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Uniqueness of Solutions of the Generalized Abel Integral Equations in Banach Spaces

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Abstract: This paper studies the uniqueness of solutions for several generalized Abel's integral equations and a related coupled system in Banach spaces. The results derived are new and based on Babenko's approach, Banach's contraction principle and the multivariate Mittag–Leffler function. We also present some examples for the illustration of our main theorems.

Keywords: Riemann–Liouville fractional integral; Banach's fixed point theorem; Babenko's approach; Wright's generalized Bessel function; multivariate Mittag–Leffler function

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1. Introduction

Let $T > 0$. The space $L[0, T]$ is given by

$$L[0,T] = \left\{ u(x) : \ \|u\| = \int_0^T |u(x)|dx < \infty \right\}.$$

Clearly, $L[0, T]$ is a Banach space. The product space $L[0, T] \times L[0, T]$ (which is also a Banach space) is defined as follows:

$$L[0,T] \times L[0,T] = \{ (u(x), v(x)) : \ u(x), v(x) \in L[0,T] \},$$

with the norm given by

$$\|(u,v)\| = \|u\| + \|v\|.$$

The Riemann–Liouville fractional integral I^α of order $\alpha \in \mathbb{R}^+$ is defined for the function $u(x)$ by (see [1,2]):

$$(I^\alpha u)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} u(t)dt.$$

In particular, we have

$$(I^0 u)(x) = u(x).$$

Let $0 \leqq \alpha_0 < \alpha_i$ for $i = 1, 2, \cdots, m$ and $\alpha_0 \leqq \alpha$. In this paper, we begin to construct an explicit solution in $L[0, T]$ to the following Abel's integral equation by using Babenko's approach and the multivariate Mittag–Leffler function:

$$I^{\alpha_0} u(x) + \sum_{i=1}^{m} a_i I^{\alpha_i} u(x) = I^\alpha f(x), \tag{1}$$

where each $a_i \quad (i = 1, 2, \cdots, m)$ is a constant and $f(x) \in L[0,T]$. We then further investigate the uniqueness of solutions in $L[0,T]$ for the following nonlinear Abel's integral equation by using Banach's fixed point theorem:

$$I^{\alpha_0} u(x) + \sum_{i=1}^{m} a_i I^{\alpha_i} u(x) = I^{\alpha} g\big(x, u(x)\big), \tag{2}$$

where g is a mapping from $[0,T] \times \mathbb{R}$ to $\mathbb{R}$ and satisfies certain conditions. Finally, the sufficient conditions are given for the uniqueness of solutions in the product space $L[0,T] \times L[0,T]$ to the associated system given by

$$\begin{cases} I^{\alpha_0} u(x) + \sum_{i=1}^{m} a_i I^{\alpha_i} u(x) = I^{\alpha} g_1\big(x, u(x), v(x)\big) \\ \\ I^{\beta_0} v(x) + \sum_{i=1}^{m} b_i I^{\beta_i} v(x) = I^{\beta} g_2\big(x, u(x), v(x)\big), \end{cases} \tag{3}$$

where g_1 and g_2 are mappings from $[0,T] \times \mathbb{R}^2$ to $\mathbb{R}$, $0 \leqq \beta_0 \leqq \beta$ and $\beta_0 < \beta_i$ for all $i = 1, 2, \cdots, m$. Equations (1)–(3) are new and, to the best of our knowledge, have never been investigated earlier.

The single-term (for $m = 1$) Equation (1) turns out to be

$$u(x) + a_1 I^{\alpha_1} u(x) = f(x) \qquad (\alpha = \alpha_0 = 0), \tag{4}$$

which is the classical Abel's integral equation of the second kind with the following solution given by Hille and Tamarkin (see, for details [3]; see also [4–6]):

$$u(x) = f(x) - a_1 \int_0^x (x-t)^{\alpha_1 - 1}\, E_{\alpha_1,\alpha_1}(-a_1 (x-t)^{\alpha_1}) f(t) dt,$$

where $E_{\alpha,\beta}(z)$ given by

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)} \qquad (\alpha, \beta > 0),$$

is the two-parameter Mittag–Leffler function.

The above solution can also be easily deduced by the Laplace transform. Indeed,

$$\mathcal{L}(u(x) + a_1 I^{\alpha_1} u(x)) = \mathcal{L} f(x) = \tilde{f}(s)$$

infers that

$$\tilde{u}(s) + \frac{a_1}{s^{\alpha_1}} \tilde{u}(s) = \tilde{f}(s).$$

Hence, we have

$$\tilde{u}(s) = \left(1 - \frac{a_1}{a_1 + s^{\alpha_1}}\right) \tilde{f}(s).$$

Using the formula (1.80) from [7]

$$\int_0^{\infty} e^{-st} t^{\alpha_1 - 1} E_{\alpha_1,\alpha_1}(-a_1 t^{\alpha_1}) dt = \frac{1}{a_1 + s^{\alpha_1}},$$

