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Chaos Control for a Fractional-Order Jerk System via Time Delay Feedback Controller and Mixed Controller

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Abstract: In this study, we propose a novel fractional-order Jerk system. Experiments show that, under some suitable parameters, the fractional-order Jerk system displays a chaotic phenomenon. In order to suppress the chaotic behavior of the fractional-order Jerk system, we design two control strategies. Firstly, we design an appropriate time delay feedback controller to suppress the chaos of the fractional-order Jerk system. The delay-independent stability and bifurcation conditions are established. Secondly, we design a suitable mixed controller, which includes a time delay feedback controller and a fractional-order PD^{σ} controller, to eliminate the chaos of the fractional-order Jerk system. The sufficient condition ensuring the stability and the creation of Hopf bifurcation for the fractional-order controlled Jerk system is derived. Finally, computer simulations are executed to verify the feasibility of the designed controllers. The derived results of this study are absolutely new and possess potential application value in controlling chaos in physics. Moreover, the research approach also enriches the chaos control theory of fractional-order dynamical system.

Keywords: fractional-order Jerk system; chaos; hopf bifurcation; stability; time delay feedback controller; fractional-order PD^{σ} controller

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1. Introduction

In the natural world, a great deal of natural phenomena display chaotic behavior. Usually, there exist chaotic phenomena in various areas such as weather, climate, economy and finance, neural networks, biological systems, fluid mechanics and so on [1–5]. The chaotic behavior sensitively depends on the initial value of the original system. That is to say, the dynamic behavior of the system will change greatly when the initial value of the system changes slightly. This kind of complex dynamical behavior may be undesirable in numerous physical sciences, biological techniques and engineering technology, since we cannot predict the long-term development law of these systems. Based on this reason, it is important for us to seek control techniques to suppress the chaotic behavior and make the system generate our desired dynamical properties. This aspect has become a problem of focus in recent years [6]. Designing valid control mechanisms to realize the target of chaos control is very vital for both theoretical study and actual applications [7,8]. For so long, there have been many techniques to suppress the chaos of the chaotic systems. For example, Zheng [8] designed an adaptive feedback controller to control the chaotic behavior of a chaotic system. Ott, Grebogi and Yorke [9] proposed an OGY control technique to suppress the chaos of a chaotic system in 1990. Li et al. [10] controlled the chaos of a brushless DC motor via a nonlinear state feedback controller. Du et al. [11] suppressed the chaos of an economic model by virtue of phase space compression. For more related works on this theme, one can see [12–15].

Physically speaking, Jerk can be regarded as the third derivative of position with regard to the time t [16]. Usually, it can be expressed as:

$$\frac{d^3 w_1(t)}{dt} = \mathcal{J}\left(w_1, \frac{dw_1(t)}{dt}, \frac{d^2 w_1(t)}{dt}\right), \tag{1}$$

which is called the Jerk equation. Set

$$\begin{cases} w_2(t) = \dfrac{dw_1(t)}{dt}, \\ w_3(t) = \dfrac{d^2 w_1(t)}{dt}, \end{cases} \tag{2}$$

then system (1) can be rewritten as the following form:

$$\begin{cases} \dfrac{dw_1(t)}{dt} = w_2(t), \\ \dfrac{dw_2(t)}{dt} = w_3(t), \\ \dfrac{dw_3(t)}{dt} = \mathcal{J}(w_1, w_2, w_3). \end{cases} \tag{3}$$

In 2021, Liu et al. [16] established the following Jerk system:

$$\begin{cases} \dfrac{dw_1(t)}{dt} = w_2(t), \\ \dfrac{dw_2(t)}{dt} = w_3(t), \\ \dfrac{dw_3(t)}{dt} = -\alpha_1 w_1(t) - \alpha_2 w_2(t) - \alpha_3 w_3(t) + \alpha_4 w_3^2 + \alpha_5 w_1(t) w_2(t), \end{cases} \tag{4}$$

where $\alpha_i (i = 1, 2, 3, 4, 5)$ represents the real number. By virtue of Hopf bifurcation theory, Lyapunov exponents and bifurcation figures, Liu et al. [16] explored the chaotic dynamics for the model (3). For details, one can see [16].

We would like to mention that the work of Liu et al. [16] merely focused on an integer-order differential system and it does not involve the fractional-order case. Recent research has demonstrated that the fractional-order dynamical model is regarded as a more rational tool for depicting the authentic natural phenomena in the object world since it has greater superiority than integer-order ones. The advantage of the fractional-order dynamical model lies in the memory trait and hereditary peculiarity of plentiful materials and evolution processes [17–20]. The fractional-order dynamical system displays an immense application prospect in many subjects in artificial intelligence, neural networks, mechanics, economics, all sorts of physical waves, bioscience and so on [21–25]. Nowadays, a number of fractional-order differential systems have been built and rich achievements have been reported. The exploration of Hopf bifurcation of fractional-order differential systems has especially attracted great attention from many scholars. For instance, Djilali et al. [26] probed the Turing–Hopf bifurcation for a fractional-order mussel-algae model; Xiao et al. [27] handled the Hopf bifurcation control issue for fractional-order small-world networks; Xu et al. [28] revealed the effect of leakage delay on bifurcation for fractional-order complex-valued neural networks; Huang et al. [29] discussed the Hopf bifurcation for fractional-order multi-delayed neural networks. In detail, we refer readers to [30–44].

Stimulated by the discussion above and based on system (4), we establish the following fractional-order Jerk system:

$$\begin{cases} \dfrac{d^{\sigma}w_1(t)}{dt^{\sigma}} = w_2(t), \\ \dfrac{d^{\sigma}w_2(t)}{dt^{\sigma}} = w_3(t), \\ \dfrac{d^{\sigma}w_3(t)}{dt^{\sigma}} = -\alpha_1 w_1(t) - \alpha_2 w_2(t) - \alpha_3 w_3(t) + \alpha_4 w_3^2 + \alpha_5 w_1(t) w_2(t), \end{cases} \tag{5}$$

where $\sigma \in (0, 1]$. The study shows that the fractional-order Jerk system (5) will display chaotic behavior when $\sigma = 0.94, \alpha_1 = 2, \alpha_2 = 1, \alpha_3 = 1.2, \alpha_4 = 0.5, \alpha_5 = 0.9$. The simulation results can be seen in Figure 1.

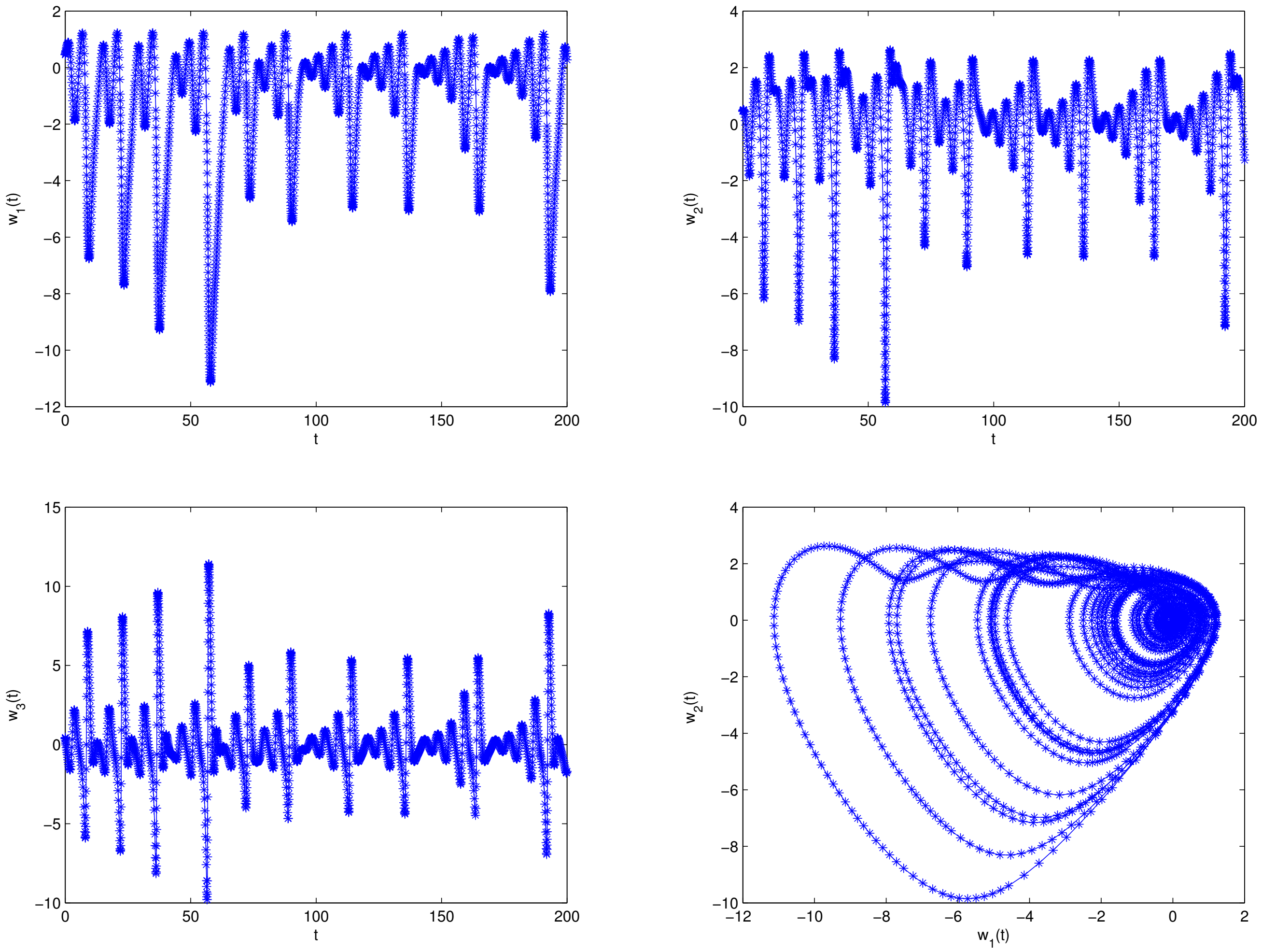

Figure 1. *Cont.*

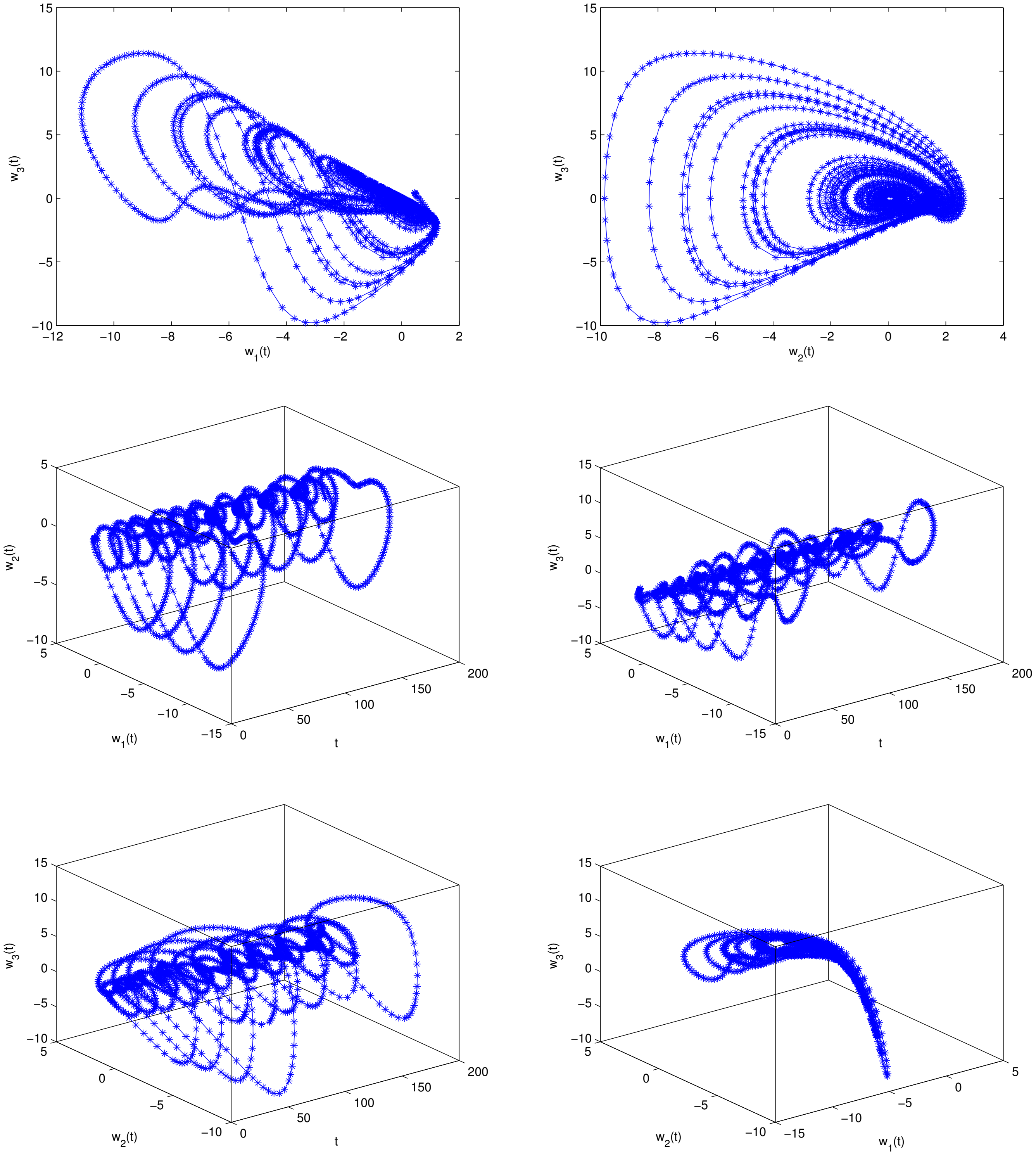

Figure 1. The numerical simulation plots of system (5) under the parameter values $\sigma = 0.94, \alpha_1 = 2, \alpha_2 = 1, \alpha_3 = 1.2, \alpha_4 = 0.5, \alpha_5 = 0.9$.

