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Solvability of a New q-Differential Equation Related to q-Differential Inequality of a Special Type of Analytic Functions

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Abstract: The current study acts on the notion of quantum calculus together with a symmetric differential operator joining a special class of meromorphic multivalent functions in the puncher unit disk. We formulate a quantum symmetric differential operator and employ it to investigate the geometric properties of a class of meromorphic multivalent functions. We illustrate a set of differential inequalities based on the theory of subordination and superordination. In this real case study, we found the analytic solutions of q-differential equations. We indicate that the solutions are given in terms of confluent hypergeometric function of the second type and Laguerre polynomial.

Keywords: fractional differential operator; symmetric operator; analytic function; subordination and superordination; univalent function; open unit disk; fractional calculus

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1. Introduction

The quantum calculus (QC) (or Jackson calculus) [1] is the richest area of research within the theory of classical mathematical analysis. It centers on a hypothetically suitable detail of the operations of differentiation and integration. It is a complete frame for study in mathematics, which has its past origins, as well as a transformed scope in the present times. It is vital to indicate that the long past of the QC eras back to the effort of Bernoulli and Euler. Nevertheless, definitely, it has strained the attention of contemporary mathematicians in the last numerous periods, which is due chiefly to its widespread fields of application. It includes multifaceted controls and computations, which make it problematic as associated with the rest of the topics in mathematics. Newly, there is a quick development in the area of the QC and its application has appeared in discrete and continuous in mathematics and physics. In the field of geometric functions, theory [2], it brought a natural extension and vision of differential and integral operators (see [3–6]).

In many investigations and research papers, the investigators and researchers faced many q-differential equations, inclusions, and inequalities, which are unfulfilled by a lack of accepting natural forms for such equations. One has operators of the category q-KP or q-KdV for instance, but even there, communicating the resulting equations looked interestingly challenging. Moreover, Laplace, heat wave, and Schrodinger operators have been formulated in numerous forms and their symmetries studied (see e.g., [7]). In addition, many operators connected with q-special functions have been sequestered and studied (see e.g., [8,9]). Nevertheless, when we investigated the development of nonlinear differential equations from zero curvature conditions on a quantum plane, for instance, we were confused about their significance, their solvability, and their relative to q-KP for sample. Thus, it appears appropriate to partially study in the area of q-differential operators and separate the more important classes while observing also for procedures of solvability [10,11].

The practical applications of fractional calculus and corresponding quantum differential operators are suggested in many sciences. Miller [12] utilized the quantum theory as a practical technique to design the devices. Cao et al. [13] utilized quantum theory as practical challenges in simulating quantum systems on classical computers. Douglas [14] presented an advanced investigation describing the quantum mechanical density matrix corresponding to a delta function, which is a model of the problem of a surface interacting polymer. Some practical applications of q-DEs to nonlocal elasticity, anomalous wave propagation, modeling of defects in solids, and even bio-engineering can be located in [15–20].

During this investigation, and by using the concept of a quantum calculus, we formulated a new symmetric differential operator (q-SDO) connected with analytic functions of meromorphic multivalent property of a complex variable. Accordingly, we propose a new formula of analytic functions utilizing the suggested q-SDO. Furthermore, we study the real situation of the considered functional containing the q-SDO, which is indicated a q-differential equation. We show that this operator is a solution of the Sturm–Liouville equation. A set of examples is given with details.

2. Methods

Our major concepts are defined in this section, as follows:

2.1. Quantum Calculus

For a non-negative integer $\aleph$, the q-integer number $\aleph([\aleph]_q)$ is organized by

$$[\aleph]_q = \frac{1-q^{\aleph}}{1-q}, \quad 0 < q < 1,$$

wherever $[0]_q = 0$, $[1]_q = 1$ besides $\lim_{q\to 1^-}[\aleph]_q = \aleph$. Accordingly, the q-derivative of any analytic function ψ in the open unit disk is given by the following arrangement

$$\mathbb{Q}_q\psi(z) = \frac{\psi(qz)-\psi(z)}{z(q-1)}, \quad z \in U := \{z \in \mathbb{C} : |z| < 1\}.$$

Obviously, a computation implies that

$$\mathbb{Q}_q\left(z^{\aleph}\right) = \left(\frac{1-q^{\aleph}}{1-q}\right)z^{\aleph-1} = [\aleph]_q z^{\aleph-1}.$$

The q-derivative is corresponding to the integral formula

$$\int \psi(z)\, d_q z = (1-q)\sum_{n=0}^{\infty} z\, q^n \psi(z\, q^n),$$

which is known as the Jackson integral of $\psi(z)$, where $d_q\,(\psi(z)) = \psi(q\,z) - \psi(z)$.

Proceeding, for a complex number κ, the q-shifted factorials are formulated by the formal [1]

$$(\kappa;q)_m = \prod_{\iota=0}^{m-1}(1-q^{\iota}\kappa), \quad m \in \mathbb{N},\ (\kappa;q)_0 = 1. \tag{1}$$

According to (1) and in terms of the gamma function, we obtain the q-shifted formula

$$(q^{\kappa};q)_m = \frac{\Gamma_q(\kappa+m)(1-q)^m}{\Gamma_q(\kappa)}, \quad \Gamma_q(\kappa) = \frac{(q;q)_\infty(1-q)^{1-\kappa}}{(q^{\kappa};q)_\infty} \tag{2}$$

where

$$\Gamma_q(\kappa+1)=\frac{\Gamma_q(\kappa)(1-q^{\kappa})}{1-q},\quad q\in(0,1).$$

and

$$(\kappa;q)_\infty=\prod_{\imath=0}^{\infty}(1-q^{\imath}\kappa). \tag{3}$$

In mathematical physics, there are special functions recognized to state q-analogs, that is deformations connecting a parameter q (see [21]). The q-hypergeometric series is formulated as

$${}_j\mathbb{F}_k\begin{bmatrix}\alpha_1 & \alpha_2 & \dots & \alpha_j \\ \beta_1 & \beta_2 & \dots & \beta_k\end{bmatrix};q,z\Big]=\sum_{n=0}^{\infty}\frac{(\alpha_1,\alpha_2,\dots,\alpha_j;q)_n}{(\beta_1,\beta_2,\dots,\beta_k,q;q)_n}\Big((-1)^n q^{\binom{n}{2}}\Big)^{1+k-j}z^n$$