we arrive at

$$\begin{aligned} u(x) &= f(x) - a_1 x^{\alpha_1 - 1} E_{\alpha_1,\alpha_1}(-a_1 x^{\alpha_1}) * f(x) \\ &= f(x) - a_1 \int_0^x (x-t)^{\alpha_1 - 1}\, E_{\alpha_1,\alpha_1}(-a_1 (x-t)^{\alpha_1}) f(t) dt, \end{aligned}$$

where $\phi * \psi$ denotes the Laplace convolution given by

$$(\phi * \psi)(x) = \int_0^x \phi(x-t)\psi(t)dt.$$

On the other hand, Babenko's approach is a potentially powerful tool for solving differential, integral and integro-differential equations by treating integral operators like variables. The method itself is similar to the Laplace transform method while dealing with such equations with constant coefficients, but it can be used in other cases as well, such as handling integral equations with variable coefficients (see [8,9]). To demonstrate this method, we are going to solve Equation (4) with Babenko's approach. Clearly, Equation (4) becomes

$$(1 + a_1 I^{\alpha_1})u(x) = f(x).$$

Therefore, we get

$$\begin{aligned} u(x) &= (1 + a_1 I^{\alpha_1})^{-1} f(x) = \sum_{k=0}^{\infty} (-1)^k (a_1 I^{\alpha_1})^k f(x) \\ &= f(x) + \sum_{k=0}^{\infty} (-1)^{k+1} (a_1 I^{\alpha_1})^{k+1} f(x) \\ &= f(x) - a_1 \sum_{k=0}^{\infty} (-a_1)^k \frac{1}{\Gamma(\alpha_1 k + \alpha_1)} \int_0^x (x-t)^{\alpha_1 k + \alpha_1 - 1} f(t)dt \\ &= f(x) - a_1 \int_0^x (x-t)^{\alpha_1 - 1} \sum_{k=0}^{\infty} (-a_1)^k \frac{1}{\Gamma(\alpha_1 k + \alpha_1)} (x-t)^{\alpha_1 k} f(t)dt \\ &= f(x) - a_1 \int_0^x (x-t)^{\alpha_1 - 1}\, E_{\alpha_1, \alpha_1}(-a_1 (x-t)^{\alpha_1}) f(t)dt. \end{aligned}$$

We now recall Wright's generalized Bessel Function $\varphi(\beta, \delta; z)$ defined as follows:

$$\varphi(\beta, \delta; z) = \sum_{j=0}^{\infty} \frac{z^j}{j!\, \Gamma(\beta j + \delta)} \qquad (\beta, \delta > 0).$$

We also define

$$S_m^{\mu}(x; z_1, \cdots, z_m; \beta_1, \cdots, \beta_m) = (h_1 * h_2 * \cdots * h_m)(x),$$

where

$$h_k = h_k(x) = x^{\mu_k - 1} \phi\Big(\beta_k, \mu_k; z_k x^{\beta_k}\Big) \qquad (x, \beta_k > 0;\ z_k \in \mathbb{R}),$$

$$\mu = \sum_{k=1}^{m} \mu_k, \quad \mu_k > 0.$$

Let

$$\begin{aligned} &G_m^{\mu}(x; \gamma_1, \cdots, \gamma_m; \beta_1, \cdots, \beta_m) \\ &\qquad = \int_0^{\infty} e^{-t} S_m^{\mu}(x; \gamma_1 t, \cdots, y_m t; \beta_1, \cdots, \beta_m)dt, \end{aligned}$$

and

$$w_\mu(x) = G_m^{\mu - \alpha_0}(x; -a_1, \cdots, -a_m; \alpha_1 - \alpha_0, \cdots, \alpha_m - \alpha_0) \qquad (\mu > \alpha_0).$$

In 2013, Pskhu [10] constructed an explicit solution for the following Abel's integral equation $\big($which is a special case of the Equation (1)$\big)$:

$$I^{\alpha_0}u(x) + \sum_{i=1}^{m} a_i I^{\alpha_i} u(x) = I^{\alpha_0} f(x) \qquad (f \in L[0,T]),$$

as follows:

$$u(x) = D^{\mu}_{0,x}(f * w_\mu)(x) \qquad (\mu > \alpha_0),$$

where the solution $u(x)$ is independent of the parameter μ and

$$D^{\mu}_{0,x}\phi(x) := \begin{cases} \dfrac{1}{\Gamma(-\mu)} \displaystyle\int_0^x \phi(t)(x-t)^{-\mu-1}\,dt = I^{-\mu}\phi(x) & (\mu < 0) \\ \phi(x) & (\mu = 0) \\ \dfrac{d^n}{dx^n}\{I^{n-\mu}\phi(x)\} & (n-1 < \mu \leqq n;\ n \in \mathbb{N}), \end{cases}$$

$\mathbb{N}$ being the set of positive integers.

