



## Article

# Existence and Kummer Stability for a System of Nonlinear $\phi$ -Hilfer Fractional Differential Equations with Application

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**Abstract:** Using Krasnoselskii's fixed point theorem and Arzela–Ascoli theorem, we investigate the existence of solutions for a system of nonlinear  $\phi$ -Hilfer fractional differential equations. Moreover, applying an alternative fixed point theorem due to Diaz and Margolis, we prove the Kummer stability of the system on the compact domains. We also apply our main results to study the existence and Kummer stability of Lotka–Volterra's equations that are useful to describe and characterize the dynamics of biological systems.

**Keywords:**  $\phi$ -Hilfer FDE; Kummer's stability; Arzela–Ascoli theorem; Krasnoselskii fixed point theorem; Lotka–Volterra's equations



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Fractional differential equations (FDEs) are important due to their applications in engineering, economics, control theory, materials sciences, physics, chemistry, and biology (see [1] and the references therein). Scientists have applied various mathematical approaches through diverse research-oriented aspects of fractional differential systems. For instance, existence, stability, and control theory for fractional differential equations were studied [2,3]. For the first time, Alsina and Ger [4] studied the Hyers–Ulam stability for differential equations. Recently, mathematicians have paid more attention to the study of stability for a wide range of differential systems [5–7].

In this paper, we begin by considering the following fractional differential equation

$$\begin{cases} {}^{\mathcal{H}}\mathbb{D}_{0+}^{\zeta, \nu; \phi} w(\eta) = \mathcal{A}(w(\eta)) + \mathfrak{g}(\eta, w(\eta)) + \int_0^t \mathfrak{h}(\eta, s, w(s)) ds, & \eta \in \omega = (0, p) \\ I_0^{1-\gamma; \phi} w(0) = w_0, & w_0 \in \mathbb{R} \end{cases} \quad (1)$$

where  ${}^{\mathcal{H}}\mathbb{D}_{0+}^{\zeta, \nu; \phi}(\cdot)$  is a  $\phi$ -Hilfer fractional derivative of order  $0 < \zeta \leq 1$  and type  $0 \leq \nu < 1$ , and  $I_0^{1-\gamma; \phi}$  is a  $\phi$ -Riemann–Liouville fractional integral of order  $1 - \gamma$  ( $\gamma = \zeta + \nu(1 - \zeta)$ ) with respect to the mapping  $\phi$ . Furthermore,  $\mathfrak{g} : \omega \times \mathbb{R} \rightarrow \mathbb{R}$  and  $\mathfrak{h} : \omega^2 \times \mathbb{R} \rightarrow \mathbb{R}$  are given mappings, and  $\mathcal{A}$  is a closed linear operator. In the following, we show the existence of solutions to Equation (1) based on the Krasnoselskii FPT and Arzela–Ascoli theorem. Using Kummer's control function, we introduce a new concept of stability and further deduce that the solution of Equation (1) is stable in Kummer's sense.

## 1. Preliminaries

In this section, we recall some fundamental definitions of the  $\phi$ -Riemann–Liouville fractional integral,  $\phi$ -Hilfer fractional derivative, and Kummer's functions. For details, please see [1,8] and the references therein.

Let  $[n, m]$  be a finite and closed interval with  $0 \leq n < m < \infty$  and  $\mathfrak{C}[n, m]$  be the space of continuous functions  $\varrho : [n, m] \rightarrow \mathbb{R}$  equipped with the following norm

$$\|\varrho\|_{\mathfrak{C}[n, m]} = \max_{\eta \in [n, m]} |\varrho(\eta)|.$$

Furthermore, the weighted space  $\mathfrak{C}_{\gamma, \phi}(n, m)$  is defined as

$$\mathfrak{C}_{1-\gamma, \phi}[n, m] = \left\{ \varrho : (n, m] \rightarrow \mathbb{R}; (\phi(\eta) - \phi(n))^{1-\gamma} \varrho(\eta) \in \mathfrak{C}[n, m] \right\} \text{ where } 0 < \gamma < 1,$$

with norm

$$\|\varrho\|_{\mathfrak{C}_{1-\gamma, \phi}[n, m]} = \max_{\eta \in [n, m]} |(\phi(\eta) - \phi(n))^{1-\gamma} \varrho(\eta)|$$

where  $\phi : [n, m] \rightarrow \mathbb{R}$  is an arbitrary function, and  $\eta \in [n, m]$ .

**Definition 1.** Let  $(n, m)$ ,  $-\infty \leq n < m \leq +\infty$  be a finite or infinite interval of the line  $\mathbb{R}$ ,  $\Gamma$  be the gamma function, and  $\zeta > 0$ . Additionally, let  $\phi(\eta)$  be a positive function defined on  $[n, m]$  so that  $\phi'(\eta) \geq 0$  on  $(n, m]$  and  $\phi'(\eta)$  is a continuous function on  $(n, m)$ . The left- and right-sided fractional integrals of a function  $\varrho$  with respect to the function  $\phi$  on  $[n, m]$  are defined by

$$I_{n^+}^{\zeta; \phi} \varrho(x) = \frac{1}{\Gamma(\zeta)} \int_n^x \phi'(\eta) (\phi(x) - \phi(\eta))^{\zeta-1} \varrho(\eta) d\eta,$$

and

$$I_{m^-}^{\zeta; \phi} \varrho(x) = \frac{1}{\Gamma(\zeta)} \int_x^m \phi'(\eta) (\phi(\eta) - \phi(x))^{\zeta-1} \varrho(\eta) d\eta$$

respectively.

The fractional integrals with the above definition have a semi-group property given by

$$I_{n^+}^{\zeta; \phi} I_{n^+}^{\nu; \phi} \varrho(x) = I_{n^+}^{\zeta+\nu; \phi} \varrho(x) \text{ and } I_{m^-}^{\zeta; \phi} I_{m^-}^{\nu; \phi} \varrho(x) = I_{m^-}^{\zeta+\nu; \phi} \varrho(x).$$

Additionally, for  $\zeta, \sigma > 0$ , we have [9]:

- (i) if  $\varrho(x) = (\phi(x) - \phi(n))^{\sigma-1}$ , then  $I_{n^+}^{\zeta; \phi} \varrho(x) = \frac{\Gamma(\sigma)}{\Gamma(\zeta + \sigma)} (\phi(x) - \phi(n))^{\zeta+\sigma-1}$ , and
- (ii) if  $\varrho(x) = (\phi(m) - \phi(x))^{\sigma-1}$ , then  $I_{m^-}^{\zeta; \phi} \varrho(x) = \frac{\Gamma(\sigma)}{\Gamma(\zeta + \sigma)} (\phi(m) - \phi(x))^{\zeta+\sigma-1}$ .

**Definition 2.** Let  $(n, m)$ ,  $-\infty \leq n < m \leq +\infty$  be a finite or infinite interval of the line  $\mathbb{R}$ ,  $\phi'(\eta) \neq 0$  for all  $\eta \in (n, m)$ , and  $\zeta > 0$ ,  $n \in \mathbb{N}$ . The left-sided Riemann–Liouville derivative of a function  $\varrho$  with respect to  $\phi$  of order  $\zeta$  correspondent to the Riemann–Liouville is defined by

$$\begin{aligned} D^{\zeta; \phi} \varrho(\eta) &= \left( \frac{1}{\phi'(\eta)} \frac{d}{dx} \right)^n I^{n-\zeta; \phi} \varrho(\eta) \\ &= \frac{1}{\Gamma(n-\zeta)} \left( \frac{1}{\phi'(\eta)} \frac{d}{dx} \right)^n \times \int_n^\eta \phi'(t) (\phi(\eta) - \phi(t))^{n-\zeta-1} \varrho(t) dt. \end{aligned}$$

**Definition 3.** Let  $n - 1 < \varsigma < n$  with  $n \in \mathbb{N}$ ,  $I = [n, m]$  ( $-\infty \leq n < m \leq \infty$ ) and  $\varrho, \phi \in \mathcal{C}^n([n, m], \mathbb{R})$  be two mappings such that  $\phi'(x) > 0$  for all  $x \in I$ . The left- and right-sided  $\phi$ -Hilfer fractional derivatives  ${}^{\mathcal{H}}\mathbb{D}_{0^+}^{\varsigma, \nu; \phi}(\cdot)$  of the arbitrary function  $\varrho$  of order  $\varsigma$  and type  $0 \leq \nu < 1$  are defined by

$${}^{\mathcal{H}}\mathbb{D}_{n^+}^{\varsigma, \nu; \phi} \varrho(x) = I_{n^+}^{\nu(n-\varsigma); \phi} \left( \frac{1}{\phi'(x)} \frac{d}{dx} \right)^n I_{n^+}^{(1-\nu)(n-\varsigma); \phi} \varrho(x),$$

and

$${}^{\mathcal{H}}\mathbb{D}_{m^-}^{\varsigma, \nu; \phi} \varrho(x) = I_{m^-}^{\nu(n-\varsigma); \phi} \left( -\frac{1}{\phi'(x)} \frac{d}{dx} \right)^n I_{m^-}^{(1-\nu)(n-\varsigma); \phi} \varrho(x)$$

respectively.