where

$$(\alpha_1,\alpha_2,\dots,\alpha_m;q)_n=(\alpha_1;q)_n(\alpha_2;q)_n\dots(\alpha_m;q)_n$$

and

$$(\alpha;q)_n=\prod_{k=0}^{n-1}(1-\alpha\, q^k)=(1-\alpha)(1-\alpha\, q)(1-\alpha\, q^2)\cdots(1-\alpha\, q^{n-1})$$

is the shifted Formula (1). The most significant special formula is suggested by assuming $j=k+1$, when it formulates

$${}_{k+1}\mathbb{F}_k\left[\begin{matrix}\alpha_1 & \alpha_2 & \dots & \alpha_k & \alpha_{k+1} \\ \beta_1 & \beta_2 & \dots & \beta_k & \end{matrix};q,z\right]=\sum_{n=0}^{\infty}\frac{(\alpha_1,\alpha_2,\dots,\alpha_{k+1};q)_n}{(\beta_1,\beta_2,\dots,\beta_k,q;q)_n}z^n.$$

The basic hypergeometric power series is a q-analog of the hypergeometric power series because

$$\lim_{q\to1}\ {}_j\mathbb{F}_k\left[\begin{matrix}q^{a_1} & q^{a_2} & \dots & q^{a_j} \\ q^{b_1} & q^{b_2} & \dots & q^{b_k}\end{matrix};q,(q-1)^{1+k-j}z\right]={}_j\mathbb{F}_k\left[\begin{matrix}a_1 & a_2 & \dots & a_j \\ b_1 & b_2 & \dots & b_k\end{matrix};z\right].$$

This calculus has been suggested to develop many classes of analytic functions. Govindaraj and Sivasubramanian [22] presented a class of analytic functions connecting with the domains bounded bconic sections. Yalcin et al. [23] studied a special class of analytic functions involving the Salagean Type q-differential operator. Hussain et al. [24] introduced an investigation in a class of multivalent univalent functions. Qadeem and Mamon [25] investigated the p-valent Salagean differential operator. Ibrahim and Darus [26] formulated a new q-differential-difference operator.

2.2. *Meromorphiclly Multivalent Functions (MMF)*

In this effort, we deal with the class of MMF denoting by $\Psi_k(\wp), k, n-\wp\in\mathbb{N}$ and structuring the power series

$$\Psi_k(\wp):=\{\psi:\psi(z)=\frac{1}{z^{\wp}}+\sum_{n=k}^{\infty}\psi_n z^{n-\wp}\}. \tag{4}$$

$$\Big(z\in\bar{U}:=\{z\in\mathbb{C}:0<|z|<1\},\ k,n-\wp\in\mathbb{N}\Big)$$

Note that $\psi(z)-z^{-\wp}$ is a holomorphic function in the open unit disk, $U:=\{z\in\mathbb{C}:|z|<1\}$ (see Komatu [27], Rogosinski [28] or Hayman [29]). Our aim is to study a subclass of $\Psi_k(\wp)$, which is expressed by a differential subordination inequality. Moreover, we examine its geometric possessions in virtue of the convolution (or Hadamard) product [30].

Definition 1. *Two functions* $u(z) = \frac{1}{z^{\wp}} + \sum_{n=k}^{\infty} u_n z^{n-\wp}$ *and* $v(z) = \frac{1}{z^{\wp}} + \sum_{n=k}^{\infty} v_n z^{n-\wp}$ *in* $\Psi_k(\wp)$ *are convoluted if they satisfy the product* $(u * v)(z) = u(z) * v(z) = \frac{1}{z^{\wp}} + \sum_{n=k}^{\infty} u_n\, v_n z^{n-\wp}$.

2.3. Q-Symmetric Differential Operator (q-SDO)

Acting the definition of QC on the class MMF $\Psi_k(\wp)$, we have the following structure:

$$\mathbb{Q}_q\psi(z) = -[\wp]_q\left(z^{-1-\wp}\right) + \sum_{n=k}^{\infty}[n-\wp]_q\psi_n z^{n-\wp-1}.$$

Definition 2. *For functions* $\psi \in \Psi_k(\wp)$, *we formulate the quantum symmetric differential operator, as follows:*

$$\begin{aligned}
\Delta_q^0\psi(z) &= \psi(z) = \frac{1}{z^{\wp}} + \sum_{n=k}^{\infty}\psi_n z^{n-\wp} \\
\Delta_q^{\alpha}\psi(z) &= \left(\frac{\alpha}{-[\wp]_q}\right)(z(\mathbb{Q}_q\psi)(z)) + \left(\frac{(1-\alpha)(-1)^{\wp+1}}{-[\wp]_q}\right)(z(\mathbb{Q}_q\psi)(-z)) \\
&= \left(\frac{\alpha}{-[\wp]_q}\right)\left(-[\wp]_q(z^{-\wp}) + \sum_{n=k}^{\infty}[n-\wp]_q\psi_n z^{n-\wp}\right) \\
&+ \left(\frac{(1-\alpha)(-1)^{\wp+1}}{-[\wp]_q}\right)\left((-[\wp]_q)(-1)^{-\wp-1}z^{-\wp} + \sum_{n=k}^{\infty}[n-\wp]_q\psi_n(-1)^{n-\wp-1}z^{n-\wp}\right) \\
&= \frac{1}{z^{\wp}} + \sum_{n=k}^{\infty}[n-\wp]_q\left(\frac{\alpha+(1-\alpha)(-1)^n}{-[\wp]_q}\right)\psi_n z^{n-\wp} \\
\Delta_q^{2\alpha}\psi(z) &= \Delta_q^{\alpha}\varphi(z)\left(\Delta_q^{\alpha}\psi(z)\right) = \frac{1}{z^{\wp}} + \sum_{n=k}^{\infty}[n-\wp]_q^2\left(\frac{\alpha+(1-\alpha)(-1)^n}{-[\wp]_q}\right)^2\psi_n z^{n-\wp} \\
&\vdots \\
\Delta_q^{m\,\alpha}\psi(z) &= \Delta_q^{\alpha}\psi(z)\left(\Delta_q^{(m-1)\alpha}\psi(z)\right) = \frac{1}{z^{\wp}} + \sum_{n=k}^{\infty}[n-\wp]_q^m\left(\frac{\alpha+(1-\alpha)(-1)^n}{-[\wp]_q}\right)^m\psi_n z^{n-\wp}
\end{aligned} \tag{5}$$

where $0 < q < 1$, $\alpha \in [0,1]$, $\wp \in \mathbb{N}$, $m \in \mathbb{N}$ *and* $z \in U$.

