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Article

Finite-Approximate Controllability of Riemann–Liouville Fractional Evolution Systems via Resolvent-Like Operators

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Abstract: This paper presents a variational method for studying approximate controllability and infinite-dimensional exact controllability (finite-approximate controllability) for Riemann–Liouville fractional linear/semilinear evolution equations in Hilbert spaces. A useful criterion for finite-approximate controllability of Riemann–Liouville fractional linear evolution equations is formulated in terms of resolvent-like operators. We also find that such a control provides finite-dimensional exact controllability in addition to the approximate controllability requirement. Assuming the finite-approximate controllability of the corresponding linearized RL fractional evolution equation, we obtain sufficient conditions for finite-approximate controllability of the semilinear RL fractional evolution equation under natural conditions. The results are a generalization and continuation of recent results on this subject. Applications to fractional heat equations are considered.

Keywords: finite-approximate controllability; evolution equation; Riemann–Liouville fractional systems

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1. Introduction

Controllability is one of the fundamental qualitative concepts in modern mathematical control theory, which plays an important role in deterministic/stochastic control theory of dynamical systems. From a mathematical point of view, exact and approximate controllability problems should be distinguished. Exact controllability allows the system to be controlled to an arbitrary final state, while approximate controllability means that the system can be controlled to an arbitrary small neighborhood of the finite state, and very often approximate controllability is quite sufficient in applications. If the semigroup generated by the linear part of the dynamical system is compact, the controllability operator is also compact and, therefore, the inverse cannot exist. Hence, the concept of complete controllability is very strong and its feasibility is limited; approximate controllability is a weaker concept that is quite appropriate in practice. We would like to mention some interesting works related to deterministic evolution systems: Triggiani [1,2], Bashirov and Mahmudov [3], Yamamoto and Park [4], Naito [5], Zhou [6,7], Seidman [8], Li and Yong [9], Mahmudov [10,11].

On the other hand, interest in fractional calculus and fractional differential equations and inclusions has increased greatly in the recent years, as they suggest more appropriate models for some real-world problems in physics, biology, and so on. This is the main reason why they are studied extensively. We note that in the case of the fractional Caputo derivative, there is a similarity of initial conditions between fractional equations and ordinary equations. However, in the case of fractional Riemann–Liouville (RL) differential equations, the initial conditions must be given in terms of initial values of the fractional derivatives of the unknown function (which is different from the ordinary case). Heymans and Podlubny [12] have shown that physical meaning can be assigned to initial conditions expressed as fractional RL derivatives or integrals over the viscoelastic field, and that such initial conditions are more suitable than the physically interpretable initial conditions.

Among the many scientific articles on exact, approximate and finite-approximate controllability, we will mention only a few that motivate this work.

- The concept of exact controllability and the existence of solutions for fractional control systems has been developed using various methods, which can be found in Balachandran and Dauer [13,14], Ndambomve and Ezzinbi [15], Diallo et al. [16], Ezzinbi et al. [17], Ezzinbi and Lalaoui [18], Kavitha et al. [19]. Controllability problems for systems of various types described by fractional differential equations of Sobolev-type were studied by, for instance, Feckan et al. [20] and Wang et al. [21].
- There are several approaches to obtain approximate controllability of an evolutionary system. Zhou [6,7] used the so-called sequential approach to obtain sufficient conditions for the approximate controllability of a nonfractional semilinear evolutionary system. Dauer and Mahmudov [22] and Mahmudov [10] have used a resolvent approach, used by Bashirov and Mahmudov [3] to study approximate controllability for linear evolution equations, and obtained some sufficient conditions for the approximate controllability of classical semilinear systems. Later, this method was adapted to study the approximate controllability of fractional semilinear evolution systems by Sakthivel et al. [23]. Thereafter, several researchers, Bora and Roy [24], Dhayal and Malik [25], Kavitha et al. [26], Haq and Sukavanam [27], Aimene [28], Bedi [29], Matar [30], Ge et al. [31], Grudzka and Rykaczewski [32], Ke et al. [33], Kumar and Sukavanam [34,35], Liu and Li [36], Sakthivel et al. [37], Wang et al. [38], Yan [39], Yang and Wang [40], Rykaczewski [41], Mahmudov and McKibben [42,43], Ndambomve and Ezzinbi [44] have used different methods to study approximate controllability for several fractional differential and integro-differential systems.
- The variational approach used by Zuazua [45,46] to study approximate and finite-approximate controllability of the heat equation was adapted by Li and Yong [9] and Mahmudov [11,47] to study the same concepts for semilinear evolution systems. Subsequently, several researchers have used this method to study the finite-approximate controllability of classical/fractional deterministic/stochastic evolution systems: Wang et al. [48], Liu [49], Ding and Li [50], Liu and Yanfang [51], Mahmudov [52–54].

In this paper, we investigate simultaneous approximate and finite-dimensional exact controllability (finite-approximate controllability) of the following RL fractional semilinear evolution system:

$$\begin{cases} {}^{RL}D^{\alpha}_{0+}y(t) = Ay(t) + Bu(t) + f(t, y(t)), \ \ t \in [0, T], \\ I^{1-\alpha}_{0+}y(0) = y_0, \ \ \frac{1}{2} < \alpha \leq 1, \end{cases} \tag{1}$$

where ${}^{RL}D^{\alpha}_{0+}$ denotes the RL fractional derivative of order α with the lower limit zero, $(\mathfrak{X}, \|\cdot\|)$ is a Hilbert space, $A : D(A) \subset \mathfrak{X} \to \mathfrak{X}$ is a closed linear operator with dense domain generating a strongly continuous semigroup $S : [0, T] \to L(\mathfrak{X})$, the control function $u \in L^2([0, T], U)$, U is a Hilbert space, $B : U \to \mathfrak{X}$ is a bounded linear operator, $f : [0, T] \times \mathfrak{X} \to \mathfrak{X}$ is a nonlinear term satisfying some assumptions to be specified later and $y_0 \in \mathfrak{X}$.

We define the controllability concepts for the system (1).

Definition 1. *For the system (1), we define the following concepts:*

(a) *The RL fractional control system (1) is approximately controllable on $[0, T]$, if for every $y_0, y_f \in \mathfrak{X}$, and for every $\varepsilon > 0$, there exists a control $u \in L^2([0, T], U)$ such that the mild solution y of the RL fractional Cauchy problem (1) satisfies $I^{1-\alpha}_{0+}y(0) = y_0$ and $\left\| y(T) - y_f \right\| < \varepsilon$.*

(b) *Let M be a finite dimensional subspace of $\mathfrak{X}$ and let us denote by π_M the orthogonal projection from $\mathfrak{X}$ into M. The RL fractional control system (1) is finite-approximately controllable on $[0, T]$, if it is approximately controllable and $\pi_M y(T) = \pi_M y_f$.*

In this paper, we introduce a new variational method to study finite-approximate RL-fractional linear evolution and apply a quasi-linearization method to study RL fractional evolution of semilinear systems. The main contributions of this work can be summarized as follows:

- We develop a constructive variational approach that is somewhat different from approaches used in the literature, and provide a necessary and sufficient condition for the finite-approximate controllability of linear classical/fractional evolution systems in terms of resolvent-like operators (Criterion (iv) of Theorem 1). We also find an explicit form of finite-approximating control that provides finite-dimensional exact controllability in addition to the approximate controllability requirement, see Equations (13) and (14). This result is new even for the classical case $\alpha = 1$. It plays a major role in the proof of Theorems 3–5 and is interesting from both theoretical and numerical points of view.
- Using the explicit form of the finite-approximating control (14) and using quasi linearization of the RL fractional semilinear evolution problem, we study the finite-approximate controllability of the RL fractional semilinear evolution system. For semilinear systems, we consider two cases: (i) the nonlinear term $f(t, y)$ is continuous and has continuous uniformly bounded Frechet derivative $f'_y(\cdot, \cdot)$ (Theorem 3), (ii) the nonlinear term $f(t, y)$ is continuous and uniformly bounded (Theorem 4). These results are new only for the fractional case $\frac{1}{2} < \alpha < 1$.

The remainder of this article is organized as follows. Section 2 contains preliminary remarks. In Section 3, we will present some results on properties of positive linear compact operators depending on parameters. We define resolvent-like operators and give necessary and sufficient conditions for finite-approximate controllability of classical/fractional linear evolution equations. In Section 4, we first use a control defined by a resolvent-like operator, define a control operator Θ_ε, and show the existence of fixed points. Next, we prove the main result about the finite-approximate controllability of an RL fractional semilinear evolution system. In Section 5, we consider the case where the nonlinear term is bounded. Finally, we present two examples to demonstrate our main results.

2. Preliminaries

Let $C((0, T], \mathfrak{X})$ be the Banach space of all continuous mappings from $(0, T]$ into $\mathfrak{X}$. Define

$$C_{1-\alpha}((0, T], \mathfrak{X}) = \left\{ y \in C((0, T], \mathfrak{X}) : \lim_{t \to 0^+} t^{1-\alpha} y(t) \text{ exists and finite} \right\}$$

with the norm $\|\cdot\|_\alpha$ defined by

$$\|y\|_\alpha = \sup_{0 \le t \le T} t^{1-\alpha} \|y(t)\|.$$

Obviously, $C_{1-\alpha}((0, T], \mathfrak{X})$ is a Banach space.

Let us recall the following known definitions in fractional calculus.

Definition 2. *The RL fractional integral of order $\alpha > 0$ with the lower limit a for a function $f : [a, \infty) \to \mathbb{R}$ is defined as*

$$I_{a^+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \ t > 0, \alpha > 0,$$

provided the right-hand side is well defined on $[0, \infty)$. Here, Γ is the gamma function.

Definition 3. *The Riemann–Liouville derivative of order $0 < \alpha < 1$ with lower limit a is defined as*

$$^{RL}D_{a^+}^\alpha f(t) = \frac{d}{dt} I_{a^+}^{1-\alpha} f(t),$$

provided that the expression on the right-hand side is well-defined.

We define the family $\{\mathfrak{U}_\alpha(t) : t \geq 0\}$ of operators by

$$\begin{aligned}
\mathfrak{U}_\alpha(t) &= \int_0^\infty \alpha\theta\Psi_\alpha(\theta)S(t^\alpha\theta)d\theta, \\
\Psi_\alpha(\theta) &= \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)!\Gamma(1-n\alpha)} \sin(n\pi\alpha), \quad \theta \in (0,\infty), \\
\Psi_\alpha(\theta) &\geq 0, \quad \int_0^\infty \Psi_\alpha(\theta)d\theta = 1, \\
\int_0^\infty \theta^\zeta \Psi_\alpha(\theta)d\theta &= \frac{\Gamma(1+\zeta)}{\Gamma(1+\alpha\zeta)}, \quad \zeta \in (-1,\infty).
\end{aligned}$$

Lemma 1. *Ref. [55] The operator $\mathfrak{U}_\alpha$ has the following properties:*

(i) For any fixed $t > 0$, $\mathfrak{U}_\alpha(t)$ are linear and bounded operators, and

$$\|\mathfrak{U}_\alpha(t)x\| \leq \frac{M_S}{\Gamma(\alpha)}\|x\|, \ \ M_S := \sup_{t\geq 0}\|S(t)\|.$$

(ii) $\{\mathfrak{U}_\alpha(t) : t > 0\}$ is compact, if $\{S(t) : t > 0\}$ is compact.

Further properties of $\mathfrak{U}_\alpha$ were explored in [55,56].

It is known that if $y(t)$ is a solution of (1), then it satisfies Equation (2), see [55]. Therefore, we present the following definition of mild solutions of system (1).

Definition 4. *For each $u \in L^2([0,T],U)$ a function $y \in C_{1-\alpha}([0,T],\mathfrak{X})$ is called a mild solution of (1) if*

$$y(t) = t^{\alpha-1}\mathfrak{U}_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1}\mathfrak{U}_\alpha(t-s)[Bu(s) + f(s,y(s))]ds, \ \ t \in [0,T]. \tag{2}$$

First, we prove several lemmas.