**Theorem 1.** If  $\varrho \in \mathcal{C}^1[n, m]$ ,  $\varsigma > 0$ ,  $0 \leq \nu < 1$ , and  $\gamma = \varsigma + \nu(1 - \varsigma)$ , then

$${}^{\mathcal{H}}\mathbb{D}_{n^+}^{\varsigma, \nu; \phi} I_{n^+}^{\varsigma; \phi} \varrho(x) = \varrho(x) \quad \text{and} \quad {}^{\mathcal{H}}\mathbb{D}_{m^-}^{\varsigma, \nu; \phi} I_{m^-}^{\varsigma; \phi} \varrho(x) = \varrho(x).$$

Additionally, we have

$$I_{n^+}^{\varsigma; \phi} {}^{\mathcal{H}}\mathbb{D}_{n^+}^{\varsigma, \nu; \phi} \varrho(x) = \varrho(x) - \frac{(\phi(x) - \phi(n))^{\gamma-1}}{\Gamma(\gamma)} I_{n^+}^{(1-\nu)(1-\varsigma); \phi} \varrho(n),$$

and

$$I_{m^-}^{\varsigma; \phi} {}^{\mathcal{H}}\mathbb{D}_{m^-}^{\varsigma, \nu; \phi} \varrho(x) = \varrho(x) - \frac{(\phi(m) - \phi(x))^{\gamma-1}}{\Gamma(\gamma)} I_{m^-}^{(1-\nu)(1-\varsigma); \phi} \varrho(m).$$

**Proof.** Ref. [9].  $\square$

The solution of a hypergeometric differential equation is called a confluent hypergeometric function [10]. There exist different standard forms of confluent hypergeometric functions, such as Kummer's functions, Tricomi's functions, Whittaker's functions, and Coulomb's wave functions. In this paper, we apply the following Kummer (confluent hypergeometric) function to study our stability:

$$\Phi(\mathfrak{P}_1, \mathfrak{P}_2; \mathfrak{z}) = {}_1F_1(\mathfrak{P}_1, \mathfrak{P}_2; \mathfrak{z}) = \frac{\Gamma(\mathfrak{P}_2)}{\Gamma(\mathfrak{P}_1)} \sum_{k=0}^{\infty} \frac{\Gamma(\mathfrak{P}_1 + k)}{\Gamma(\mathfrak{P}_2 + k)} \frac{\mathfrak{z}^k}{k!}, \quad (2)$$

which is the solution of the differential equation

$$\mathfrak{z} \frac{d^2 u}{dz^2} + (\mathfrak{P}_2 - \mathfrak{z}) \frac{du}{dz} - \mathfrak{P}_1 u(\mathfrak{z}) = 0,$$

where  $\mathfrak{z}, \mathfrak{P}_1 \in \mathbb{C}$  and  $\mathfrak{P}_2 \in \mathbb{C} \setminus \mathbb{Z}_0^-$ . Kummer's function was introduced by Kummer in 1837. The series (2) is also known as the confluent hyper-geometric function of the first kind, and is convergent for any  $\mathfrak{z} \in \mathbb{C}$ . In this article, we apply it on the real line  $\mathbb{R}$  as our control function. Clearly, for  $\mathfrak{P}_1 = \mathfrak{P}_2$ , we have

$$\Phi(\mathfrak{P}_1, \mathfrak{P}_2; \mathfrak{z}) = {}_1F_1(\mathfrak{P}_1, \mathfrak{P}_1; \mathfrak{z}) = \frac{\Gamma(\mathfrak{P}_1)}{\Gamma(\mathfrak{P}_1)} \sum_{k=0}^{\infty} \frac{\Gamma(\mathfrak{P}_1 + k)}{\Gamma(\mathfrak{P}_1 + k)} \frac{\mathfrak{z}^k}{k!} = \sum_{k=0}^{\infty} \frac{\mathfrak{z}^k}{k!} = e^{\mathfrak{z}}.$$

Letting  $\varsigma, \nu \in \omega$ , we consider the following inequality for  $\epsilon > 0$

$$\begin{aligned} & |{}^{\mathcal{H}}\mathbb{D}_{0+}^{\varsigma, \nu; \phi} v(\eta) - \mathcal{A}(v(\eta)) + \mathfrak{g}(\eta, v(\eta)) - \int_0^t \mathfrak{h}(\eta, s, v(s)) ds| \\ & \leq \epsilon \Phi(\varsigma, \nu; (\phi(\eta) - \phi(0))^{\varsigma}) \end{aligned} \quad (3)$$

where  $\Phi$  is the Kummer's function (see [10]), to define a new stability concept called Kummer's stability.

**Definition 4.** For a positive constant  $C_{\Phi}$ , for all  $\epsilon > 0$ , and every solution  $v \in (\mathcal{C}[0, p], \mathbb{R})$  to inequality (3), if we can find a solution  $w \in (\mathcal{C}[0, p], \mathbb{R})$  to Equation (1), with the following property:

$$|w(\eta) - v(\eta)| \leq C_{\Phi} \epsilon \Phi(\varsigma, \nu; (\phi(\eta) - \phi(0))^{\varsigma}), \text{ for all } \eta \in [0, p],$$

then we say that Equation (1) has Kummer's stability with respect to  $\Phi(\varsigma, \nu; (\phi(\eta) - \phi(0))^{\varsigma})$ .

Our approach is motivated by the fact that inversion of a perturbed differential operator may result from the sum of a compact operator and a contraction mapping (see [11–13] and the references therein). We begin by stating the following Krasnoselskii FPT, which has many applications in studying the existence of solutions to differential equations:

**Theorem 2.** (Krasnoselskii FPT) Let  $X$  be a Banach space and  $\mathfrak{M} \subseteq X$  be a closed, convex, and non-empty set. Additionally, let  $\mathfrak{T}, \mathfrak{S}$  be mappings so that:

- $\mathfrak{T}u + \mathfrak{S}v \in \mathfrak{M}$  whenever  $u, v \in \mathfrak{M}$ ,
- The operator  $\mathfrak{T}$  is continuous and compact, and
- Mapping  $\mathfrak{S}$  is a contraction.

Then, there exists a  $w \in \mathfrak{M}$  so that  $w = \mathfrak{T}w + \mathfrak{S}w$ .

In addition, we mention an alternative FPT presented by Diaz and Margolis in 1967, and it plays a crucial role in proving our stability result [14].

**Theorem 3.** Consider the generalized complete metric space  $(\mathfrak{X}, Y)$  and let  $\Theta$  be a self-map operator which is a strictly contraction mapping with the Lipschitz constant  $\kappa < 1$ . Then, we have two options: (i) either for every  $n \in \mathbb{N}$ ,  $Y(\Theta^{n+1}\mathfrak{z}, \Theta^n\mathfrak{z}) = +\infty$ ; or (ii) if there exists  $n \in \mathbb{N}$  so that the operator  $\Theta$  satisfies  $Y(\Theta^{n+1}\mathfrak{z}, \Theta^n\mathfrak{z}) < \infty$  for some  $\mathfrak{z} \in \mathfrak{X}$ , then the sequence  $\{\Theta^n\mathfrak{z}\}$  tends to a unique fixed point  $\mathfrak{z}^*$  of  $\Theta$  in the set  $\mathfrak{X}^* = \{\mathfrak{v} \in \mathfrak{X} : Y(\Theta^n\mathfrak{v}, \Theta^n\mathfrak{v}) < \infty\}$ . Furthermore, for all  $\mathfrak{z} \in \mathfrak{X}$ :

$$Y(\mathfrak{z}, \mathfrak{z}^*) \leq \frac{1}{1 - \kappa} Y(\mathfrak{z}, \Theta\mathfrak{z}).$$

Now, we are ready to prove that Equation (1) is equivalent to an integral equation. Then, by the above theorem, we infer that a fixed point exists for the integral equation, so Equation (1) has at least one solution.