Note that when $q \to 1$, we obtain the original symmetric operator [31]. Obviously, the q-SDO $\Delta_q^{m\,\alpha}\psi(z) \in \Psi_k(\wp)$; also, for two functions φ and $\psi \in \Psi_k(\wp)$, we obtain

$$\begin{aligned}
&\Delta_q^{\alpha}[A_1\,\varphi(z) + A_2\,\psi(z)] \\
&= \left(\frac{\alpha}{-[\wp]_q}\right)(z\mathbb{Q}_q[A_1\,\varphi(z) + A_2\,\psi(z)]) + \left(\frac{(1-\alpha)(-1)^{\wp+1}}{-[\wp]_q}\right)(z\mathbb{Q}_q[A_1\,\varphi(-z) + A_2\,\psi(-z)]) \\
&= A_1\left(\left(\frac{\alpha}{-[\wp]_q}\right)(z\mathbb{Q}_q\varphi(z)) + \left(\frac{(1-\alpha)(-1)^{\wp+1}}{-[\wp]_q}\right)(z\mathbb{Q}_q\varphi(-z))\right) \\
&+ A_2\left(\left(\frac{\alpha}{-[\wp]_q}\right)(z\mathbb{Q}_q\psi(z)) + \left(\frac{(1-\alpha)(-1)^{\wp+1}}{-[\wp]_q}\right)(z\mathbb{Q}_q\psi(-z))\right) \\
&= A_1\Delta_q^{\alpha}\varphi(z) + A_2\Delta_q^{\alpha}\psi(z);\; A_1,\; A_2 \in \mathbb{R}.
\end{aligned}$$

Generally, we can prove the following proposition.

Proposition 3. *Let* φ *and* ψ *in* $\Psi_k(\wp)$. *Then*

$$\Delta_q^{m\,v}[A_1\,\varphi(z) + A_2\,\psi(z)] = A_1\Delta_q^{m\,\alpha}\varphi(z) + A_2\Delta_q^{m\,\alpha}\psi(z).$$

Two analytic functions η_1 and η_2 are subordinated denoting by $\eta_1 \prec \eta_2$, if there is an analytic function κ satisfying $\kappa(0) = 0$, $|\kappa(z)| \leq |z| < 1$ and $\eta_1(z) = \eta_2(\kappa(z)), z$ in U (see [32]).

Definition 4. *Let $-1 \leq \nu < \mu \leq 1$ and $\tau < 0$. A function $\psi \in \Psi_k(\wp)$ is selected to be in the class $\Psi^{\alpha}_{k,q}(\mu,\nu,\tau,\wp)$, when it fulfilled the first order differential subordination inequality*

$$(1-\tau)z^{\wp}\,[\Delta_q^{m\,\alpha}\psi(z)] - \left(\frac{\tau}{\wp}\right) z^{1+\wp}[\Delta_q^{m\,\alpha}\psi(z)]' \prec \frac{1+\mu\,z}{1+\nu\,z} := J_{\mu,\nu}(z). \tag{6}$$

The functional

$$J_{\mu,\nu}(\ell(z)) := \frac{1+\mu\ell(z)}{1+\nu\ell(z)}$$

and its special case of the form

$$J_{\mu,\nu}(z) = \frac{1+\mu z}{1+\nu z}$$

are important because $J_{\mu,\nu}(\ell(z))$ is the class of Caratheodory analytic functions of order $\frac{1-\mu}{1-\nu}$, that is, $\Re\,\{J_{\mu,\nu}(\ell(z))\} > \frac{1-\mu}{1-\nu}$ (see Janowski [33] or Jahangiri et al. [34]). The classes of q-Janowski starlike and q-Janowski convex functions and other formulas are investigated by many researchers Ahuja et al. [35], Ibrahim et al. [36], Srivastava et al. [37], Srivastava and Deeb [38], and Srivastava [39].

2.4. Lemmas

We request the following preliminaries, which can be located in [32].

Lemma 5 ([32])**.** *Let $g_1(z)$ analytic in U and $g_2(z)$ convex univalent in U with $g_1(0) = g_2(0)$. If*

$$g_1(z) + \frac{1}{c}(zg_1'(z)) \prec g_2(z)$$

for a non-zero complex constant number c with $\Re(c) \geq 0$, then

$$g_1(z) \prec g_2(z).$$

Lemma 6 ([32] (Theorem 3.1c. P73))**.**

Assume the class of holomorphic functions

$$\mathbb{H}[\varrho, n] = \{g : g(z) = \varrho + \varrho_n z^n + \varrho_{n+1}z^{n+1} + \ldots\},$$

where $\varrho \in \mathbb{C}$ and positive integer n.
Let $\complement > 0$ and $\epsilon = \epsilon(\complement, n)$ be the solution of the equation

$$\epsilon_2 = \frac{3\pi/2 - \tan^{-1}(n\complement\epsilon_2)}{\pi}.$$

Moreover, let

$$\epsilon_1 = \epsilon_1(\epsilon_2, \complement, n) = \epsilon_2 + (2/\pi)\tan^{-1}(n\complement\epsilon_2), \quad 0 < \epsilon_2 \leq \epsilon.$$

If $g \in \mathbb{H}[1,n]$, then

$$g(z) + \complement z g'(z) \prec \left[\frac{z+1}{1-z}\right]^{\epsilon_1} \quad \Rightarrow \quad g(z) \prec \left[\frac{z+1}{1-z}\right]^{\epsilon_2}.$$

Lemma 7 (see [40]). *Let $\hbar, p \in \mathbb{H}[\varrho, n]$, where p is convex univalent in U and for $\Bbbk_1, \Bbbk_2 \in \mathbb{C}, \Bbbk_2 \neq 0$, then*

$$\Bbbk_1 \hbar(z) + \Bbbk_2 z \hbar'(z) \prec \Bbbk_1 p(z) + \Bbbk_2 z\, p'(z) \to \hbar(z) \prec p(z).$$

Lemma 8 (see [41]). *Let $g, p \in \mathbb{H}[\varrho, n]$, where p is convex univalent in U such that $g(z) + \Bbbk\, z g'(z)$ is univalent then*

$$p(z) + \Bbbk z\, p'(z) \prec g(z) + \Bbbk\, z g'(z) \to p(z) \prec g(z).$$

3. Results

Our main results are stated in this section concerning the class $\Psi^{\alpha}_{k,q}(\mu, \nu, \tau, \wp)$. This section is devoted into two subsections including q-differential inequalities, which deals with the complex studies and q-differential equations, which investigates real cases.