In the present study, we are going to focus on the following two aspects:

1. Control the chaotic behavior of system (5) via designing a suitable time delay feedback controller;
2. Control the chaotic behavior of system (5) via designing an appropriate mixed controller which includes time delay feedback controller and fractional-order PD$^{\sigma}$ con-

troller. Up to now, there have been very few papers that deal with the chaos control via this mixed controller.

The main highlights of this research can be summarized as follows:

- Based on the previous publications, we build a novel Jerk system.
- A suitable time delay feedback controller is successfully designed to suppress the chaotic behavior of Jerk system (5);
- A suitable mixed controller which includes time delay feedback controller and fractional-order PD$^{\sigma}$ controller is successfully designed to suppress the chaotic behavior of the Jerk system (5);
- The research idea can also be applied to deal with the chaos control issue for numerous other fractional-order differential systems in many areas.

This work can be planned as follows. Basic knowledge about fractional-order dynamical systems is presented in Section 2. In Section 3, we investigate the chaos control of system (5) via a time delay feedback controller. In Section 4, we discuss the chaos control of system (5) via designing a mixed controller, which includes a time delay feedback controller and fractional-order PD$^{\sigma}$ controller. In Section 5, Matlab simulation figures are given to support the derived results. Section 6 ends this study.

Remark 1. *The fractional-order system (5) is derived from (4) by replacing the integer-order derivatives with fractional orders. System (5) can describe the memory trait and hereditary peculiarity of the state variables more precisely.*

2. Preliminary Theory

In this segment, we state the necessary definitions and lemmas about the fractional-order dynamical system.

Definition 1 ([35])**.** *Define the Riemann–Liouville fractional integral of order σ for the function $g(\epsilon)$ as follows:*

$$\mathcal{I}^{\sigma} g(\epsilon) = \frac{1}{\Gamma(\sigma)} \int_{\epsilon_0}^{\epsilon} (\epsilon - \varsigma)^{\sigma-1} g(\chi) d\varsigma,$$

where $\epsilon > \epsilon_0, \sigma > 0$ and $\Gamma(\varsigma) = \int_0^{\infty} s^{\varsigma-1} e^{-s} ds$.

Definition 2 ([35])**.** *The Caputo-tpye fractional-order derivative of order σ for the function $g(\varsigma) \in ([\varsigma_0, \infty), R)$ is given by:*

$$\mathcal{D}^{\sigma} g(\varsigma) = \frac{1}{\Gamma(\iota - \sigma)} \int_{\varsigma_0}^{\varsigma} \frac{g^{(\iota)}(s)}{(r - s)^{\sigma - \iota + 1}} ds,$$

where $\varsigma \geq \varsigma_0$ and ι stands for a positive integer ($\sigma \in [\iota - 1, \iota)$). Especially, when $\sigma \in (0,1)$, then

$$\mathcal{D}^{\sigma} g(\varsigma) = \frac{1}{\Gamma(1 - \sigma)} \int_{\varsigma_0}^{\varsigma} \frac{g'(s)}{(\varsigma - s)^{\sigma}} ds.$$

Lemma 1 ([36])**.** *Give the fractional-order model: $\mathcal{D}^{\sigma} u = \mathcal{P} u, u(0) = u_0$, where $\sigma \in (0,1)$, $u \in R^m, \mathcal{H} \in R^{m \times m}$. Let $\mu_j (j = 1, 2, \cdots, m)$ be the root of the characteristic equation of $\mathcal{D}^{\sigma} u = \mathcal{P} u$, then the equilibrium point of system $\mathcal{D}^{\sigma} u = \mathcal{P} u$ is locally asymptotically stable provided that $|arg(\mu_j)| > \frac{\sigma\pi}{2} (j = 1, 2, \cdots, m)$ and the equilibrium point of system $\mathcal{D}^{\sigma} u = \mathcal{P} u$ is stable, provided that $|arg(\mu_j)| > \frac{\sigma\pi}{2} (j = 1, 2, \cdots, m)$ and each critical eigenvalue satisfying $|arg(\mu_j)| = \frac{\sigma\pi}{2} (j = 1, 2, \cdots, m)$ has geometric multiplicity one.*

3. Chaos Control via Time Delay Feedback Controller

In this segment, we shall design a suitable controller to suppress the chaotic behavior of the chaotic Jerk system. Following the idea of Yu and Chen [37], we design the time delay feedback controller as follows:

$$\zeta(t) = \varrho_1[w_3(t-\vartheta) - w_3(t)], \tag{6}$$

where ϱ_1 represents feedback gain coefficient and ϑ is a delay. Adding the time delay feedback controller (6) to the second equation of system (5), one can get:

$$\begin{cases} \dfrac{d^\sigma w_1(t)}{dt^\sigma} = w_2(t), \\ \dfrac{d^\sigma w_2(t)}{dt^\sigma} = w_3(t) + \varrho_1[w_3(t-\vartheta) - w_3(t)], \\ \dfrac{d^\sigma w_3(t)}{dt^\sigma} = -\alpha_1 w_1(t) - \alpha_2 w_2(t) - \alpha_3 w_3(t) + \alpha_4 w_3^2 + \alpha_5 w_1(t) w_2(t). \end{cases} \tag{7}$$

System (7) comes from adding a perturbation term to the second equation of system (5). Clearly, system (7) owns one equilibrium point $W_1(0,0,0)$. Clearly, we can easily obtain the following linear system of (7) near the equilibrium point $W_1(0,0,0)$:

$$\begin{cases} \dfrac{d^\sigma w_1(t)}{dt^\sigma} = w_2(t), \\ \dfrac{d^\sigma w_2(t)}{dt^\sigma} = (1-\varrho_1) w_3(t) + \varrho_1 w_3(t-\vartheta), \\ \dfrac{d^\sigma w_3(t)}{dt^\sigma} = -\alpha_1 w_1(t) - \alpha_2 w_2(t) - \alpha_3 w_3(t). \end{cases} \tag{8}$$

The characteristic equation of (8) owns the following expression:

$$\det \begin{bmatrix} s^\sigma & -1 & 0 \\ 0 & s^\sigma & (\varrho_1 - 1) + \varrho_1 e^{-s\vartheta} \\ \alpha_1 & \alpha_2 & s^\varrho + \alpha_3 \end{bmatrix} = 0. \tag{9}$$

Then,

$$\mathcal{Q}_1(s) + \mathcal{Q}_2(s) e^{-s\vartheta} = 0, \tag{10}$$

where

$$\begin{cases} \mathcal{Q}_1(s) = s^{3\sigma} + c_1 s^{2\sigma} + c_2 s^\sigma + c_3, \\ \mathcal{Q}_2(s) = c_4 s^\sigma + c_5, \end{cases} \tag{11}$$

where

$$\begin{cases} c_1 = \alpha_3, \\ c_2 = \alpha_2(1-\varrho_1), \\ c_3 = \alpha_1(\varrho_1 - 1), \\ c_4 = -\alpha_2 \varrho_1, \\ c_5 = -\alpha_1 \varrho_1. \end{cases} \tag{12}$$

If $\vartheta = 0$, then (10) becomes:

$$\lambda^3 + c_1 \lambda^2 + (c_2 + c_4)\lambda + c_3 + c_5 = 0. \tag{13}$$

Suppose that

$$(\mathcal{S}_1) \begin{cases} c_1 > 0, \\ c_1(c_2 + c_4) > c_3 + c_5, \\ (c_3 + c_5)[c_1(c_2 + c_4) - (c_3 + c_5)] > 0 \end{cases}$$

is true, then all the roots $\lambda_1, \lambda_2, \lambda_3$ of (13) obey $|\arg(\lambda_1)| > \frac{\sigma\pi}{2}, |\arg(\lambda_2)| > \frac{\sigma\pi}{2}$ and $|\arg(\lambda_3)| > \frac{\sigma\pi}{2}$. Applying Lemma 1, we can conclude that the zero equilibrium point $W_1(0,0,0)$ of system (7) is locally asymptotically stable for $\vartheta = 0$.

Suppose that $s = iv = v(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$ is the root of Equation (10). Then,

$$\begin{cases} \mathcal{C}_1 \cos v\vartheta + \mathcal{C}_2 \sin v\vartheta = \mathcal{D}_1, \\ \mathcal{C}_2 \cos v\vartheta - \mathcal{C}_1 \sin v\vartheta = \mathcal{D}_2, \end{cases} \tag{14}$$

where

$$\begin{cases} \mathcal{C}_1 = \rho_1 v^{\sigma} + \rho_2, \\ \mathcal{C}_2 = \rho_3 v^{\sigma}, \\ \mathcal{D}_1 = \rho_4 v^{3\sigma} + \rho_5 v^{2\sigma} + \rho_6 v^{\sigma} + \rho_7, \\ \mathcal{D}_2 = \rho_8 v^{3\sigma} + \rho_9 v^{2\sigma} + \rho_{10} v^{\sigma}, \end{cases} \tag{15}$$

where

$$\begin{cases} \rho_1 = c_4 \cos\dfrac{\sigma\pi}{2}, \\ \rho_2 = c_5, \\ \rho_3 = c_4 \sin\dfrac{\sigma\pi}{2}, \\ \rho_4 = -\cos\dfrac{3\sigma\pi}{2}, \\ \rho_5 = -c_1 \cos\sigma\pi, \\ \rho_6 = -c_2 \cos\dfrac{\sigma\pi}{2}, \\ \rho_7 = -c_3, \\ \rho_8 = -\sin\dfrac{3\sigma\pi}{2}, \\ \rho_9 = -c_1 \sin\sigma\pi, \\ \rho_{10} = -c_2 \sin\dfrac{\sigma\pi}{2}. \end{cases} \tag{16}$$

By virtue of (14), one has

$$\cos v\vartheta = \frac{\mathcal{D}_1\mathcal{C}_1 + \mathcal{D}_2\mathcal{C}_2}{\mathcal{C}_1^2 + \mathcal{C}_2^2} \tag{17}$$

and

$$\mathcal{C}_1^2 + \mathcal{C}_2^2 = \mathcal{D}_1^2 + \mathcal{D}_2^2. \tag{18}$$

It follows from (18) that

$$\tau_1 v^{6\sigma} + \tau_2 v^{5\sigma} + \tau_3 v^{4\sigma} + \tau_4 v^{3\sigma} + \tau_5 v^{2\sigma} + \tau_6 v^{\sigma} + \tau_7 = 0, \tag{19}$$

where

$$\begin{cases} \tau_1 = \rho_4^2 + \rho_8^2, \\ \tau_2 = 2(\rho_4\rho_5 + \rho_8\rho_9), \\ \tau_3 = \rho_5^2 + \rho_9^2 + 2(\rho_4\rho_6 + \rho_8\rho_{10}), \\ \tau_4 = 2(\rho_4\rho_7 + \rho_5\rho_6 + \rho_9\rho_{10}), \\ \tau_5 = \rho_6^2 + \rho_{10}^2 - \rho_1^2 - \rho_3^2 + 2\rho_5\rho_7, \\ \tau_6 = 2(\rho_6\rho_7 - \rho_1\rho_2), \\ \tau_7 = \rho_7^2 - \rho_2^2. \end{cases} \tag{20}$$

Denote

$$\Xi_1(v) = \tau_1 v^{6\sigma} + \tau_2 v^{5\sigma} + \tau_3 v^{4\sigma} + \tau_4 v^{3\sigma} + \tau_5 v^{2\sigma} + \tau_6 v^{\sigma} + \tau_7. \tag{21}$$

If

$$(\mathcal{S}_2) \;\; |\rho_7| < |\rho_2|$$

holds, since $\lim_{\upsilon\to+\infty}\Xi_1(\upsilon)=+\infty$, then Equation (19) has at least one positive real root. Thus, Equation (10) has at least one pair of pure roots. By means of Sun et al. [40], we can easily build the following conclusion.

Lemma 2. *(1) Assume that $\tau_k>0(k=1,2,\cdots,7)$, then Equation (10) has no root with zero real parts for $\vartheta\geq 0$. (2) Assume that $(\mathcal{S}_2)$ is fulfilled and $\tau_k>0(k=1,2,\cdots,6)$, then Equation (10) has a pair of purely imaginary roots $\pm i\upsilon_0$ if $\vartheta=\vartheta_0^{(i)}(i=1,2,\cdots,)$ where*

$$\vartheta_0^{(i)}=\frac{1}{\upsilon_0}\left[\arccos\left(\frac{\mathcal{D}_1\mathcal{C}_1+\mathcal{D}_2\mathcal{C}_2}{\mathcal{C}_1^2+\mathcal{C}_2^2}\right)+2i\pi\right], \tag{22}$$

where $i=0,1,2,\cdots$, and $\upsilon_0>0$ represents the unique zero of $\Xi_1(\upsilon)$.