Lemma 2. *For any $G \in C([0,T],L(\mathfrak{X}))$, there exists a unique strongly continuous function $\mathfrak{T}_\alpha : \Delta \to L(\mathfrak{X})$, $\Delta = \{(t,s) : 0 \leq s \leq t \leq T\}$, such that*

$$\begin{aligned}
\mathfrak{T}_\alpha(t,s;G)y &= (t-s)^{\alpha-1}\mathfrak{U}_\alpha(t-s)y + \int_s^t (t-r)^{\alpha-1}\mathfrak{U}_\alpha(t-r)G(r)\mathfrak{T}_\alpha(r,s;G)ydr \\
&= (t-s)^{\alpha-1}\mathfrak{U}_\alpha(t-s)y + \int_s^t (r-s)^{\alpha-1}\mathfrak{T}_\alpha(t,r;G)G(r)\mathfrak{U}_\alpha(r-s)ydr.
\end{aligned}$$

Proof. The proof is similar to that of [9]. □

It is clear that

$$\begin{aligned}
\mathfrak{T}^*_\alpha(t,s;G)y &= (t-s)^{\alpha-1}\mathfrak{U}^*_\alpha(t-s)y + \int_s^t (t-r)^{\alpha-1}\mathfrak{T}^*_\alpha(r,s;G)G^*(r)\mathfrak{U}^*_\alpha(t-r)ydr \\
&= (t-s)^{\alpha-1}\mathfrak{U}^*_\alpha(t-s)y + \int_s^t (t-r)^{\alpha-1}\mathfrak{U}^*_\alpha(r-s)G^*(r)\mathfrak{T}^*_\alpha(t,r;G)ydr.
\end{aligned}$$

The following lemma for the left fractional operator is given in [57]. For the right fractional operator, the following result can be proved in a similar way.

Lemma 3. *Let $\frac{1}{2} < \alpha \leq 1$, $a > 0$, $b(t)$ be a nonnegative, nondecreasing continuous function on $0 \leq t \leq T$, $b(t) \leq M$, M is a positive constant. Further, suppose that $u(t)$ is nonnegative and $u(t)$ is locally integrable on $0 \leq t \leq T$ with*

$$u(t) \leq a + b(t) \int_t^T (T-s)^{\alpha-1}(s-t)^{\alpha-1}u(s)ds, \;\; 0 \leq t \leq T.$$

Then, one has

$$u(t) \leq aF_\alpha\big(\Gamma(\alpha)b(t)(T-t)^\alpha\big), \;\; 0 \leq t \leq T,$$

where $F_\alpha(z) = \sum_{k=0}^{\infty} c_k z^k$, $c_{k+1} = \frac{\Gamma(k(2\alpha-1)+\alpha)}{\Gamma(k(2\alpha-1)+2\alpha)} c_k$.

Lemma 4. *For $G \in C([0,T], L(\mathfrak{X}))$ and $0 \leq s \leq t \leq T$, we have*

$$(t-s)^{1-\alpha}\|\mathfrak{T}_\alpha(t,s;G)\| = (t-s)^{1-\alpha}\|\mathfrak{T}^*_\alpha(t,s;G)\| \leq M_{\mathfrak{T}},$$

where $L_G := \|G\|_{C([0,T],L(\mathfrak{X}))}$, $M_{\mathfrak{T}} := \frac{M_S}{\Gamma(\alpha)} F_\alpha(M_S L_G T^\alpha)$.

Proof. Existence of $T_\alpha(t,s;G)$ follows from Lemma 2. From Lemma 1, it follows that

$$\begin{aligned}
&(t-s)^{1-\alpha}\|\mathfrak{T}_\alpha(t,s;G)y\| \\
&\leq \|\mathfrak{U}_\alpha(t-s)y\| + \int_s^t (t-r)^{\alpha-1}(r-s)^{\alpha-1}\|\mathfrak{U}_\alpha(t-r)\|\|G(r)\|(r-s)^{1-\alpha}\|\mathfrak{T}_\alpha(r,s;G)y\|dr \\
&\leq \frac{M_S}{\Gamma(\alpha)}\|y\| + \frac{M_S L_G}{\Gamma(\alpha)} \int_s^t (t-r)^{\alpha-1}\|\mathfrak{T}_\alpha(r,s;G)y\|dr \\
&\leq \frac{M_S}{\Gamma(\alpha)}\|y\| + \frac{M_S L_G}{\Gamma(\alpha)} \int_s^t (t-r)^{\alpha-1}(r-s)^{\alpha-1}\|\mathfrak{T}_\alpha(r,s;G)y\|dr.
\end{aligned}$$

Applying Lemma 3, we have

$$(t-s)^{1-\alpha}\|\mathfrak{T}_\alpha(t,s;G)\| \leq \frac{M_S}{\Gamma(\alpha)} F_\alpha\big(M_S L_G (t-s)^\alpha\big).$$

□

3. Finite-Approximate Controllability of Linear Systems

In the present section, we investigate the finite-approximate controllability of the linear evolution system:

$$\begin{cases} {}^{RL}D^\alpha_{0+}y(t) = Ay(t) + Bu(t), \quad t \in [0,T], \\ I^{1-\alpha}_{0+}y(0) = y_0. \end{cases} \tag{3}$$

The finite-approximate controllability concept was introduced in [45]. This property not only means that the distance from $y(T)$ to the target y_f is small but also that the projections of $y(T)$ and y_f over M coincide.

It is known that the resolvent operator $\big(\varepsilon I + \Gamma_0^T\big)^{-1}$ is useful in studying the controllability properties of linear and semilinear systems, see [3,11]. In this regard, we state a new criteria of the finite-approximate controllability for RL fractional linear evolution system (3) in terms of the resolvent-like operator $\big(\varepsilon(I - \pi_M) + \Gamma_0^T\big)^{-1}$. We show that for the RL fractional linear evolution system (3), the approximate controllability on $[0,T]$ is equivalent to the finite-approximate controllability on $[0,T]$. We present necessary and sufficient conditions for the finite-approximate controllability concept of RL fractional linear evolution systems in Hilbert spaces in terms of resolvent-like operators. Moreover, we find an explicit form of the finite-approximating control in terms of the resolvent-like operator $\left(\varepsilon(I-\pi_M)+\Gamma_0^T\right)^{-1}$.

Firstly, we present two results on the resolvent operators.

Lemma 5. *Let $(\mathfrak{X},\|\cdot\|)$ be a Hilbert space. Assume that $\Gamma(\varepsilon),\Gamma:\mathfrak{X}\to\mathfrak{X}$, $\varepsilon>0$, are linear positive operators.*

(a) *If*

$$\lim_{\varepsilon\to 0^+}\|\Gamma(\varepsilon)h-\Gamma h\|=0,\ h\in\mathfrak{X},$$

then for any sequence $\{\varepsilon_n>0\}$ converging to 0 as $n\to\infty$, we have

$$\lim_{n\to\infty}\left\|\varepsilon_n(\varepsilon_n I+\Gamma(\varepsilon_n))^{-1}\pi_M\right\|=0.$$

(b) *For any $\varepsilon>0$, we have $\left\|\varepsilon(\varepsilon I+\Gamma(\varepsilon))^{-1}\pi_M\right\|<1$.*

Proof. (a) It is clear that $(\varepsilon I+\Gamma(\varepsilon))^{-1}\pi_M$ maps $\mathfrak{X}$ into finite-dimensional space $\operatorname{Im}\left((\varepsilon I+\Gamma(\varepsilon))^{-1}\pi_M\right)$ and

$$0\le\left\|\varepsilon(\varepsilon I+\Gamma(\varepsilon))^{-1}\pi_M\right\|\le 1.$$

Then, for any sequence $\{\varepsilon_n>0\}$ converging to 0 as $n\to\infty$, we have

$$0\le\rho:=\lim_{n\to\infty}\left\|\varepsilon_n(\varepsilon_n I+\Gamma(\varepsilon_n))^{-1}\pi_M\right\|\le 1.$$

Show that $\rho=0$. Let $\left\|\varepsilon_n(\varepsilon_n I+\Gamma(\varepsilon_n))^{-1}\pi_M\right\|:=\gamma_n$. Then, $0\le\lim_{n\to\infty}\gamma_n=\rho\le 1$ and, by the definition of γ_n, there exists a sequence $\{h_{n,m}\in\mathfrak{X}:\|h_{n,m}\|=1\}$ such that

$$\varepsilon_n(\varepsilon_n I+\Gamma(\varepsilon_n))^{-1}\pi_M h_{n,m}=:z_{n,m},$$
$$0\le\|z_{n,m}\|\le 1,\quad \|z_{n,m}\|\to\gamma_n\ \text{ as }\ m\to\infty.$$

It follows that

$$\varepsilon_n\pi_M h_{n,m}=\varepsilon_n z_{n,m}+\Gamma(\varepsilon_n)z_{n,m}. \tag{4}$$

Since $\{\pi_M h_{n,m}\}$ and $\{z_{n,m}\}$ are bounded sequences of finite dimensional vectors, without loss of generality, we assume that

$$z_{n,m}\to z_n\ \text{ and } \pi_M h_{n,m}\to h_n \text{ strongly as } m\to\infty.$$

Taking limit as $m\to\infty$ in (4), we obtain

$$\varepsilon_n h_n=\varepsilon_n z_n+\Gamma(\varepsilon_n)z_n,\quad \|h_n\|\le 1,\ 0\le\|z_n\|=\gamma_n\le 1. \tag{5}$$

Next, having in mind that $z_n\to z$ along some subsequence, we take the limit as $n\to\infty$ of Equation (5) to obtain

$$0=\lim_{n\to\infty}\Gamma(\varepsilon_n)z_n=\lim_{n\to\infty}(\Gamma(\varepsilon_n)-\Gamma)z+\lim_{n\to\infty}\Gamma(\varepsilon_n)(z_n-z)+\Gamma z=\Gamma z=0,$$
$$\Gamma z=0\Longrightarrow z=0.$$

By definition of the positive operator, $\Gamma z=0$ implies that $z=0$. Thus,

$$\rho=\lim_{n\to\infty}\left\|\varepsilon_n(\varepsilon_n I+\Gamma(\varepsilon_n))^{-1}\pi_M\right\|=\lim_{n\to\infty}\gamma_n=\lim_{n\to\infty}\|z_n\|=\|z\|=0.$$

(b) It is clear that $(\varepsilon I+\Gamma(\varepsilon))^{-1}\pi_M$ maps $\mathfrak{X}$ into a finite-dimensional subspace of $\mathfrak{X}$ and

$$\left\|\varepsilon(\varepsilon I+\Gamma(\varepsilon))^{-1}\pi_M\right\|\leq 1.$$

Let us show that $\left\|\varepsilon(\varepsilon I+\Gamma(\varepsilon))^{-1}\pi_M\right\|<1$. On the contrary, assume that there exists a sequence $\{h_n\in\mathfrak{X}:\|h_n\|=1\}$ such that

$$\varepsilon(\varepsilon I+\Gamma(\varepsilon))^{-1}\pi_M h_n=:z_n,\quad \|z_n\|\to 1 \text{ as } n\to\infty. \tag{6}$$

It follows that $\{z_n\}$ is a sequence of finite dimensional vectors and

$$\varepsilon\pi_M h_n=\varepsilon z_n+\Gamma(\varepsilon)z_n \text{ and } z_n\to z_0 \text{ strongly in } \mathfrak{X}. \tag{7}$$

$$\langle\pi_M h_n,z_n\rangle=\langle z_n,z_n\rangle+\frac{1}{\varepsilon}\langle\Gamma(\varepsilon)z_n,z_n\rangle,$$
$$\|z_n\|^2<\langle z_n,z_n\rangle+\frac{1}{\varepsilon}\langle\Gamma(\varepsilon)z_n,z_n\rangle=\langle\pi_M h_n,z_n\rangle\leq\|\pi_M h_n\|\|z_n\|\leq\|z_n\|.$$

Taking limit as $n\to\infty$, we obtain

$$1\leq 1+\frac{1}{\varepsilon}\langle\Gamma(\varepsilon)z_0,z_0\rangle\leq 1,$$
$$\langle\Gamma(\varepsilon)z_0,z_0\rangle=0\Longrightarrow z_0=0.$$

Now, from Equation (7), it follows that $\|z_n\|\to 0$ as $n\to\infty$. Contradiction. The lemma is proved. □

The next lemma gives a relationship between $(\varepsilon I+\Gamma)^{-1}$ and the resolvent-like operator $(\varepsilon(I-\pi_M)+\Gamma)^{-1}$.