3.1. q-Differential Inequalities

Inclusion property is indicated in the next result:

Theorem 9. *Let $\psi \in \Psi_k(\wp)$. If $\tau_2 < \tau_1 < 0$ then*

$$\Psi^{\alpha}_{k,q}(\mu, \nu, \tau_2, \wp) \subset \Psi^{\alpha}_{k,q}(\mu, \nu, \tau_1, \wp).$$

Proof. Let $\psi \in \Psi^{\alpha}_{k,q}(\mu, \nu, \tau_2, \wp)$. Formulate an analytic function $W \in U$ as follows:

$$W(z) = z^{\wp}[\Delta_q^{m\,\alpha} \psi(z)],$$

achieving $W(0) = 1$. A computation gives

$$(1 - \tau_2) z^{\wp}\, [\Delta_q^{m\alpha} \psi(z)] - \left(\frac{\tau_2}{\wp}\right) z^{1+\wp} [\Delta_q^{m\,\alpha} \psi(z)]' = W(z) - \frac{\tau_2}{\wp}(z\, W'(z)).$$

Immediately, we have the inequality

$$W(z) - \frac{\tau_2}{\wp}(zW'(z)) \prec \frac{\mu\, z + 1}{\nu\, z + 1}.$$

Employing Lemma 5 given that $c = -\dfrac{\tau_2}{\wp} > 0$, we have

$$W(z) \prec \frac{\mu\, z + 1}{\nu\, z + 1}, \quad z \in U.$$

Since, $0 < \tau_1/\tau_2 < 1$ and since $J_{\mu,\nu}(z)$ is convex univalent in U, we obtain the following arrangement:

$$\begin{aligned}
&(1 - \tau_1) z^{\wp}\, [\Delta_q^{m\,\alpha} \psi(z)] - \left(\frac{\tau_1}{\wp}\right) z^{1+\wp} [\Delta_q^{m\,\alpha} \psi(z)]' \\
&= (1 - \tau_1) W(z) - \left(\frac{\tau_1}{\wp}\right) (zW'(z) - \wp W(z)) + \left(\frac{\tau_1}{\tau_2} W(z) - \frac{\tau_1}{\tau_2} W(z)\right) \\
&= \frac{\tau_1}{\tau_2}\left((1 - \tau_2) W(z) - \left(\frac{\tau_2}{\wp}\right) (zW'(z) - \wp W(z))\right) + \left(1 - \frac{\tau_1}{\tau_2}\right) W(z) \\
&= \frac{\tau_1}{\tau_2}\left((1 - \tau_2) z^{\wp}\, [\Delta_q^{m\,\alpha} \psi(z)] - \left(\frac{\tau_2}{\wp}\right) z^{1+\wp} [\Delta_q^{m\,\alpha} \psi(z)]'\right) + \left(1 - \frac{\tau_1}{\tau_2}\right) W(z) \\
&\prec J_{\mu,\nu}(z).
\end{aligned}$$

Hence, by Definition 4, we receive $\psi \in \Psi^{\alpha}_{k,q}(\mu, \nu, \tau_1, \wp)$. □

Next, we deal with results concerning differential inequalities.

Theorem 10. *Define the functional*

$$G(z) = (1-\tau)z^{\wp}\,[\Delta_q^{m\,\alpha}\psi(z)] - \left(\frac{\tau}{\wp}\right) z^{1+\wp}[\Delta_q^{m\,\alpha}\psi(z)]'.$$

If

$$v_1(1+\wp)z^{\wp}\Delta_q^{m\,\alpha}\psi(z) + [v_1 - v_2(1+\wp) - v_2]z^{1+\wp}\Big(\Delta_q^{m\,\alpha}\psi(z)\Big)' - v_2 z^{2+\wp}\Big(\Delta_q^{m\,\alpha}\psi(z)\Big)''$$

$$\prec \left(\frac{1+z}{1-z}\right)^{\vartheta_1}$$

then

$$G(z) \prec \left(\frac{1+z}{1-z}\right)^{\vartheta_2}$$

$$\Big(\vartheta_1 > 0,\ \vartheta_2 > 0, v_1 = 1-\tau,\ v_2 = \frac{\tau}{\wp},\ \wp < 0\Big),$$

where for a constant $\vartheta = \vartheta(n)$ satisfies the equation

$$\vartheta_2 = \frac{3\pi/2 - \tan^{-1}(n\vartheta_2)}{\pi},$$

the constants ϑ_1 and ϑ_2 satisfy the relation

$$\vartheta_1 = \vartheta_1(\vartheta_2, n) = \vartheta_2 + (2/\pi)\tan^{-1}(n\vartheta_2), \quad 0 < \vartheta_2 \le \vartheta.$$

Proof. Is clear that $G \in \mathbb{H}[1, n]$. A computation yields

$$\begin{aligned}
G(z) + zG'(z) &= (1-\tau)z^{\wp}\,[\Delta_q^{m\,\alpha}\psi(z)] - \left(\frac{\tau}{\wp}\right) z^{1+\wp}[\Delta_q^{m\,\alpha}\psi(z)]' \\
&\quad + z\left((1-\tau)z^{\wp}\,[\Delta_q^{m\,\alpha}\psi(z)] - \left(\frac{\tau}{\wp}\right) z^{1+\wp}[\Delta_q^{m\,\alpha}\psi(z)]'\right)' \\
&= v_1(1+\wp)z^{\wp}\Delta_q^{m\,\alpha}\psi(z) + [v_1 - v_2(1+\wp) - v_2]z^{1+\wp}\Big(\Delta_q^{m\,\alpha}\psi(z)\Big)' \\
&\quad - v_2 z^{2+\wp}\Big(\Delta_q^{m\,\alpha}\psi(z)\Big)'' \\
&\prec \left(\frac{1+z}{1-z}\right)^{\vartheta_1}.
\end{aligned}$$

