\lambda),\ \text{for } n \geq 1. \tag{1}$$

The sequence $(C_n)_{n\geq 0}$ of the *Catalan numbers*, which is called the sequence A000108 in the *OEIS* [18], is often defined with the help of the central binomial coefficient $\binom{2n}{n}$ by

$$C_n = \frac{1}{n+1}\binom{2n}{n},$$

thus, its first terms are in Table 1.

Table 1. Values of C_n for n from 0 to 14.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
C_n	1	1	2	5	14	42	132	429	1430	4862	16,796	58,786	208,012	742,900	2,674,440

which can lead us to the following recurrence relation (it was first discovered by Euler in 1761; for more facts, see [19])

$$C_n = \frac{4n-2}{n+1}C_{n-1},\ \text{ for } n \geq 1,$$

with the initial condition $C_0 = 1$. Sometimes the sequence $(C_n)_{n\geq 0}$ is defined on the basis of the generating function $(1-\sqrt{1-4x})/(2x)$, as the following holds (for $|x| < 1/4$)

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{n+1}x^n = \frac{2}{1+\sqrt{1-4x}}.$$

The aim of this paper is to obtain a (quantitative) result for the coefficients of the power series of $h_n(\lambda)$ which implies the Mork–Ulness' conjecture (qualitative version). More precisely,

Theorem 1. *For all $n \geq 1$, we have*

$$h_n(\lambda) = \frac{1-(-1)^n}{2\lambda} + (-1)^n\sum_{i=1}^{\lfloor n/2\rfloor} C_{i-1}\lambda^{3i-1} + O(\lambda^{3\lfloor n/2\rfloor+2}), \tag{2}$$

where C_n is the n-th Catalan number.

Remark 1. *We remark that Mork and Ulness [16] posed a slightly different conjecture. In fact, we can express their question by defining $h_\infty^{(1)}(\lambda)$ and $h_\infty^{(2)}(\lambda)$ as*

$$h_\infty^{(1)}(\lambda) := \lim_{n\to\infty} h_{2n+1}(\lambda) = \frac{1}{\lambda} - \sum_{i\geq 0} C_i\lambda^{3i+2}$$

and

$$h_\infty^{(2)}(\lambda) := \lim_{n\to\infty} h_{2n}(\lambda) = \sum_{i\geq 0} C_i\lambda^{3i+2}.$$

They also asserted that these functions should converge in the whole unit disk (or the punctured one for $h_\infty^{(1)}(\lambda)$). However, this is not true (this is expected because of the exponential nature of Catalan numbers). For example, the simple bound $\binom{2n}{n} \geq 4^n/(2n+1)$, which comes from the fact that $4^n = (1+1)^{2n} = \sum_{k=0}^{2n}\binom{2n}{k}$, implies that $C_n > 4^n/(3n^2)$ (some other bounds can be found in ([19] Chapter 2) and [20]) and so if $|\lambda| > 1/\sqrt[3]{2} \approx 0.793$ then $|C_n\lambda^{3n+2}| \geq$

$3^{-1}|\lambda|^2(2^n/n^2)(\sqrt[3]{2}|\lambda|)^{3n} > |\lambda|^2/3$ *(for* $n \geq 4$*) yielding the divergence of* $h_\infty^{(2)}(\lambda)$*. In order to compute the radius of convergence, say* r*, of* $h_\infty^{(2)}(\lambda)$*, one can write this function as* $h_\infty^{(2)}(\lambda) = \sum_{n\geq 0} a_n\lambda^n$*, where*

$$a_n = \begin{cases} C_{(n-2)/3}, & \text{if } n \equiv 2 \pmod 3, \\ 0, & \text{if } n \not\equiv 2 \pmod 3. \end{cases}$$

Thus, $1/r = \limsup_{n\to\infty} \sqrt[n]{a_n}$ *and, by using* $C_n \approx 4^n/(n^{3/2}\sqrt{\pi})$ *(which comes from the Stirling formula* $n! \approx \sqrt{2\pi n}(n/e)^n$*), we obtain*

$$\begin{aligned} \frac{1}{r} &= \limsup_{n\to\infty} \sqrt[n]{a_n} \\ &= \limsup_{n\to\infty} \sqrt[3n+2]{C_n} \\ &= \limsup_{n\to\infty} \sqrt[3n+2]{\frac{4^n}{n^{3/2}\sqrt{\pi}}} \\ &= \sqrt[3]{4}. \end{aligned}$$

Therefore, $B(0, 1/\sqrt[3]{4})$ *is the disk of convergence of* $h_\infty^{(2)}(\lambda)$ *(observe that* $r = 1/\sqrt[3]{4} \approx 0.6299$*).*

2. Auxiliary Results

Before proceeding further, we shall present some useful tools related to the previous sequences.