We would also like to add that Gorenflo and Luchko [11] established an explicit solution to the following generalized Abel integral equation of the second kind, which was based on a modification of the Mikusiński operational calculus and the Mittag–Leffler function of several variables (see, for details, [12]):

$$u(x) - \sum_{j=1}^{m} a_j I^{\alpha_j \mu} u(x) = f(x) \qquad (\alpha_j > 0;\ m \geqq 1;\ \mu > 0;\ x > 0),$$

which is also a special case of Equation (1).

There are many analytic and numerical studies on Abel's integral equation and its variants in distribution, as well as the existence and uniqueness of the corresponding solutions by using fixed point theorems [7,8,13–15]. For example, Brunner et al. [16] considered numerical solutions of Abel's integral equation of the second kind:

$$u(x) = f(x) + \int_0^x (x-t)^{-\alpha}\, \kappa\big(x, t, u(t)\big) dt \qquad (x \in [0,T]),$$

where $0 < \alpha < 1$ and $f \in C[0,T]$, and the kernel κ is continuous on $\mathbb{S} \times \mathbb{R}$, with

$$\mathbb{S} = \{(t,s) : 0 \leqq s \leqq t \leqq T\},$$

and satisfies the Lipschitz conditions in the third argument.

The multivariate Mittag–Leffler function was studied by (among others) Hadid and Luchko [17] for solving linear fractional differential equations with constant coefficients by applying the operational calculus:

$$E_{(\alpha_1,\cdots,\alpha_m),\beta}(z_1,\cdots,z_m) = \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,\cdots,k_m} \frac{z_1^{k_1}\cdots z_m^{k_m}}{\Gamma(\alpha_1 k_1 + \cdots + \alpha_m k_m + \beta)},$$

where $\alpha_i > 0 \ \ (i = 1,2,\cdots,m)$ and $\beta > 0$.

2. A Set of Main Results

In this section, we begin to establish an explicit solution to Equation (1) by using Babenko's approach in [18].

Theorem 1. *Assume that* $f \in L[0,T]$, $0 \leqq \alpha_0 < \alpha_i$ *for* $i = 1,2,\cdots,m$ *and* $\alpha_0 \leqq \alpha$. *Then, Equation* (1) *has a unique solution in* $L[0,T]$ *given by*

$$u(x) = \sum_{k=0}^{\infty}(-1)^k \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,\cdots,k_m} a_1^{k_1}\cdots a_m^{k_m} \cdot I^{k_1(\alpha_1-\alpha_0)+\cdots+k_m(\alpha_m-\alpha_0)+\alpha-\alpha_0} f(x), \tag{5}$$

where each a_i $(i = 1,2,\cdots,m)$ *is a constant.*

Proof. Equation (1) becomes

$$u(x) + \sum_{i=1}^{m} a_i I^{\alpha_i-\alpha_0} u(x) = I^{\alpha-\alpha_0} f(x),$$

by applying the operator $D_{0,x}^{\alpha_0}$ to both sides of Equation (1). This implies that, by Babenko's method, we have

$$\begin{aligned} u(x) &= \left(1 + \sum_{i=1}^{m} a_i I^{\alpha_i-\alpha_0}\right)^{-1} I^{\alpha-\alpha_0} f(x) \\ &= \sum_{k=0}^{\infty}(-1)^k \left(\sum_{i=1}^{m} a_i I^{\alpha_i-\alpha_0}\right)^{k} I^{\alpha-\alpha_0} f(x) \\ &= \sum_{k=0}^{\infty}(-1)^k \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,\cdots,k_m} \left(a_1 I^{\alpha_1-\alpha_0}\right)^{k_1} \cdots \left(a_m I^{\alpha_m-\alpha_0}\right)^{k_m} I^{\alpha-\alpha_0} f(x) \\ &= \sum_{k=0}^{\infty}(-1)^k \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,\cdots,k_m} a_1^{k_1}\cdots a_m^{k_m} \\ &\quad \cdot I^{k_1(\alpha_1-\alpha_0)+\cdots+k_m(\alpha_m-\alpha_0)+\alpha-\alpha_0} f(x). \end{aligned}$$

Let

$$\Phi_\gamma(x) = \frac{x_+^\gamma}{\Gamma(\gamma)},$$

where $\gamma \in \mathbb{R}$ and

$$x_+^\alpha = \begin{cases} x^\alpha & (x > 0) \\ 0 & (\text{otherwise}). \end{cases}$$

We then find from [19] that

$$\Phi_{\gamma_1} * \Phi_{\gamma_2} = \Phi_{\gamma_1+\gamma_2}.$$

Clearly, we have

$$I^{k_1(\alpha_1-\alpha_0)+\cdots+k_m(\alpha_m-\alpha_0)+\alpha-\alpha_0} f(x) = \Phi_{k_1(\alpha_1-\alpha_0)+\cdots+k_m(\alpha_m-\alpha_0)+\alpha-\alpha_0} * f.$$