Lemma 6. *Let $(\mathfrak{X},\|\cdot\|)$ be a Hilbert space. If $\Gamma:\mathfrak{X}\to\mathfrak{X}$ is a linear nonnegative operator, then the operator $\varepsilon(I-\pi_M)+\Gamma:\mathfrak{X}\to\mathfrak{X}$ is invertible and*

$$\left\|(\varepsilon(I-\pi_M)+\Gamma)^{-1}h\right\|\leq\frac{1}{\min(\varepsilon,\delta)}\|h\|,\ \ h\in\mathfrak{X}, \tag{8}$$

where $\delta=\min\{\langle\pi_M\Gamma\pi_M\varphi,\varphi\rangle:\|\pi_M\varphi\|=1\}$. Moreover, if $\Gamma:\mathfrak{X}\to\mathfrak{X}$ is a linear positive operator, then

$$(\varepsilon(I-\pi_M)+\Gamma)^{-1}=\left(I-\varepsilon(\varepsilon I+\Gamma)^{-1}\pi_M\right)^{-1}(\varepsilon I+\Gamma)^{-1}. \tag{9}$$

Proof. We write $\varepsilon(I-\pi_M)+\Gamma$ as follows.

$$\varepsilon(I-\pi_M)+\Gamma=\varepsilon(I-\pi_M)+(I-\pi_M)\Gamma+\pi_M\Gamma.$$

It is clear that

$$\begin{aligned}&\langle(\varepsilon(I-\pi_M)+\Gamma)\varphi,\varphi\rangle\\&=\langle(\varepsilon(I-\pi_M)+(I-\pi_M)\Gamma)\varphi,\varphi\rangle+\langle\pi_M\Gamma\varphi,\varphi\rangle\\&\geq\begin{cases}\langle\pi_M\Gamma\pi_M\varphi,\varphi\rangle, & \varphi\in M,\\ \langle\varepsilon(I-\pi_M)\varphi+(I-\pi_M)\Gamma(I-\pi_M)\varphi,\varphi\rangle, & \varphi\in\mathfrak{X}\ominus M\end{cases}\\&\geq\min(\varepsilon,\delta)\|\varphi\|^2.\end{aligned}$$

It follows that $\varepsilon(I-\pi_M)+\Gamma$ is invertible and Equation (8) is satisfied. If $\Gamma:\mathfrak{X}\to\mathfrak{X}$ is a linear positive operator, then, by Lemma 5, $\left(I-\varepsilon(\varepsilon I+\Gamma)^{-1}\pi_M\right)^{-1}$ exists. On the other hand, since $(\varepsilon I+\Gamma)$ is invertible and

$$\varepsilon(I-\pi_M)+\Gamma=(\varepsilon I+\Gamma)\left(I-\varepsilon(\varepsilon I+\Gamma)^{-1}\pi_M\right),$$

the operator $\varepsilon(I-\pi_M)+\Gamma$ is boundedly invertible and Equation (9) is satisfied. □

Next, we present new criteria for the finite-approximate controllability of fractional linear evolution equations.

A controllability operator is the bounded linear operator $L_0^T:L^2([0,T],U)\to\mathfrak{X}$ defined by

$$L_0^T u:=\int_0^T (T-s)^{\alpha-1}\mathfrak{U}_\alpha(T-s)Bu(s)ds.$$

The controllability Gramian is defined by

$$\Gamma_0^T:=L_0^T\left(L_0^T\right)^*=\int_0^T (T-s)^{2(\alpha-1)}\mathfrak{U}_\alpha(T-s)BB^*\mathfrak{U}_\alpha^*(T-s)ds:\mathfrak{X}\to\mathfrak{X}. \tag{10}$$

Theorem 1. *The following statements are equivalent:*

(i) *The system (3) is approximately controllable on* $[0,T]$;

(ii) Γ_0^T *is positive, that is* $\left\langle\Gamma_0^T x,x\right\rangle>0$ *for all* $0\neq x\in\mathfrak{X}$;

(iii) $\varepsilon\left(\varepsilon I+\Gamma_0^T\right)^{-1}\to 0$ *as* $\varepsilon\to 0^+$ *in the strong operator topology;*

(iv) $\varepsilon\left(\varepsilon(I-\pi_M)+\Gamma_0^T\right)^{-1}\to 0$ *as* $\varepsilon\to 0^+$ *in the strong operator topology;*

(v) *The system (3) is finite-approximately controllable on* $[0,T]$.

Proof. The equivalences (i)⇔(ii)⇔(iii) are well known, see [3]. For the equivalence (iii)⟺(v), for any $\varepsilon>0$, $h\in\mathfrak{X}$, consider the following functional $J_\varepsilon(\cdot,h):\mathfrak{X}\to R:$

$$\begin{aligned}J_\varepsilon(\varphi,h)=&\frac{1}{2}\int_0^T\left\|(T-s)^{\alpha-1}B^*\mathfrak{U}_\alpha^*(T-s)\varphi\right\|^2ds\\&+\frac{\varepsilon}{2}\langle(I-\pi_M)\varphi,\varphi\rangle-\left\langle\varphi,h-T^{\alpha-1}\mathfrak{U}_\alpha(T)y_0\right\rangle.\end{aligned}$$

Assume that (iii)(⟺(ii)) is satisfied. It is clear that $J_\varepsilon(\cdot,h)$ is Gateaux differentiable, $J_\varepsilon'(\varphi,h)=\Gamma_0^T\varphi+\varepsilon(I-\pi_M)\varphi-h+T^{\alpha-1}\mathfrak{U}_\alpha(T)y_0$ is strictly monotonic and, consequently, $J_\varepsilon(\cdot,h)$ is strictly convex, since Γ_0^T is positive. Thus, $J_\varepsilon(\cdot,h)$ has a unique minimum and can be found as follows:

$$\begin{aligned}&\Gamma_0^T\varphi+\varepsilon(I-\pi_M)\varphi-h+T^{\alpha-1}\mathfrak{U}_\alpha(T)y_0=0,\\&\varphi_{\min}=-\left(\varepsilon(I-\pi_M)+\Gamma_0^T\right)^{-1}\left(T^{\alpha-1}\mathfrak{U}_\alpha(T)y_0-h\right).\end{aligned}$$

It follows that for the control $u_\varepsilon(s) = (T-s)^{\alpha-1}B^*\mathfrak{U}_\alpha^*(T-s)\varphi_{\min}$

$$\begin{aligned}
x_\varepsilon(T) - h &= T^{\alpha-1}\mathfrak{U}_\alpha(T)y_0 + \int_0^T (T-s)^{\alpha-1}\mathfrak{U}_\alpha(T-s)Bu(s)ds - h \\
&= T^{\alpha-1}\mathfrak{U}_\alpha(T)y_0 - h - \Gamma_0^T\Big(\varepsilon(I-\pi_M) + \Gamma_0^T\Big)^{-1}\Big(T^{\alpha-1}\mathfrak{U}_\alpha(T)y_0 - h\Big) \\
&= T^{\alpha-1}\mathfrak{U}_\alpha(T)y_0 - h - \Big(\Gamma_0^T + \varepsilon(I-\pi_M) - \varepsilon(I-\pi_M)\Big) \\
&\times \Big(\varepsilon(I-\pi_M) + \Gamma_0^T\Big)^{-1}\Big(T^{\alpha-1}\mathfrak{U}_\alpha(T)y_0 - h\Big) \\
&= \varepsilon(I-\pi_M)\Big(\varepsilon(I-\pi_M) + \Gamma_0^T\Big)^{-1}\Big(T^{\alpha-1}\mathfrak{U}_\alpha(T)y_0 - h\Big).
\end{aligned} \tag{11}$$

Thus,

$$\begin{aligned}
&\lim_{\varepsilon\to0^+}\|x_\varepsilon(T) - h\| = \lim_{\varepsilon\to0^+}\varepsilon\left\|(I-\pi_M)\Big(\varepsilon(I-\pi_M) + \Gamma_0^T\Big)^{-1}\Big(T^{\alpha-1}\mathfrak{U}_\alpha(T)y_0 - h\Big)\right\| = 0, \\
&\pi_M(x_\varepsilon(T) - h) = 0,
\end{aligned}$$

that is, the system (3) is finite-approximately controllable on $[0,T]$. Thus, (iii)$\Longrightarrow$(v). The implication (v)$\Rightarrow$(iii) is obvious, since finite-approximate controllability implies the approximate controllability. For the implication (iii)$\Rightarrow$(iv), suppose that for any $h \in \mathfrak{X}$

$$\lim_{\varepsilon\to0^+}\left\|\Big(\varepsilon I + \Gamma_0^T\Big)^{-1}h\right\| = 0.$$

From Equation (9), it follows that for any $h \in \mathfrak{X}$

$$\begin{aligned}
\left\|\varepsilon\big(\varepsilon(I-\pi_M) + \Gamma_0^T\big)^{-1}h\right\| &\le \left\|\Big(I - \varepsilon\big(\varepsilon I + \Gamma_0^T\big)^{-1}\pi_M\Big)^{-1}\right\|\left\|\varepsilon\big(\varepsilon I + \Gamma_0^T\big)^{-1}h\right\| \\
&\le \frac{1}{1 - \left\|\varepsilon\big(\varepsilon I + \Gamma_0^T\big)^{-1}\pi_M\right\|}\left\|\varepsilon\big(\varepsilon I + \Gamma_0^T\big)^{-1}h\right\|.
\end{aligned} \tag{12}$$

On the other hand, from

$$\begin{aligned}
&\varepsilon_1(\varepsilon_1 I + \Gamma)^{-1}\pi_M - \varepsilon(\varepsilon I + \Gamma)^{-1}\pi_M \\
&= \varepsilon_1(\varepsilon_1 I + \Gamma)^{-1}\Big(I + \varepsilon^{-1}\Gamma - I - \varepsilon_1^{-1}\Gamma\Big)\varepsilon(\varepsilon I + \Gamma)^{-1}\pi_M \\
&= \varepsilon_1(\varepsilon_1 I + \Gamma)^{-1}\Big(\varepsilon^{-1}\Gamma - \varepsilon_1^{-1}\Gamma\Big)\varepsilon(\varepsilon I + \Gamma)^{-1}\pi_M \\
&= (\varepsilon_1 I + \Gamma)^{-1}(\varepsilon_1\Gamma - \varepsilon\Gamma)(\varepsilon I + \Gamma)^{-1}\pi_M \\
&= (\varepsilon_1 I + \Gamma)^{-1}(\varepsilon_1 - \varepsilon)\Gamma(\varepsilon I + \Gamma)^{-1}\pi_M,
\end{aligned}$$

it follows that $\varepsilon(\varepsilon I + \Gamma)^{-1}\pi_M$ is continuous in ε. Indeed,

$$\left\|\varepsilon_1(\varepsilon_1 I + \Gamma)^{-1}\pi_M - \varepsilon(\varepsilon I + \Gamma)^{-1}\pi_M\right\| \le \frac{|\varepsilon_1 - \varepsilon|}{\varepsilon_1} \to 0 \quad \text{as} \quad \varepsilon_1 \to \varepsilon.$$

By Equation (12), continuity of $\varepsilon(\varepsilon I + \Gamma)^{-1}\pi_M$ and Lemma 5, we have

$$\begin{aligned}
\gamma &= \max_{0\le\varepsilon\le1}\left\|\varepsilon\Big(\varepsilon I + \Gamma_0^T\Big)^{-1}\pi_M\right\| < 1, \\
\left\|\varepsilon\Big(\varepsilon(I-\pi_M) + \Gamma_0^T\Big)^{-1}h\right\| &\le \frac{1}{1-\gamma}\left\|\varepsilon\Big(\varepsilon I + \Gamma_0^T\Big)^{-1}h\right\|.
\end{aligned}$$

Thus, $\varepsilon\left(\varepsilon(I-\pi_M)+\Gamma_0^T\right)^{-1}$ converges to zero as $\varepsilon \to 0^+$ in the strong operator topology. The implication (iv)⇒(v) follows from Equation (11). □

Remark 1. *The control*

$$u_\varepsilon(s) = (T-s)^{\alpha-1}B^*\mathfrak{U}_\alpha^*(T-s)\left(\varepsilon(I-\pi_M)+\Gamma_0^T\right)^{-1}\left(h - T^{\alpha-1}\mathfrak{U}_\alpha(T)y_0\right) \tag{13}$$

in addition to the requirement of approximate controllability provides finite-dimensional exact controllability of Equation (3).