Set $\vartheta_0=\vartheta_0^{(0)}$. Now we make the following hypothesis:

$$(\mathcal{S}_3)\ \ \mathcal{A}_{1R}\mathcal{A}_{2R}+\mathcal{A}_{1I}\mathcal{A}_{2I}>0,$$

where

$$\begin{cases}\mathcal{A}_{1R}=3\sigma\upsilon_0^{3\sigma-1}\cos\dfrac{(3\sigma-1)\pi}{2}+2\sigma c_1\upsilon_0^{2\sigma-1}\cos\dfrac{(2\sigma-1)\pi}{2}\\ \qquad +\sigma c_2\upsilon_0^{\sigma-1}\cos\dfrac{(\sigma-1)\pi}{2}+\sigma c_4\Big[\upsilon_0^{\sigma-1}\cos\dfrac{(\sigma-1)\pi}{2}\cos\upsilon_0\vartheta_0\\ \qquad +\upsilon_0^{\sigma-1}\sin\dfrac{(\sigma-1)\pi}{2}\sin\upsilon_0\vartheta_0\Big],\\ \mathcal{A}_{1I}=3\sigma\upsilon_0^{3\sigma-1}\sin\dfrac{(3\sigma-1)\pi}{2}+2\sigma c_1\upsilon_0^{2\sigma-1}\sin\dfrac{(2\sigma-1)\pi}{2}\\ \qquad +\sigma c_2\upsilon_0^{\sigma-1}\sin\dfrac{(\sigma-1)\pi}{2}+\sigma c_4\Big[\upsilon_0^{\sigma-1}\sin\dfrac{(\sigma-1)\pi}{2}\cos\upsilon_0\vartheta_0\\ \qquad -\upsilon_0^{\sigma-1}\cos\dfrac{(\sigma-1)\pi}{2}\sin\upsilon_0\vartheta_0\Big],\\ \mathcal{A}_{2R}=\Big(c_4\upsilon_0^{\sigma}\cos\dfrac{\sigma\pi}{2}+c_5\Big)\upsilon_0\sin\upsilon_0\vartheta_0-\Big(c_4\upsilon_0^{\sigma}\sin\dfrac{\sigma\pi}{2}\Big)\upsilon_0\cos\upsilon_0\vartheta_0,\\ \mathcal{A}_{2I}=\Big(c_4\upsilon_0^{\sigma}\cos\dfrac{\sigma\pi}{2}+c_5\Big)\upsilon_0\cos\upsilon_0\vartheta_0+\Big(c_4\upsilon_0^{\sigma}\sin\dfrac{\sigma\pi}{2}\Big)\upsilon_0\sin\upsilon_0\vartheta_0.\end{cases} \tag{23}$$

Lemma 3. *Suppose that $s(\vartheta)=\eta_1(\vartheta)+i\eta_2(\vartheta)$ is the root of (10) near $\vartheta=\vartheta_0$ satisfying $\eta_1(\vartheta_0)=0,\eta_2(\vartheta_0)=\upsilon_0$, then $Re\left(\frac{ds}{d\vartheta}\right)\Big|_{\vartheta=\vartheta_0,\upsilon=\upsilon_0}>0$.*

Proof. In view of (10), one gets

$$\left(\frac{ds}{d\vartheta}\right)^{-1}=\frac{\mathcal{A}_1(s)}{\mathcal{A}_2(s)}-\frac{\vartheta}{s}, \tag{24}$$

where

$$\begin{cases}\mathcal{A}_1(s)=3\sigma s^{3\sigma-1}+2\sigma c_1s^{2\sigma-1}+\sigma c_2s^{\sigma-1}+\sigma c_4s^{\sigma-1}e^{-s\vartheta},\\ \mathcal{A}_2(s)=se^{-s\vartheta}(c_4s^{\sigma}+c_5).\end{cases} \tag{25}$$

By (24), we have

$$\mathrm{Re}\left[\left(\frac{ds}{d\vartheta}\right)^{-1}\right]_{\vartheta=\vartheta_0,\upsilon=\upsilon_0}=\mathrm{Re}\left[\frac{\mathcal{A}_1(s)}{\mathcal{A}_2(s)}\right]_{\vartheta=\vartheta_0,\upsilon=\upsilon_0}=\frac{\mathcal{A}_{1R}\mathcal{A}_{2R}+\mathcal{A}_{1I}\mathcal{A}_{2I}}{\mathcal{A}_{2R}^2+\mathcal{A}_{2I}^2}. \tag{26}$$

Applying $(\mathcal{S}_3)$, one derives

$$\mathrm{Re}\left[\left(\frac{ds}{d\vartheta}\right)^{-1}\right]_{\vartheta=\vartheta_0, v=v_0} > 0,$$

which finishes the proof. □

By virtue of Lemma 1, the following conclusion can be easily derived.

Theorem 1. *Suppose that $(\mathcal{S}_1)$–$(\mathcal{S}_3)$ hold true, then the zero equilibrium point $W_1(0,0,0)$ of system (5) is locally asymptotically stable provided that ϑ lies in the interval $[0, \vartheta_0)$ and system (5) will generate a Hopf bifurcation around the zero equilibrium point $W_1(0,0,0)$ for $\vartheta = \vartheta_0$.*

Remark 2. *From Theorem 1, we can easily know that the delay stability region of system (5) is $[0, \vartheta_{0*})$ and the critical value of the onset of Hopf bifurcation of system (5) is ϑ_0.*

4. Chaos Control via Fractional-Order PD$^\sigma$ Controller

In this segment, we shall design an appropriate controller to suppress the chaotic behavior of the chaotic Jerk system. Following the idea of Ding et al. [37–39], we design a mixed controller which includes time delay feedback controller and fractional-order PD$^\sigma$ controller as follows:

The time delay feedback controller is given by

$$\zeta(t) = \varrho_2[w_3(t-\vartheta) - w_3(t)], \tag{27}$$

where ϱ_2 represents feedback gain coefficient and ϑ is a delay.

The fractional-order PD$^\sigma$ controller is given by:

$$\varsigma(t) = \mu_p w_1(t-\vartheta) + \mu_d \frac{d^\sigma w_1(t)}{dt^\sigma}, \tag{28}$$

where μ_p and $\mu_d \neq 1$ stands for the proportional control coefficient and the derivative control coefficient, respectively, ϑ stands for the time delay. Adding (27) and (28) to the second equation and the first equation of system (5), we get

$$\begin{cases} \dfrac{d^\sigma w_1(t)}{dt^\sigma} = w_2(t) + \mu_p w_1(t-\vartheta) + \mu_d \dfrac{d^\sigma w_1(t)}{dt^\sigma}, \\ \dfrac{d^\sigma w_2(t)}{dt^\sigma} = w_3(t) + \varrho_2[w_3(t-\vartheta) - w_3(t)], \\ \dfrac{d^\sigma w_3(t)}{dt^\sigma} = -\alpha_1 w_1(t) - \alpha_2 w_2(t) - \alpha_3 w_3(t) + \alpha_4 w_3^2 + \alpha_5 w_1(t) w_2(t). \end{cases} \tag{29}$$

System (29) comes from adding two perturbation terms to the first equation and the second equation of system (5). It follows from (29) that

$$\begin{cases} \dfrac{d^\sigma w_1(t)}{dt^\sigma} = \dfrac{1}{1-\mu_d} w_2(t) + \dfrac{\mu_p}{1-\mu_d} w_1(t-\vartheta), \\ \dfrac{d^\sigma w_2(t)}{dt^\sigma} = w_3(t) + \varrho_2[w_3(t-\vartheta) - w_3(t)], \\ \dfrac{d^\sigma w_3(t)}{dt^\sigma} = -\alpha_1 w_1(t) - \alpha_2 w_2(t) - \alpha_3 w_3(t) + \alpha_4 w_3^2 + \alpha_5 w_1(t) w_2(t). \end{cases} \tag{30}$$

The linear system of (30) near the zero equilibrium point is given by:

$$\begin{cases} \dfrac{d^{\sigma}w_1(t)}{dt^{\sigma}} = \dfrac{1}{1-\mu_d}w_2(t) + \dfrac{\mu_p}{1-\mu_d}w_1(t-\vartheta), \\ \dfrac{d^{\sigma}w_2(t)}{dt^{\sigma}} = (1-\varrho_2)w_3(t) + \varrho_2 w_3(t-\vartheta), \\ \dfrac{d^{\sigma}w_3(t)}{dt^{\sigma}} = -\alpha_1 w_1(t) - \alpha_2 w_2(t) - \alpha_3 w_3(t) + \alpha_4 w_3^2 + \alpha_5 w_1(t)w_2(t). \end{cases} \tag{31}$$

The characteristic equation of (31) is given by:

$$\det\begin{bmatrix} s^{\sigma} - \frac{\mu_p}{1-\mu_d}e^{-s\vartheta} & -\frac{1}{1-\mu_d} & 0 \\ 0 & s^{\sigma} & (\varrho_2-1) - \varrho_2 e^{-s\vartheta} \\ \alpha_1 & \alpha_2 & s^{\sigma}+\alpha_3 \end{bmatrix} = 0. \tag{32}$$

Then,

$$\mathcal{V}_1(s) + \mathcal{V}_2(s)e^{-s\vartheta} + \mathcal{V}_3(s)e^{-2s\vartheta} = 0, \tag{33}$$

where

$$\begin{cases} \mathcal{V}_1(s) = s^{3\sigma} + d_1 s^{2\sigma} + d_2 s^{\sigma} + d_3, \\ \mathcal{V}_2(s) = d_4 s^{2\sigma} + d_5 s^{\sigma} + d_6, \\ \mathcal{V}_3(s) = d_7, \end{cases} \tag{34}$$

where

$$\begin{cases} d_1 = \alpha_3, \\ d_2 = \alpha_2(1-\varrho_2), \\ d_3 = \dfrac{\alpha_1(\varrho_2-1)}{1-\nu_d}, \\ d_4 = -\dfrac{\nu_p}{1-\nu_d}, \\ d_5 = \alpha_2\varrho_2 - \dfrac{\alpha_3}{1-\nu_d}, \\ d_6 = \dfrac{\alpha_2(\varrho_2-1)\mu_p - \alpha_1\varrho_2}{1-\nu_d}, \\ d_7 = \dfrac{\alpha_2\varrho_2\mu_p}{1-\nu_p}. \end{cases} \tag{35}$$

Equation (33) can be rewritten as:

$$\mathcal{V}_1(s)e^{s\vartheta} + \mathcal{V}_2(s) + \mathcal{V}_3(s)e^{-s\vartheta} = 0, \tag{36}$$

When $\vartheta = 0$, then (36) takes the following form:

$$\lambda^3 + (d_1+d_4)\lambda^2 + (d_2+d_5)\lambda + d_3 + d_6 + d_7 = 0. \tag{37}$$

If

$$(\mathcal{S}_4)\begin{cases} d_1 + d_4 > 0, \\ (d_1+d_4)(d_2+d_5) > d_3 + d_6 + d_7, \\ (d_3+d_6+d_7)[(d_1+d_4)(d_2+d_5) - (d_3+d_6+d_7)] > 0 \end{cases}$$

holds, then all the roots $\lambda_1, \lambda_2, \lambda_3$ of (37) obey $|\arg(\lambda_1)| > \frac{\sigma\pi}{2}, |\arg(\lambda_2)| > \frac{\sigma\pi}{2}$ and $|\arg(\lambda_3)| > \frac{\sigma\pi}{2}$. Applying Lemma 1, we get that the zero equilibrium point $W_1(0,0,0)$ of system (29) is locally asymptotically stable for $\vartheta = 0$.