**Proposition 1.** Assume that  $\mathfrak{g} : \omega \times \mathbb{R} \rightarrow \mathbb{R}$  and  $\mathfrak{h} : \omega^2 \times \mathbb{R} \rightarrow \mathbb{R}$  are real-valued continuous mappings, and  $\mathcal{A}$  is a closed operator, then the following integral equation is equivalent to Equation (1):

$$w(\eta) = \frac{(\phi(\eta) - \phi(0))^{\gamma-1}}{\Gamma(\gamma)} w_0 + I_{0+}^{\varsigma; \phi} [\mathfrak{g}(\eta, w(\eta), w(\eta)) + \int_0^s \mathfrak{h}(\eta, \tau, u(\tau)) d\tau + \mathcal{A}(w(\eta))] \quad (4)$$

where  $\gamma \geq 0$  and we obtain from  $\gamma = \varsigma + \nu(1 - \varsigma)$  for  $0 < \varsigma \leq 1$  and  $0 \leq \nu < 1$  in (1).

**Proof.** Using the properties of the  $\phi$ -Hilfer fractional derivative outlined in the preliminaries, we have

$${}^H\mathbb{D}^{\zeta,\nu;\phi}w(\eta) = I^{\nu(n-\zeta);\phi}D^{\gamma;\phi}w(\eta) = I^{\gamma-\zeta;\phi}D^{\gamma;\phi}w(\eta),$$

where  $\gamma = \zeta + \nu(1 - \zeta)$ . So, by the above equality, we have

$$I^{\gamma-\zeta;\phi}D^{\gamma;\phi}w(\eta) = \mathcal{A}(w(\eta)) + \mathfrak{g}(\eta, w(\eta)) + \int_0^\eta \mathfrak{h}(\eta, s, w(s))ds.$$

Now, applying  $I^{\zeta;\phi}$  to both sides of the above equation and using Theorem 1, we obtain

$$I^{\zeta;\phi}I^{\gamma-\zeta;\phi}D^{\gamma;\phi}w(\eta) = I^{\zeta;\phi}\left(\mathcal{A}(w(\eta)) + \mathfrak{g}(\eta, w(\eta)) + \int_0^s \mathfrak{h}(\eta, \tau, w(\tau))d\tau\right),$$

and

$$I^{\gamma;\phi}D^{\gamma;\phi}w(\eta) = I^{\zeta;\phi}\left(\mathcal{A}(w(\eta)) + \mathfrak{g}(\eta, w(\eta)) + \int_0^s \mathfrak{h}(\eta, \tau, w(\tau))d\tau\right).$$

Then,

$$w(\eta) = \frac{(\phi(\eta) - \phi(0))^{\gamma-1}}{\Gamma(\gamma)}\mathfrak{w}_0 + I^{\zeta;\phi}\left(\mathcal{A}(w(\eta)) + \mathfrak{g}(\eta, w(\eta)) + \int_0^s \mathfrak{h}(\eta, \tau, u(\tau))d\tau\right).$$

Conversely, assuming that  $w \in \mathfrak{C}[0, \mathfrak{p}]$  satisfies Equation (4), we claim that the fractional differential Equation (1) holds. We apply  ${}^H\mathbb{D}^{\zeta,\nu;\phi}$  to the Equation (4) and imply by Theorem 1 that

$$\begin{aligned} & {}^H\mathbb{D}^{\zeta,\nu;\phi}w(\eta) \\ &= {}^H\mathbb{D}^{\zeta,\nu;\phi}\left(\frac{(\phi(\eta) - \phi(0))^{\gamma-1}}{\Gamma(\gamma)}\mathfrak{w}_0 + I_{0^+}^{\zeta;\phi}\mathfrak{g}(\eta, w(\eta)) + I_{0^+}^{\zeta;\phi}\left[\int_0^s \mathfrak{h}(\eta, \tau, w(\tau))d\tau\right] + I_{0^+}^{\zeta;\phi}(\mathcal{A}(w(\eta)))\right). \end{aligned}$$

From  ${}^H\mathbb{D}^{\zeta,\nu;\phi}\mathfrak{w}_0 = 0$ , we obtain

$${}^H\mathbb{D}_{0^+}^{\zeta,\nu;\phi}w(\eta) = \mathcal{A}(w(\eta)) + \mathfrak{g}(\eta, w(\eta)) + \int_0^\eta \mathfrak{h}(t, s, u(s))ds.$$

This completes the proof.  $\square$

**Remark 1.** Let  $w \in \mathcal{C}(\omega, \mathbb{R})$  satisfy inequality (3). Then the following integral inequality holds

$$\begin{aligned}
& \left| w(t) - \frac{(\phi(\eta) - \phi(0))^{\gamma-1}}{\Gamma(\gamma)} w_0 \right. \\
& \quad \left. - I_{0+}^{\zeta; \phi} \mathfrak{g}(\eta, w(\eta)) - I_{0+}^{\zeta; \phi} \left[ \int_0^s \mathfrak{h}(\eta, \tau, w(\tau)) d\tau \right] - I_{0+}^{\zeta; \phi} (\mathcal{A}(w(\eta))) \right| \\
& \leq \frac{\epsilon}{\Gamma(\zeta)} \int_0^\eta \phi'(t) (\phi(x) - \phi(\eta))^{\zeta-1} \Phi(\zeta, \nu; (\phi(\eta) - \phi(0))^\zeta) ds \\
& = \frac{\epsilon}{\Gamma(\zeta)} \int_0^\eta \phi'(\eta) (\phi(x) - \phi(\eta))^{\zeta-1} \frac{\Gamma(\nu)}{\Gamma(\zeta)} \sum_{\mathfrak{k}=0}^{\infty} \frac{\Gamma(\zeta + \mathfrak{k})}{\Gamma(\nu + \mathfrak{k})} \frac{(\phi(\eta) - \phi(0))^{\mathfrak{k}\zeta}}{\mathfrak{k}!} ds \\
& = \frac{\epsilon}{\Gamma(\zeta)} \frac{\Gamma(\nu)}{\Gamma(\zeta)} \sum_{\mathfrak{k}=0}^{\infty} \frac{\Gamma(\zeta + \mathfrak{k})}{\Gamma(\nu + \mathfrak{k})} \frac{1}{\mathfrak{k}!} \int_0^\eta \phi'(t) (\phi(x) - \phi(\eta))^{\zeta-1} (\phi(\eta) - \phi(0))^{\mathfrak{k}\zeta} ds \\
& = \frac{\epsilon}{\Gamma(\zeta)} \frac{\Gamma(\nu)}{\Gamma(\zeta)} \sum_{\mathfrak{k}=0}^{\infty} \frac{\Gamma(\zeta + \mathfrak{k})}{\Gamma(\nu + \mathfrak{k})} \frac{1}{\mathfrak{k}!} \int_0^\eta (\phi(x) - \phi(\eta))^{\zeta-1} (\phi(\eta) - \phi(0))^{\mathfrak{k}\zeta} d\phi(s) \\
& = \frac{\epsilon}{\Gamma(\zeta)} \frac{\Gamma(\nu)}{\Gamma(\zeta)} \sum_{\mathfrak{k}=0}^{\infty} \frac{\Gamma(\zeta + \mathfrak{k})}{\Gamma(\nu + \mathfrak{k})} \frac{1}{\mathfrak{k}!} \int_0^{\phi(\eta) - \phi(0)} (\phi(\eta) - \phi(0) - w)^{\zeta-1} w^{\mathfrak{k}\zeta} dw \\
& = \frac{\epsilon}{\Gamma(\zeta)} \frac{\Gamma(\nu)}{\Gamma(\zeta)} \sum_{\mathfrak{k}=0}^{\infty} \frac{\Gamma(\zeta + \mathfrak{k})}{\Gamma(\nu + \mathfrak{k})} \frac{1}{\mathfrak{k}!} (\phi(\eta) - \phi(0))^{\zeta(\mathfrak{k}+1)} \int_0^{\phi(\eta) - \phi(0)} \left( 1 - \frac{w}{\phi(\eta) - \phi(0)} \right)^{\zeta-1} w^{\mathfrak{k}\zeta} dw \\
& = \frac{\epsilon}{\Gamma(\zeta)} \frac{\Gamma(\nu)}{\Gamma(\zeta)} \sum_{\mathfrak{k}=0}^{\infty} \frac{\Gamma(\zeta + \mathfrak{k})}{\Gamma(\nu + \mathfrak{k})} \frac{1}{\mathfrak{k}!} (\phi(\eta) - \phi(0))^{\zeta(\mathfrak{k}+1)} \int_0^1 (1 - \nu)^{\zeta-1} \nu^{\mathfrak{k}\zeta} d\nu \\
& = \frac{\epsilon}{\Gamma(\zeta)} \frac{\Gamma(\nu)}{\Gamma(\zeta)} \sum_{\mathfrak{k}=0}^{\infty} \frac{\Gamma(\zeta + \mathfrak{k})}{\Gamma(\nu + \mathfrak{k})} \frac{1}{\mathfrak{k}!} (\phi(\eta) - \phi(0))^{\zeta(\mathfrak{k}+1)} \frac{\Gamma(\mathfrak{k}\zeta + 1) \Gamma(\zeta)}{\Gamma((\mathfrak{k} + 1)\zeta + 1)} \\
& \leq \epsilon \frac{\Gamma(\nu)}{\Gamma(\zeta)} \sum_{\mathfrak{k}=0}^{\infty} \frac{\Gamma(\zeta + \mathfrak{k})}{\Gamma(\nu + \mathfrak{k})} \frac{(\phi(\eta) - \phi(0))^{\zeta(\mathfrak{k}+1)}}{\mathfrak{k}!} \\
& \leq \epsilon \Phi(\zeta, \nu; (\phi(\eta) - \phi(0))^\zeta).
\end{aligned}$$