Our the first ingredient provides a useful form to the Laurent series of $h_n(\lambda)$.

Lemma 1. *For any* $n \geq 1$*, there exists a power series* $P_n(\lambda)$ *such that*

$$h_n(\lambda) = \begin{cases} \lambda^2 + \lambda^5 P_n(\lambda^3), & \text{if } n \text{ is even;} \\ \frac{1}{\lambda} + \lambda^2 P_n(\lambda^3), & \text{if } n \text{ is odd.} \end{cases}$$

Proof. By definition in (1), $h_n(\lambda)$ satisfies the following recurrence relation

$$h_{n+1}(\lambda) = \frac{1}{(h_n(\lambda))^2 + \lambda},$$

with $h_0(\lambda) = 0$ (since $h_1(\lambda) = H(0, \lambda) = 1/\lambda$). Now, by defining $f_n(\lambda) := \lambda\, h_n(\lambda)$ and using the previous recurrence, we obtain

$$f_{n+1}(\lambda)/\lambda = \frac{1}{(f_n(\lambda)/\lambda)^2 + \lambda}$$

and so

$$f_{n+1}(\lambda) = \frac{\lambda^3}{(f_n(\lambda))^2 + \lambda^3}, \tag{3}$$

with $f_0(\lambda) = 0$. We claim that $f_n(\lambda) = g_n(\lambda^3)$ for some rational function $g_n(\lambda)$, where n is any positive integer. Indeed, we can proceed by induction on n. For $n = 1$, we can take $g_1(\lambda) = 1$. Suppose (by induction hypothesis) that $f_n(\lambda) = g_n(\lambda^3)$, for some formal power series $g_n(\lambda)$, then, by (3), we have

$$f_{n+1}(\lambda) = \frac{\lambda^3}{(g_n(\lambda^3))^2 + \lambda^3} = g_{n+1}(\lambda^3),$$

where $g_{n+1}(\lambda)$ can be chosen by satisfying the recurrence

$$g_{n+1}(\lambda) = \frac{\lambda}{(g_n(\lambda))^2 + \lambda},$$

with $g_0(\lambda) = 0$. The inductive process is finished. Observe that, since $h_n(\lambda) = \lambda^{-1} g_n(\lambda^3)$, then it suffices to prove that

$$g_n(\lambda) = \begin{cases} \lambda + O(\lambda^2), & \text{if } n \text{ is even;} \\ 1 + O(\lambda), & \text{if } n \text{ is odd.} \end{cases} \tag{4}$$

The proof is also by induction on n (more precisely, a double induction). For the basis cases, we have $g_1(\lambda) = 1 = 1 + O(\lambda)$ and

$$g_2(\lambda) = \frac{\lambda}{1+\lambda} = \lambda(1 - \lambda + \lambda^2 - \cdots) = \lambda + O(\lambda^2),$$

where we used that for $|\lambda| < 1$, one has $(1+\lambda)^{-1} = \sum_{k \geq 0} (-\lambda)^n$ (in general, it holds that $(1 + O(1))^{-1} = 1 + O(1)$). Suppose that (4) is valid for all $n \in [1, 2k]$. Then,

$$\begin{aligned} g_{2k+1}(\lambda) &= \frac{\lambda}{(g_{2k}(\lambda))^2 + \lambda} \\ &= \frac{\lambda}{(\lambda + O(\lambda^2))^2 + \lambda} \\ &= \frac{\lambda}{\lambda + O(\lambda^2)} \\ &= \frac{1}{1 + O(\lambda)} = 1 + O(\lambda), \end{aligned}$$

where we used $O(\lambda^r) + O(\lambda^s) = O(\lambda^{\min\{r,s\}})$, since $|\lambda| < 1$.

Now, we use the previous fact

$$\begin{aligned} g_{2k+2}(\lambda) &= \frac{\lambda}{(g_{2k+1}(\lambda))^2 + \lambda} \\ &= \frac{\lambda}{(1 + O(\lambda))^2 + \lambda} \\ &= \frac{\lambda}{1 + O(\lambda)} \\ &= \lambda(1 + O(\lambda)) = \lambda + O(\lambda^2). \end{aligned}$$

This completes the induction proof of (4).