Then according to Lemma 6 with $\complement = 1$, we obtain $G(z) \prec \left(\frac{1+z}{1-z}\right)^{\vartheta_2}$. □

Note that when $n = \vartheta_2 = 1, \vartheta_1 = 3/2$, we obtain the next outcome

Corollary 11. *Let the assumptions of Theorem 10 hold. If the differential inequality*

$$v_1(1+\wp)z^{\wp}\Delta_q^{m\,\alpha}\psi(z) + [v_1 - v_2(1+\wp) - v_2]z^{1+\wp}\Big(\Delta_q^{m\,\alpha}\psi(z)\Big)' - v_2 z^{2+\wp}\Big(\Delta_q^{m\,\alpha}\psi(z)\Big)''$$

$$\prec \left(\frac{1+z}{1-z}\right)^{3/2}$$

$$\Big(v_1 = 1-\tau,\ v_2 = \frac{\tau}{\wp},\ \wp < 0\Big)$$

occurs, then $\psi \in \Psi^{\alpha}_{q,k}(1,-1,\tau,\wp)$.

Theorem 12. *Let* $\psi \in \Psi^{\alpha}_{k,q}(\mu,\nu,\tau,\wp)$ *and* $g \in \Psi_k(\wp)$. *Then* $\psi * g \in \Psi^{\alpha}_{k,q}(\mu,\nu,\tau,\wp)$ *if*

$$\Re\Big(z^{\wp}\Delta_q^{m\,\alpha} g(z)\Big) > \frac{1}{2}. \tag{7}$$

Proof. In virtue of the convolution's behavior, a computation gives

$$\begin{aligned}
&(1-\tau)z^{\wp}\,[\Delta_q^{m\,\alpha}(\psi * g)(z)] - \left(\frac{\tau}{\wp}\right) z^{1+\wp}[\Delta_q^{m\,\alpha}(\psi * g)(z)]' \\
&= (1-\tau)\Big(z^{\wp}[\Delta_q^{m\,\alpha}\psi(z)] * z^{\wp}[\Delta_q^{m\,\alpha}g(z)]\Big) - \left(\frac{\tau}{\wp}\right)\Big(z^{1+\wp}[\Delta_q^{m\alpha}g(z)]' * (z^{\wp}[\Delta_q^{m\,\alpha}g(z)])\Big) \\
&= \left((1-\tau)z^{\wp}\,[\Delta_q^{m\,\alpha}\psi(z)] - \left(\frac{\tau}{\wp}\right) z^{1+\wp}[\Delta_q^{m\,\alpha}g(z)]'\right) * \Big(z^{\wp}\Delta_q^{m\,\alpha}g(z)\Big) \\
&= G(z) * \Big(z^{\wp}\Delta_q^{m\,\alpha}g(z)\Big),
\end{aligned}$$

where $G(z) \prec J_{\mu,\nu}(z)$. By the condition (7) yields $\Big(z^{\wp}\Delta_q^{m\,\alpha}g(z)\Big)$ admits the Herglotz integral expression [42]

$$\Big(z^{\wp}\Delta_q^{m\,\alpha}g(z)\Big) = \int_{|\chi|=1}\left(\frac{d\varpi(\chi)}{1-\chi\, z}\right),$$

where $d\,\varpi$ defines the probability measure on the unit circle $|\chi| = 1$ and

$$\int_{|\chi|=1} d\varpi(\chi) = 1.$$

By the convexity of $J_{\mu,\nu}(z)$ in U, we obtain

$$\begin{aligned}
&(1-\tau)z^{\wp}\,[\Delta_q^{m\,\alpha}(\psi * g)(z)] - \left(\frac{\tau}{\wp}\right) z^{1+\wp}[\Delta_q^{m\,\alpha}(\psi * g)(z)]' \\
&= G(z) * \Big(z^{\wp}\Delta_q^{m\,\alpha}g(z)\Big) \\
&= \int_{|\chi|=1} G(\chi\, z)d\varpi\,(\chi) \\
&\prec J_{\mu,\nu}(z).
\end{aligned}$$

Hence, $\psi * g \in \Psi^{\alpha}_{k,q}(\mu,\nu,\tau,\wp)$. □

More differential and integral inequalities are presented in the next results.

Theorem 13. *Define the functional G as follows:*

$$\begin{aligned}
G(z) &= (1-\tau)z^{\wp}\,[\Delta_q^{m\,\alpha}\psi(z)] - \left(\frac{\tau}{\wp}\right) z^{1+\wp}[\Delta_q^{m\,\alpha}\psi(z)]', \quad \tau < 0 \\
&= 1 + \sum_{n=1}^{\infty}\gamma_n z^n, \quad z \in U.
\end{aligned}$$

If $\Re(G(z)) > 0$ *then the coefficient bounds are determined by the integral inequality*

$$|\gamma_n| \le 2\int_0^{2\pi} |e^{-in\,v}|\, d\omega(v),$$

where $d\omega$ *is a probability measure. Moreover, if* $\Re\big(e^{i\sigma}G(z)\big) > 0$, $\sigma \in \mathbb{R}$ *then* $G(z)$ *is convex in* U.

Proof. To prove the first result, we assume that

$$\Re(G(z)) = \Re\left(1 + \sum_{n=1}^{\infty} \gamma_n z^n\right) > 0.$$

Then in virtue of the Carathéodory positivist theorem in the class of analytic functions, we obtain

$$|\gamma_n| \leq 2\int_0^{2\pi} |e^{-inv}|\, d\omega(\theta),$$

where $d\omega$ is a probability measure.