Moreover, it follows from [20] that

$$\begin{aligned} &\left\|\Phi_{k_1(\alpha_1-\alpha_0)+\cdots+k_m(\alpha_m-\alpha_0)+\alpha-\alpha_0} * f\right\| \\ &\quad \leqq \left\|\Phi_{k_1(\alpha_1-\alpha_0)+\cdots+k_m(\alpha_m-\alpha_0)+\alpha-\alpha_0}\right\| \|f\| \\ &\quad \leqq \frac{T^{k_1(\alpha_1-\alpha_0)+\cdots+k_m(\alpha_m-\alpha_0)+\alpha-\alpha_0}}{\Gamma(k_1(\alpha_1-\alpha_0)+\cdots+k_m(\alpha_m-\alpha_0)+\alpha-\alpha_0+1)} \|f\|. \end{aligned}$$

This implies that

$$
\begin{aligned}
\|u(x)\| &\leqq \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,\cdots,k_m} |a_1|^{k_1}\cdots|a_m|^{k_m} \\
&\quad \cdot \frac{T^{k_1(\alpha_1-\alpha_0)+\cdots+k_m(\alpha_m-\alpha_0)+\alpha-\alpha_0}}{\Gamma(k_1(\alpha_1-\alpha_0)+\cdots+k_m(\alpha_m-\alpha_0)+\alpha-\alpha_0+1)}\|f\| \\
&= T^{\alpha-\alpha_0}E_{(\alpha_1-\alpha_0,\cdots,\alpha_m-\alpha_0),\,\alpha-\alpha_0+1}\left(|a_1|T^{\alpha_1-\alpha_0},\cdots,|a_m|T^{\alpha_m-\alpha_0}\right)\|f\| \\
&< \infty.
\end{aligned}
$$

Hence, we get $u(x) \in L[0,T]$, and that the series on the right-hand side of Equation (5) is absolutely convergent in $L[0,T]$.

The uniqueness of the solution follows immediately from the fact that the following fractional integral equation:

$$
I^{\alpha_0}u(x) + \sum_{i=1}^{m} a_i I^{\alpha_i}u(x) = 0,
$$

has only the zero solution by Babenko's method.

It remains to show that the series on the right-hand side of Equation (5) is a solution of Equation (1). Indeed, we have

$$
\begin{aligned}
& I^{\alpha_0}u(x) + \sum_{i=1}^{m} a_i I^{\alpha_i}u(x) \\
&\quad = \sum_{k=0}^{\infty}(-1)^k \sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m} a_1^{k_1}\cdots a_m^{k_m}\cdot I^{k_1(\alpha_1-\alpha_0)+\cdots+k_m(\alpha_m-\alpha_0)+\alpha}f(x) \\
&\qquad + \sum_{k=0}^{\infty}(-1)^k \sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\sum_{i=1}^{m} a_1^{k_1}\cdots a_i^{k_i+1}\cdots a_m^{k_m} \\
&\qquad \cdot I^{k_1(\alpha_1-\alpha_0)+\cdots+(k_i+1)(\alpha_i-\alpha_0)+\cdots+k_m(\alpha_m-\alpha_0)+\alpha}f(x) \\
&\quad = I^{\alpha}f(x) + \sum_{k=1}^{\infty}(-1)^k \sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m} a_1^{k_1}\cdots a_m^{k_m} \\
&\qquad \cdot I^{k_1(\alpha_1-\alpha_0)+\cdots+k_m(\alpha_m-\alpha_0)+\alpha}f(x) \\
&\qquad + \sum_{k=0}^{\infty}(-1)^k \sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\sum_{i=1}^{m} a_1^{k_1}\cdots a_i^{k_i+1}\cdots a_m^{k_m} \\
&\qquad \cdot I^{k_1(\alpha_1-\alpha_0)+\cdots+(k_i+1)(\alpha_i-\alpha_0)+\cdots+k_m(\alpha_m-\alpha_0)+\alpha}f(x) = I^{\alpha}f(x),
\end{aligned}
$$

by noting that

$$
\begin{aligned}
& \sum_{k=1}^{\infty}(-1)^k \sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m} a_1^{k_1}\cdots a_m^{k_m}\cdot I^{k_1(\alpha_1-\alpha_0)+\cdots+k_m(\alpha_m-\alpha_0)+\alpha}f(x) \\
&\qquad + \sum_{k=0}^{\infty}(-1)^k \sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\sum_{i=1}^{m} a_1^{k_1}\cdots a_i^{k_i+1}\cdots a_m^{k_m} \\
&\qquad \cdot I^{k_1(\alpha_1-\alpha_0)+\cdots+(k_i+1)(\alpha_i-\alpha_0)+\cdots+k_m(\alpha_m-\alpha_0)+\alpha}f(x) = 0,
\end{aligned}
$$