## 2. Existence Result

In this section, we study Equation (1) under the following hypotheses:

**Hypothesis 1 (H1).**  $\mathfrak{g} \in \mathcal{C}(\omega \times \mathbb{R}, \mathbb{R})$ . Moreover, there exists  $q_1$  such that

$$|\mathfrak{g}(\eta, w)| \leq q_1 \mathfrak{M}_1,$$

where  $\eta \in \omega$ ,  $w \in \mathcal{C}([0, p], \mathbb{R})$  and  $\mathfrak{M}_1 = \|w\|_{\mathcal{C}([0, p])}$ .

**Hypothesis 2 (H2).** There exists  $q_2$  such that  $|\mathfrak{h}(\eta, s, w)| \leq q_2 |w(\eta)|$  for all  $\eta \in \omega$  and  $w \in \mathcal{C}([0, p], \mathbb{R})$ .

**Hypothesis 3 (H3).** The operator  $\mathcal{A}$  is bounded and  $\|\mathcal{A}\| < \frac{\Gamma(\zeta + 1)}{\Gamma(\gamma) (\phi(p) - \phi(0))^\gamma}$ .

**Hypothesis 4 (H4).** The function  $\phi(\eta)$  is uniformly continuous for all  $\eta \in \omega$ .

**Lemma 1.** Let the operator  $\mathcal{T} : \mathfrak{C}[0, \mathfrak{p}] \rightarrow \mathfrak{C}[0, \mathfrak{p}]$  given as

$$\begin{aligned}
 (\mathcal{T}w)(\eta) &= \frac{(\phi(\eta) - \phi(0))^{\gamma-1}}{\Gamma(\gamma)} \mathfrak{w}_0 + I_{0^+}^{\zeta; \phi} \mathfrak{g}(\eta, w(\eta)) \\
 &\quad + I_{0^+}^{\zeta; \phi} \left[ \int_0^s \mathfrak{h}(\eta, \tau, w(\tau)) d\tau \right] + I_{0^+}^{\zeta; \phi} (\mathcal{A}(w(\eta)))
 \end{aligned}$$

and assume that the hypotheses (H1)–(H3) are satisfied. Then, the operator  $\mathcal{T}$  maps the closed ball  $\mathfrak{B}_\tau = \{w \in \mathfrak{C}([0, \mathfrak{p}]) : \|w\| \leq \tau\}$  into itself, if

$$\tau \geq \frac{\Gamma(\zeta + \gamma) |\mathfrak{w}_0|}{\Gamma(\zeta + \gamma) - \frac{\Gamma(\gamma) \mathfrak{C}_\phi^\zeta}{\Gamma(\zeta + \gamma)} \left[ \mathfrak{q}_2 \mathfrak{r} \mathfrak{p} + \mathfrak{q}_1 \mathfrak{M}_1 + \frac{\Gamma(\zeta + 1)}{\Gamma(\gamma) \mathfrak{C}_\phi^\gamma} \right]}, \tag{5}$$

where  $\mathfrak{C}_\phi := (\phi(\mathfrak{p}) - \phi(0))$ .

**Proof.** Clearly, we need to prove that if  $w(\eta) \in \mathfrak{B}_\tau$  then  $(\mathcal{T}w)(\eta) \in \mathfrak{B}_\tau$ . For all  $\eta \in [0, \mathfrak{p}]$ , we have

$$\begin{aligned}
 &|(\mathcal{T}w)(\eta)| \\
 &\leq |\mathfrak{w}_0| + \frac{1}{\Gamma(\zeta)} \int_0^\eta \frac{\phi'(s)(\phi(s) - \phi(0))^{\gamma-1}}{(\phi(s) - \phi(0))^{1-\zeta}} \max_{s \in [0, \eta]} |(\phi(s) - \phi(0))^{1-\gamma} \mathfrak{g}(s, w(s))| ds \\
 &\quad + \frac{1}{\Gamma(\zeta)} \int_0^\eta \frac{\phi'(s)(\phi(s) - \phi(0))^{\gamma-1}}{(\phi(\eta) - \phi(s))^{1-\zeta}} \max_{s \in [0, \eta]} (\phi(s) - \phi(0))^{1-\gamma} \left[ \int_0^s |\mathfrak{h}(\eta, \tau, w(\tau))| d\tau \right] ds \\
 &\quad + \frac{\|\mathcal{A}\|}{\Gamma(\zeta)} \int_0^\eta \frac{\phi'(s)(\phi(s) - \phi(0))^{\gamma-1}}{(\phi(\eta) - \phi(s))^{1-\zeta}} \max_{s \in [0, \eta]} |(\phi(s) - \phi(0))^{1-\gamma} w(s)| ds \\
 &\leq |\mathfrak{w}_0| + \frac{\mathfrak{q}_1 \mathfrak{M}_1 \tau}{\Gamma(\zeta)} \int_0^\eta \frac{\phi'(s)(\phi(s) - \phi(0))^{\gamma-1}}{(\phi(\eta) - \phi(s))^{1-\zeta}} ds + \frac{\mathfrak{q}_2 \mathfrak{r} \mathfrak{p}}{\Gamma(\zeta)} \int_0^\eta \frac{\phi'(s)(\phi(s) - \phi(0))^{\gamma-1}}{(\phi(\eta) - \phi(s))^{1-\zeta}} ds \\
 &\quad + \frac{\|\mathcal{A}\| \tau}{\Gamma(\zeta)} \int_0^\eta \frac{\phi'(s)(\phi(s) - \phi(0))^{\gamma-1}}{(\phi(\eta) - \phi(s))^{1-\zeta}} ds \\
 &\leq |\mathfrak{w}_0| + \frac{1}{\Gamma(\zeta)} [\|\mathcal{A}\| \tau + \mathfrak{q}_2 \mathfrak{r} \mathfrak{p} + \mathfrak{q}_1 \mathfrak{M}_1 \tau] \int_0^\eta \frac{\phi'(s)(\phi(s) - \phi(0))^{\gamma-1}}{(\phi(\eta) - \phi(s))^{1-\zeta}} ds \\
 &\leq |\mathfrak{w}_0| + \frac{(\phi(\eta) - \phi(0))^{\zeta + \gamma - 1}}{\Gamma(\zeta)} [\|\mathcal{A}\| \tau + \mathfrak{q}_2 \mathfrak{r} \mathfrak{p} + \mathfrak{q}_1 \mathfrak{M}_1 \tau] B(\zeta, \gamma)
 \end{aligned}$$

where  $B$  is the beta function. From the formula

$$B(\zeta, \gamma) = \frac{\Gamma(\zeta) \Gamma(\gamma)}{\Gamma(\zeta + \gamma)},$$

we have

$$|(\mathcal{T}w)(\eta)| \leq |\mathfrak{w}_0| + \frac{\Gamma(\gamma)(\phi(\mathfrak{p}) - \phi(0))^\zeta}{\Gamma(\zeta + \gamma)} [\|\mathcal{A}\| \tau + \mathfrak{q}_2 \mathfrak{r} \mathfrak{p} + \mathfrak{q}_1 \mathfrak{M}_1 \tau].$$