The second result comes as follows: since

$$\Re\left(e^{i\sigma}G(z)\right) > 0, \quad z \in U, \sigma \in \mathbb{R}$$

then according to [30], Theorem 1.6-P22, and $\sigma \in \mathbb{R}$, we obtain

$$G(z) \approx \frac{\mu z + 1}{\nu z + 1}, \quad z \in U.$$

Since $\frac{\mu z + 1}{\nu z + 1}$ is convex in U, then majority fact implies that $G(z)$ is convex in U. □

The conclusion of Theorem 13 yields the sufficient conditions for functions $\psi \in \Psi_k(\wp)$ to be in $\Psi^{\alpha}_{k,q}(\mu, \nu, \tau, \wp)$.

Theorem 14. *Let* $\psi \in \Psi_k(\wp)$ *and*

$$\natural(z) := z^{\wp+1}\Delta_q^{m\,\alpha}\psi(z) \prec \frac{z}{(1+z)^2}, \quad z \in U.$$

Then $\natural(z)$ *is starlike univalent in* U *satisfying the integral inequalities*

$$\left(\int_0^z \frac{\sqrt{\natural(\zeta)}}{\chi} d\chi\right)^2 \prec \left(2\tan^{-1}(z^{1/2})\right)^2,$$

where for $0 < |z| = \varrho < 1$,

$$-\frac{\pi}{2} < -2\tan^{-1}\sqrt{\varrho} \leq \Re\left(\int_0^z \frac{\sqrt{\flat(\chi)}}{\chi} d\chi\right) < 2\tan^{-1}\sqrt{\varrho} \leq \frac{\pi}{2}.$$

Proof. Consider

$$\natural(z) = z^{\wp+1}\Delta_q^{m\,\alpha}\varphi(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad z \in U.$$

Clearly,

$$\mathcal{B}(z) := \left(2\tan^{-1}(z^{1/2})\right)^2 = 4z - \left(\frac{8}{3}\right)z^2 + \left(\frac{92}{45}\right)z^3 + O(z^4).$$

By the starlikeness of the function (see [32]-P177)

$$\varphi(z) = \frac{z}{(1+z)^2} = z - 2z^2 + 3z^3 - 4z^4 + 5z^5 + O(z^6),$$

together with the majority theory, we obtain the starlikeness of $\natural(z) \in U$.

The second and third outcomes are direct applications of [32], Corollary 3.6a.1. □

Theorem 15. *Define the functional*

$$G(z) = (1-\tau)z^{\wp}\,[\Delta_q^{m\,\alpha}\varphi(z)] - \left(\frac{\tau}{\wp}\right) z^{1+\wp}[\Delta_q^{m\,\alpha}\varphi(z)]'$$

Let the following differential inequalities hold

$$\frac{-\Big(z^{\wp}(z(\tau z[\Delta_q^{m\,\alpha}\varphi(z)]'' + (2\tau(\wp+1)-\wp)[\Delta_q^{m\,\alpha}\varphi(z)]') + (\tau-1)\wp(\wp+1)[\Delta_q^{m\,\alpha}\varphi(z)])\Big)}{\wp} \prec p_2(z) + zp_2'(z), \quad (8)$$

where $p_2(0) = 1$ *and convex in* U*. In addition, let* $G(z)$ *be univalent in* U *such that* $G \in \mathbb{H}[p_1(0), 1] \cap \mathbb{Q}$*, where* $\mathbb{Q}$ *presents the set of all (1-1) analytic functions* g *such that*

$$\lim_{z\in\partial U} g \neq \infty$$

and

$$p_1(z) + z\,p_1'(z) \prec \frac{-\Big(z^{\wp}(z(\tau z[\Delta_q^{m\,\alpha}\varphi(z)]'' + (2\tau(\wp+1)-\wp)[\Delta_q^{m\,\alpha}\varphi(z)]') + (\tau-1)\wp(\wp+1)[\Delta_q^{m\,\alpha}\varphi(z)])\Big)}{\wp}. \quad (9)$$

Then

$$p_1(z) \prec G(z) \prec p_2(z)$$

and $p_1(z)$ *is the best sub-dominant and* $p_2(z)$ *is the best dominant.*

Proof. Since,

$$G(z) + zG'(z) = \prec \frac{-\Big(z^{\wp}(z(\tau z[\Delta_q^{m\,\alpha}\varphi(z)]'' + (2\tau(\wp+1)-\wp)[\Delta_q^{m\,\alpha}\varphi(z)]') + (\tau-1)\wp(\wp+1)[\Delta_q^{m\,\alpha}\varphi(z)])\Big)}{\wp}$$

then we have the bi-subordination

$$\begin{aligned} p_1(z) + zp_1'(z) &\prec G(z) + zG'(z) \\ &\prec p_2(z) + zp_2'(z). \end{aligned}$$

Thus, Lemmas 7 and 8 imply the desired assertion. □

3.2. q-Differential Equations

In this part, we deal with the real formula

$$\begin{aligned} \Re\big(G(z) + zG'(z)\big) &= \Re\Big(v_1(1+\wp)z^{\wp}\Delta_q^{m\,\alpha}\psi(z) + [v_1 - v_2(1+\wp) - v_2]z^{1+\wp}\Big(\Delta_q^{m\,\alpha}\psi(z)\Big)' \\ &\quad - v_2 z^{2+\wp}\Big(\Delta_q^{m\,\alpha}\psi(z)\Big)''\Big) \\ &= v_1(1+\wp)xy_q + \left(\frac{(1+\wp)(2v_1-1)-1}{\wp}\right)x^{1-\wp}y_q' - \left(\frac{1-v_1}{\wp}\right)x^{1-2\wp}\,y_q'', \end{aligned}$$

such that $x := \Re(z^{\wp}), v_1 = 1-\tau > 0, v_2 = (1-v_1)/\wp$ and $\Re(\Delta_q^{m\,\alpha}\psi(z)) := y_q(x)$. By approximate $v_1 \to 2$, we obtain

$$\Re\big(G(z) + zG'(z)\big) = 2(1+\wp)xy_q + \left(\frac{3(1+\wp)-1}{\wp}\right)x^{1-\wp}y_q' + \left(\frac{1}{\wp}\right)x^{1-2\wp}y_q''.$$

The analytic result of $\Re(G(z) + zG'(z)) = 0$ is indicated by finding the outcome of the following second order differential equation:

$$2(1+\wp)xy_q + \left(\frac{3(1+\wp)-1}{\wp}\right)x^{1-\wp}y_q' + \left(\frac{1}{\wp}\right)x^{1-2\wp}y_q'' = 0. \tag{10}$$

The analytic outcome of Equation (10) is given in the following result.