Suppose that $s = i\theta = \theta(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$ is the root of Equation (36). Then,

$$\begin{cases} \mathcal{M}_1\cos\theta\vartheta + \mathcal{M}_2\sin\theta\vartheta = \mathcal{M}_3, \\ \mathcal{N}_1\cos\theta\vartheta + \mathcal{N}_2\sin\theta\vartheta = \mathcal{N}_3, \end{cases} \tag{38}$$

where

$$\begin{cases} \mathcal{M}_1 = e_1\theta^{3\sigma} + e_2\theta^{2\sigma} + e_3\theta^{\sigma} + e_4, \\ \mathcal{M}_2 = e_5\theta^{3\sigma} + e_6\theta^{2\sigma} + e_7\theta^{\sigma} + e_8, \\ \mathcal{M}_3 = e_9\theta^{2\sigma} + e_{10}\theta^{\sigma} + e_{11}, \\ \mathcal{N}_1 = f_1\theta^{3\sigma} + f_2\theta^{2\sigma} + f_3\theta^{\sigma} + f_4, \\ \mathcal{N}_2 = f_5\theta^{3\sigma} + f_6\theta^{2\sigma} + f_7\theta^{\sigma} + f_8, \\ \mathcal{N}_3 = f_9\theta^{2\sigma} + f_{10}\theta^{\sigma}, \end{cases} \tag{39}$$

where

$$\begin{cases} e_1 = \cos\dfrac{3\theta\pi}{2}, \\ e_2 = d_1\cos\sigma\pi, \\ e_3 = d_2\cos\dfrac{\sigma\pi}{2}, \\ e_4 = d_3 + d_7, \\ e_5 = -\sin\dfrac{3\sigma\pi}{2}, \\ e_6 = -d_1\sin\sigma\pi, \\ e_7 = -d_2\sin\dfrac{\sigma\pi}{2}, \\ e_8 = d_7, \\ e_9 = -d_4\cos\theta\pi, \\ e_{10} = -d_5\cos\dfrac{\sigma\pi}{2}, \\ e_{11} = -d_6, \\ f_1 = \sin\dfrac{3\sigma\pi}{2} \\ f_2 = d_1\sin\sigma\pi, \\ f_3 = d_2\sin\dfrac{\sigma\pi}{2}, \\ f_4 = d_7, \\ f_5 = \cos\dfrac{3\theta\pi}{2}, \\ f_6 = d_1\cos\sigma\pi, \\ f_7 = d_2\cos\dfrac{\sigma\pi}{2}, \\ f_8 = d_3 - d_7, \\ f_9 = -d_4\sin\theta\pi,, \\ f_{10} = -d_5\sin\dfrac{\sigma\pi}{2}. \end{cases} \tag{40}$$

It follows from (38) that:

$$\begin{cases} \cos\theta\vartheta = \dfrac{\mathcal{M}_3\mathcal{N}_2 - \mathcal{N}_3\mathcal{M}_2}{\mathcal{M}_1\mathcal{N}_2 - \mathcal{N}_1\mathcal{M}_2}, \\ \sin\theta\vartheta = \dfrac{\mathcal{N}_3\mathcal{M}_1 - \mathcal{N}_1\mathcal{M}_3}{\mathcal{M}_1\mathcal{N}_2 - \mathcal{N}_1\mathcal{M}_2}. \end{cases} \tag{41}$$

By means of (41), one derives:

$$(\mathcal{M}_3\mathcal{N}_2 - \mathcal{N}_3\mathcal{M}_2)^2 + (\mathcal{N}_3\mathcal{M}_1 - \mathcal{N}_1\mathcal{M}_3)^2 = (\mathcal{M}_1\mathcal{N}_2 - \mathcal{N}_1\mathcal{M}_2)^2, \tag{42}$$

which leads to

$$\begin{aligned} &\xi_1\theta^{12\sigma} + \xi_2\theta^{11\sigma} + \xi_3\theta^{10\sigma} + \xi_4\theta^{9\sigma} + \xi_5\theta^{8\sigma} + \xi_6\theta^{7\sigma} + \xi_7\theta^{6\sigma} \\ &+\xi_8\theta^{5\sigma} + \xi_9\theta^{4\sigma} + \xi_{10}\theta^{3\sigma} + \xi_{11}\theta^{2\sigma} + \xi_{12}\theta^{\sigma} + \xi_{13} = 0, \end{aligned} \tag{43}$$

where

$$
\begin{cases}
\xi_1 = (e_1 f_5 - f_1 e_5)^2, \\
\xi_2 = 2(e_1 f_5 - f_1 e_5)(e_1 f_6 + e_2 f_5 - f_1 e_6 - f_2 e_5), \\
\xi_3 = (e_1 f_6 + e_2 f_5 - f_1 e_6 - f_2 e_5)^2 + 2(e_1 f_5 - f_1 e_5) \\
\quad \times (e_1 f_7 + e_2 f_6 + e_3 f_5 - f_1 e_7 - f_2 e_6 - f_3 e_5), \\
\quad - (e_9 f_5 - f_9 e_5)^2 - (e_1 f_9 - f_9 e_1)^2, \\
\xi_4 = 2(e_1 f_5 - f_1 e_5)(e_1 f_8 + e_2 f_7 + e_3 f_6 + e_4 f_3 - f_1 e_8 \\
\quad - f_2 e_7 - f_3 e_6 - f_4 e_5) + 2(e_1 f_6 + e_2 f_5 - f_1 e_6 - f_2 e_5) \\
\quad \times (e_1 f_7 + e_2 f_6 + e_3 f_5 - f_1 e_7 - f_2 e_6 - f_3 e_5) \\
\quad - 2(e_9 f_5 - f_9 e_5)(e_9 f_6 + e_{10} f_5 - f_9 e_6 - f_{10} e_5) \\
\quad - 2(e_1 f_9 - f_9 e_1)(e_2 f_9 + e_1 f_{10} - f_1 e_{10} - f_2 e_9), \\
\xi_5 = (e_1 f_7 + e_2 f_6 + e_3 f_5 - f_1 e_7 - f_2 e_6 - f_3 e_5)^2 \\
\quad + 2(e_1 f_5 - f_1 e_5)(e_2 f_8 + e_3 f_7 + e_4 f_6 - f_2 e_8 \\
\quad - f_3 e_7 - f_4 e_6) + 2(e_1 f_6 + e_2 f_5 - f_1 e_6 - f_2 e_5) \\
\quad \times (e_1 f_8 + e_2 f_7 + e_3 f_6 + e_4 f_3 - f_1 e_8 - f_2 e_7 \\
\quad - f_3 e_6 - f_4 e_5) - (e_9 f_6 + e_{10} f_5 - f_9 e_6 - f_{10} e_5)^2 \\
\quad - 2(e_9 f_5 - f_9 e_5)(e_9 f_7 + e_{10} f_6 + e_{11} f_5 - f_9 e_7 - f_{10} e_6) \\
\quad - (e_2 f_9 + e_1 f_{10} - f_1 e_{10} - f_2 e_9)^2 - 2(e_1 f_9 - f_9 e_1) \\
\quad \times (e_3 f_9 + e_2 f_{10} - f_1 e_{11} - f_2 e_{10} - f_3 e_9), \\
\xi_6 = 2(e_1 f_5 - f_1 e_5)(e_3 f_8 + e_4 f_7 - f_3 e_8 - f_4 e_7) \\
\quad + 2(e_1 f_6 + e_2 f_5 - f_1 e_6 - f_2 e_5)(e_2 f_8 + e_3 f_7 + e_4 f_6 \\
\quad - f_2 e_8 - f_3 e_7 - f_4 e_6) + 2(e_1 f_7 + e_2 f_6 + e_3 f_5 \\
\quad - f_1 e_7 - f_2 e_6 - f_3 e_5)(e_1 f_8 + e_2 f_7 + e_3 f_6 + e_4 f_3 - f_1 e_8 \\
\quad - f_2 e_7 - f_3 e_6 - f_4 e_5) - 2(e_9 f_5 - f_9 e_5)(e_9 f_8 + e_{10} f_7 \\
\quad + e_{11} f_6 - f_9 e_8 - f_{10} e_7) - 2(e_9 f_7 + e_{10} f_6 + e_{11} f_5 \\
\quad - f_9 e_7 - f_{10} e_6)(e_9 f_8 + e_{10} f_7 + e_{11} f_6 - f_9 e_8 - f_{10} e_7) \\
\quad - 2(e_1 f_9 - f_9 e_1)(e_4 f_9 + e_3 f_{10} - f_2 e_{11} - f_3 e_{10}) \\
\quad - 2(e_2 f_9 + e_1 f_{10} - f_1 e_{10} - f_2 e_9)(e_3 f_9 + e_2 f_{10} \\
\quad - f_1 e_{11} - f_2 e_{10} - f_3 e_9), \\
\xi_7 = (e_1 f_8 + e_2 f_7 + e_3 f_6 + e_4 f_3 - f_1 e_8 - f_2 e_7 - f_3 e_6 - f_4 e_5)^2 \\
\quad + 2(e_1 f_5 - f_1 e_5)(e_4 f_8 - f_4 e_8) + 2(e_1 f_6 + e_2 f_5 - f_1 e_6 - f_2 e_5) \\
\quad \times (e_3 f_8 + e_4 f_7 - f_3 e_8 - f_4 e_7) + 2(e_1 f_7 + e_2 f_6 + e_3 f_5 - f_1 e_7 \\
\quad - f_2 e_6 - f_3 e_5)(e_2 f_8 + e_3 f_7 + e_4 f_6 - f_2 e_8 - f_3 e_7 - f_4 e_6) \\
\quad - (e_9 f_7 + e_{10} f_6 + e_{11} f_5 - f_9 e_7 - f_{10} e_6)^2 \\
\quad - 2(e_9 f_5 - f_9 e_5)(e_{10} f_8 + e_{11} f_7 - f_{10} e_8) \\
\quad - 2(e_9 f_6 + e_{10} f_5 - f_9 e_6 - f_{10} e_5)(e_9 f_8 + e_{10} f_7 \\
\quad + e_{11} f_6 - f_9 e_8 - f_{10} e_7) \\
\quad - (e_3 f_9 + e_2 f_{10} - f_1 e_{11} - f_2 e_{10} - f_3 e_9)^2 \\
\quad - 2(e_1 f_9 - f_9 e_1)(e_4 f_{10} - f_3 e_{11} - f_4 e_{10}) \\
\quad - 2(e_2 f_9 + e_1 f_{10} - f_1 e_{10} - f_2 e_9) \\
\quad \times (e_4 f_9 + e_3 f_{10} - f_2 e_{11} - f_3 e_{10}),
\end{cases}
\tag{44}
$$

$$
\begin{cases}
\xi_8 = 2(e_1f_6+e_2f_5-f_1e_6-f_2e_5)(e_4f_8-e_8f_4) \\
\quad +2(e_1f_7+e_2f_6+e_3f_5-f_1e_7-f_2e_6-f_3e_5) \\
\quad \times(e_3f_8+e_4f_7-f_3e_8-f_4e_7)+2(e_1f_8+e_2f_7 \\
\quad +e_3f_6+e_4f_3-f_1e_8-f_2e_7-f_3e_6-f_4e_5)(e_2f_8+e_3f_7 \\
\quad +e_4f_6-f_2e_8-f_3e_7-f_4e_6)-2e_{11}f_8(e_9f_5-f_9e_5) \\
\quad -2(e_9f_6+e_{10}f_5-f_9e_6-f_{10}e_5)(e_9f_8+e_{10}f_7 \\
\quad +e_{11}f_6-f_9e_8-f_{10}e_7)-2(e_9f_7+e_{10}f_6+e_{11}f_5 \\
\quad -f_9e_7-f_{10}e_6)(e_9f_8+e_{10}f_7+e_{11}f_6-f_9e_8-f_{10}e_7) \\
\quad +2e_{11}f_4(e_1f_9-f_9e_1)-2(e_2f_9+e_1f_{10}-f_1e_{10}-f_2e_9) \\
\quad \times(e_4f_{10}-e_{11}f_3-f_4e_{10})-2(e_3f_9+e_2f_{10} \\
\quad -f_1e_{11}-f_2e_{10}-f_3e_9)(e_4f_9+e_3f_{10} \\
\quad -f_2e_{11}-f_3e_{10}), \\
\xi_9 = (e_2f_8+e_3f_7+e_4f_6-f_2e_8-f_3e_7-f_4e_6)^2 \\
\quad +2(e_3f_8+e_4f_7-f_3e_8-f_4e_7)(e_1f_8+e_2f_7+e_3f_6+e_4f_3 \\
\quad -f_1e_8-f_2e_7-f_3e_6-f_4e_5)+2(e_4f_8-f_4e_8) \\
\quad \times(e_1f_7+e_2f_6+e_3f_5-f_1e_7-f_2e_6-f_3e_5) \\
\quad +2(e_3f_8+e_4f_7-f_3e_8-f_4e_7)(e_1f_8+e_2f_7+e_3f_6+e_4f_3 \\
\quad -f_1e_8-f_2e_7-f_3e_6-f_4e_5)-2(e_{10}f_8+e_{11}f_7-f_{10}e_8) \\
\quad \times(e_9f_7+e_{10}f_6+e_{11}f_5-f_9e_7-f_{10}e_6) \\
\quad -(e_9f_8+e_{10}f_7+e_{11}f_6-f_9e_8-f_{10}e_7)^2 \\
\quad -(e_4f_9+e_3f_{10}-f_2e_{11}-f_3e_{10})^2 \\
\quad +2f_4e_{11}(e_2f_9+e_1f_{10}-f_1e_{10}-f_2e_9) \\
\quad -2(e_3f_9+e_2f_{10}-f_1e_{11}-f_2e_{10}-f_3e_9) \\
\quad \times(e_4f_{10}-f_3e_{11}-f_4e_{10}), \\
\xi_{10} = 2(e_4f_8-f_4e_8)(e_1f_8+e_2f_7+e_3f_6 \\
\quad +e_4f_3-f_1e_8-f_2e_7-f_3e_6-f_4e_5)+2(e_2f_8 \\
\quad +e_3f_7+e_4f_6-f_2e_8-f_3e_7-f_4e_6) \\
\quad \times(e_3f_8+e_4f_7-f_3e_8-f_4e_7)+2e_{11}f_4 \\
\quad \times(e_3f_9+e_2f_{10}-f_1e_{11}-f_2e_{10}-f_3e_9) \\
\quad -2(e_4f_{10}-e_{11}f_3-f_4e_{10})(e_4f_9+f_{10}e_3 \\
\quad -f_2e_{11}-f_3e_{10}), \\
\xi_{11} = (e_3f_8+e_4f_7-f_3e_8-f_4e_7)^2+2(e_4f_8-f_4e_8) \\
\quad \times(e_2f_8+e_3f_7+e_4f_6-f_2e_8-f_3e_7-f_4e_6)-2e_{11}f_8 \\
\quad \times(e_9f_8+e_{10}f_7+e_{11}f_6-f_9e_8-f_{10}e_7) \\
\quad -(e_{10}f_8+e_{11}f_7-f_{10}e_8)^2 \\
\quad -(e_4f_{10}-e_{11}f_3-f_4e_{10})^2 \\
\quad -2f_4e_{11}(e_4f_9+e_3f_{10}-f_2e_{11}-f_{10}e_3), \\
\xi_{12} = 2(e_4f_8-f_4e_8)(e_3f_8+e_4f_7-f_3e_8-f_4e_7) \\
\quad -2e_{11}f_8(e_{10}f_8+e_{11}f_7-f_{10}e_8) \\
\quad +2e_{11}f_4(e_4f_{10}-e_{11}f_3-f_4e_{10}), \\
\xi_{13} = (e_4f_8-e_8f_4)^2-(e_{11}f_8)^2-(e_{11}f_4)^2.
\end{cases} \tag{45}
$$

Define

$$
\begin{aligned}
&\Xi_2(\theta) = \xi_1\theta^{12\sigma}+\xi_2\theta^{11\sigma}+\xi_3\theta^{10\sigma}+\xi_4\theta^{9\sigma}+\xi_5\theta^{8\sigma}+\xi_6\theta^{7\sigma} \\
&+\xi_7\theta^{6\sigma}+\xi_8\theta^{5\sigma}+\xi_9\theta^{4\sigma}+\xi_{10}\theta^{3\sigma}+\xi_{11}\theta^{2\sigma}+\xi_{12}\theta^{\sigma}+\xi_{13}.
\end{aligned} \tag{46}
$$

Suppose that:

$$(\mathcal{S}_5)\ \ (e_4f_8-e_8f_4)^2 < (e_{11}f_8)^2+(e_{11}f_4)^2$$

holds, because $\lim_{\theta\to\infty}\Xi_2(\theta) = +\infty$, then Equation (43) has at least one positive real root. Thus Equation (33) owns at least one pair of purely roots. Applying Sun et al. [40], we obtain the following conclusion.