after the sign changes and cancellations. Obviously, it is true that

$$(-1)^1 \sum_{k_1+\cdots+k_m=1} \binom{1}{k_1, \cdots, k_m} a_1^{k_1} \cdots a_m^{k_m} \cdot I^{k_1(\alpha_1-\alpha_0)+\cdots+k_m(\alpha_m-\alpha_0)+\alpha} f(x)$$
$$+ (-1)^0 \sum_{k_1+\cdots+k_m=0} \binom{0}{k_1, \cdots, k_m} \sum_{i=1}^{m} a_1^{k_1} \cdots a_i^{k_i+1} \cdots a_m^{k_m}$$
$$\cdot I^{k_1(\alpha_1-\alpha_0)+\cdots+(k_i+1)(\alpha_i-\alpha_0)+\cdots+k_m(\alpha_m-\alpha_0)+\alpha} f(x) = 0.$$

This completes the proof of Theorem 1. □

As an example, we can deduce that the following integral equation:

$$I^{0.5}u(x) + Iu(x) + I^{1.5}u(x) = \frac{1}{2}x^2$$

has a unique solution given by

$$u(x) = \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+k_2=k} \binom{k}{k_1, k_2} \frac{x^{0.5k_1+k_2+1.5}}{\Gamma(0.5k_1 + k_2 + 2.5)},$$

in the space $L[0, T]$ by using the relation:

$$\frac{1}{2}x^2 = \Phi_3(x) = \Phi_2 * \Phi_1 = I^2 1,$$

and Theorem 1.

Using Banach's fixed point theorem, we are now ready to show the uniqueness of solutions to Equation (2) in the space $L[0, T]$.

Theorem 2. *Assume that $0 \leqq \alpha_0 < \alpha_i$ for $i = 1, 2, \cdots, m$ and $\alpha_0 \leqq \alpha$. Suppose also that there exists a constant $C \geqq 0$ such that*

$$|g(x, y_1) - g(x, y_2)| \leqq C|y_1 - y_2|,$$

for all $x \in [0, T]$, y_1, $y_2 \in \mathbb{R}$, $g(x, 0) \in L[0, T]$ and

$$CT^{\alpha-\alpha_0} E_{(\alpha_1-\alpha_0,\cdots,\alpha_m-\alpha_0),\, \alpha-\alpha_0+1}\left(|a_1|T^{\alpha_1-\alpha_0}, \cdots, |a_m|T^{\alpha_m-\alpha_0}\right) < 1.$$

Then the Equation (2) has a unique solution in the space $L[0, T]$.

Proof. Let $u \in L[0, T]$. Then $g\big(x, u(x)\big) \in L[0, T]$. Indeed, we have

$$\big|g\big(x, u(x)\big)\big| = \big|g\big(x, u(x)\big) - g(x, 0) + g(x, 0)\big| \leqq \big|g\big(x, u(x)\big) - g(x, 0)\big| + |g(x, 0)|$$
$$\leqq C|u(x)| + |g(x, 0)|,$$

which implies that

$$\int_0^T \big|g\big(x, u(x)\big)\big| dx \leqq C \int_0^T |u(x)| dx + \int_0^T |g(x, 0)| dx < \infty.$$

We now define a nonlinear mapping S on $L[0, T]$ as follows:

$$(Su)(x) = \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1, \cdots, k_m} a_1^{k_1} \cdots a_m^{k_m}$$
$$\cdot I^{k_1(\alpha_1-\alpha_0)+\cdots+k_m(\alpha_m-\alpha_0)+\alpha-\alpha_0} g\big(x, u(x)\big).$$

It follows from the lines of the proof of Theorem 1 that

$$\begin{aligned}\|S(u)\| &\leqq T^{\alpha-\alpha_0} E_{(\alpha_1-\alpha_0,\cdots,\alpha_m-\alpha_0),\,\alpha-\alpha_0+1}\left(|a_1|T^{\alpha_1-\alpha_0},\cdots,|a_m|T^{\alpha_m-\alpha_0}\right)\\ &\quad\cdot \left\|g\left(x,u(x)\right)\right\| < \infty,\end{aligned}$$

which shows that S is a mapping from $L[0,T]$ to itself.