Remark 2. *Theorem 1(iv) is new, even for the classical case $\alpha = 1$. The equivalences of Theorem 1 (i)⇔(ii)⇔(iii) was proved in [3,10].*

Remark 3. *Theorem is true for*

$$\Gamma_0^T(G) := \int_0^T \mathfrak{T}_\alpha(T,s;G)BB^*\mathfrak{T}_\alpha^*(T,s;G)ds : \mathfrak{X} \to \mathfrak{X},$$

which coincides with Equation (10) if $G = 0$. In this case, the control which provides the finite-approximate controllability has the following form

$$u_\varepsilon(t,z) = B^*\mathfrak{T}_\alpha^*(T,t;F(z))\left(\varepsilon(I-\pi_M)+\Gamma_0^T(F(z))\right)^{-1}(h - \mathfrak{T}_\alpha(T,0;F(z))y_0). \tag{14}$$

Remark 4. *An Analogue of Theorem 1 is true for different kinds of equations, such as fractional linear differential equations with Caputo derivative, fractional linear differential equations with Hadamard derivative, Fredholm-type linear integral equations, and so on.*

4. Finite-Approximate Controllability of Semilinear System

We impose the following assumptions:

- **(S)** $\mathfrak{X}$ and U are separable Hilbert spaces, $\mathfrak{U}(t), t > 0$ is a compact semigroup on $\mathfrak{X}$ and $B \in L(U,\mathfrak{X})$.
- **(F)** $f : [0,T] \times \mathfrak{X} \to \mathfrak{X}$ is continuous and has continuous uniformly bounded Frechet derivative $f_z'(\cdot,\cdot)$, that is, for some $L > 0$,

$$\|f_z'(t,z)\|_{L(\mathfrak{X})} \le L, \quad \forall (t,z) \in [0,T] \times \mathfrak{X}.$$

It is clear that under the conditions (S), (F) for any $y_0 \in \mathfrak{X}$ and $u(\cdot) \in L_2(0,T;U)$, the system (2) admits a solution $y(\cdot) = y(\cdot, y_0, u)$.

Define

$$F(t,z) = \int_0^1 f'(t,rz)dr, \quad z \in \mathfrak{X}. \tag{15}$$

Thanks to the assumption (F), there exists a constant $L > 0$ such that the operator F defined by Equation (15) has the following properties:

$$\begin{gathered} F : [0,T] \times \mathfrak{X} \to L(\mathfrak{X}), \\ f(t,z) = F(t,z)z + f(t,0), \\ \|F(t,z(t))\|_{L(\mathfrak{X})} \le L, \quad z(\cdot) \in C([0,T],\mathfrak{X}),\ t \in [0,T], \\ F(\cdot,\cdot) \in C([0,T] \times \mathfrak{X}, L(\mathfrak{X})). \end{gathered}$$

For simplicity, we assume that $f(t,0) \equiv 0$. Then, the system (2) can be rewritten as follows:

$$y(t) = t^{\alpha-1}\mathfrak{U}_\alpha(t)y_0 + \int_0^t (t-s)^{\alpha-1}\mathfrak{U}_\alpha(t-s)[Bu(s) + F(s,y(s))y(s)]ds.$$

For any fixed $z(\cdot) \in C([0,T],\mathfrak{X})$, let $y(\cdot) = y(\cdot, y_0, z, u)$ be the solution of

$$y(t) = t^{\alpha-1}\mathfrak{U}_\alpha(t)y_0 + \int_0^t (t-s)^{\alpha-1}\mathfrak{U}_\alpha(t-s)[Bu(s) + F(s,z(s))y(s)]ds. \quad (16)$$

It is not hard to see that both of the above equations admit unique solutions.

(AC) Linearized system

$$y(t) = t^{\alpha-1}\mathfrak{U}_\alpha(t)y_0 + \int_0^t (t-s)^{\alpha-1}\mathfrak{U}_\alpha(t-s)[Bu(s) + F(s,z(s))y(s)]ds \quad (17)$$

is approximately controllable for any $z(\cdot) \in C([0,T],\mathfrak{X})$.

Remark 5. *The condition (AC) for the heat equation is related to the unique continuation principle for the linear backward heat equation.*

The following lemma gives the variation of constants formula for Equation (17). Note that this lemma is only true for the Riemann–Liouville case, for the Caputo fractional case, it fails.

Lemma 7. *Let $\mathfrak{T}_\alpha(\cdot,\cdot;F)$ be the operator defined by Equation (16). Then, the solution of Equation (16) can be represented by*

$$y(t) = \mathfrak{T}_\alpha(t,0;F(z))y_0 + \int_0^t \mathfrak{T}_\alpha(t,s;F(z))Bu(s)ds, \quad (18)$$

where

$$\mathfrak{T}_\alpha(t,s;F(z))y = (t-s)^{\alpha-1}\mathfrak{U}_\alpha(t-s)y + \int_s^t (t-s)^{\alpha-1}\mathfrak{U}_\alpha(t-r)F(r,z(r))\mathfrak{T}_\alpha(r,s;F(z))ydr,$$

for $0 \le s \le t \le T$.

Proof. The proof of this lemma in the classical case can be found, for example, in [9]. To show that Equation (18) is valid, it is enough to repeat the classical proof. □

In this section, we first show that for every $\varepsilon > 0$ and every final state $y_f \in \mathfrak{X}$, the integral equation

$$z(t) = \mathfrak{T}_\alpha(t,0;F(z))y_0 + \int_0^t \mathfrak{T}_\alpha(t,s;F(z))Bu_\varepsilon(s,z)ds,$$

with the control (14)

$$u_\varepsilon(t,z) = B^*\mathfrak{T}_\alpha^*(T,t;F(z))\Big(\varepsilon(I-\pi_M) + \Gamma_0^T(F(z))\Big)^{-1}\Big(y_f - \mathfrak{T}_\alpha(T,0;F(z))y_0\Big)$$

has at least one solution, say y_ε^*. Then, we can finite-approximate any point $y_f \in \mathfrak{X}$ by using these solutions y_ε^*, $\varepsilon > 0$.

Next, we prove the existence of the fixed point of the operator $\Theta_\varepsilon : C_{1-\alpha}((0,T],\mathfrak{X}) \to C_{1-\alpha}((0,T],\mathfrak{X})$ defined Equation by (25).

Define

$$\Gamma_0^T(G)y = \int_0^T \mathfrak{T}_\alpha(T,s;G)BB^*\mathfrak{T}_\alpha^*(T,s;G)yds,$$

where

$$\mathfrak{T}_\alpha^*(t,s;G)y = (t-s)^{\alpha-1}\mathfrak{U}_\alpha^*(t-s)y + \int_s^t (t-r)^{\alpha-1}\mathfrak{T}_\alpha^*(r,s;G)G^*(r)\mathfrak{U}_\alpha^*(t-r)ydr$$
$$= (t-s)^{\alpha-1}\mathfrak{U}_\alpha^*(t-s)y + \int_s^t (r-s)^{\alpha-1}\mathfrak{U}_\alpha^*(r-s)G^*(r)\mathfrak{T}_\alpha^*(t,r;G)ydr.$$

Lemma 8. *Let $G_n \in C([0,T], L(\mathfrak{X}))$ and $\eta_n, \eta \in \mathfrak{X}$ such that*

$$\begin{cases} G_n & \text{is uniformly bounded in} \quad C([0,T],L(\mathfrak{X})), \\ \eta_n \rightharpoonup \eta & \text{weakly in} \quad \mathfrak{X}, \text{ as } n\to\infty, \end{cases} \tag{19}$$

then there exists $G \in C([0,T], L(\mathfrak{X}))$ such that

$$\mathfrak{T}_\alpha^*(T,\cdot;G_n)\eta_n \to \mathfrak{T}_\alpha^*(T,\cdot;G)\eta \quad \text{in} \quad C_{1-\alpha}((0,T],\mathfrak{X}),$$
$$\mathfrak{T}_\alpha(T,\cdot;G_n)\eta_n \to \mathfrak{T}_\alpha(T,\cdot;G)\eta \quad \text{in} \quad C_{1-\alpha}((0,T],\mathfrak{X}),$$

as $n \to \infty$.

Proof. Let $\{e_m : m \geq 1\}$ be a basis of $\mathfrak{X}$. By our assumption, there exists $C > 0$ such that for all $n \geq 1$

$$\int_0^T \|G_n(t)\|^2_{L(\mathfrak{X})}dt \leq \max_{0\leq t\leq T}\|G_n(t)\|^2_{L(\mathfrak{X})} \leq L_G.$$

It follows that

$$\int_0^T \|G_n(t)e_m\|^2_{\mathfrak{X}}dt \leq L_G.$$

By the "diagonal argument", we know that there exists a subsequence, denoted again by $\{G_n(\cdot)e_m : n \geq 1\}$, which is weakly convergent in $L^2(0,T;\mathfrak{X})$ for all $m \geq 1$. Since $\{e_m : m \geq 1\}$ is dense in $\mathfrak{X}$, we know that the sequence $\{G_n(\cdot)x\}$ is weakly convergent in $L^2(0,T;\mathfrak{X})$ for all $x \in \mathfrak{X}$ to some $G(\cdot)x \in L^2(0,T;\mathfrak{X})$. It is clear that $G(\cdot) \in L^2(0,T;L(\mathfrak{X}))$. Denote

$$\xi_n(t) = (T-t)^{1-\alpha}\mathfrak{T}_\alpha^*(T,t;G_n)\eta_n, \;\; \xi(t) = (T-t)^{1-\alpha}\mathfrak{T}_\alpha^*(T,t;G)\eta, \;\; t\in(0,T].$$

It is easily seen that

$$\xi_n(t) = \mathfrak{U}_\alpha^*(T-t)\eta_n + \int_t^T (r-t)^{\alpha-1}(T-r)^{\alpha-1}\mathfrak{U}_\alpha^*(r-t)G_n^*(r)\xi_n(r)dr, \;\; t\in(0,T], \tag{20}$$

and

$$\|\xi_n(t)\| \leq \frac{M_S}{\Gamma(\alpha)} + \frac{M_S L_G}{\Gamma(\alpha)}\int_t^T (r-t)^{\alpha-1}(T-r)^{\alpha-1}\|\xi_n(r)\|dr.$$