Therefore, since $|\lambda| < 1$, we can write

$$g_n(\lambda) = \begin{cases} \lambda + \sum_{i \geq 2} c_i \lambda^i, & \text{if } n \text{ is even;} \\ 1 + \sum_{i \geq 1} c_i \lambda^i, & \text{if } n \text{ is odd} \end{cases}$$

and so

$$h_n(\lambda) = \frac{1}{\lambda} g_n(\lambda^3) = \begin{cases} \lambda^2 + \sum_{i \geq 2} c_i \lambda^{3i-1}, & \text{if } n \text{ is even;} \\ \dfrac{1}{\lambda} + \sum_{i \geq 1} c_i \lambda^{3i-1}, & \text{if } n \text{ is odd.} \end{cases}$$

This completes the proof. □

Remark 2. *Note that, by using Lemma 1, we can write*

$$h_n(\lambda) = \alpha_{-1,n}\lambda^{-1} + \alpha_{0,n}\lambda^2 + \alpha_{1,n}\lambda^5 + \cdots = \sum_{k=-1}^{\infty} \alpha_{k,n}\lambda^{3k+2} \in \mathbb{R}[[\lambda]], \tag{5}$$

where $\alpha_{-1,n} = (1-(-1)^n)/2$, *i.e.,* $\alpha_{-1,n}$ *is* 1 *if* n *is odd and* 0 *if* n *is even. In particular,* $h_n(\lambda)$ *is an analytic function in some neighborhood of* $\lambda = 0$*, when* n *is even, and for* n *odd,* $h_n(\lambda)$ *has a simple pole at origin (with residue equal to* 1*).*

Remark 3. *Another viewpoint of Lemma 1 (and consequently, of Remark 2) is that the k-th derivative of* $h_n(\lambda) = 0$ *as* $\lambda \to 0$*, for any* $k \equiv 0$ *or* $1 \pmod 3$*. This fact can also be proved by a harder (but maybe theoretically useful) combination of induction, the generalized Chain Rule (Faà di Bruno's formula) and the fact that all odd order derivatives of* $H_\lambda(z) := H(z,\lambda)$ *vanish (for fixed* λ*) at* $z = 0$*. This last assertion follows from Cauchy's integral formula. Indeed, we have*

$$H_\lambda^{(2n+1)}(0) = \frac{(2n+1)!}{2\pi i}\int_{\gamma_R} \frac{H_\lambda(\omega)}{\omega^{2n+2}}d\omega = \frac{(2n+1)!}{2\pi i}\int_{\gamma_R} \frac{1}{(\omega^2+\lambda)\omega^{2n+2}}d\omega,$$

where γ_R *is the circle* $\gamma(t) := Re^{it}$*, for* $t \in [0, 2\pi]$ *and* $0 < R < |\lambda|$*. Now, we can use the partial fraction decomposition to deduce that*

$$\frac{1}{(\omega^2+\lambda)\omega^{2n+2}} = \frac{A}{\omega+\sqrt{|\lambda|}} - \frac{A}{\omega-\sqrt{|\lambda|}} + \frac{B}{\omega^{2n+2}},$$

for computable constants A *and* B*. Hence, again by the Cauchy integral formula, we have*

$$\int_{\gamma_R} \frac{1}{(\omega^2+\lambda)\omega^{2n+2}}d\omega = 2A\pi i f(0) - 2A\pi i f(0) + 2B\pi i f^{(2n+1)}(0),$$

where $f(z) = 1$*, for all* z*. Thus,* $H_\lambda^{(2n+1)}(0)$ *is equal to zero as claimed.*

Now we show the important connection of the sequence $(\alpha_{k,n})_{k\geq 0}$ to the Catalan numbers. For the simplicity of notation, we use the following notation in the rest of the text:

$$\alpha_{k,n} = \begin{cases} d_k, & \text{for odd } n; \\ e_k, & \text{for even } n. \end{cases}$$

Lemma 2. *Let* $(C_k)_{k\geq 0}$ *be the Catalan sequence. We have*

(i) *If* $(d_k)_{k\geq 0}$ *is defined by the recurrence,*

$$d_{k+1} = -(C_1 d_k + \cdots + C_{k+1} d_0) - C_{k+2},$$

with $d_0 = -C_0$*, then* $d_k = -C_k$*, for all* $k \geq 0$*.*

(ii) *If* $(e_k)_{k\geq 1}$ *is defined by the recurrence,*

$$e_{k+1} = C_0\, e_k + \cdots + C_{k-1}\, e_1 + C_k,$$

with $e_1 = C_1$*, then* $e_k = C_k$*, for all* $k \geq 1$*.*

Proof. Let us recall that Catalan numbers satisfy the Segner recurrence relation (see, e.g., [19], p. 117)

$$C_{i+1} = \sum_{j=0}^{n} C_j\, C_{i-j}, \tag{6}$$

with $C_0 = 1$.