It now remains to show that the mapping S is contractive. In fact, we have

$$\int_0^T \left|g\left(x,u(x)\right) - g\left(x,v(x)\right)\right| dx \leqq C \int_0^T |u(x) - v(x)| dx$$

and

$$\begin{aligned}\|S(u) - S(v)\| &\leqq T^{\alpha-\alpha_0} E_{(\alpha_1-\alpha_0,\cdots,\alpha_m-\alpha_0),\,\alpha-\alpha_0+1}\left(|a_1|T^{\alpha_1-\alpha_0},\cdots,|a_m|T^{\alpha_m-\alpha_0}\right)\\ &\quad\cdot \left\|g\left(x,u(x)\right) - g\left(x,v(x)\right)\right\|\\ &\leqq CT^{\alpha-\alpha_0} E_{(\alpha_1-\alpha_0,\cdots,\alpha_m-\alpha_0),\,\alpha-\alpha_0+1}\left(|a_1|T^{\alpha_1-\alpha_0},\cdots,|a_m|T^{\alpha_m-\alpha_0}\right)\|u-v\|\\ &= q\|u-v\|,\end{aligned}$$

where

$$q = CT^{\alpha-\alpha_0} E_{(\alpha_1-\alpha_0,\cdots,\alpha_m-\alpha_0),\,\alpha-\alpha_0+1}\left(|a_1|T^{\alpha_1-\alpha_0},\cdots,|a_m|T^{\alpha_m-\alpha_0}\right) < 1.$$

This completes the proof of Theorem 2. □

Finally, we present the sufficient conditions for the uniqueness of the solution of Equation (3) in the product space $L[0,T] \times L[0,T]$.

Theorem 3. *Assume that* $0 \leqq \alpha_0 < \alpha_i$, $0 \leqq \beta_0 < \beta_i$ *for* $i = 1,2,\cdots,m$ *and* $\alpha_0 \leqq \alpha$, $\beta_0 \leqq \beta$. *Suppose also that there exist constants* C_1, C_2, C_3 *and* C_4 *such that*

$$|g_1(x,y_1,y_2) - g_1(x,z_1,z_2)| \leqq C_1|y_1 - z_1| + C_2|y_2 - z_2|$$

and

$$|g_2(x,y_1,y_2) - g_2(x,z_1,z_2)| \leqq C_3|y_1 - z_1| + C_4|y_2 - z_2|,$$

for all $x \in [0,T]$, $y_1,y_2,z_1,z_2 \in \mathbb{R}$, $g_1(x,0,0), g_2(x,0,0) \in L[0,T]$ *and*

$$\begin{aligned}q &= \max\{C_1,C_2\}T^{\alpha-\alpha_0} E_{(\alpha_1-\alpha_0,\cdots,\alpha_m-\alpha_0),\,\alpha-\alpha_0+1}\left(|a_1|T^{\alpha_1-\alpha_0},\cdots,|a_m|T^{\alpha_m-\alpha_0}\right)\\ &\quad+ \max\{C_3,C_4\}T^{\beta-\beta_0} E_{(\beta_1-\beta_0,\cdots,\beta_m-\beta_0),\,\beta-\beta_0+1}\\ &\qquad\cdot\left(|b_1|T^{\beta_1-\beta_0},\cdots,|b_m|T^{\beta_m-\beta_0}\right)\\ &< 1.\end{aligned}$$

Then, Equation (3) *has a unique solution in the space* $L[0,T] \times L[0,T]$.

Proof. Let $u, v \in L[0,T]$. Then $g_1\left(x,u(x),v(x)\right) \in L[0,T]$. Indeed, we have

$$\begin{aligned}&\left|g_1\left(x,u(x),v(x)\right)\right|\\ &\quad= \left|g_1\left(x,u(x),v(x)\right) - g_1(x,0,0) + g_1(x,0,0)\right|\\ &\quad\leqq \left|g_1\left(x,u(x),v(x)\right) - g_1(x,0,0)\right| + |g_1(x,0,0)|\\ &\quad\leqq C_1|u(x)| + C_2|v(x)| + |g_1(x,0,0)|.\end{aligned}$$

This implies that

$$\int_0^T |g_1(x, u(x), v(x))|dx \leqq C_1 \int_0^T |u(x)|dx + C_2 \int_0^T |v(x)|dx + \int_0^T |g(x, 0, 0)|dx < \infty.$$

Similarly, we can see that $g_2(x, u(x), v(x)) \in L[0, T]$.

Let us now define the mappings S_1, S_2 on $L[0, T] \times L[0, T]$ as follows:

$$S_1(u, v)(x) = \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1, \cdots, k_m} a_1^{k_1} \cdots a_m^{k_m} \cdot I^{k_1(\alpha_1-\alpha_0)+\cdots+k_m(\alpha_m-\alpha_0)+\alpha-\alpha_0} g_1(x, u(x), v(x))$$

and

$$S_2(u, v)(x) = \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1, \cdots, k_m} a_1^{k_1} \cdots a_m^{k_m} \cdot I^{k_1(\alpha_1-\alpha_0)+\cdots+k_m(\alpha_m-\alpha_0)+\alpha-\alpha_0} g_2(x, u(x), v(x)).$$