Applying H3 and condition (5), we have

$$|(\mathcal{T}w)(\eta)| \leq |\mathfrak{w}_0| + \frac{\Gamma(\gamma) \mathfrak{C}_\phi^\zeta}{\Gamma(\zeta + \gamma)} \left[ \mathfrak{q}_2 \mathfrak{r} \mathfrak{p} + \mathfrak{q}_1 \mathfrak{M}_1 \tau + \frac{\Gamma(\zeta + 1) \tau}{\Gamma(\gamma) \mathfrak{C}_\phi^\gamma} \right] \leq \tau.$$

This completes the proof.  $\square$

The following theorem shows the existence of solutions to the fractional differential Equation (1) using Krasnoselskii's FPT listed above.

**Theorem 4.** Assume that hypotheses H1–H4 are satisfied. Then, Equation (1) has a solution.

**Proof.** Define  $\mathfrak{T} : \mathfrak{C}[0, \mathfrak{p}] \rightarrow \mathfrak{C}[0, \mathfrak{p}]$  as

$$(\mathfrak{T}w)(\eta) = (\mathfrak{T}_1w)(\eta) + (\mathfrak{T}_2w)(\eta),$$

where

$$(\mathfrak{T}_1w)(\eta) := \frac{(\phi(\eta) - \phi(0))^{\gamma-1}}{\Gamma(\gamma)} w_0 + I_{0+}^{\zeta; \phi} \mathfrak{g}(\eta, w(\eta), w(\eta)) + I_{0+}^{\zeta; \phi} \left[ \int_0^s \mathfrak{h}(\eta, \tau, w(\tau)) d\tau \right],$$

and

$$(\mathfrak{T}_2w)(\eta) := I_{0+}^{\zeta; \phi} (\mathcal{A}(w(\eta))).$$

From Proposition 1, solving Equation (1) is equivalent to finding a fixed point for the operator  $\mathfrak{T}$  defined on the space  $\mathfrak{C}[0, \mathfrak{p}]$ .

Suppose that  $\tau$  satisfies condition (5) and  $\mathfrak{B}_\tau = \{w \in \mathfrak{C}([0, \mathfrak{p}]) : \|w\| \leq \tau\}$ . Due to Lemma 1, the operator  $\mathfrak{T}$  maps  $\mathfrak{B}_\tau$  into itself. Now, we use Krasnoselskii FPT to show that  $\mathfrak{T}$  has a fixed point.

**Claim 1.** The operator  $\mathfrak{T}_1$  is continuous on  $\mathfrak{B}_\tau$ .

Let  $\{w_n\}$  be a sequence in  $\mathfrak{B}_\tau$  that converges to  $w$ . We need to prove that  $\mathfrak{T}_1w_n \rightarrow \mathfrak{T}_1w$ . For each  $\eta \in [0, \mathfrak{p}]$ , we have

$$\begin{aligned} & |(\mathfrak{T}_1w_n)(\eta) - (\mathfrak{T}_1w)(\eta)| \\ & \leq \frac{1}{\Gamma(\zeta)} \int_0^\eta \frac{\phi'(s)(\phi(s) - \phi(0))^{\gamma-1}}{(\phi(\eta) - \phi(0))^{1-\zeta}} \left| \max_{s \in [0, \eta]} (\phi(s) - \phi(0))^{1-\gamma} \mathfrak{g}(s, w_n(s), (w_n(s)) \right. \\ & \quad \left. - \mathfrak{g}(s, w(s), (w(s))) \right| ds \\ & \quad + \frac{1}{\Gamma(\zeta)} \int_0^\eta \frac{\phi'(s)(\phi(s) - \phi(0))^{\gamma-1}}{(\phi(\eta) - \phi(0))^{1-\zeta}} \max_{s \in [0, \eta]} (\phi(s) - \phi(0))^{1-\gamma} \left[ \int_0^s |\mathfrak{h}(\eta, \tau, w_n(\tau)) \right. \\ & \quad \left. - \mathfrak{h}(\eta, \tau, w(\tau))| d\tau \right] ds. \end{aligned}$$

Since  $\mathfrak{g}$  and  $\mathfrak{h}$  are continuous, and  $w_n \rightarrow w$  as  $n \rightarrow +\infty$  in  $\mathfrak{B}_\tau$ , we can conclude that  $|(\mathfrak{T}_1w_n)(\eta) - (\mathfrak{T}_1w)(\eta)| \rightarrow 0$  as  $n \rightarrow +\infty$  by Lebesgue dominated convergence theorem.



**Claim 2.**  $\mathfrak{T}_1$  is an equicontinuous operator.

To prove our second claim, we let  $\eta_1, \eta_2 \in \omega$  with  $\eta_2 < \eta_1$  and  $w \in \mathfrak{B}_\tau$ ,

$$\begin{aligned} & |(\mathfrak{T}_1 w)(\eta_1) - (\mathfrak{T}_1 w)(\eta_2)| \\ & \leq \frac{(\phi(\eta_1) - \phi(\eta_2))^{1-\gamma} (\phi(s) - \phi(0))^{1-\gamma}}{\Gamma(\gamma)} |w_0| \\ & + \frac{(\phi(\eta_1) - \phi(\eta_2))^{1-\zeta} q_1 \mathfrak{M}_1 \tau}{\Gamma(\zeta)} \int_{\eta_2}^{\eta_1} \phi'(s) (\phi(s) - \phi(0))^{1-\gamma} ds \\ & + \frac{(\phi(\eta_1) - \phi(\eta_2))^{1-\zeta} q_2 \tau \mathfrak{p}}{\Gamma(\zeta)} \int_{\eta_2}^{\eta_1} \phi'(s) (\phi(s) - \phi(0))^{1-\gamma} ds \\ & \leq \frac{(\phi(\eta_1) - \phi(\eta_2))^{1-\gamma} (\phi(s) - \phi(0))^{1-\gamma}}{\Gamma(\gamma)} |w_0| \\ & + \frac{(\phi(\eta_1) - \phi(\eta_2))^{1-\zeta}}{\Gamma(\zeta)} (q_1 \mathfrak{M}_1 \tau + q_2 \tau) \int_0^{\eta_1} \phi'(s) (\phi(s) - \phi(0))^{1-\gamma} ds \\ & \leq \frac{(\phi(\eta_1) - \phi(\eta_2))^{1-\gamma} (\phi(\mathfrak{p}) - \phi(0))^{1-\gamma}}{\Gamma(\gamma)} |w_0| \\ & + \frac{\Gamma(\gamma) (\phi(\eta_1) - \phi(\eta_2))^{1-\zeta+\gamma}}{\Gamma(\zeta+1)} (q_2 \tau + q_1 \mathfrak{M}_1 \tau). \end{aligned}$$

Hence, we have

$$\begin{aligned} & |(\mathfrak{T}_1 w)(\eta_1) - (\mathfrak{T}_1 w)(\eta_2)| \\ & \leq \frac{(\phi(\eta_1) - \phi(\eta_2))^{1-\gamma}}{\Gamma(\gamma)} |w_0| \\ & + \frac{\Gamma(\gamma) (\phi(\eta_1) - \phi(\eta_2))^{1-\zeta+\gamma}}{\Gamma(\zeta+1) (\phi(b) - \phi(0))^{1-\gamma}} (q_2 \tau + q_1 \mathfrak{M}_1 \tau), \end{aligned}$$

regarding Hypothesis 4, the right-hand side of the above inequality tends to zero whenever  $\eta_1 \rightarrow \eta_2$ , so it clearly claims that  $\mathfrak{T}_1$  is equicontinuous. Furthermore, using the previous lemma, it is uniformly bounded. Therefore, by Arzela–Ascoli Theorem,  $\mathfrak{T}_1$  is compact on  $\mathfrak{B}_\tau$ .