Theorem 16. *The analytic solvability of Equation (10) is presented as follows:*

$$\begin{aligned} y_q(x) \approx &\left(\frac{2^{\frac{\wp}{2(\wp+1)}}}{e^{2x^{\wp+1}}}\right)\left(x^{\frac{\wp(\wp+1)}{2(\wp+1)}-\frac{\wp}{2}}\right) \\ &\times\left\{c_1\mathbb{U}\left(\frac{2\wp}{2+\wp},\frac{\wp}{1+\wp},\frac{\wp+2x^{\wp+1}}{\wp+1}\right) + c_2\mathbb{L}^{(-1/(\wp+1))}_{(-2\wp/(\wp+2)}\left(\frac{(\wp+2)x^{1+\wp}}{\wp+1}\right)\right\} \end{aligned} \tag{11}$$

where $\mathbb{U}$ indicates the second type of confluent function and $\mathbb{L}$ represents the Laguerre polynomial.

Proof. Equation (10) takes the Sturm–Liouville formula (SLF). Thus, we come to the conclusion

$$\frac{d}{dx}\left(e^{\frac{(2+3\wp)x^{1+\wp}}{1+\wp}}y_q'(x)\right) + 2e^{\frac{(2+3\wp)x^{1+\wp}}{1+\wp}}\left(\wp(1+\wp)x^{2\wp}\right)y_q(x) = 0. \tag{12}$$

A computation yields the analytic solution (11). □

3.3. Numerical Examples

We illustrate the following numerical examples.

Example 17. *Consider $\wp = 1$, then Equation (11) takes the next SLF*

$$\frac{d}{dx}\left(\exp(\frac{5x^2}{2})y_q'(x)\right) + 4\exp(\frac{5x^2}{2})x^2\,y_q(x) = 0. \tag{13}$$

Equation (13) yields the structure

$$e^{2x^2}\sqrt{e^{x^2}}y_q''(x) + 5e^{2x^2}\sqrt{e^{x^2}}xy_q'(x) + 4e^{2x^2}\sqrt{e^{x^2}}x^2y_q(x) = 0.$$

This leads to the formula equation

$$e^{(5x^2)/2}\left(4x^2y_q(x) + 5xy_q'(x) + y_q''(x)\right) = 0$$

Thus, we attain the outcome (see Figure 1)

$$y_q(x) \approx \exp(-2x^2)\left\{c_1\mathbb{H}_{-4/3}(\sqrt{\frac{3}{2}}x) + c_2\,({}_1F_1)\left(\frac{2}{3};\frac{1}{2};\frac{3x^2}{2}\right)\right\},$$

where $\mathbb{H}_n(\chi)$ indicates the Hermite function and $({}_1F_1)$ is the hypergeometric function. Obviously, the outcome (13) is proposed at ∂U (see Figure 1, left column). Hence, we have

$$\Re(\Delta_q^{m\,\alpha}\psi(z)) \approx y_q(x), x \to 1.$$

Assume that $y_q(0) = 1$, which yields the outcome (see Figure 1, right column)

$$y_q(x) \approx \frac{\exp(-2x^2)}{3.7}\Big(3.7c_1 H_{-4/3}1.22x - \left(1.5\sqrt{\pi}c_1 \;\; - 3.7\right)\,({}_1F_1)(\frac{2}{3};\frac{1}{2};\frac{3x^2}{2})\Big).$$

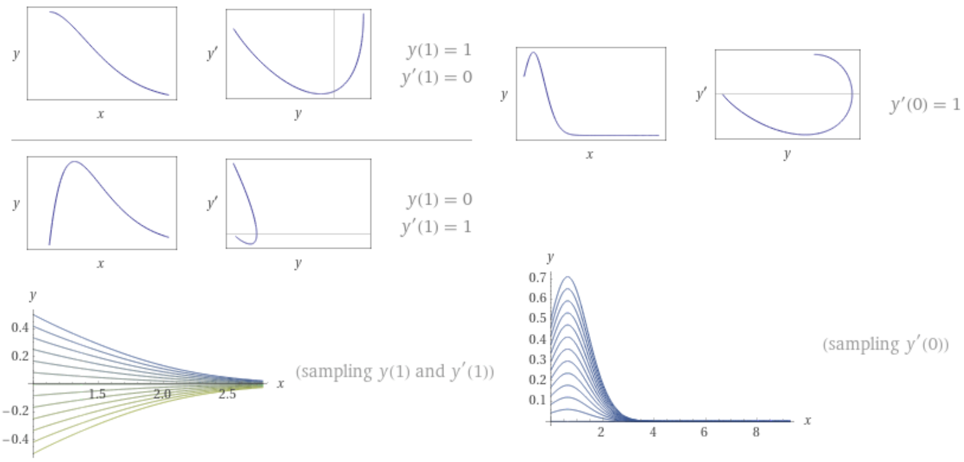

Figure 1. The solution of (13) for $\wp = 1$.