(i). We shall proceed by induction on k. For $k = 0$, one has $d_0 = -C_0$ (by definition). Suppose $d_t = C_t$, for all $t \in [0, k]$. Then,

$$\begin{aligned} d_{k+1} &= -(C_1 d_k + \cdots + C_{k+1} d_0) - C_{k+2} \\ &= C_1 C_k + \cdots + C_{k+1} C_0 - C_{k+2} \\ &= \underbrace{C_1 C_k + \cdots + C_{k+1} C_0}_{C_{k+2} - C_{k+1}} - C_{k+2} \\ &= -C_{k+1} \end{aligned}$$

which completes the proof (where we used (6)).

(ii). Again by induction on k, the basis case $e_1 = C_1$ follows by definition. Assume now that $e_t = C_t$, for all $t \in [1, k]$. Then, by the recurrence for $(e_k)_k$ together with the induction hypothesis, we obtain

$$\begin{aligned} e_{k+1} &= C_0 e_k + \cdots + C_{k-1} e_1 + C_k \\ &= C_0 C_k + \cdots + C_{k-1} C_1 + C_k \\ &= \underbrace{C_0 C_k + \cdots + C_{k-1} C_1}_{C_{k+1} - C_k} + C_k \\ &= C_{k+1} \end{aligned}$$

which finishes the proof (where we used again (6)). □

The next lemma gives a helpful recurrence for C_n, depending on the parity of n. The proof follows by induction together with (6) (we leave the details to the readers).

Lemma 3. *Let $(C_n)_{n \geq 0}$ be the Catalan sequence. Then,*

$$C_{2n} = 2 \sum_{j=1}^{n} C_{j-1} C_{2n-j} \text{ and } C_{2n+1} = C_n^2 + 2 \sum_{j=1}^{n} C_{j-1} C_{2n-j+1},$$

for all $n \geq 0$ (with $C_0 = 1$).

Now, we are ready to deal with the proof.

3. The Proof of the Theorem 1

First, observe that (2) can be rewritten for any as

$$h_{2j}(\lambda) = C_0 \lambda^2 + \cdots + C_{j-1} \lambda^{3j-1} + O(\lambda^{3j+2}) \tag{7}$$

and

$$h_{2j+1}(\lambda) = \frac{1}{\lambda} - (C_0 \lambda^2 + \cdots + C_{j-1} \lambda^{3j-1}) + O(\lambda^{3j+2}), \tag{8}$$

where we adopt the convention that $C_0 \lambda^2 + \cdots + C_{j-1} \lambda^{3j-1} = 0$ for $j = 0$.

Now, we want to prove the following fact:

Claim. It holds that

$$(C_0 \lambda^2 + \cdots + C_{n-1} \lambda^{3n-1})^2 = C_1 \lambda^4 + \cdots + C_n \lambda^{3n+1} + O(\lambda^{3n+2}), \tag{9}$$

for a non-negative integer n.

Proof. The proof is by induction on n. The identity is true for $n = 0$, since $C_1 = C_0^2$. Suppose that (9) holds, then one has

$$
\begin{aligned}
(C_0\lambda^2 + \cdots + C_{n-1}\lambda^{3n-1} + C_n\lambda^{3n+2})^2 &= (C_0\lambda^2 + \cdots + C_{n-1}\lambda^{3n-1})^2 \\
&\quad +2(C_0\lambda^2 + \cdots + C_{n-1}\lambda^{3n-1})C_n\lambda^{3n+2} + (C_n\lambda^{3n+2})^2.
\end{aligned}
$$

Since we desire to evaluate the identity up to $O(\lambda^{3n+5})$, then

$$
2(C_0\lambda^2 + \cdots + C_{n-1}\lambda^{3n-1})C_n\lambda^{3n+2} + (C_n\lambda^{3n+2})^2 = 2C_0C_n\lambda^{3n+4} + O(\lambda^{3n+7}). \tag{10}
$$