Lemma 4. *Assume that $\xi_k > 0 (k = 1, 2, \cdots, 13)$, Equation (33) possesses no root with zero real parts for $\vartheta \geq 0$. (2) Assume that $(\mathcal{S}_5)$ is fulfilled and $\xi_k > 0\,(k = 1, 2, \cdots, 12)$, then Equation (33) has a pair of purely imaginary roots $\pm i\theta_0$ if $\vartheta = \vartheta_0^{(h)} (h = 1, 2, \cdots,)$ where*

$$\vartheta_0^{(h)} = \frac{1}{\theta_0}\left[\arccos\left(\frac{\mathcal{M}_3\mathcal{N}_2 - \mathcal{N}_3\mathcal{M}_2}{\mathcal{M}_1\mathcal{N}_2 - \mathcal{N}_1\mathcal{M}_2}\right) + 2h\pi\right], \tag{47}$$

where $h = 0, 1, 2, \cdots$, and $\varsigma_0 > 0$ denotes the unique zero of $\Xi_2(\theta)$.

Set $\vartheta_{0*} = \vartheta_0^{(0)}$. Now we make the hypothesis as follows:

$$(\mathcal{S}_6)\ \ \mathcal{G}_{1R}\mathcal{G}_{2R} + \mathcal{G}_{1I}\mathcal{G}_{2I} > 0,$$

where

$$\begin{cases} \mathcal{G}_{1R} = \left[3\sigma\theta_0^{3\sigma-1}\cos\frac{(3\sigma-1)\pi}{2} + 2\sigma d_1\theta_0^{2\sigma-1}\cos\frac{(2\sigma-1)\pi}{2}\right. \\ \qquad \left.+\sigma d_2\theta_0^{\sigma-1}\cos\frac{(\sigma-1)\pi}{2}\right]\cos\theta_0\vartheta_{0*} - \left[3\sigma\theta_0^{3\sigma-1}\sin\frac{(3\sigma-1)\pi}{2}\right. \\ \qquad \left.+2\sigma d_1\theta_0^{2\sigma-1}\sin\frac{(2\sigma-1)\pi}{2} + \sigma d_2\theta_0^{\sigma-1}\sin\frac{(\sigma-1)\pi}{2}\right]\sin\theta_0\vartheta_{0*} \\ \qquad + 2\sigma d_4\theta_0^{2\sigma-1}\cos\frac{(2\sigma-1)\pi}{2} + \sigma d_5\theta_0^{\sigma-1}\cos\frac{(\sigma-1)\pi}{2}, \\ \mathcal{G}_{1I} = \left[3\sigma\theta_0^{3\sigma-1}\cos\frac{(3\sigma-1)\pi}{2} + 2\sigma d_1\theta_0^{2\sigma-1}\cos\frac{(2\sigma-1)\pi}{2}\right. \\ \qquad \left.+\sigma d_2\theta_0^{\sigma-1}\cos\frac{(\sigma-1)\pi}{2}\right]\sin\theta_0\vartheta_{0*} + \left[3\sigma\theta_0^{3\sigma-1}\sin\frac{(3\sigma-1)\pi}{2}\right. \\ \qquad \left.+2\sigma d_1\theta_0^{2\sigma-1}\sin\frac{(2\sigma-1)\pi}{2} + \sigma d_2\theta_0^{\sigma-1}\sin\frac{(\sigma-1)\pi}{2}\right]\cos\theta_0\vartheta_{0*} \\ \qquad + 2\sigma d_4\theta_0^{2\sigma-1}\sin\frac{(2\sigma-1)\pi}{2} + \sigma d_5\theta_0^{\sigma-1}\sin\frac{(\sigma-1)\pi}{2}, \\ \mathcal{G}_{2R} = \left(\sigma_0^{3\sigma} + d_1\theta_0^{2\sigma}\cos\sigma\pi + d_2\theta_0^{\sigma}\cos\frac{\sigma\pi}{2} + d_3\right)\theta_0\sin\theta_0\vartheta_{0*} \\ \qquad - \left(\sigma_0^{3\sigma} + d_1\theta_0^{2\sigma}\sin\sigma\pi + d_2\theta_0^{\sigma}\sin\frac{\sigma\pi}{2} + d_3\right)\theta_0\cos\theta_0\vartheta_{0*} \\ \qquad + d_7\theta_0\sin\theta_0\vartheta_{0*}, \\ \mathcal{G}_{2I} = \left(\sigma_0^{3\sigma} + d_1\theta_0^{2\sigma}\cos\sigma\pi + d_2\theta_0^{\sigma}\cos\frac{\sigma\pi}{2} + d_3\right)\theta_0\cos\theta_0\vartheta_{0*} \\ \qquad + \left(\sigma_0^{3\sigma} + d_1\theta_0^{2\sigma}\sin\sigma\pi + d_2\theta_0^{\sigma}\sin\frac{\sigma\pi}{2} + d_3\right)\theta_0\sin\theta_0\vartheta_{0*} \\ \qquad + d_7\theta_0\cos\theta_0\vartheta_{0*}. \end{cases} \tag{48}$$

Lemma 5. *Suppose that $s(\vartheta) = \varpi_1(\vartheta) + i\varpi_2(\vartheta)$ is the root of (36) near $\vartheta = \vartheta_{0*}$ satisfying $\varpi_1(\vartheta_{0*}) = 0, \varpi_2(\vartheta_{0*}) = \theta_0$, then $Re\left(\frac{ds}{d\vartheta}\right)\Big|_{\vartheta=\vartheta_{0*},\theta=\theta_0} > 0$.*

Proof. By virtue of (36), one gets

$$\left(\frac{ds}{d\vartheta}\right)^{-1} = \frac{\mathcal{G}_1(s)}{\mathcal{G}_2(s)} - \frac{\vartheta}{s}, \tag{49}$$

where

$$\begin{cases} \mathcal{G}_1(s) = \left(3\sigma s^{3\sigma-1} + 2\sigma d_1 s^{2\sigma-1} + \sigma d_2 s^{\sigma-1}\right)e^{s\vartheta} \\ \qquad\quad + 2\sigma d_4 s^{2\sigma-1} + \sigma d_5 s^{\sigma-1}, \\ \mathcal{G}_2(s) = -se^{-s\vartheta}\left(s^{3\sigma} + d_1 s^{2\sigma} + d_2 s^{\sigma} + d_3\right) \\ \qquad\quad + d_7 s e^{-s\vartheta}. \end{cases} \tag{50}$$

Then,

$$\mathrm{Re}\left[\left(\frac{ds}{d\vartheta}\right)^{-1}\right]_{\vartheta=\vartheta_{0*},\theta=\theta_0} = \mathrm{Re}\left[\frac{\mathcal{G}_1(s)}{\mathcal{G}_2(s)}\right]_{\vartheta=\vartheta_{0*},\theta=\theta_0} = \frac{\mathcal{G}_{1R}\mathcal{G}_{2R} + \mathcal{G}_{1I}\mathcal{G}_{2I}}{\mathcal{G}_{2R}^2 + \mathcal{G}_{2I}^2}. \tag{51}$$

By virtue of $(\mathcal{S}_6)$, one derives:

$$\mathrm{Re}\left[\left(\frac{ds}{d\vartheta}\right)^{-1}\right]_{\vartheta=\vartheta_{0*},\theta=\theta_0} > 0.$$

The proof finishes. □

Applying Lemma 1, one can derive the following result.

Theorem 2. *Assume that $(\mathcal{S}_4)$–$(\mathcal{S}_6)$ are satisfied, then the zero equilibrium point $W_1(0,0,0)$ of system (29) is locally asymptotically stable provided that $\vartheta \in [0, \vartheta_{0*})$ and system (29) will generate a Hopf bifurcation around the zero equilibrium point $W_1(0,0,0)$, when $\vartheta = \vartheta_{0*}$.*

Remark 3. *Liu et al. [16] investigated the chaotic dynamics for some quadratic Jerk system (4), which only involves the integer-order operator. This present research is concerned with chaos control issue for Jerk system (5), which only involves the fractional-order operator. The research approach of [16] can not be applied to model (5) to derive chaos control results of this study. Based on this viewpoint, we think that the derived results of this study replenish the work of [16]. In addition, the investigation idea enriches the chaos control theory of fractional-order chaotic dynamical system.*

Remark 4. *Although there are many works that deal with the chaos control via time delay feedback controller, In this paper, we deal with the chaos control by two methods. One is the classical time delay feedback control, another is mixed control including time delay feedback control and fractional-order PD^{σ} control. Based on this viewpoint, we think that this paper has some novelties.*

Remark 5. *From Theorem 2, we can easily know that the delay stability region of system (29) is $[0, \vartheta_{0*})$ and the critical value of the onset of Hopf bifurcation of system (29) is ϑ_{0*}.*

Remark 6. *In this paper, we choose $\sigma = 0.94, \alpha_1 = 2, \alpha_2 = 1, \alpha_3 = 1.2, \alpha_4 = 0.5, \alpha_5 = 0.9$; through computer simulations, we know that the fractional-order Jerk system (5) displays chaotic behavior. If we choose another set of values, we also know whether system (5) will generate chaos via computer simulations. Of course, we can deal with the chaos control via the proposed controller.*

Remark 7. *In [37], Yu and Chen explored the Hopf bifurcation control of integer-order system via a time delay feedback controller. In [38], Ding et al. investigated the bifurcation control of integer-order complex networks by PD controller. In [39], Tang et al. dealt with the Hopf bifurcation of a congestion system via fractional-order PD control. In this work, we control the chaos of the Jerk system (5) via a mixed controller including a time delay feedback controller and a fractional-order PD^{σ} controller, which owns more adjustable parameters. Thus, our work generalizes the works of [37–39].*

5. Examples

Example 1. *Consider the following fractional-order controlled Jerk system:*

$$\begin{cases} \dfrac{d^{0.94}w_1(t)}{dt^{0.94}} = w_2(t), \\ \dfrac{d^{0.94}w_2(t)}{dt^{0.94}} = w_3(t) + 0.5[w_3(t-\vartheta) - w_3(t)], \\ \dfrac{d^{0.94}w_3(t)}{dt^{0.94}} = -\alpha_1 w_1(t) - \alpha_2 w_2(t) - \alpha_3 w_3(t) + \alpha_4 w_3^2 + \alpha_5 w_1(t) w_2(t). \end{cases} \tag{52}$$

One can easily get that system (52) has a zero equilibrium point $W_1(0,0,0)$. By virtue of Matlab software, one gets $v_0 = 3.8872, \vartheta_0 = 1.3$. The hypotheses $(\mathcal{S}_1)$–$(\mathcal{S}_3)$ in Theorem 1 are fulfilled. Let $\vartheta = 1.25 < \vartheta_0 = 1.3$, which implies that ϑ lies in the interval $[0, 1.3)$. The corresponding Matlab simulation plots are given in Figure 2. From Figure 2, one can easily find that all the physical state variables w_1, w_2, w_3 will tend to 0 when the time $t \to \infty$. Let $\vartheta = 1.45 > \vartheta_0 = 1.3$, which manifests that ϑ crosses the critical numerical value 1.3. The corresponding numerical simulation results are presented in Figure 3. Figure 3 shows very clearly that all the physical state variables w_1, w_2, w_3 are to preserve a periodic vibrational situation around 0 when the time $t \to \infty$. Both cases illustrate the disappearance of chaos of the fractional-order chaotic Jerk system (5). In addition, the bifurcation diagrams, which can be seen in Figures 4–6, are given to demonstrating that the bifurcation value of system (52) is approximately equal to 1.3. The numerical figures strongly support the effectiveness of the designed time delay feedback controller.