Then, by the Gronwall inequality Lemma 3, one obtains the uniform boundedness of $\{\mathfrak{T}_\alpha^*(T,t;G_n)\eta_n\}$ in $C_{1-\alpha}((0,T],\mathfrak{X})$ and the uniform boundedness of $\{G_n(\cdot)\mathfrak{T}_\alpha^*(T,t;G_n)\eta_n\}$ in $C_{1-\alpha}((0,T],\mathfrak{X})$. Thus, having in mind compactness of $\mathfrak{U}_\alpha(t)$, $t>0$, one can show that $\{\mathfrak{T}_\alpha^*(T,t;G_n)\eta_n\}$ is relatively compact in $C_{1-\alpha}((0,T],\mathfrak{X})$. Let $\xi(\cdot)$ be any limit point of $\{\xi_n(\cdot)\}$ in $C(0,T;\mathfrak{X})$. On the other hand, for any $r\in[0,T]$

$$\mathfrak{U}_\alpha^*(r-t)G_n^*(r)\xi_n(r) \to \mathfrak{U}_\alpha^*(r-t)G^*(r)\xi(r) \text{ in } \mathfrak{X}.$$

Passing to the limit in Equation (20) along some proper subsequence, we see that $\xi(\cdot)$ satisfies

$$\xi(t) = \mathfrak{U}_\alpha^*(T-t)\eta + \int_t^T (r-t)^{\alpha-1}(T-r)^{\alpha-1}\mathfrak{U}_\alpha^*(r-t)G^*(r)\xi(r)dr, \;\; t\in[0,T], \tag{21}$$

By uniqueness of the solutions of Equation (21), we obtain that the whole sequence $\{\mathfrak{T}_\alpha^*(T,t;G_n)\eta_n\}$ converges to $\mathfrak{T}_\alpha^*(T,t;G)\eta$ in $C_{1-\alpha}((0,T],\mathfrak{X})$. Similarly, we may prove that

$$\mathfrak{T}_\alpha(T,\cdot;G_n)\eta_n \to \mathfrak{T}_\alpha(T,\cdot;G)\eta \text{ in } C_{1-\alpha}((0,T],\mathfrak{X}) \text{ as } n\to\infty.$$

□

Lemma 9. *Let $G_n \in C([0,T],L(\mathfrak{X}))$ and $\eta_n,\eta\in\mathfrak{X}$ such that*

$$\begin{cases} G_n & \text{is uniformly bounded in} \quad C([0,T],L(\mathfrak{X})),\\ \eta_n \rightharpoonup \eta & \text{weakly in} \quad \mathfrak{X}, \text{ as } n\to\infty, \end{cases}$$

then there exists $G\in C([0,T],L(\mathfrak{X}))$ such that

$$\Gamma_0^T(G_n)\eta_n \to \Gamma_0^T(G)\eta \quad \text{in } \mathfrak{X}, \text{ as } n\to\infty,$$

where

$$\Gamma_0^T(G_n)\eta_n = \int_0^T \mathfrak{T}_\alpha(T,s;G_n)BB^*\mathfrak{T}_\alpha^*(T,s;G_n)\eta_n ds,$$
$$\Gamma_0^T(G)\eta = \int_0^T \mathfrak{T}_\alpha(T,s;G)BB^*\mathfrak{T}_\alpha^*(T,s;G)\eta ds.$$

Proof. The desired convergence follows from Lemma 8, boundedness of $\mathfrak{T}_\alpha(T,s;G_n)$ and from the following inequality

$$\begin{aligned}
&\left\|\Gamma_0^T(G_n)\eta_n - \Gamma_0^T(G)\eta\right\|\\
&= \left\|\int_0^T \mathfrak{T}_\alpha(T,s;G_n)BB^*\mathfrak{T}_\alpha^*(T,s;G_n)\eta_n ds - \int_0^T \mathfrak{T}_\alpha(T,s;G)BB^*\mathfrak{T}_\alpha^*(T,s;G)\eta ds\right\|\\
&\le \left\|\int_0^T \mathfrak{T}_\alpha(T,s;G_n)BB^*\mathfrak{T}_\alpha^*(T,s;G_n)\eta_n ds - \int_0^T \mathfrak{T}_\alpha(T,s;G_n)BB^*\mathfrak{T}_\alpha^*(T,s;G)\eta ds\right\|\\
&+ \left\|\int_0^T \mathfrak{T}_\alpha(T,s;G_n)BB^*\mathfrak{T}_\alpha^*(T,s;G)\eta ds - \int_0^T \mathfrak{T}_\alpha(T,s;G)BB^*\mathfrak{T}_\alpha^*(T,s;G)\eta ds\right\|\\
&\le \int_0^T \|\mathfrak{T}_\alpha(T,s;G_n)BB^*\|\|\mathfrak{T}_\alpha^*(T,s;G_n)\eta_n - \mathfrak{T}_\alpha^*(T,s;G)\eta\| ds\\
&+ \int_0^T \|[\mathfrak{T}_\alpha(T,s;G_n) - \mathfrak{T}_\alpha(T,s;G)]BB^*\mathfrak{T}_\alpha^*(T,s;G)\eta\| ds.
\end{aligned}$$

□

Lemma 10. *Let $G_n \in C([0,T],L(\mathfrak{X}))$ such that*

$$G_n \quad \text{is uniformly bounded in} \quad C([0,T],L(\mathfrak{X})),$$

then there exists $G\in C([0,T],L(\mathfrak{X}))$ such that

$$\lim_{n\to\infty}\left\|\varepsilon\left(\varepsilon I + \Gamma_0^T(G_n)\right)^{-1}\pi_M - \varepsilon\left(\varepsilon I + \Gamma_0^T(G)\right)^{-1}\pi_M\right\| = 0, \tag{22}$$

$$\lim_{n\to\infty}\left\|\left(\varepsilon(I-\pi_M) + \Gamma_0^T(G_n)\right)^{-1}h \to \left(\varepsilon(I-\pi_M) + \Gamma_0^T(G)\right)^{-1}h\right\| = 0, \tag{23}$$

for any $h\in\mathfrak{X}$.

Proof. Set $\gamma_n := \left\| \varepsilon(\varepsilon I + \Gamma_0^T(G_n))^{-1}\pi_M - \varepsilon(\varepsilon I + \Gamma_0^T(G))^{-1}\pi_M \right\|_{L(\mathfrak{X})}$. There exists $\{h_m \in \mathfrak{X} : \|h_m\| = 1\}$ such that

$$\gamma_{n,m} := \left\| \left(\varepsilon\left(\varepsilon I + \Gamma_0^T(G_n)\right)^{-1}\pi_M - \varepsilon\left(\varepsilon I + \Gamma_0^T(G)\right)^{-1}\pi_M \right) h_m \right\|, \ \ \gamma_{n,m} \to \gamma_n$$

as $m \to \infty$. Since $\{\pi_M h_m\}$ is a sequence of finite-dimensional vectors and $\|\pi_M h_m\| \le 1$, then there is a subsequence denoted by $\{\pi_M h_m\}$again, such that $\pi_M h_m \to h_0 \in M$ as $m \to \infty$. It follows that

$$\begin{aligned} z_{n,m} &:= \left(\varepsilon\left(\varepsilon I + \Gamma_0^T(G_n)\right)^{-1}\pi_M - \varepsilon\left(\varepsilon I + \Gamma_0^T(G)\right)^{-1}\pi_M \right) h_m \\ &= \varepsilon\left(\varepsilon I + \Gamma_0^T(G_n)\right)^{-1}\left(\Gamma_0^T(G) - \Gamma_0^T(G_n)\right)\left(\varepsilon I + \Gamma_0^T(G)\right)^{-1}\pi_M h_m \\ &\to \varepsilon\left(\varepsilon I + \Gamma_0^T(G_n)\right)^{-1}\left(\Gamma_0^T(G) - \Gamma_0^T(G_n)\right)\left(\varepsilon I + \Gamma_0^T(G)\right)h_0 := z_n \end{aligned}$$

as $m \to \infty$, and $\lim_{m\to\infty} \gamma_{n,m} = \|z_n\| = \gamma_n, \quad \|z_{n,m}\| \le 2$. By Lemma 9, we have

$$\begin{aligned} \gamma_n = \|z_n\| &= \left\| \varepsilon\left(\varepsilon I + \Gamma_0^T(G_n)\right)^{-1}\left(\Gamma_0^T(G) - \Gamma_0^T(G_n)\right)\left(\varepsilon I + \Gamma_0^T(G)\right)^{-1}h_0 \right\| \\ &\le \left\| \varepsilon\left(\varepsilon I + \Gamma_0^T(G_n)\right)^{-1} \right\| \left\| \left(\Gamma_0^T(G) - \Gamma_0^T(G_n)\right)\left(\varepsilon I + \Gamma_0^T(G)\right)^{-1}h_0 \right\| \\ &\le \left\| \left(\Gamma_0^T(G) - \Gamma_0^T(G_n)\right)\left(\varepsilon I + \Gamma_0^T(G)\right)^{-1}h_0 \right\| \to 0 \ \text{ as } n \to \infty. \end{aligned}$$

So, $\lim_{n\to\infty}\lim_{m\to\infty} \gamma_{n,m} = \lim_{n\to\infty} \gamma_n = 0$. For every $h \in \mathfrak{X}$, we have

$$\begin{aligned} &\left(\varepsilon(I - \pi_M) + \Gamma_0^T(G)\right)^{-1}h - \left(\varepsilon(I - \pi_M) + \Gamma_0^T(G_n)\right)^{-1}h \\ &= \left(\varepsilon(I - \pi_M) + \Gamma_0^T(G_n)\right)^{-1} \\ &\times \left[\left(\varepsilon(I - \pi_M) + \Gamma_0^T(G) - \Gamma_0^T(G) + \Gamma_0^T(G_n)\right)\left(\varepsilon(I - \pi_M) + \Gamma_0^T(G)\right)^{-1} - I \right] h \\ &= \left(\varepsilon(I - \pi_M) + \Gamma_0^T(G_n)\right)^{-1}\left[I + \left(\Gamma_0^T(G_n) - \Gamma_0^T(G)\right)\left(\varepsilon(I - \pi_M) + \Gamma_0^T(G)\right)^{-1} - I \right] h \\ &= \left(\varepsilon(I - \pi_M) + \Gamma_0^T(G_n)\right)^{-1}\left(\Gamma_0^T(G_n) - \Gamma_0^T(G)\right)\left(\varepsilon(I - \pi_M) + \Gamma_0^T(G)\right)^{-1}h. \end{aligned}$$

By Equation (22) and Lemma 5, we have

$$\begin{aligned} &\lim_{n\to\infty} \left\| \left(\varepsilon(I - \pi_M) + \Gamma_0^T(G_n)\right)^{-1} \right\| \\ &\le \lim_{n\to\infty} \left\| \left(I - \varepsilon\left(\varepsilon I + \Gamma_0^T(G_n)\right)^{-1}\pi_M \right)^{-1} \right\| \left\| \left(\varepsilon I + \Gamma_0^T(G_n)\right)^{-1} \right\| \\ &\le \frac{1}{\varepsilon} \frac{1}{1 - \lim_{n\to\infty} \left\| \varepsilon(\varepsilon I + \Gamma_0^T(G_n))^{-1}\pi_M \right\|} \\ &= \frac{1}{\varepsilon} \frac{1}{1 - \left\| \varepsilon(\varepsilon I + \Gamma_0^T(G))^{-1}\pi_M \right\|} := \delta(\varepsilon) \end{aligned} \tag{24}$$