On the other hand, in the induction hypothesis

$$
(C_0\lambda^2 + \cdots + C_{n-1}\lambda^{3n-1})^2 = C_1\lambda^4 + \cdots + C_n\lambda^{3n+1} + O(\lambda^{3n+2}),
$$

the terms of order λ^{3n+4} were neglected (since we were interested in $O(\lambda^{3n+2})$). Thus, we can improve the previous identity by considering these terms (note that this procedure does not affect the induction hypothesis). Additionally, since the sum of two numbers, which are congruent to 2 modulo 3, is congruent to 1 modulo 3, there is no term of magnitude λ^{3n+3} in $(C_0\lambda^2 + \cdots + C_{n-1}\lambda^{3n-1})^2$. Let us also suppose that n is odd (the even case is carried out along the same lines). We then have

$$
\begin{aligned}
(C_0\lambda^2 + \cdots + C_{n-1}\lambda^{3n-1})^2 &= C_1\lambda^4 + \cdots + C_n\lambda^{3n+1} \\
&\quad +2(C_1C_{n-1} + \cdots + C_{(n-1)/2}C_{(n+1)/2})\lambda^{3n+4} + O(\lambda^{3n+5}).
\end{aligned} \tag{11}
$$

Now, we combine (10) and (11) together with Lemma 3 to arrive at

$$
\begin{aligned}
(C_0\lambda^2 + \cdots + C_{n-1}\lambda^{3n-1} + C_n\lambda^{3n+2})^2 &= (C_0\lambda^2 + \cdots + C_{n-1}\lambda^{3n-1})^2 \\
&\quad +2(C_0\lambda^2 + \cdots + C_{n-1}\lambda^{3n-1})C_n\lambda^{3n+2} + (C_n\lambda^{3n+2})^2 \\
&= C_1\lambda^4 + \cdots + C_n\lambda^{3n+1} \\
&\quad +2(C_1C_{n-1} + \cdots + C_{(n-1)/2}C_{(n+1)/2})\lambda^{3n+4} + O(\lambda^{3n+5}) \\
&\quad +2C_0C_n\lambda^{3n+4} + O(\lambda^{3n+7}) \\
&= C_1\lambda^4 + \cdots + C_{n+1}\lambda^{3n+4} + O(\lambda^{3n+5})
\end{aligned}
$$

which finishes the proof of the claim.

Now, we return to the proof of (2). Again, the proof is by induction on n. For the basis case, we have

$$
h_1(\lambda) = \frac{1}{\lambda}
$$

and, by Lemma 1,

$$
h_2(\lambda) = \lambda^2 + O(\lambda^5).
$$

Suppose that (2) is true for $h_n(\lambda)$ with $n \in [1, 2j]$. Then, by the recurrence relation for $(h_n(\lambda))_n$ together with the induction hypothesis, we infer that

$$
h_{2j+1}(\lambda) = \frac{1}{(h_{2j}(\lambda))^2 + \lambda} = \frac{1}{(C_0\lambda^2 + \cdots + C_{j-1}\lambda^{3j-1} + O(\lambda^{3j+2}))^2 + \lambda}.
$$

However, we can use (9) to write

$$
(C_0\lambda^2 + \cdots + C_{j-1}\lambda^{3j-1} + O(\lambda^{3j+2}))^2 + \lambda = C_0\lambda + C_1\lambda^4 + \cdots + C_j\lambda^{3j+1} + O(\lambda^{3j+2}). \tag{12}
$$

From Lemma 1 and Remark 2, one has

$$
h_{2j+1}(\lambda) = \frac{1}{\lambda} + d_0\lambda^2 + d_1\lambda^5 + \cdots + d_{j-1}\lambda^{3j-1} + O(\lambda^{3j+2}).
$$

Thus, the coefficients $d_0, d_1, \ldots, d_{j-1}$ satisfy the following equality

$$1 \equiv (C_0\lambda + C_1\lambda^4 + \cdots + C_k\lambda^{3j+1} + O(\lambda^{3j+2}))\left(\frac{1}{\lambda} + d_0\lambda^2 + d_1\lambda^5 + \cdots + d_{j-1}\lambda^{3j-1} + O(\lambda^{3j+2})\right)$$

and so

$$1 \equiv 1 + \lambda\left(\sum_{i=0}^{j-1} d_i\lambda^{3i+2}\right) + C_1\lambda^4\left(\frac{1}{\lambda} + \sum_{i=0}^{j-2} d_i\lambda^{3i+2}\right) + \cdots + C_k\lambda^{3j+1}\left(\frac{1}{\lambda}\right) + O(\lambda^{3j+1}).$$