Example 2. *Consider the following fractional-order controlled Jerk system:*

$$\begin{cases} \dfrac{d^{0.94}w_1(t)}{dt^{0.94}} = w_2(t) - 0.5w_1(t-\vartheta) - 0.8\dfrac{d^{0.94}w_1(t)}{dt^{0.94}}, \\ \dfrac{d^{0.94}w_2(t)}{dt^{0.94}} = w_3(t) + 0.3[w_3(t-\vartheta) - w_3(t)], \\ \dfrac{d^{0.94}w_3(t)}{dt^{0.94}} = -2w_1(t) - w_2(t) - 1.2w_3(t) - 0.5w_3^2 + 0.9w_1(t) w_2(t). \end{cases} \tag{53}$$

One can easily get that system (53) has a zero equilibrium point $W_1(0,0,0)$. By virtue of Matlab software, one gets $\theta_0 = 2.0231, \vartheta_{0*} = 1.21$. The hypotheses $(\mathcal{S}_4)$–$(\mathcal{S}_6)$ in Theorem 2 are fulfilled. Let $\vartheta = 0.98 < \vartheta_{0*} = 1.21$, which implies that ϑ lies in the interval $[0, 1.21)$. The corresponding Matlab simulation plots are given in Figure 7. From Figure 7, one can easily find that all the physical state variables w_1, w_2, w_3 will tend to 0 when the time $t \to \infty$. Let $\vartheta = 1.3 > \vartheta_{0*} = 1.21$, which manifests that ϑ crosses the critical numerical value 1.21. The corresponding numerical simulation results are presented in Figure 8. Figure 8 shows very clearly that all the physical state variables w_1, w_2, w_3 are to preserve a periodic vibrational situation around 0 when the time $t \to \infty$. Both cases illustrate the disappearance of chaos of fractional-order chaotic Jerk system (5). In addition, the bifurcation diagrams, which can be seen in Figures 9–11, are given to demonstrate that the bifurcation value of system (53) is approximately equal to 1.21. The numerical figures strongly support the effectiveness of the designed mixed controller, which includes the time delay feedback controller and fractional-order PD^{σ} controller.

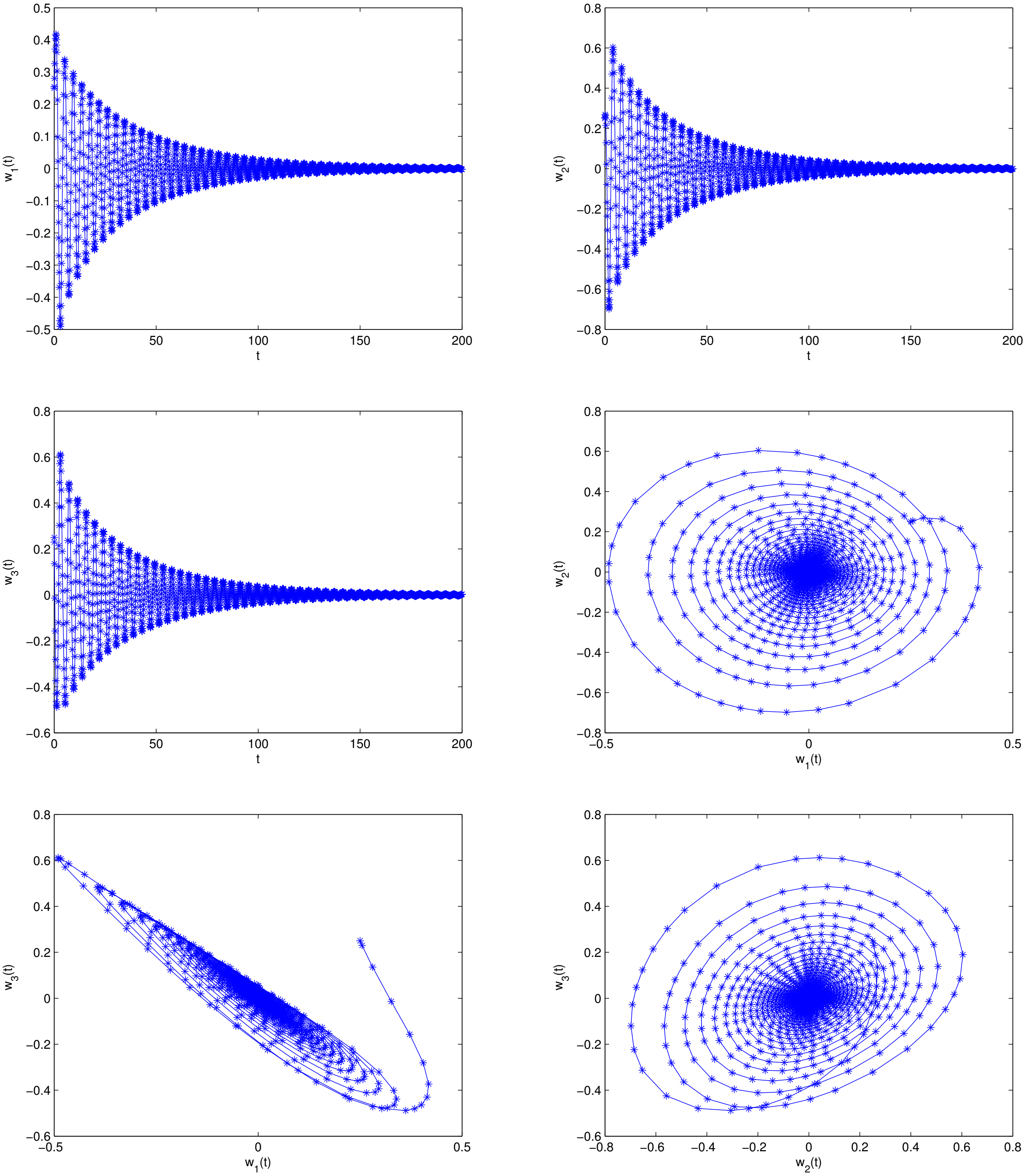

Figure 2. *Cont*.

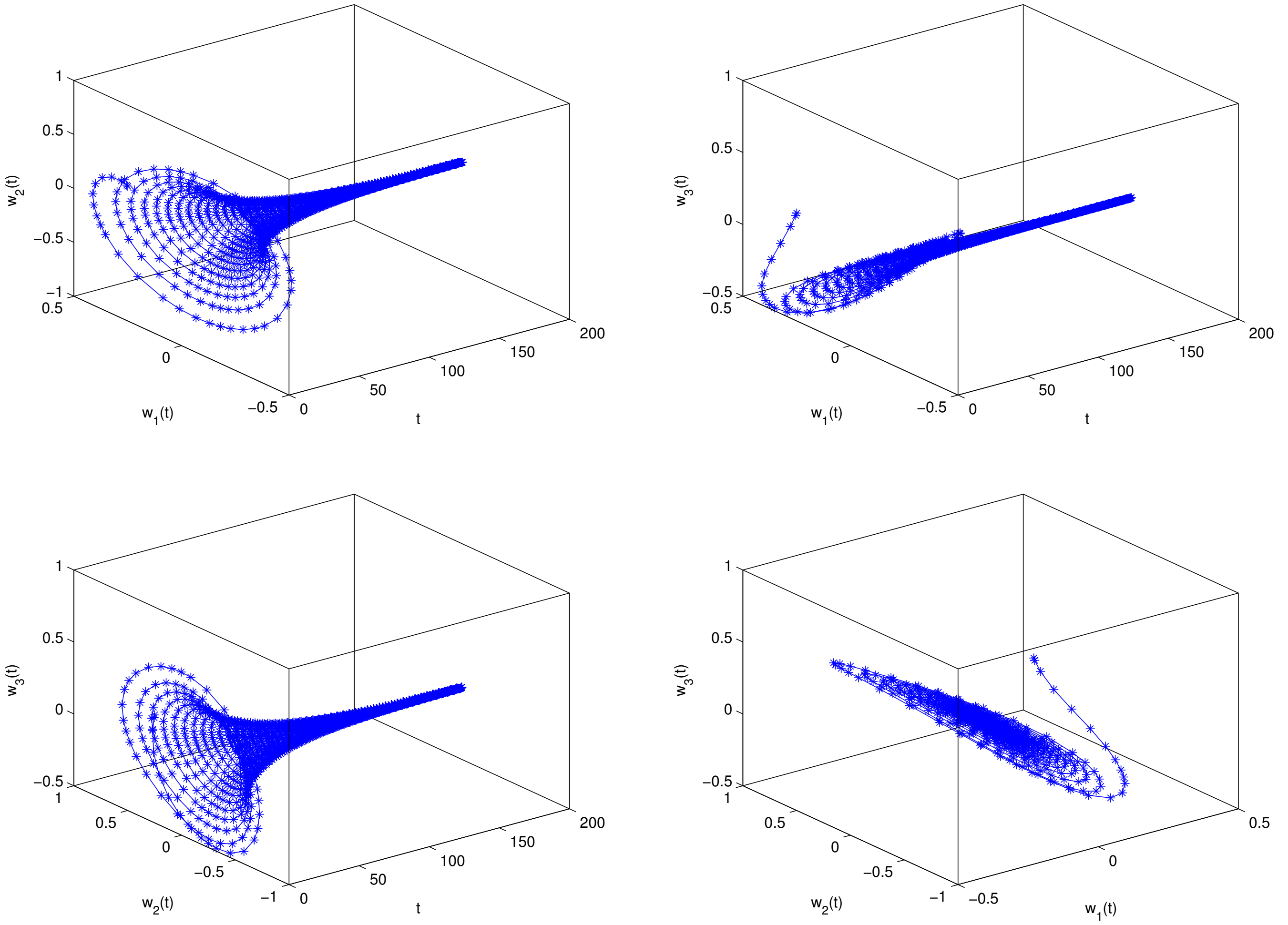

Figure 2. The Matlab simulation results of the controlled Jerk system (52) with $\vartheta = 1.25 < \vartheta_0 = 1.3$ and the initial value $(0.25, 0.25, 0.25)$.