**Claim 3.** The operator  $\mathfrak{T}_2$  is a contraction.

Let  $w_1, w_2 \in \mathfrak{C}_{1-\gamma, \zeta}([0, \mathfrak{p}])$ ; then, we have

$$\begin{aligned} & |(\mathfrak{T}_2 w_1)(t) - (\mathfrak{T}_2 w_2)(t)| \\ & \leq \frac{\|A\|}{\Gamma(\zeta)} \int_0^\eta \frac{\phi'(s)}{(\phi(\eta) - \phi(s))^{1-\zeta}} |w_1(s) - w_2(s)| ds \\ & \leq \frac{\|A\| \Gamma(\gamma) (\phi(\mathfrak{p}) - \phi(0))^\gamma}{\Gamma(\zeta+1)} |w_1(\eta) - w_2(\eta)|. \end{aligned}$$

By Hypothesis 3 (H3), we infer that  $\|A\| \Gamma(\gamma) (\phi(\mathfrak{p}) - \phi(0))^\gamma < \Gamma(\zeta+1)$ . Thus,  $\mathfrak{T}_2$  is a contraction mapping. By Theorem 2, the mapping  $\mathcal{T}$  has at least a fixed point, which directly implies that Equation (1) has a solution. This completes the proof.  $\square$

### 3. Stability Analysis

In this section, we present the Kummer stability with respect to  $\Phi(\zeta, \nu; (\phi(\eta) - \phi(0))^\zeta)$  for Equation (1) based on Theorem 3. We begin by assuming the following hypotheses:

(K1)  $g \in \mathcal{C}(\omega \times \mathbb{R}, \mathbb{R})$ . Moreover, there exists  $\mathcal{L}_g > 0$  such that

$$|g(\eta, w_1) - g(\eta, w_2)| \leq \mathcal{L}_g |w_1 - w_2|, \tag{6}$$

for all  $\eta \in [0, p]$ .

(K2)  $h : \omega^2 \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function which satisfies a Lipschitz condition in the third argument, i.e., there exists  $\mathcal{L}_h > 0$  such that

$$|h(\eta, s, w) - h(\eta, s, v)| \leq \mathcal{L}_h |w - v|, \tag{7}$$

for all  $s, \eta \in \omega$  and  $w, v \in \mathbb{R}$ .

**Theorem 5.** Suppose that  $g$  and  $h$  satisfy K1 and K2. Additionally, let

$$\|\mathcal{A}\| < \frac{\Gamma(\zeta + 1) - (2\mathcal{L}_g + p\mathcal{L}_h)\Gamma(\gamma)(\phi(p) - \phi(0))^\zeta}{\Gamma(\gamma)(\phi(p) - \phi(0))^\zeta}. \tag{8}$$

If a continuously differentiable function  $w : \omega \rightarrow \mathbb{R}$  for  $\epsilon \geq 0$  satisfies

$$\begin{aligned} & |{}^{\mathcal{H}}\mathbb{D}_{0^+}^{\zeta, \nu; \phi} w(\eta) - \mathcal{A}(w(\eta)) - g(\eta, w(\eta)) - \int_0^\eta h(\eta, s, w(s)) ds| \\ & \leq \epsilon \Phi(\zeta, \nu; (\phi(\eta) - \phi(0))^\zeta), \end{aligned}$$

for all  $\eta \in \omega$ , then there exists a unique continuous function  $v_0 : \omega \rightarrow \mathbb{R}$  that satisfies Equation (1) and

$$\begin{aligned} & |w(\eta) - v_0(\eta)| \\ & \leq \frac{\Gamma(\zeta + 1)\epsilon}{\Gamma(\zeta + 1) - (2\mathcal{L}_g + p\mathcal{L}_h + \|\mathcal{A}\|)\Gamma(\gamma)(\phi(p) - \phi(0))^\zeta} \Phi(\zeta, \nu; (\phi(\eta) - \phi(0))^\zeta), \end{aligned} \tag{9}$$

for all  $\eta \in \omega$ .

**Proof.** Let  $\mathfrak{Y} := \mathcal{C}_{1-\gamma, \phi}(0, p]$  be endowed with the following generalized metric, defined by

$$\begin{aligned} & d^*(w, v) \\ & = \inf\{\mathcal{C} \geq 0 : |w(\eta) - v(\eta)| \leq \mathcal{C}\epsilon\Phi(\zeta, \nu; (\phi(\eta) - \phi(0))^\zeta), \text{ for all } \eta \in \omega\}, \end{aligned} \tag{10}$$

for all  $w, v \in \mathfrak{Y}$ . It is not difficult to see that  $(\mathfrak{Y}, d^*)$  is a complete generalized metric space [5]. Define the operator  $\mathcal{S} : \mathfrak{Y} \rightarrow \mathfrak{Y}$  by

$$\begin{aligned} & (\mathcal{S}w)(\eta) := \\ & \frac{(\phi(\eta) - \phi(0))^{\gamma-1}}{\Gamma(\gamma)} w_0 + I_{0^+}^{\zeta; \phi}(\mathcal{A}(w(\eta))) + I_{0^+}^{\zeta; \phi}g(\eta, w(\eta)) \\ & + I_{0^+}^{\zeta; \phi} \left[ \int_0^s h(\eta, \tau, w(\tau)) d\tau \right], \end{aligned}$$

for all  $\eta \in \omega$  and  $w \in \mathfrak{Y}$ . For any  $w, v \in \mathfrak{Y}$ , choose a constant  $\mathcal{K}$  so that  $d^*(w, v) \leq \mathcal{K}$ , i.e.,

$$|w(t) - v(t)| \leq \mathcal{K}\epsilon\Phi(\zeta, \nu; (\phi(\eta) - \phi(0))^\zeta) \tag{11}$$

for all  $\eta \in \omega$ . So, using Remark 1, we have

$$\begin{aligned}
 & |(\mathcal{S}\mathfrak{w})(\eta) - (\mathcal{S}\mathfrak{v})(\eta)| \\
 & \leq \frac{1}{\Gamma(\zeta)} \int_0^\eta \frac{\phi'(s)}{(\phi(\eta) - \phi(s))^\zeta} |\mathfrak{g}(\eta, \mathfrak{w}(\eta)) - \mathfrak{g}(\eta, \mathfrak{v}(\eta))| \\
 & + \frac{1}{\Gamma(\zeta)} \int_0^\eta \frac{\phi'(s)}{(\phi(\eta) - \phi(s))^\zeta} \left[ \int_0^s |\mathfrak{h}(\eta, \tau, \mathfrak{w}(\tau)) - \mathfrak{h}(\eta, \tau, \mathfrak{v}(\tau))| d\tau \right] \\
 & + \frac{\|\mathcal{A}\|}{\Gamma(\zeta)} \int_0^\eta \frac{\phi'(s)}{(\phi(\eta) - \phi(s))^\zeta} |\mathfrak{w}(\eta) - \mathfrak{v}(\eta)| \\
 & \leq \frac{(2\mathcal{L}_g + \mathfrak{p}\mathcal{L}_h + \|\mathcal{A}\|)\mathcal{K}\epsilon}{\Gamma(\zeta)} \int_0^\eta \frac{\phi'(s)}{(\phi(\eta) - \phi(s))^\zeta} \Phi(\zeta, \nu; (\phi(\eta) - \phi(0))^\zeta) \\
 & \leq \frac{(2\mathcal{L}_g + \mathfrak{p}\mathcal{L}_h + \|\mathcal{A}\|)\Gamma(\gamma)(\phi(\mathfrak{p}) - \phi(0))^\zeta}{\Gamma(\zeta + 1)} \mathcal{K}\epsilon \Phi(\zeta, \nu; (\phi(\eta) - \phi(0))^\zeta),
 \end{aligned}$$