Now, the desired convergence (24) follows from Lemma 9:

$$
\begin{aligned}
&\lim_{n\to\infty}\left\|\left(\varepsilon(I-\pi_M)+\Gamma_0^T(G)\right)^{-1}h-\left(\varepsilon(I-\pi_M)+\Gamma_0^T(G_n)\right)^{-1}h\right\| \\
&\le \lim_{n\to\infty}\left\|\left(\varepsilon(I-\pi_M)+\Gamma_0^T(G_n)\right)^{-1}\right\|\lim_{n\to\infty}\left\|\left(\Gamma_0^T(G_n)-\Gamma_0^T(G)\right)\left(\varepsilon(I-\pi_M)+\Gamma_0^T(G)\right)^{-1}h\right\| \\
&\le \delta(\varepsilon)\lim_{n\to\infty}\left\|\left(\Gamma_0^T(G_n)-\Gamma_0^T(G)\right)\left(\varepsilon(I-\pi_M)+\Gamma_0^T(G)\right)^{-1}h\right\| = 0.
\end{aligned}
$$

□

Lemma 11. *Let $z(\cdot)\in C_{1-\alpha}((0,T],\mathfrak{X})$ and $\mathfrak{T}_\alpha(t,s;F(z))$ be the operator generated by $A+F(z)$, where F is defined by Equation (15) and let*

$$
\begin{aligned}
u_\varepsilon(t,z) = {} & B^*\mathfrak{T}_\alpha^*(T,t;F(z))\left(\varepsilon(I-\pi_M)+\Gamma_0^T(F(z))\right)^{-1} \\
& \times\left(y_f-\mathfrak{T}_\alpha(T,0;F(z))y_0\right).
\end{aligned}
$$

The control $z\to u_\varepsilon(t,z):C_{1-\alpha}((0,T],\mathfrak{X})\to C_{1-\alpha}((0,T],\mathfrak{X})$ is continuous and

$$(T-t)^{1-\alpha}\|u_\varepsilon(t,z)\|\le R_\varepsilon := \frac{1}{\varepsilon}M_BM_{\mathfrak{T}}\left(\left\|y_f\right\|+T^{\alpha-1}M_{\mathfrak{T}}\|y_0\|\right).$$

Proof. To prove continuity of $u_\varepsilon(\cdot,z)$, let $\{z_n\}\subset C_{1-\alpha}((0,T],\mathfrak{X})$ with $z_n\to z$ in $C_{1-\alpha}((0,T],\mathfrak{X})$. By assumption (F), the function $F(s,z(s))$ is continuous. It follows that $\mathfrak{T}(T,s;F(z))$ and

$$h(z):=y_f-\mathfrak{T}_\alpha(T,0;F(z))y_0$$

are continuous in z. Then, from the following equality

$$
\begin{aligned}
& u_\varepsilon(t,z_n)-u_\varepsilon(t,z) \\
& = B^*\mathfrak{T}_\alpha^*(T,t;F(z_n))\left(\varepsilon(I-\pi_M)+\Gamma_0^T(F(z_n))\right)^{-1}(h(z_n)-h(z)) \\
& + B^*\mathfrak{T}_\alpha^*(T,t;F(z_n)) \\
& \times\left[\left(\varepsilon(I-\pi_M)+\Gamma_0^T(F(z_n))\right)^{-1}-\left(\varepsilon(I-\pi_M)+\Gamma_0^T(F(z))\right)^{-1}\right]h(z) \\
& + [B^*\mathfrak{T}_\alpha^*(T,t;F(z_n))-B^*\mathfrak{T}_\alpha^*(T,t;F(z))]\left(\varepsilon(I-\pi_M)+\Gamma_0^T(F(z))\right)^{-1}h(z)
\end{aligned}
$$

and Lemmas 8 and 10, it follows that $u_\varepsilon(t,z_n)\to u_\varepsilon(t,z)$ as $n\to\infty$ in $C(0,T;\mathfrak{X})$. Moreover, by Lemmas 2 and 4, we have

$$
\begin{aligned}
(T-t)^{1-\alpha}\|u_\varepsilon(t,z)\| &\le M_BM_{\mathfrak{T}}\frac{1}{\varepsilon}\left\|y_f-\mathfrak{T}_\alpha(T,0;F(z))y_0\right\| \\
&\le \frac{1}{\varepsilon}M_BM_{\mathfrak{T}}\left(\left\|y_f\right\|+T^{\alpha-1}M_{\mathfrak{T}}\|y_0\|\right) := R_\varepsilon.
\end{aligned}
$$

□

Now, we define the operator $\Theta_\varepsilon:C_{1-\alpha}(0,T;\mathfrak{X})\to C_{1-\alpha}(0,T;\mathfrak{X})$ as follows

$$
\begin{aligned}
(\Theta_\varepsilon z)(t) &= \mathfrak{T}_\alpha(t,0;F(z))y_0+\int_0^t\mathfrak{T}_\alpha(t,s;F(z))Bu_\varepsilon(s,z)ds \\
&= t^{\alpha-1}\mathfrak{U}_\alpha(t-s)y_0+\int_0^t(t-s)^{\alpha-1}\mathfrak{U}_\alpha(t-s)[F(s,z(s))(\Theta_\varepsilon z)(s)+Bu_\varepsilon(s,z)]ds
\end{aligned}
\tag{25}
$$

Theorem 2. *For any $\varepsilon>0$, the operator $\Theta_\varepsilon(z)$ has a fixed point in $C_{1-\alpha}((0,T],\mathfrak{X})$.*

Proof. Claim 1. The operator $\Theta_\varepsilon(z)$ sends $C_{1-\alpha}((0,T],\mathfrak{X})$ into a bounded set.

We need to show that, for any $\varepsilon > 0$, there exists $k(\varepsilon) > 0$ such that $\|(\Theta_\varepsilon z)(t)\| \leq k(\varepsilon)$ for all $z(\cdot) \in C_{1-\alpha}((0,T],\mathfrak{X})$. Indeed, by Lemmas 11 and 4, we have

$$\begin{aligned}
&t^{1-\alpha}\|(\Theta_\varepsilon z)(t)\| \\
&\leq t^{1-\alpha}\|\mathfrak{T}_\alpha(t,0;F(z))\|\|y_0\| \\
&+ t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}(t-s)^{1-\alpha}\|\mathfrak{T}_\alpha(t,s;F(z))\|\|B\|\|u_\varepsilon(s,z)\|ds \\
&\leq M_{\mathfrak{T}}\|y_0\| + M_{\mathfrak{T}}M_B R_\varepsilon t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}ds \\
&\leq M_{\mathfrak{T}}\|y_0\| + M_{\mathfrak{T}}M_B R_\varepsilon \frac{T}{\alpha}.
\end{aligned}$$

Claim 2. The operator $\Theta_\varepsilon : C_{1-\alpha}((0,T],\mathfrak{X}) \to C_{1-\alpha}((0,T],\mathfrak{X})$ is continuous.

Assume that the sequence $\{z_n\} \subset C_{1-\alpha}((0,T],\mathfrak{X})$ such that $z_n \to z$ in $C_{1-\alpha}((0,T],\mathfrak{X})$. Then, using the triangle inequality, we have

$$\begin{aligned}
t^{1-\alpha}\|(\Theta_\varepsilon z_n)(t) - (\Theta_\varepsilon z)(t)\| \leq &\left\|\left(t^{1-\alpha}\mathfrak{T}_\alpha(t,0;F(z_n))y_0 - \mathfrak{T}_\alpha(t,0;F(z))\right)y_0\right\| \\
&+ t^{1-\alpha}\int_0^t \|\mathfrak{T}_\alpha(t,s;F(z_n))\|\|B\|\|u_\varepsilon(s,z_n) - u_\varepsilon(s,z)\|ds \\
&+ t^{1-\alpha}\int_0^t \|\mathfrak{T}_\alpha(t,0;F(z_n)) - \mathfrak{T}_\alpha(t,0;F(z))\|\|B\|\|u_\varepsilon(s,z)\|ds
\end{aligned}$$

Now, from the continuity of $u_\varepsilon(s,\cdot), \mathfrak{T}_\alpha(t,s;F(\cdot))$, we obtain the desired continuity of Θ_ε.

Claim 3. The family of functions $\{\Theta_\varepsilon z : z \in C_{1-\alpha}((0,T],\mathfrak{X})\}$ is equi-continuous.

Claim 4. The set $V(t) = \{(\Theta_\varepsilon z)(t) : z(\cdot) \in C_{1-\alpha}((0,T],\mathfrak{X})\}$ is relatively compact in $\mathfrak{X}$.

Proof of Claims 3 and 4 is similar to that reported in [23,38].

Thus, thanks to the Arzela–Ascoli theorem, the operator Θ_ε is a compact operator for any $\varepsilon > 0$. Consequently, the operator $\Theta_\varepsilon : C_{1-\alpha}((0,T],\mathfrak{X}) \to C_{1-\alpha}((0,T],\mathfrak{X})$ is continuous and compact with uniformly bounded image. By the Schauder fixed point theorem, the operator Θ_ε has at least one fixed point in $C_{1-\alpha}((0,T],\mathfrak{X})$. □

Assume that $y_\varepsilon^*(\cdot)$ is a fixed point of Θ_ε. Next, we will show that the fixed point $y_\varepsilon^*(\cdot)$ and the corresponding control

$$\begin{aligned}
u^*(t,y_\varepsilon^*) &= B^*\mathfrak{T}_\alpha^*(T,t;F(y_\varepsilon^*))\left(\varepsilon(I-\pi_M) + \Gamma_0^T(F(y_\varepsilon^*))\right)^{-1}(h(y_\varepsilon^*)), \\
h(y_\varepsilon^*) &= y_f - \mathfrak{T}_\alpha(T,0;F(y_\varepsilon^*))y_0, \\
\varphi_\varepsilon &= \left(\varepsilon(I-\pi_M) + \Gamma_0^T(F(y_\varepsilon^*))\right)^{-1}(h(y_\varepsilon^*)),
\end{aligned}$$

satisfies

$$\left\|y_\varepsilon^*(T;y_0,u_\varepsilon^*) - y_f\right\| < \varepsilon, \quad \pi_M y_\varepsilon^*(T;y_0,u_\varepsilon^*) = \pi_M y_f \text{ for any } y_f \in \mathfrak{X}.$$

Now, we are ready to state and prove the main result on finite-approximate controllability of semilinear evolution systems in this paper.

Theorem 3. *Let (S), (F), and (AC) hold. Then, the RL fractional evolution system (1) is finite-approximately controllable on* $[0,T]$.

Proof. Let $y_\varepsilon^* \in C_{1-\alpha}((0,T],\mathfrak{X})$ be a fixed point of Θ_ε. Then, by Theorem 1, we have

$$y_\varepsilon^*(T) - y_f = -\varepsilon(I-\pi_M)\varphi_\varepsilon = -\varepsilon(I-\pi_M)\left(\varepsilon(I-\pi_M) + \Gamma_0^T(F(y_\varepsilon^*))\right)^{-1}(h(y_\varepsilon^*)).$$

By the assumption (F), $\|F(s, y_\varepsilon^*(s))\|_{L(\mathfrak{X})} \leq L_F$. Then, by Lemma 8, there exists $\widetilde{F} \in C(0, T; L(\mathfrak{X}))$ such that

$$\mathfrak{T}_\alpha(T, 0; F(y_\varepsilon^*))y_0 \to \mathfrak{T}_\alpha\left(T, 0; \widetilde{F}\right)y_0.$$

Denote $\widetilde{h} = y_f - \mathfrak{T}_\alpha\left(T, 0; \widetilde{F}\right)y_0$. Then,

$$\left\|h(y_\varepsilon^*) - \widetilde{h}\right\| \leq \left\|\mathfrak{T}_\alpha(T, 0; F(y_\varepsilon^*))y_0 - \mathfrak{T}_\alpha\left(T, 0; \widetilde{F}\right)y_0\right\| \to 0 \quad \text{as} \quad \varepsilon \to 0^+. \tag{26}$$

To prove the strong convergence, we recall that

$$\begin{aligned}
\left\|y_\varepsilon^*(T) - y_f\right\| &\leq \varepsilon\left\|(I - \pi_M)\left(\varepsilon(I - \pi_M) + \Gamma_0^T(F(y_\varepsilon^*))\right)^{-1}\left(h(y_\varepsilon^*) - \widetilde{h}\right)\right\| \\
&+\varepsilon\left\|(I - \pi_M)\left[\left(\varepsilon(I - \pi_M) + \Gamma_0^T(F(y_\varepsilon^*))\right)^{-1} - \left(\varepsilon(I - \pi_M) + \Gamma_0^T\left(\widetilde{F}\right)\right)^{-1}\right]\left(\widetilde{h}\right)\right\| \\
&+\varepsilon\left\|(I - \pi_M)\left(\varepsilon(I - \pi_M) + \Gamma_0^T\left(\widetilde{F}\right)\right)^{-1}\left(\widetilde{h}\right)\right\|.
\end{aligned} \tag{27}$$

By Equation (26) and the uniform boundedness of $\left\|\varepsilon\left(\varepsilon(I - \pi_M) + \Gamma_0^T(F(y_\varepsilon^*))\right)^{-1}\right\|$, the first term goes to zero. The second term approaches zero thanks to Lemma 10. The last term converges to zero according to Theorem 1. Thus, taking limit in Equation (27), we complete the proof. □

5. Bounded Nonlinear Term

Next, we consider the case when the nonlinear term is bounded and the semigroup $\{S(t) : t > 0\}$ is compact and analytic.