Example 18. *Assume that $\wp = 2$, then Equation (11) becomes the SLF*

$$\frac{d}{dx}\left(\exp(\frac{8x^3}{3})y_q'(x)\right) + 12\exp(\frac{8x^3}{3})x^4 y_q(x) = 0, \tag{14}$$

which is equivalent to solve the differential equation

$$12x^4 y_q(x) + 8x^2 y_q'(x) + y_q''(x) = 0.$$

Hence, with the outcome at ∂U (see Figure 2, first row)

$$y_q(x) = c_1\exp(\frac{-2x^3}{3})\,x + \frac{2^{2/3}c_2\exp(\frac{-2x^3}{3})(x^3)^{1/3}\;\Gamma(\frac{-1}{3},\frac{4x^3}{3})}{3^{1/3}}.$$

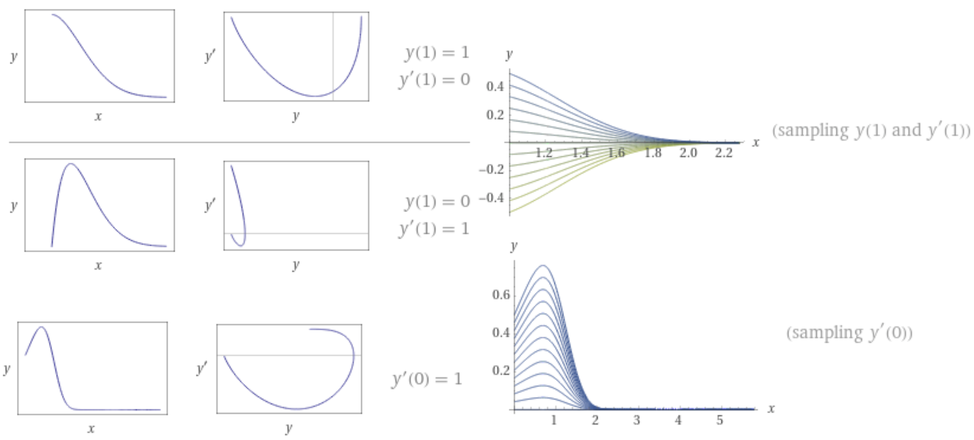

Figure 2. The solution of (14) for $\wp = 2$.

In addition, the outcome, when $y_q(0) = 1$ is formulated by the construction (see Figure 2, second row)

$$y_q(x) = \frac{1}{9}\exp(\frac{-2x^3}{3})\Big(c_1 x + 3.3x\Gamma_1(\frac{-1}{3},\frac{4x^3}{3})\Big).$$

Example 19. *Suppose that* $\wp = 3$, *then Equation* (11) *becomes*

$$\frac{d}{dx}\left(\exp(\frac{5x^4}{2})y_q'(x)\right) + 12\exp(\frac{5x^4}{2})\,x^6 y_q(x) = 0, \tag{15}$$

which is equivalent to

$$12x^6 y_q(x) + 10x^3 y_q'(x) + y_q''(x) = 0.$$

Thus, with the outcome approximating to the boundary of U (see Figure 3, first column)

$$y_q(x) \approx c_1 \exp(\frac{-x^4}{2})x + c_2 \exp(\frac{-x^4}{2})\,x\,\Gamma(\frac{-1}{4}, \frac{3x^4}{2})$$

$$y_q(x) \approx\ c_1 2.718^{(-0.5x^4)}x + 1.1066\ c_2\, 2.718^{(-0.5x^4)}(x^4)^{(1/4)}\Gamma(-0.25, 1.5x^4).$$

In addition, the outcome when $y_q(0) = 1$ *is formulated by the construction (see Figure 3, second column)*

$$y_q(x) \approx \frac{1}{8}\exp(\frac{-x^4}{2})\left(c_1 x + 1.68(3x^4)^{1/4}\Gamma(\frac{-1}{4}, \frac{3x^4}{2})\right).$$

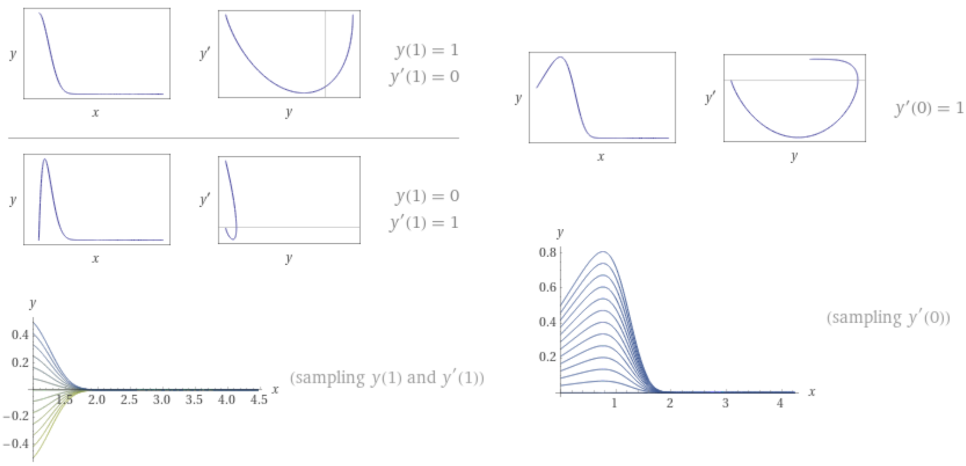

Figure 3. The solution of (15) for $\wp = 3$.

Proposition 20. *If*

$$\Re\big(G(z) + zG'(z)\big) > 0, \quad z \in U \tag{16}$$

then the equation

$$2(1+\wp)xy_q + \left(\frac{3(1+\wp)-1}{\wp}\right)x^{1-\wp}y_q' + \left(\frac{1}{\wp}\right)x^{1-2\wp}y_q'' = k, \quad k > 0 \tag{17}$$

has a positive solution.

Proof. In view of the assumption (16) together with Lemma 6 (the first part), we obtain $\Re(G) > 0$. This yields that $\Re(\Delta_q^{m\,\alpha}\psi(z)) = y_q(x) > 0$, which means that Equation (17) has a positive outcome. □

4. Conclusions

From above, we expressed a new quantum symmetric differential operator (q-SDO) related to a class of analytic function with the property of multivalent meromorphically in the open unit disk. We investigated two different concepts, q-differential inequalities

$$p_1(z) \prec G(z) + zG'(z) \prec p_2(z)$$

of a complex variable and the real cases of q-differential equations corresponding to the same class of analytic functions

$$\Re(G(z) + zG'(z)) = 0.$$

The functional $G(z)$ is defined by using a symmetric differential operator in terms of quantum calculus formulating by

$$(1-\tau)z^{\wp}\,[\Delta_q^{m\,\alpha}\psi(z)] - \left(\frac{\tau}{\wp}\right) z^{1+\wp}[\Delta_q^{m\,\alpha}\psi(z)]'.$$

Some geometric properties are investigated For future investigations, we suggest another class of analytic functions to define a new symmetric differential operator, such as harmonic and multivalent harmonic functions.

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