Furthermore, we define a mapping S on $L[0, T] \times L[0, T]$ as follows:

$$S(u, v) = (S_1(u, v), S_2(u, v)),$$

with

$$\|S(u, v)\| = \|S_1(u, v)\| + \|S_2(u, v)\|.$$

Thus, clearly, we see that

$$\|S_1(u, v)\| \leqq T^{\alpha-\alpha_0} E_{(\alpha_1-\alpha_0,\cdots,\alpha_m-\alpha_0),\, \alpha-\alpha_0+1}(|a_1|T^{\alpha_1-\alpha_0}, \cdots, |a_m|T^{\alpha_m-\alpha_0}) \cdot \|g_1(x, u(x), v(x))\| < \infty$$

and

$$\|S_2(u, v)\| \leqq T^{\beta-\beta_0} E_{(\beta_1-\beta_0,\cdots,\beta_m-\beta_0),\, \beta-\beta_0+1}\Big(|b_1|T^{\beta_1-\beta_0}, \cdots, |b_m|T^{\beta_m-\beta_0}\Big) \cdot \|g_2(x, u(x), v(x))\| < \infty.$$

This implies that S is a mapping from $L[0, T] \times L[0, T]$ to itself.

It remains to be shown that S is contractive. Indeed, we have

$$\|S(u_1, v_1) - S(u_2, v_2)\| = \|S_1(u_1, v_1) - S_1(u_2, v_2)\| + \|S_2(u_1, v_1) - S_2(u_2, v_2)\|,$$

and

$$\begin{aligned}
&\|S_1(u_1,v_1) - S_1(u_2,v_2)\| \\
&\quad \leqq T^{\alpha-\alpha_0} E_{(\alpha_1-\alpha_0,\cdots,\alpha_m-\alpha_0),\,\alpha-\alpha_0+1}\left(|a_1|T^{\alpha_1-\alpha_0},\cdots,|a_m|T^{\alpha_m-\alpha_0}\right) \\
&\qquad \cdot \|g_1(x,u_1,v_1) - g_1(x,u_2,v_2)\| \\
&\quad \leqq T^{\alpha-\alpha_0} E_{(\alpha_1-\alpha_0,\cdots,\alpha_m-\alpha_0),\,\alpha-\alpha_0+1}\left(|a_1|T^{\alpha_1-\alpha_0},\cdots,|a_m|T^{\alpha_m-\alpha_0}\right) \\
&\qquad \cdot \left(C_1 \int_0^T |u_1(x) - u_2(x)|dx + C_2 \int_0^T |v_1(x) - v_2(x)|dx\right) \\
&\quad \leqq \max\{C_1, C_2\} T^{\alpha-\alpha_0} E_{(\alpha_1-\alpha_0,\cdots,\alpha_m-\alpha_0),\,\alpha-\alpha_0+1}\left(|a_1|T^{\alpha_1-\alpha_0},\cdots,|a_m|T^{\alpha_m-\alpha_0}\right) \\
&\qquad \cdot \|(u_1,v_1) - (u_2,v_2)\|.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\|S_2(u_1,v_1) - S_2(u_2,v_2)\| \\
&\quad \leqq \max\{C_3, C_4\} T^{\beta-\beta_0} E_{(\beta_1-\beta_0,\cdots,\beta_m-\beta_0),\,\beta-\beta_0+1}\left(|b_1|T^{\beta_1-\beta_0},\cdots,|b_m|T^{\beta_m-\beta_0}\right) \\
&\qquad \cdot \|(u_1,v_1) - (u_2,v_2)\|.
\end{aligned}$$

We thus find that

$$\begin{aligned}
&\|S(u_1,v_1) - S(u_2,v_2)\| \\
&\quad \leqq \max\{C_1, C_2\} T^{\alpha-\alpha_0} E_{(\alpha_1-\alpha_0,\cdots,\alpha_m-\alpha_0),\,\alpha-\alpha_0+1}\left(|a_1|T^{\alpha_1-\alpha_0},\cdots,|a_m|T^{\alpha_m-\alpha_0}\right) \\
&\qquad \cdot \|(u_1,v_1) - (u_2,v_2)\| \\
&\qquad + \max\{C_3, C_4\} T^{\beta-\beta_0} E_{(\beta_1-\beta_0,\cdots,\beta_m-\beta_0),\,\beta-\beta_0+1}\left(|b_1|T^{\beta_1-\beta_0},\cdots,|b_m|T^{\beta_m-\beta_0}\right) \\
&\qquad \cdot \|(u_1,v_1) - (u_2,v_2)\| \\
&\quad = q\|(u_1,v_1) - (u_2,v_2)\|,
\end{aligned}$$

where q is defined above and $q < 1$. This completes the proof of Theorem 3. □

3. An Illustrative Example

In this section, we present the following example to illustrate the use of Theorem 3.

Example 1. *The integral system given by*

$$\begin{cases} I^{0.5}u(x) - I^{1.6}u(x) + I^{1.5}u(x) = \dfrac{1}{8} I^{1.7} \sin v(x) \\ I^{2.3}v(x) + I^{3.3}v(x) - I^{3.4}v(x) - I^{3.5}v(x) = \dfrac{1}{53} I^{2.3} \cos u(x), \end{cases}$$

has a unique solution in $L[0,1] \times L[0,1]$.