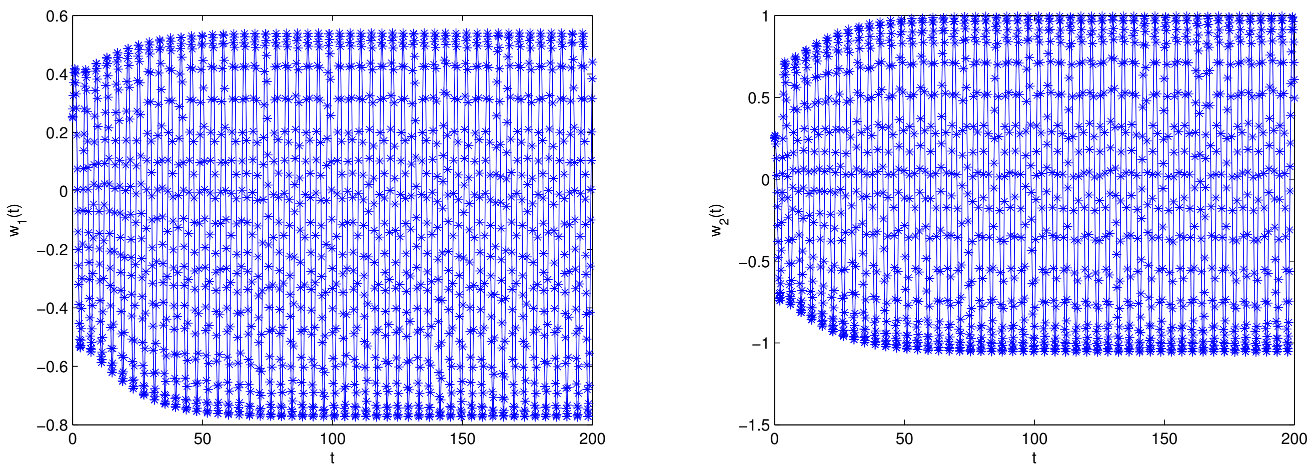

Figure 3. *Cont.*

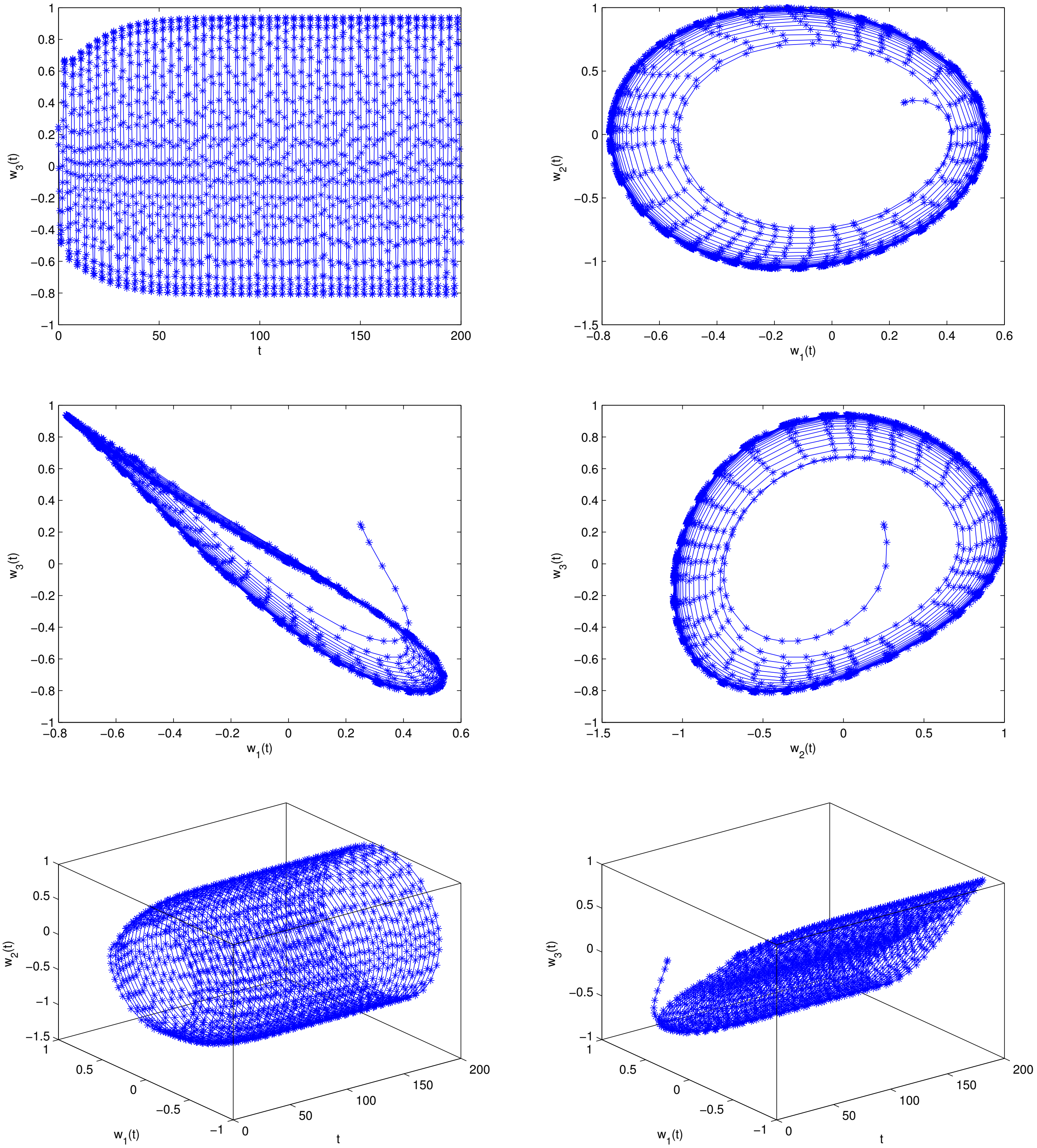

Figure 3. *Cont.*

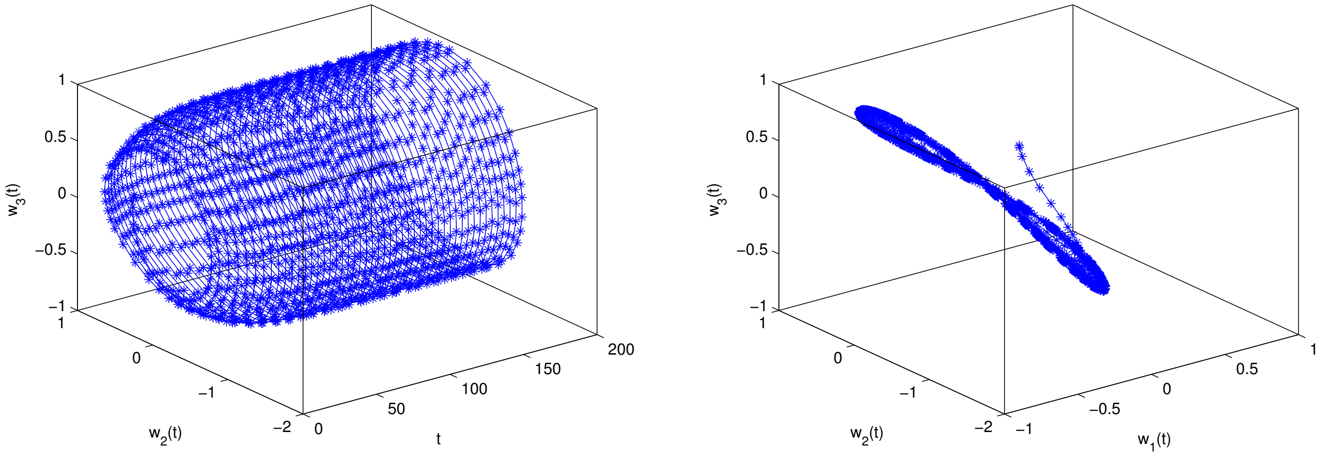

Figure 3. The Matlab simulation results of the controlled Jerk system (52) with $\vartheta = 1.45 > \vartheta_0 = 1.3$ and the initial value $(0.25, 0.25, 0.25)$.

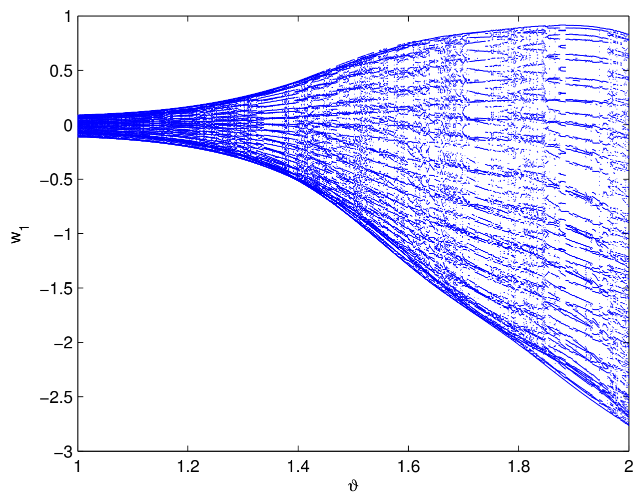

Figure 4. The bifurcation diagram of the controlled Jerk system (52): $\vartheta - w_1$.

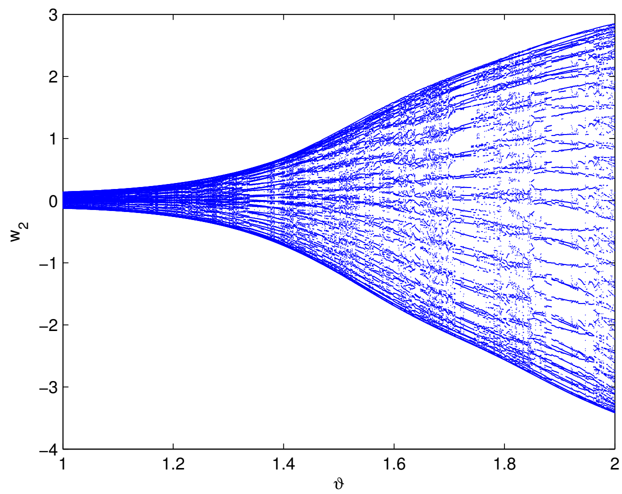

Figure 5. The bifurcation diagram of the controlled Jerk system (52): $\vartheta - w_2$.

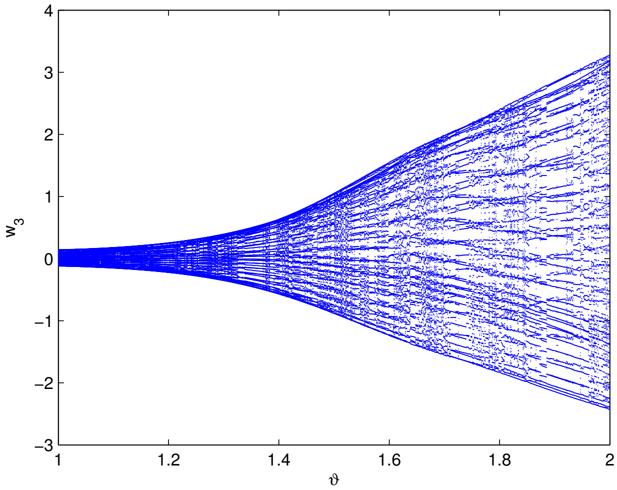

Figure 6. The bifurcation diagram of the controlled Jerk system (52): $\vartheta - w_3$.