which indicates that

$$d^*((\mathcal{S}\mathfrak{w}), (\mathcal{S}\mathfrak{v})) \leq \frac{(2\mathcal{L}_g + \mathfrak{p}\mathcal{L}_h + \|\mathcal{A}\|)\Gamma(\gamma)(\phi(\mathfrak{p}) - \phi(0))^\zeta}{\Gamma(\zeta + 1)} d^*(\mathfrak{w}, \mathfrak{v}),$$

for all  $\mathfrak{w}, \mathfrak{v} \in (\mathfrak{Y}, d^*)$ . From (8), we have  $(2\mathcal{L}_g + \mathfrak{p}\mathcal{L}_h + \|\mathcal{A}\|)\Gamma(\gamma)(\phi(\mathfrak{p}) - \phi(0))^\zeta < \Gamma(\zeta + 1)$ . Hence, the operator  $\mathcal{S}$  is a strict contraction. Moreover, for element  $\mathfrak{v}_0 \in (\mathfrak{Y}, d^*)$ , we have

$$\begin{aligned}
 & |(\mathcal{S}\mathfrak{v}_0)(\eta) - \mathfrak{v}_0(\eta)| \\
 & \leq |\mathfrak{v}_0(\eta) - \frac{(\phi(\eta) - \phi(0))^{\gamma-1}}{\Gamma(\gamma)} \mathfrak{v}_0| \\
 & - I_{0+}^{\zeta;\phi} \mathfrak{g}(\eta, \mathfrak{v}_0(\eta)) - I_{0+}^{\zeta;\phi} \left[ \int_0^s \mathfrak{h}(\eta, \tau, \mathfrak{v}_0(\tau)) d\tau \right] - I_{0+}^{\zeta;\phi} (\mathcal{A}(\mathfrak{v}_0(\eta)))| \\
 & \leq \epsilon \Phi(\alpha, \beta; (\phi(\eta) - \phi(0))^\alpha),
 \end{aligned}$$

for all  $\eta \in \omega$ . In summary,  $d^*(\mathcal{S}\mathfrak{v}_0, \mathfrak{v}_0) \leq 1$  and  $d^*(\mathcal{S}^n \mathfrak{v}_0, \mathcal{S}^{n+1} \mathfrak{v}_0) < +\infty$  for all  $n \in \mathbb{N}$ . According to Theorem 3, there exists a unique continuous function  $\mathfrak{w} : \omega \rightarrow \mathbb{R}$  such that  $\mathcal{S}\mathfrak{w} = \mathfrak{w}$ ,  $\mathfrak{w}$  satisfies Equation (1) for all  $\eta \in \omega$  and

$$\begin{aligned}
 & \mathfrak{w}(\eta) \\
 & = \frac{(\phi(\eta) - \phi(0))^{\gamma-1}}{\Gamma(\gamma)} \mathfrak{w}_0 \\
 & + I_{0+}^{\zeta;\phi} \mathfrak{g}(\eta, \mathfrak{v}_0(\eta)) + I_{0+}^{\zeta;\phi} \left[ \int_0^s \mathfrak{h}(\eta, \tau, \mathfrak{v}_0(\tau)) d\tau \right] + I_{0+}^{\zeta;\phi} (\mathcal{A}(\mathfrak{v}_0(\eta))),
 \end{aligned}$$

for every  $\eta \in \omega$ . In addition, it follows from the above calculations that

$$d^*(\mathfrak{w}, \mathfrak{v}_0) \leq \frac{\Gamma(\zeta + 1)}{\Gamma(\zeta + 1) - (2\mathcal{L}_g + \mathfrak{p}\mathcal{L}_h + \|\mathcal{A}\|)\Gamma(\gamma)(\phi(\mathfrak{p}) - \phi(0))^\zeta},$$

which justifies inequality (9).  $\square$

### 3.1. The System of FDEs with Initial Conditions

Based on the results obtained in the previous section, we consider the following system of FDEs with initial conditions:

$$\begin{cases} \mathcal{H}\mathbb{D}_{0+}^{\zeta,\nu;\phi} w_i(\eta) = \mathfrak{g}_i(w_1(\eta), \dots, w_n(\eta)) + \int_0^\eta \mathfrak{h}_i(s, w_1(s), \dots, w_n(s)) ds, & \eta \in \omega = (0, \mathfrak{p}] \\ I_0^{1-\gamma;\phi} w_i(0) = \mathfrak{w}_i(0), & \mathfrak{w}_0 \in \mathbb{R}, \end{cases} \quad (12)$$

where  $i = 1, \dots, n$ ,  $\mathcal{H}\mathbb{D}_{0+}^{\zeta,\nu;\phi}(\cdot)$  is a  $\phi$ -Hilfer fractional derivative of order  $0 < \zeta \leq 1$  and type  $0 \leq \nu < 1$ , and  $I_0^{1-\gamma;\phi}$  is  $\phi$ -Riemann-Liouville fractional integral of order  $1 - \gamma$  ( $\gamma = \zeta + \nu(1 - \zeta)$ ) with respect to the mapping  $\phi$ . Furthermore,  $\mathfrak{g}_i : \omega \times \mathbb{R} \rightarrow \mathbb{R}$  and  $\mathfrak{h}_i : \omega^2 \times \mathbb{R} \rightarrow \mathbb{R}$  are given mappings.

We can rewrite the above equation as follows

$$\begin{cases} \mathcal{H}\mathbb{D}_{0+}^{\zeta,\nu;\phi} W(\eta) = \mathfrak{G}(W(\eta)) + \int_0^\eta \mathfrak{H}(W(s)) ds, & \eta \in \omega = (0, \mathfrak{p}] \\ I_0^{1-\gamma;\phi} W(0) = \mathfrak{W}_0, \end{cases} \quad (13)$$

where

$$W(\eta) = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}, \quad \mathfrak{W}_0 = \begin{bmatrix} \mathfrak{w}_1(0) \\ \mathfrak{w}_2(0) \\ \vdots \\ \mathfrak{w}_n(0) \end{bmatrix}$$

and

$$\mathfrak{G}(W(\eta)) + \int_0^\eta \mathfrak{H}(W(s)) ds = \begin{bmatrix} \mathfrak{g}_1(\eta, w_1(\eta), \dots, w_n(\eta)) + \int_0^\eta \mathfrak{h}_1(\eta, s, w_1(s), \dots, w_n(s)) ds \\ \mathfrak{g}_2(\eta, w_1(\eta), \dots, w_n(\eta)) + \int_0^\eta \mathfrak{h}_2(\eta, s, w_1(s), \dots, w_n(s)) ds \\ \vdots \\ \mathfrak{g}_n(\eta, w_1(\eta), \dots, w_n(\eta)) + \int_0^\eta \mathfrak{h}_n(\eta, s, w_1(s), \dots, w_n(s)) ds \end{bmatrix}.$$

From Theorem 1, we obtain the equivalent matrix equation

$$W(\eta) = \mathfrak{W}_0 + I_0^{\zeta,\phi} \left( \mathfrak{G}(W(\eta)) + \int_0^\eta \mathfrak{H}(W(s)) ds \right). \quad (14)$$

Changing Equation (13) as

$$\begin{cases} \mathcal{H}\mathbb{D}_{0+}^{\zeta,\nu;\phi} w(\eta) = \mathfrak{g}(\eta, w(\eta)) + \int_0^\eta \mathfrak{h}(w(s)) ds, & \eta \in \omega = (0, \mathfrak{p}] \\ I_0^{1-\gamma;\phi} w(0) = \mathfrak{w}_0. \end{cases} \quad (15)$$

A continuous function  $w$  is said to be a solution of Equation (15) if it satisfies

$$w(\eta) = \mathfrak{w}_0 + I^{\zeta,\phi} \left( \mathfrak{g}(\eta, w(\eta)) + \int_0^\eta \mathfrak{h}(s, w(s)) ds \right). \quad (16)$$

Using Theorem 5, under conditions (K1), (K2) and (8), Equation (15) has a unique solution.

#### 4. Application of $\phi$ -Hilfer Fractional Derivative

In this section, we propose the proof of the existence of the solution of the Lotka–Volterra model by considering it being ruled by a  $\phi$ -Hilfer fractional derivative of the model, as an application.