Demonstration of Example 1 Clearly, we have

$$g_1\big(x, u(x), v(x)\big) = \frac{1}{8} \sin v(x)$$

and

$$g_2\big(x, u(x), v(x)\big) = \frac{1}{53} \cos u(x),$$

and $C_1 = 0, C_2 = 1/8, C_3 = 1/53$ and $C_4 = 0$, by noting that

$$|\sin z_1 - \sin z_2| \leqq |z_1 - z_2|$$

and

$$|\cos z_1 - \cos z_2| \leqq |z_1 - z_2|,$$

for all $z_1, z_2 \in \mathbb{R}$. Furthermore, we have

$$T = 1 \qquad \text{and} \qquad |a_1| = |a_2| = |b_1| = |b_2| = |b_3| = 1.$$

Hence, we get

$$\begin{aligned} q &= \max\{C_1, C_2\} T^{\alpha-\alpha_0} E_{(\alpha_1-\alpha_0,\cdots,\alpha_m-\alpha_0),\, \alpha-\alpha_0+1}\left(|a_1|T^{\alpha_1-\alpha_0}, \cdots, |a_m|T^{\alpha_m-\alpha_0}\right) \\ &\quad + \max\{C_3, C_4\} T^{\beta-\beta_0} E_{(\beta_1-\beta_0,\cdots,\beta_m-\beta_0),\, \beta-\beta_0+1}\left(|b_1|T^{\beta_1-\beta_0}, \cdots, |b_m|T^{\beta_m-\beta_0}\right) \\ &= \frac{1}{8} E_{(1.1,1),2.2}(1,1) + \frac{1}{53} E_{(1,1.1,1.2),1}(1,1,1). \end{aligned}$$

It is now evident that

$$\begin{aligned} & E_{(1.1,1),2.2}(1,1) \\ &\quad = \sum_{k=0}^{\infty} \sum_{k_1+k_2=k} \binom{k}{k_1, k_2} \frac{1}{\Gamma(1.1k_1 + k_2 + 2.2)} \\ &\quad \leqq \sum_{k=0}^{\infty} \sum_{k_1+k_2=k} \binom{k}{k_1, k_2} \frac{1}{\Gamma(k+2)} = \sum_{k=0}^{\infty} \frac{2^k}{(k+1)!} \\ &\quad = 1 + \frac{2}{2!} + \frac{2\cdot 2}{1\cdot 2\cdot 3} + \frac{2\cdot 2\cdot 2}{1\cdot 2\cdot 3\cdot 4} + \frac{2\cdot 2\cdot 2\cdot 2}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{2\cdot 2\cdot 2\cdot 2\cdot 2}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \cdots \\ &\quad \leqq 1 + 1 + 1 + \frac{2}{15}\left(1 + \frac{2}{5} + \left(\frac{2}{5}\right)^2 + \cdots\right) \\ &\quad = 3 + \frac{2}{15}\frac{1}{1-\frac{2}{5}} \leqq 3.23. \end{aligned}$$

On the other hand, we have

$$E_{(1,1.1,1.2),1}(1,1,1) = \sum_{k=0}^{\infty} \sum_{k_1+k_2+k_3=k} \binom{k}{k_1, k_2, k_3} \frac{1}{\Gamma(k_1 + 1.1k_2 + 1.2k_3 + 1)}.$$

It follows from [21] that

$$\frac{1}{\Gamma(k_1 + 1.1k_2 + 1.2k_3 + 1)} \leqq \frac{1}{\Gamma(k+1)}$$

for all $k = 0, 1, \cdots$. This implies that

$$\begin{aligned} & E_{(1,1.1,1.2),1}(1,1,1) \\ &\quad \leqq \sum_{k=0}^{\infty} \frac{3^k}{k!} \\ &\quad = 1 + \frac{3}{1} + \frac{3\cdot 3}{1\cdot 2} + \frac{3\cdot 3\cdot 3}{1\cdot 2\cdot 3} + \frac{3\cdot 3\cdot 3\cdot 3}{1\cdot 2\cdot 3\cdot 4} + \frac{3\cdot 3\cdot 3\cdot 3\cdot 3}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \cdots \\ &\quad \leqq 1 + 3 + 4.5 + 4.5 + 3.375 + 2.025 + \left(\frac{1}{80} + 1\right) + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \\ &\quad = 20.4125. \end{aligned}$$

Clearly, therefore, we get $q < 1$. This completes our demonstration of Example 1.

4. Conclusions

By using Babenko's approach, Banach's contraction principle and the multivariate Mittag–Leffler function, we have studied several generalized forms of Abel's integral equations and a related coupled system with constant coefficients in Banach spaces. The results, which we have presented in this article, are new and provide interesting generalizations of the existing results in the literature. We have also included some examples, including one example that shows the application of our main theorem (Theorem 3).

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