By reordering this sum, we obtain

$$0 \equiv (d_0 + C_1)\lambda^3 + (d_1 + C_1 d_0 + C_2)\lambda^6 + \cdots + (d_{j-1} + C_1 d_{j-2} + \cdots + C_j)\lambda^{3j} + O(\lambda^{3j+1}).$$

Therefore, $d_0 = -C_1 = -C_0$ and

$$d_t = -(C_1 d_{t-1} + \cdots + C_t d_0) - C_t,$$

for all $t \in [1, j-1]$. By Lemma 3 (i), we conclude that $d_t = -C_t$, for all $t \in [1, j-1]$ which yields that

$$h_{2j+1}(\lambda) = \frac{1}{\lambda} - (C_0\lambda^2 + C_1\lambda^5 + \cdots + C_{k-1}\lambda^{3j-1}) + O(\lambda^{3j+2})$$

as desired.

Thus, we determine that (2) holds for $h_n(\lambda)$ for all $n \in [1, 2j+1]$. To finish the proof, we must prove that (2) is also true for $n = 2j+2$. First, one has that

$$h_{2j+2}(\lambda) = \frac{1}{(h_{2j+1}(\lambda))^2 + \lambda} = \frac{1}{((1/\lambda) - (C_0\lambda^2 \cdots + C_{j-1}\lambda^{3j-1}) + O(\lambda^{3j+2}))^2 + \lambda}.$$

However, by (9) and after a straightforward calculation, we arrive at

$$(h_{2j+1}(\lambda))^2 + \lambda = \frac{1}{\lambda^2} - (C_0\lambda + C_1\lambda^4 + \cdots + C_{j-1}\lambda^{3j-2}) + C_k\lambda^{3j+1} + O(\lambda^{3j+2}). \quad (13)$$

Now, we use Lemma 1 (and Remark 2) to write

$$h_{2j+2}(\lambda) = e_0\lambda^2 + e_1\lambda^5 + \cdots + e_j\lambda^{3j+2} + O(\lambda^{3j+5}),$$

where $e_0 = 1$. Hence,

$$1 \equiv \left(\frac{1}{\lambda^2} - (C_0\lambda + \cdots + C_{j-1}\lambda^{3j-2}) + C_k\lambda^{3j+1} + O(\lambda^{3j+2})\right)(\lambda^2 + e_1\lambda^5 + \cdots + e_j\lambda^{3j+2} + O(\lambda^{3j+5})).$$

Thus,

$$1 \equiv 1 + \lambda^{-2}\left(\sum_{i=1}^{j} e_i\lambda^{3i+2}\right) - C_0\lambda\left(\sum_{i=0}^{j} e_i\lambda^{3i+2}\right) + \cdots + C_{j-1}\lambda^{3j-2}\cdot\lambda^2 + O(\lambda^{3j+3})$$

which can be re-written as

$$0 \equiv (e_1 - 1)\lambda^3 + (e_2 - C_0 e_1 - C_1)\lambda^6 + \cdots + (e_j - e_{j-1} - C_1 e_{j-2} - \cdots - C_{j-2}e_1 + C_{j-1})\lambda^{3j} + O(\lambda^{3j+3}).$$

We then deduce that $e_1 = 1 = C_1$ and

$$e_t = C_0 e_{j-1} + C_1 e_{j-2} + \cdots + C_{j-2}e_1 - C_{j-1},$$

for all $t \in [1, j]$. By Lemma 3 (ii), we have $e_t = C_t$, for all $t \in [1, j]$, yielding that

$$h_{2j+2}(\lambda) = C_0\lambda^2 + C_1\lambda^5 + \cdots + C_j\lambda^{3j+2} + O(\lambda^{3j+5}).$$

The proof is then complete. □

4. Conclusions

This paper is devoted to the proof of a conjecture formulated by Mork and Ulness ([16], Conjecture 4.2). Roughly speaking, they computationally observed the relation between the coefficients of $h_n(\lambda)$ (the n-th iteration of $1/(z^2 + \lambda)$ at $z = 0$) and the Catalan sequence $(C_k)_k$. Indeed, we prove a quantitative version of their conjecture by showing that the sequence $\left(h_n(\lambda) - (\frac{1-(-1)^n}{2\lambda} + (-1)^n \sum_{i=1}^{\lfloor n/2 \rfloor} C_{i-1}\lambda^{3i-1})\right)_n$ tends to zero (with order $|\lambda|^{3\lfloor n/2 \rfloor+2}$) as $n \to \infty$.

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