First, we state the Lotka–Volterra model, which was introduced by Lotka and Volterra [15] independently. This model is known as the predator–prey equations or the Lotka–Volterra equations, and it is given by

$$\begin{cases} \dot{X} = \alpha(X - \frac{X}{N_1}) - \beta XY, & \text{on } [n, m], \\ \dot{Y} = \delta XY - \sigma(Y - \frac{Y}{N_2}), & \text{on } [n, m], \\ X(n) = X_n \\ Y(n) = Y_n \end{cases} \tag{17}$$

where  $X$  and  $Y$  are population size or the population density of different species;  $X_n, Y_n$  are the initial conditions;  $\alpha, \beta, \delta,$  and  $\sigma$  represent different growth or decay rates; and  $N_1, N_2$  are the carrying capacities. The above system shows an interaction between the logistic growth and decay of two different species.

Based on the definitions in the previous sections, we can restate the model (17) in the sense of the  $\phi$ -Hilfer fractional derivative. Taking  $\alpha = \zeta, \beta = \nu$  and  $\sigma = \gamma$ , where  $\gamma = \zeta + \nu(1 - \zeta)$ , we will apply the model

$$\begin{cases} {}^{\mathcal{H}}\mathbb{D}_{0^+}^{\zeta, \nu; \phi} X(\eta) = \zeta(X - \frac{X}{N_1}) - \nu XY, & \text{on } [n, m] \\ {}^{\mathcal{H}}\mathbb{D}_{0^+}^{\zeta, \nu; \phi} Y(\eta) = \delta XY - \gamma(Y - \frac{Y}{N_2}), & \text{on } [n, m] \\ I_0^{1-\gamma; \phi} X(0) = \mathfrak{w}_X \\ I_0^{1-\gamma; \phi} Y(0) = \mathfrak{w}_Y \end{cases} \tag{18}$$

where  $\mathfrak{w}_X, \mathfrak{w}_Y \in \mathbb{R}^+$  to analyze the existence, uniqueness, and stability of solutions.

**Proposition 2.** Equation (18) has a unique solution  $(X, Y)$  on the ball with radius  $r$ , and

$$L < \frac{1}{2} \left( \frac{\Gamma(\zeta + 1)}{\Gamma(\gamma)(\phi(p) - \phi(0))^\zeta} - \|\mathcal{A}\| - p\mathcal{L}_h \right) \tag{19}$$

where  $L = \nu r^k + \delta r^k + \max \{ \zeta(1 - \frac{1}{N_1}), \gamma(1 - \frac{1}{N_2}) \}$  for some  $0 < k \leq \zeta$ . The solutions are of the following form:

$$X(\eta) = \frac{(\phi(\eta) - \phi(0))^{\gamma-1}}{\Gamma(\gamma)} \mathfrak{w}_X + I_{0^+}^{\zeta; \phi} [g(\eta, X(\eta), X(\eta))] + \int_0^\eta h(\eta, \tau, u(\tau)) d\tau + \mathcal{A}(X(\eta))$$

and

$$Y(\eta) = \frac{(\phi(\eta) - \phi(0))^{\gamma-1}}{\Gamma(\gamma)} \mathfrak{w}_Y + I_{0^+}^{\zeta; \phi} [g(\eta, Y(\eta), Y(\eta))] + \int_0^\eta h(\eta, \tau, u(\tau)) d\tau + \mathcal{A}(Y(\eta))$$

where  $\gamma \geq 0$  and we obtain from  $\gamma = \zeta + \nu(1 - \zeta)$  for  $0 < \zeta \leq 1$  and  $0 \leq \nu < 1$  in (1).

**Proof.** Assume that  $\mathfrak{F}_i : \Omega \rightarrow \mathbb{R}$  is a continuous function on close set  $\Omega \subseteq \mathbb{R}_n^+$ , where  $\mathfrak{F}_i = \mathfrak{g}_i(w_1(\eta), \dots, w_n(\eta)) + \int_0^\eta \mathfrak{h}_i(s, w_1(s), \dots, w_n(s)) ds$  and  $i = 1, \dots, n$ . Let  $X_1 = (x_1, y_1)$  and  $X_2 = (x_2, y_2)$ , then

$$J_1 + J_2 := \left| \zeta \left( x_1 - \frac{x_1}{N_1} \right) - \nu x_1 y_1 - \zeta \left( x_2 - \frac{x_2}{N_1} \right) + \nu x_2 y_2 \right| + \left| \delta x_1 y_1 - \gamma \left( y_1 - \frac{y_1}{N_2} \right) - \delta x_2 y_2 - \gamma \left( y_2 - \frac{y_2}{N_2} \right) \right|,$$

but we have

$$J_1 = \left| \zeta \left( x_1 - \frac{x_1}{N_1} \right) - \nu x_1 y_1 - \zeta \left( x_2 - \frac{x_2}{N_1} \right) + \nu x_2 y_2 \right| \leq \zeta |x_1 - x_2| \left( 1 - \frac{1}{N_1} \right) + \nu |x_1| |y_1 - y_2| + \nu |y_2| |x_1 - x_2|$$

and

$$J_2 = \left| \delta x_1 y_1 - \gamma \left( y_1 - \frac{y_1}{N_2} \right) - \delta x_2 y_2 - \gamma \left( y_2 - \frac{y_2}{N_2} \right) \right| \leq \delta |x_1| |y_1 - y_2| + \delta |y_2| |x_1 - x_2| + \gamma |y_1 - y_2| \left( 1 - \frac{1}{N_2} \right).$$

Now, assume that  $\Omega = B_r$ , so for some  $0 < k \leq \zeta$ , we have

$$J_1 + J_2 \leq \left( \zeta \left( 1 - \frac{x_1}{N_1} \right) + \nu r^k + \delta r^k \right) |x_1 - x_2| + \left( \nu r^k + \delta r^k + \gamma \left( 1 - \frac{1}{N_2} \right) \right) |y_1 - y_2| \leq L (|x_1 - x_2| + |y_1 - y_2|)$$

where  $L = \nu r^k + \delta r^k + \max \left\{ \zeta \left( 1 - \frac{1}{N_1} \right), \gamma \left( 1 - \frac{1}{N_2} \right) \right\}$ , and using the discussion in Section 3.1 and Theorem 5, our proof is complete.  $\square$

**Example 1.** Let  $K : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$  be a continuous function and  $w(\eta)$  be a continuous function on  $[0, 1]$  so that  $|K(\eta, \lambda)w(\eta)| < \frac{\Gamma(1/3)}{3\Gamma(2/9)}(e-1)^{-\frac{2}{9}}$ . Consider the following fractional system

$$\begin{cases} \mathcal{H}\mathbb{D}_{0^+}^{\frac{1}{3}, \frac{2}{3}; e^\eta} w(\eta) = \int_0^1 K(\eta, \lambda)w(\eta)d\eta + \frac{1}{5} \sin(w(\eta)) + \int_0^1 \sin\left(\frac{3}{5}\eta w(s)\right)ds, & \eta \in (0, 1) \\ I_0^{\frac{2}{9}, \phi} w(0) = w_0, & w_0 \in \mathbb{R} \end{cases} \quad (20)$$

For all  $\eta \in [0, 1]$  and continuous real-valued functions  $w$  on  $[0, 1]$ , we have  $|\frac{1}{5} \sin(w(\eta))| \leq \frac{1}{5}|w(\eta)|$ . Moreover,

$$\left| \int_0^1 \sin\left(\frac{3}{5}\eta w(s)\right)ds \right| \leq \frac{3}{5}|w(\eta)|,$$

for all  $\eta \in [0, 1]$ . Furthermore, by assumption, the operator  $3 \int_0^1 K(\eta, \lambda)w(\eta)d\eta$  is bounded and we have  $|\int_0^1 K(\eta, \lambda)w(\eta)d\eta| \leq \frac{\Gamma(4/3)}{\Gamma(2/9)(e-1)^{2/9}}$ , for all  $\eta, \lambda \in [0, 1]$  and continuous functions  $w$ . So H1–H3 are satisfied for  $q_1 = 1/5$  and  $q_2 = 3/5$ . Therefore, Theorem 4 proved that Equation (20) has at least one solution.

## 5. Conclusions

In this paper, we considered a class of fractional differential equations including a closed linear operator. Next, we used the Krasnoselskii fixed-point theorem to investigate the existing result under some mild conditions. Moreover, we introduced and then proved the Kummer stability of  $\phi$ -Hilfer fractional differential equations on the compact domains.

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