For any $\varepsilon > 0$, we define the operator

$$\Xi_\varepsilon y(t) = t^{\alpha-1}\mathfrak{U}_\alpha(t)y_0 + \int_0^t (t - s)^{\alpha-1}\mathfrak{U}_\alpha(t - s)[Bu_\varepsilon(s, y) + f(s, y(s))]ds,$$

where

$$\begin{aligned}
u_\varepsilon(t, y) &= B^*\mathfrak{U}_\alpha^*(T - t)\left(\varepsilon(I - \pi_M) + \Gamma_0^T\right)^{-1}\left(h - T^{\alpha-1}\mathfrak{U}_\alpha(T)\right) \\
&\quad - B^*\mathfrak{U}_\alpha^*(T - t)\int_0^t (T - r)^{\alpha-1}\left(\varepsilon(I - \pi_M) + \Gamma_r^T\right)^{-1}\mathfrak{U}_\alpha(T - r)f(r, y(r))dr, \\
\Gamma_r^T &:= \int_r^T (T - s)^{\alpha-1}\mathfrak{U}_\alpha(T - s)BB^*\mathfrak{U}_\alpha^*(T - s)ds.
\end{aligned}$$

It should be stressed that if $\mathfrak{U}_\alpha(t)$, $t > 0$, is analytic, then the controllability of the corresponding linear system on $[0, T]$ implies (finite-)approximate controllability on any $[r, T]$, $0 \leq r < T$. $\left(\varepsilon(I - \pi_M) + \Gamma_r^T\right)^{-1}$ is well-defined. To show that Ξ_ε has a fixed point, we employ the contraction mapping principle in $C_{1-\alpha}((0, T]; \mathfrak{X})$.

Lemma 12. *For all $\varepsilon > 0$, the operator Ξ_ε has a unique fixed point in $C_{1-\alpha}([0, T]; \mathfrak{X})$.*

Proof. It is not difficult to see that Ξ_ε maps $C_{1-\alpha}([0, T]; \mathfrak{X})$ into itself. We show that for every $\varepsilon > 0$, there exists $n > 1$ such that Ξ_ε^n is a contraction mapping. For each $y, z \in C_{1-\alpha}([0, T]; \mathfrak{X})$ and $t \in [0, T]$, we have

$$\begin{aligned}
\|u_\varepsilon(t, y) - u_\varepsilon(t, z)\| &\leq \frac{M_B M_S^2}{\varepsilon\Gamma^2(\alpha)}\int_0^t (T - r)^{\alpha-1}\|f(r, y(r)) - f(r, z(r))\|dr \\
&\leq \frac{M_B L_f M_S^2}{\varepsilon\Gamma^2(\alpha)}\int_0^t (T - r)^{\alpha-1}\|y(r) - z(r)\|dr, \quad \text{with} \quad L_u := \frac{M_B L_f M_S^2}{\varepsilon\Gamma^2(\alpha)}.
\end{aligned}$$

Next, using the inequality

$$\int_0^t (t-s)^{\alpha-1} \int_0^s (T-r)^{\alpha-1} \|y(r)-z(r)\| dr ds \le \int_0^t \int_r^t (t-s)^{\alpha-1} ds (T-r)^{\alpha-1} \|y(r)-z(r)\| dr$$
$$\le \frac{1}{\alpha} \int_0^t (t-r)^{\alpha} (T-r)^{\alpha-1} \|y(r)-z(r)\| dr \le \frac{T^\alpha}{\alpha} \int_0^t (t-r)^{\alpha-1} \|y(r)-z(r)\| dr,$$

we obtain

$$\begin{aligned} t^{1-\alpha}\|\Xi_\varepsilon y(t) - \Xi_\varepsilon z(t)\| &\le t^{1-\alpha} \int_0^t (t-s)^{\alpha-1} \|\mathfrak{U}_\alpha(t-s)\| \|B\| \|u_\varepsilon(s,y) - u_\varepsilon(s,z)\| ds \\ &+ t^{1-\alpha} \int_0^t (t-s)^{\alpha-1} \|\mathfrak{U}_\alpha(t-s)\| \|f(s,y(s)) - f(s,z(s))\| ds \\ &\le t^{1-\alpha} \frac{L_u M_B M_S}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \int_0^s (T-r)^{\alpha-1} \|y(r)-z(r)\| dr ds \\ &+ t^{1-\alpha} \frac{L_f M_S}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y(s)-z(s)\| ds \\ &\le \frac{t^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{T^\alpha L_u M_B M_S}{\alpha} + L_f M_S \right) \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} s^{1-\alpha} \|y(s)-z(s)\| ds \\ &\le \frac{t^{1-\alpha}}{\Gamma(\alpha)} \gamma(\alpha) t^{2\alpha-1} B(\alpha,\alpha) \|y-z\|_\alpha = \Gamma(\alpha) \frac{t^{2\alpha}\gamma(\alpha)}{\Gamma(2\alpha)} \|y-z\|_\alpha. \end{aligned}$$

By induction on n, it follows easily that

$$t^{1-\alpha}\|\Xi_\varepsilon^n y(t) - \Xi_\varepsilon^n z(t)\| \le \frac{\Gamma(\alpha) t^{n\alpha} \gamma^n(\alpha)}{\Gamma((n+1)\alpha)} \|y-z\|_\alpha.$$

Therefore, we obtain

$$\|\Xi_\varepsilon^n y - \Xi_\varepsilon^n z\|_\alpha \le \frac{\Gamma(\alpha) T^{n\alpha} \gamma^n(\alpha)}{\Gamma((n+1)\alpha)} \|y-z\|_\alpha.$$

We know that $\dfrac{T^{n\alpha}\gamma^n(\alpha)}{\Gamma((n+1)\alpha)}$ is the general term of the Mittag–Leffler series $E_{\alpha,\alpha}(\gamma(\alpha)T^\alpha) := \sum_{n=0}^\infty \dfrac{T^{n\alpha}\gamma^n(\alpha)}{\Gamma((n+1)\alpha)}$ and this series converges uniformly on $[0,T]$. Then, for a large enough n, one can obtain $\frac{\Gamma(\alpha)T^{n\alpha}\gamma^n(\alpha)}{\Gamma((n+1)\alpha)} < 1$. Hence, Ξ_ε^n is a contraction operator on $C_{1-\alpha}([0,T];\mathfrak{X})$. We can deduce that Ξ_ε has a unique fixed point on $C_{1-\alpha}([0,T];\mathfrak{X})$. □

Theorem 4. *Assume that*

(i) *the RL fractional linear system corresponding to (1) is approximately controllable;*
(ii) *f is uniformly bounded;*
(iii) *$\{S(t) : t > 0\}$ is compact and analytic,*
then the RL fractional semilinear evolution system (1) is finite-approximately controllable on $[0,T]$.

Proof. Let y_ε be a fixed point of Ξ_ε :

$$y_\varepsilon(t) = t^{\alpha-1}\mathfrak{U}_\alpha(t)y_0 + \int_0^t (t-s)^{\alpha-1}\mathfrak{U}_\alpha(t-s)[Bu_\varepsilon(s,y_\varepsilon) + f(s,y_\varepsilon(s))]ds.$$

Simple calculations show that

$$\begin{aligned}
y_\varepsilon(T) &= T^{\alpha-1}\mathfrak{U}_\alpha(T)y_0 + \Gamma_0^T\Big(\varepsilon(I-\pi_M)+\Gamma_0^T\Big)^{-1}\Big(h - T^{\alpha-1}\mathfrak{U}_\alpha(T)\Big) \\
&- \int_0^T (T-s)^{\alpha-1}\mathfrak{U}_\alpha(T-s)BB^*\mathfrak{U}_\alpha^*(T-s) \\
&\times \int_0^s (T-r)^{\alpha-1}\Big(\varepsilon(I-\pi_M)+\Gamma_r^T\Big)^{-1}\mathfrak{U}_\alpha(T-r)f(r,y(r))dr \\
&= T^{\alpha-1}\mathfrak{U}_\alpha(T)y_0 + \Gamma_0^T\Big(\varepsilon(I-\pi_M)+\Gamma_0^T\Big)^{-1}\Big(h - T^{\alpha-1}\mathfrak{U}_\alpha(T)\Big) \\
&- \int_0^T \int_r^T (T-s)^{\alpha-1}\mathfrak{U}_\alpha(T-s)BB^*\mathfrak{U}_\alpha^*(T-s)ds \\
&\times (T-r)^{\alpha-1}\Big(\varepsilon(I-\pi_M)+\Gamma_r^T\Big)^{-1}\mathfrak{U}_\alpha(T-r)f(r,y(r))dr \\
&= T^{\alpha-1}\mathfrak{U}_\alpha(T)y_0 + \Gamma_0^T\Big(\varepsilon(I-\pi_M)+\Gamma_0^T\Big)^{-1}\Big(h - T^{\alpha-1}\mathfrak{U}_\alpha(T)\Big) \\
&- \int_0^T (T-r)^{\alpha-1}\Gamma_r^T\Big(\varepsilon(I-\pi_M)+\Gamma_r^T\Big)^{-1}\mathfrak{U}_\alpha(T-r)f(r,y_\varepsilon(r))dr \\
&+ \int_0^T (T-s)^{\alpha-1}\mathfrak{U}_\alpha(T-s)f(s,y_\varepsilon(s))ds \\
&= h - \varepsilon(I-\pi_M)\Big(\varepsilon(I-\pi_M)+\Gamma_0^T\Big)^{-1}\Big(h - T^{\alpha-1}\mathfrak{U}_\alpha(T)\Big) \\
&+ \varepsilon(I-\pi_M)\int_0^T (T-r)^{\alpha-1}\Big(\varepsilon(I-\pi_M)+\Gamma_r^T\Big)^{-1}\mathfrak{U}_\alpha(T-r)f(r,y_\varepsilon(r))dr.
\end{aligned}$$

It follows from the boundedness of f that there exists a subsequence still denoted by $\{f(r,y_\varepsilon(r))\}$ which converges weakly to $f(r)$ in $\mathfrak{X}$. From the above equality, we have

$$\begin{aligned}
&\|y_\varepsilon(T)-h\| \\
&\le \varepsilon\Big\|(I-\pi_M)\Big(\varepsilon(I-\pi_M)+\Gamma_0^T\Big)^{-1}\Big(h - T^{\alpha-1}\mathfrak{U}_\alpha(T)\Big)\Big\| \\
&+ \varepsilon\int_0^T (T-r)^{\alpha-1}\Big\|(I-\pi_M)\Big(\varepsilon(I-\pi_M)+\Gamma_r^T\Big)^{-1}\Big\|\|\mathfrak{U}_\alpha(T-r)f(r,y_\varepsilon(r)) - f(r)\|dr \\
&+ \varepsilon\int_0^T (T-r)^{\alpha-1}\Big\|(I-\pi_M)\Big(\varepsilon(I-\pi_M)+\Gamma_r^T\Big)^{-1}f(r)\Big\|dr.
\end{aligned}$$