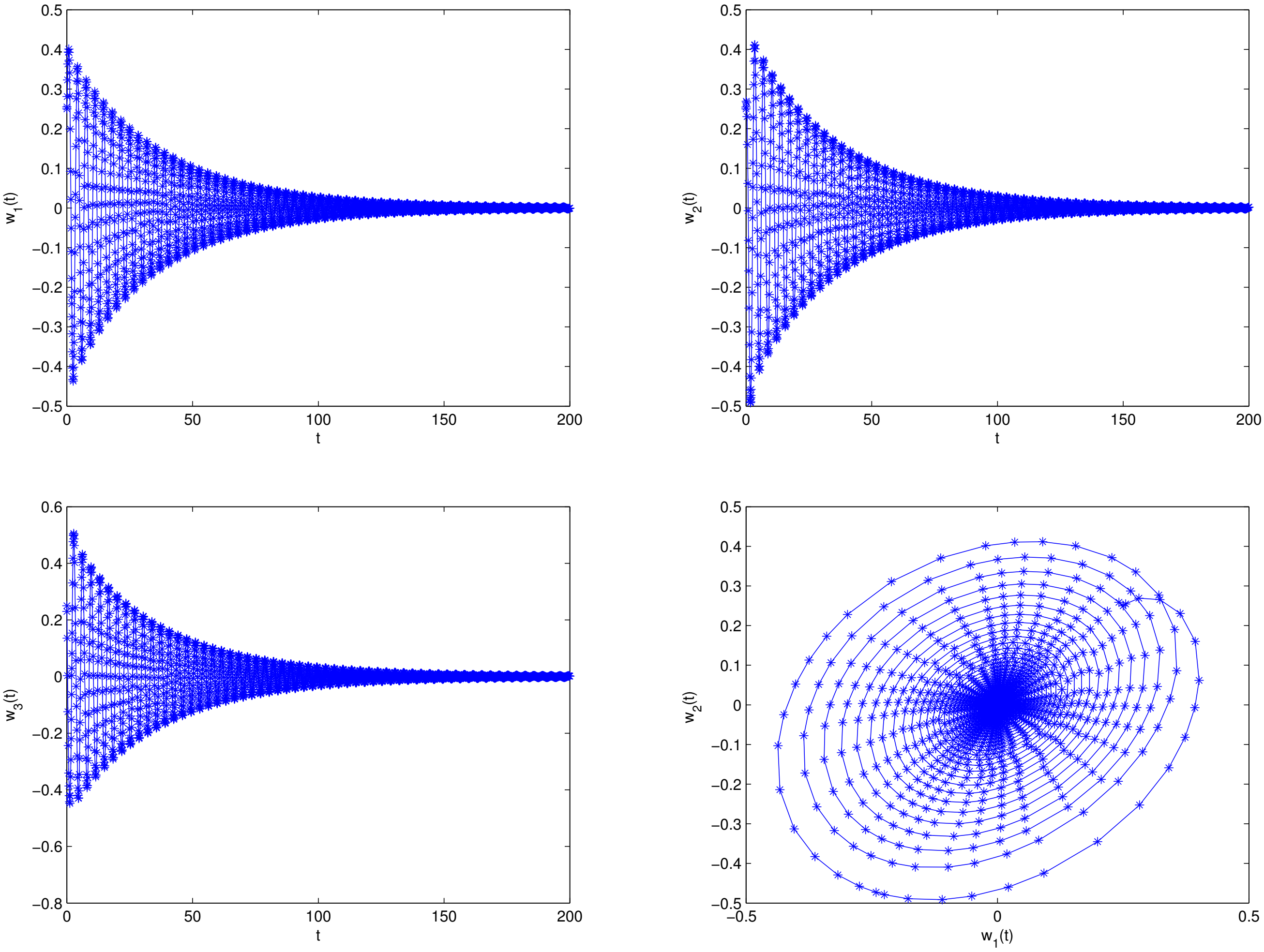

Figure 7. *Cont.*

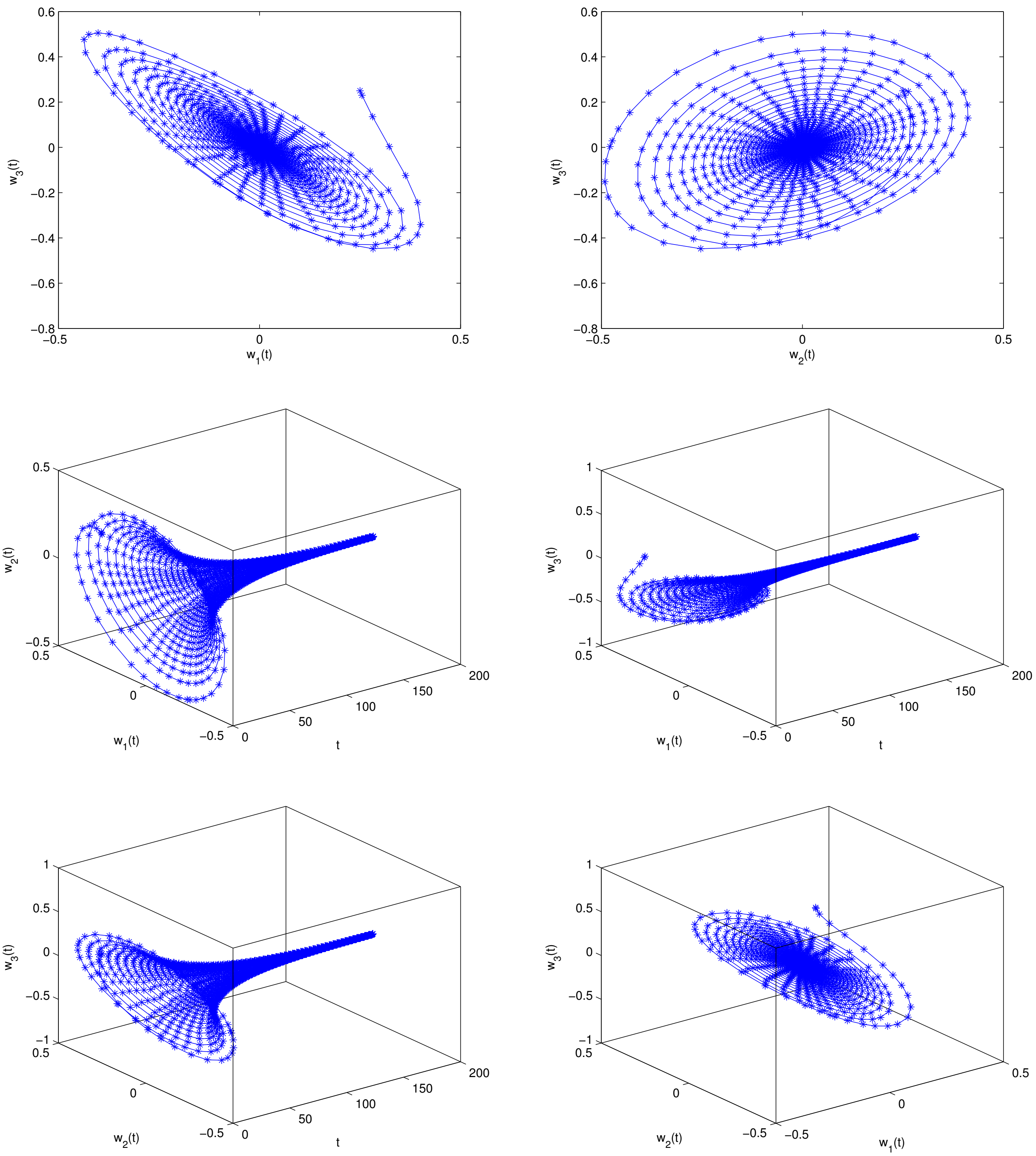

Figure 7. The Matlab simulation results of the controlled Jerk system (53) with $\vartheta = 0.98 < \vartheta_{0*} = 1.21$ and the initial value $(0.25, 0.25, 0.25)$.

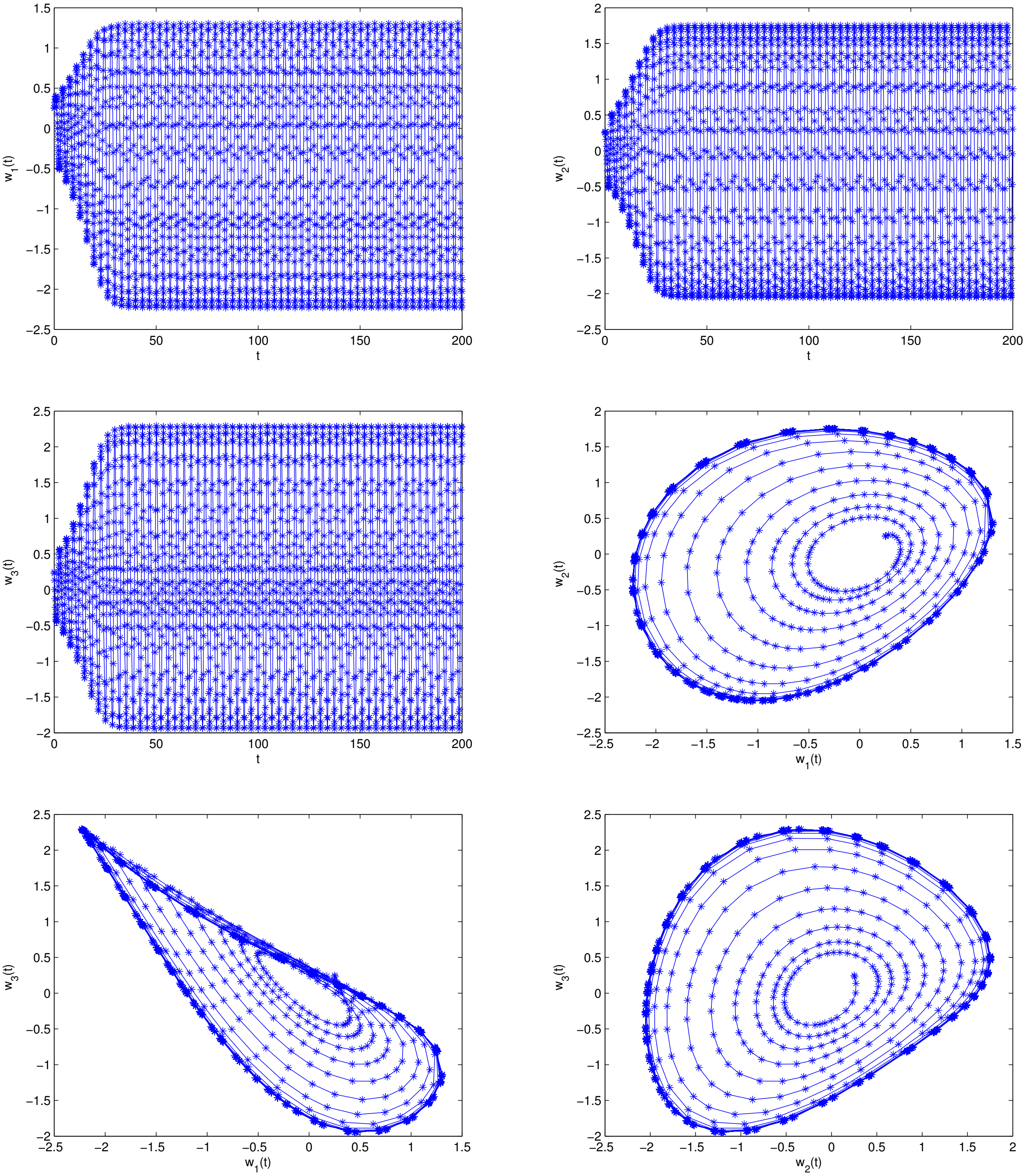

Figure 8. *Cont.*

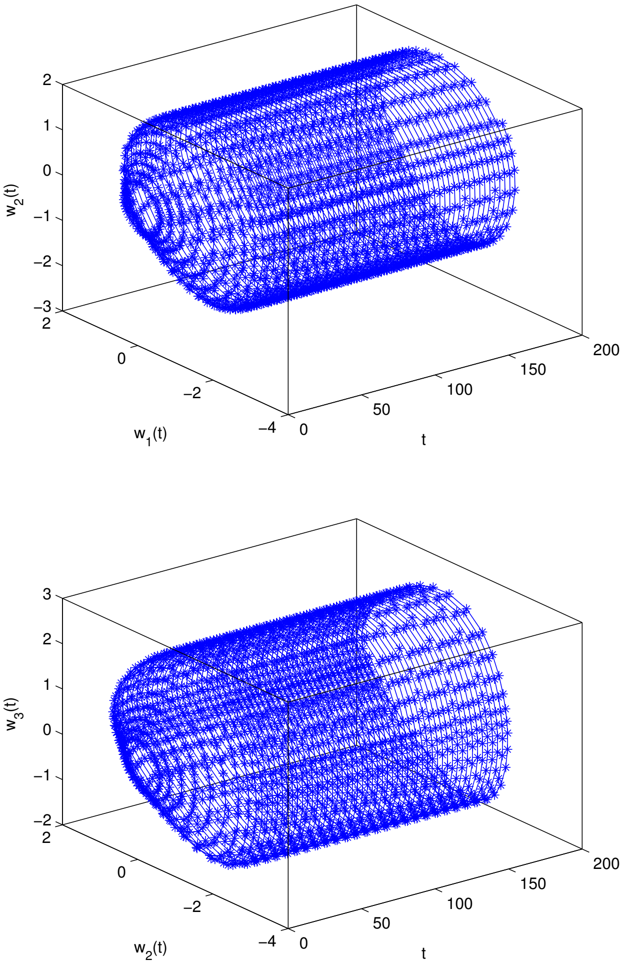

Figure 8. The Matlab simulation results of the controlled Jerk system (53) with $\vartheta = 1.3 > \vartheta_{0*} = 1.21$ and the initial value $(0.25, 0.25, 0.25)$.

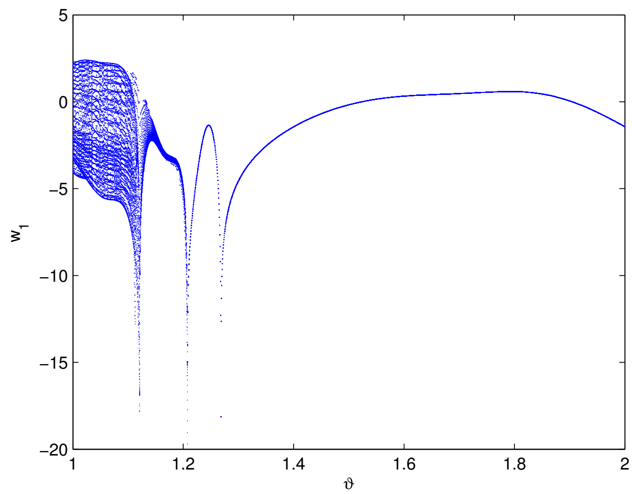

Figure 9. The bifurcation diagram of the controlled Jerk system (53): $\vartheta - w_1$.

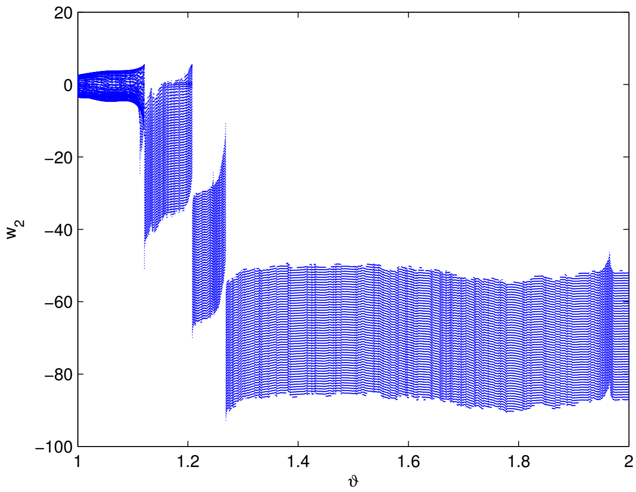

Figure 10. The bifurcation diagram of the controlled Jerk system (53): $\vartheta - w_2$.

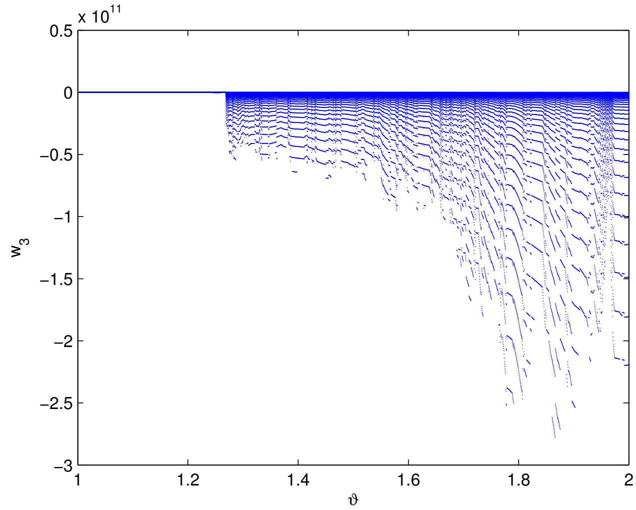

Figure 11. The bifurcation diagram of the controlled Jerk system (53): $\vartheta - w_3$.

Remark 8. *In term of the Matlab simulation results of Examples 1 and 2, we can see that the stability domain of the controlled Jerk system (53) is narrowed and the time of creation of the Hopf bifurcation of system (53) is advanced (in the controlled Jerk system (52), $\vartheta_0 = 1.3$, but in the controlled Jerk system (53), $\vartheta_{0*} = 1.21$).*

6. Conclusions

Chaos control is an ancient and classic problem. During the past decades, the chaos control has attracted great interest in scientific and technological circles. In this current work, on the basis of the previous literature, we build a new fractional-order chaotic Jerk system. By means of a reasonable time delay feedback controller, we can effectively control the chaotic phenomenon of the established fractional-order chaotic Jerk system. By virtue of a suitable mixed controller, which includes a time delay feedback controller and a fractional-order PD$^\sigma$ controller, we can successfully suppress the chaotic behavior of the established fractional-order chaotic Jerk system. The investigation indicates that the time delay in the time delay feedback controller and the mixed controller is a very momentous parameter in controlling the chaos of the fractional-order chaotic Jerk system. The established results of this work are completely novel and the investigation idea of this work can also be applied to probe many chaos control issues of fractional-order differential models in lots of disciplines. In the near future, we will try to deal with the chaos control of fractional-order dynamical models via other mixed controllers (for example, the combination of a nonlinear time delay feedback controller and a fractional-order PD^{σ} controller, etc.).

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