On the other hand, by the analyticity of $\mathfrak{U}_\alpha$, for all $0 \le s < T$, the operator $\varepsilon(\varepsilon(I-\pi_M)+\Gamma_r^T)^{-1} \to 0$ strongly converges as $\varepsilon \to 0^+$, $\Big\|\varepsilon\big(\varepsilon(I-\pi_M)+\Gamma_r^T\big)^{-1}\Big\| \le 1$, by compactness of $\mathfrak{U}_\alpha(T-r)$, $0 \le r < T$ $\|\mathfrak{U}_\alpha(T-r)f(r,y_\varepsilon(r)) - f(r)\| \to 0$ as $\varepsilon \to 0^+$. Thus, by the Lebesgue-dominated convergence theorem, we have

$$\begin{aligned}
\|y_\varepsilon(T)-h\| &\to 0 \text{ as } \varepsilon \to 0^+, \\
\pi_M y_\varepsilon(T) &= \pi_M h.
\end{aligned}$$

This gives the finite-approximate controllability of (1). □

Using the control

$$\begin{aligned}
u_\varepsilon(t,y) &= B^*\mathfrak{U}_\alpha^*(T-t)\Big(\varepsilon(I-\pi_M)+\Gamma_0^T\Big)^{-1} \\
&\times\left(h - T^{\alpha-1}\mathfrak{U}_\alpha(T) - \int_0^T (T-r)^{\alpha-1}\mathfrak{U}_\alpha(T-r)f(r,y(r))dr\right)
\end{aligned}$$

define the following operator

$$\Xi_\varepsilon y(t) = t^{\alpha-1}\mathfrak{U}_\alpha(t)y_0 + \int_0^t (t-s)^{\alpha-1}\mathfrak{U}_\alpha(t-s)[Bu_\varepsilon(s,y) + f(s,y(s))]ds.$$

In the following theorem, we remove analyticity of the semigroup $\{S(t) : t > 0\}$.

Theorem 5. *Assume that*

- **(i)** *the RL fractional linear system corresponding to (1) is approximately controllable;*
- **(ii)** *f is uniformly bounded;*
- **(iii)** *$\{S(t) : t > 0\}$ is compact.*
  Then, the RL fractional semilinear evolution system (1) is finite-approximately controllable on $[0, T]$.

Proof. The proof is similar to that reported in [23], Theorem 3.3. □

6. Applications

Example 1. *We consider a system governed by the semilinear heat equation with lumped control*

$$\begin{cases} {}^{RL}D^\alpha_{0+}y(t,\theta) = \dfrac{\partial^2 y(t,\theta)}{\partial\theta^2} + \chi_{(\alpha_1,\alpha_2)}(\theta)u(t) + f(y(t,\theta)), \\ y(t,0) = y(t,\pi) = 0, \quad 0 < t < T, \\ I^{1-\alpha}_{0+}y(0,\theta) = y_0(\theta), \quad 0 \le \theta \le \pi, \end{cases} \tag{28}$$

where $\chi_{(\alpha_1,\alpha_2)}(\theta)$ is the characteristic function of $(\alpha_1, \alpha_2) \subset (0, \pi)$. Let $\mathfrak{X} = L^2[0, \pi]$, $U = R$, and $A = d^2/d\theta^2$ with $D(A) = H^1_0[0, \pi] \cap H^2[0, \pi]$. We define the bounded linear operator $B : R \to L^2[0, \pi]$ by $(Bu)(t) = \chi_{(\alpha_1,\alpha_2)}(\theta)u(t)$, and the nonlinear operator f is assumed to be bounded.

Set $M = L^2_K[0, \pi] := \left\{\varphi : \varphi(\theta) = \sum_{i=1}^K \alpha_i e_i(\theta),\ \alpha_i \in R\right\}$ and denote by π_M the operator of the orthogonal projection $L^2[0, \pi]$ onto $L^2_K[0, \pi]$. Define

$$\begin{aligned}
\mathfrak{U}_\alpha(t)h &= \sum_{n=1}^\infty E_{\alpha,\alpha}\left(-n^2\pi^2 t^{\alpha n}\right)\langle h, e_n\rangle e_n, \quad E_{\alpha,\alpha}(t) = \sum_{n=0}^\infty \frac{t^n}{\Gamma(n\alpha+\alpha)} \quad \text{(Mittag–Leffler series)},\\
L^T_0 u &= \int_0^T (T-t)^{\alpha-1}\mathfrak{U}_\alpha(T-t)(Bu)(t)dt\\
&= \sum_{n=1}^\infty \int_0^T (T-t)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-t)^\alpha)\left\langle \chi_{(\alpha_1,\alpha_2)}(\theta), e_n\right\rangle u(t)dte_n,\\
\left(L^T_0\right)^* h &= \sum_{n=1}^\infty \int_0^T (T-t)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-t)^\alpha)\left\langle \chi_{(\alpha_1,\alpha_2)}(\theta), e_n\right\rangle\langle h, e_n\rangle dt,\\
\Gamma^T_0 h = L^T_0\left(L^T_0\right)^* h &= \sum_{n=1}^\infty \int_0^T (T-t)^{2(\alpha-1)}E^2_{\alpha,\alpha}(-\lambda_n(T-t)^\alpha)dt\left\langle \chi_{(\alpha_1,\alpha_2)}(\theta), e_n\right\rangle^2\langle h, e_n\rangle e_n.
\end{aligned}$$

Subsequently, we attain

$$\left(\varepsilon(I-\pi_M)+\Gamma_0^T\right)^{-1}g$$

$$=\sum_{n=1}^{\infty}\frac{1}{\left(\varepsilon(I-\pi_M)+\int_0^T(T-t)^{2(\alpha-1)}E_{\alpha,\alpha}^2\left(-\lambda_n(T-t)^\alpha\right)dt\left\langle\chi_{(\alpha_1,\alpha_2)}(\theta),e_n\right\rangle^2\right)}\langle g,e_n\rangle e_n$$

$$=\sum_{n=1}^{K}\frac{1}{\int_0^T(T-t)^{2(\alpha-1)}E_{\alpha,\alpha}^2\left(-\lambda_n(T-t)^\alpha\right)dt\left\langle\chi_{(\alpha_1,\alpha_2)}(\theta),e_n\right\rangle^2}\langle g,e_n\rangle e_n$$

$$+\sum_{n=K+1}^{\infty}\frac{1}{\left(\varepsilon+\int_0^T(T-t)^{2(\alpha-1)}E_{\alpha,\alpha}^2\left(-\lambda_n(T-t)^\alpha\right)dt\left\langle\chi_{(\alpha_1,\alpha_2)}(\theta),e_n\right\rangle^2\right)}\langle g,e_n\rangle e_n$$

It is clear that $\varepsilon\left(\varepsilon(I-\pi_M)+\Gamma_0^T\right)^{-1}g\to 0$ as $\varepsilon\to 0^+$ if $\left\langle\chi_{(\alpha_1,\alpha_2)}(\theta),e_n\right\rangle=\int_{\alpha_1}^{\alpha_2}\sqrt{2}\sin(n\pi\theta)d\theta=-\frac{\sqrt{2}}{n\pi}\cos(n\pi\theta)\,|_{\alpha_1}^{\alpha_2}\neq 0$, which holds whenever $\alpha_1\pm\alpha_2$ is an irrational number.

If $\alpha_1\pm\alpha_2$ is an irrational number, then the fractional linear system corresponding to Equation (28) is finite-approximately controllable, and by Theorem 5, the system (28) is finite-approximately controllable on $[0,T]$.

Example 2. *Let $H^l(\Omega)$ and $H_0^m(\Omega)$, $l,m\in N$, denote the Sobolev space. Define the differential operator $\mathfrak{L}$ by*

$$\mathfrak{L}y(\theta)=-\sum_{i,j=1}^{d}\frac{\partial}{\partial\theta_i}\left(a_i(\theta)\frac{\partial y}{\partial\theta_j}(\theta)\right)+c(\theta)y(\theta),\ \ \theta\in\Omega,$$

where $a_{ij}=a_{ji}\in C^1(\overline{\Omega})$, $1\leq i,j\leq d$, $c\in C(\overline{\Omega})$, $c(\theta)\geq 0$, $\theta\in\overline{\Omega}$, and

$$\sum_{i,j=1}^{d}a_{ij}(\theta)\eta_i\eta_j\geq l|\eta|^2,\ \ l>0,\ \ \theta\in\overline{\Omega},\ \eta\in R^d.$$

Hence, $\mathfrak{L}:D(\mathfrak{L})=H^2(\Omega)\cap H_0^1(\Omega)\subset L^2(\Omega)\to L^2(\Omega)$.

Let Ω be a bounded domain in R^d with smooth boundary $\partial\Omega$. We consider the following initial value/boundary value problem of the fractional differential equation:

$$\begin{aligned}{}^{RL}D_{0^+}^{\alpha}y(t,\theta)+\mathfrak{L}y(t,\theta)&=\chi_\omega v(t,\theta)+f(y(t,\theta))\quad(\theta,t)\in\Omega\times(0,T),\\ y(t,\theta)&=0,\quad(\theta,t)\in\partial\Omega\times(0,T),\\ D_{0^+}^{\alpha-1}y(0,\theta)&=y_0,\quad v\in C_0^\infty(\omega\times(0,T)),\ \ \omega\subset\Omega.\end{aligned}\tag{29}$$

For any $y_0\in L^2(\Omega)$, Equation (29) has a unique solution y given by

$$y(t,\theta)=\sum_{n=1}^{\infty}(y_0,e_n)t^\alpha E_{\alpha,\alpha}(-\lambda_n t^\alpha)e_n(\theta)+\sum_{n=1}^{\infty}\int_0^t(f(\cdot,t-r),e_n)r^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n r^\alpha)dre_n(\theta),$$

$$L_0^Tf=\sum_{n=1}^{\infty}\int_0^T(f(\cdot,T-r),e_n)r^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n r^\alpha)dre_n(\theta),$$

$$\left(L_0^T\right)^*g=\chi_\omega(T-t)^{\alpha-1}\sum_{n=1}^{\infty}E_{\alpha,\alpha}\left(-\lambda_n(T-t)^\alpha\right)(g,e_n)e_n(\theta).$$

Assume that $L^*g=0$. Then, $(T-t)^{\alpha-1}\sum_{n=1}^{\infty}E_{\alpha,\alpha}\left(-\lambda_n(T-t)^\alpha\right)(g,e_n)e_n(\theta)=0$ on $\omega\times(0,T)$. By Proposition 4.2 [58], $g=0$ on $\Omega\times(0,T)$, which is equivalent to the linear system associated with Equation (29). Thus, the system (29) is finite-approximately controllable on $[0,T]$, provided that the nonlinear term f is bounded.

7. Conclusions

In this paper, we focused on:

- Establishing necessary and sufficient conditions for finite-approximate controllability of Riemann–Liouville (and classical) linear evolution systems in Hilbert spaces in terms of resolvent-like operators. The criterion of Theorem 1(iv) is new even for nonfractional linear evolution equations. Moreover, we found an explicit form of the control which, in addition to the approximate controllability requirement, ensures finite dimensional exact controllability.
- The variational method is used to prove simultaneous approximate and exact finite-dimensional controllability for Riemann–Liouville fractional semilinear evolution system (1) under different conditions for the nonlinear term.

One can assume that the results of this work apply to a class of problems determined by various types of fractional (impulsive) evolution systems, such as Caputo FDEs, Hadamard-type FDEs, stochastic FDEs, Sobolev-type FDEs, and so on.

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