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Article

Fox H-Functions in Self-Consistent Description of a Free-Electron Laser

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Abstract: A fractional calculus concept is considered in the framework of a Volterra type integro-differential equation, which is employed for the self-consistent description of the high-gain free-electron laser (FEL). It is shown that the Fox H-function is the Laplace image of the kernel of the integro-differential equation, which is also known as a fractional FEL equation with Caputo–Fabrizio type fractional derivative. Asymptotic solutions of the equation are analyzed as well.

Keywords: Volterra type equation; Caputo–Fabrizio fractional derivative; Laplace transform; free-electron laser; Fox H-function

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1. Introduction

In this paper, we discuss a fractional calculus concept for the classical electrodynamics of free-electron lasers (FEL)s. It is well known that a self-consistent description of FELs is presented in the framework of an integro-differential equation. The latter can also be considered as a specific form of a fractional integro-differential equation. We study the kernel of this equation and show that it can be presented in the form of the Fox H-function. The Fox H-functions is widely used in fractional calculus, and it plays an essential role in a variety of applications of fractional calculus [1,2]. Between many of these examples, a new one is in the field of fractional electrodynamics of the FEL, recently considered in ref. [3]. Experimental implementation and theoretical description of the FEL is a long-lasting problem that started in the seventies of the last century. This extensively studied phenomenon is well described and reviewed [4–7], to mention a few. Contemporary studies are also reflected in recent publications and related to both classical and quantum descriptions [8,9]. In a paradoxical way, the classical electrodynamics of electrons interacting with electromagnetic fields explains this quantum lasing phenomenon [4–7,10]. In particular, the self-consistent evolution in the small-signal slow-varying amplitude approximation (of the electromagnetic field) is accounted by the Volterra-type integro-differential equation [3].

$$\frac{d}{d\tau}E(\tau) = -i\pi g_0 \int_0^\tau E(\tau-\tau')e^{-iv\tau'}e^{-\mu\tau'^2}\tau' d\tau' , \qquad E(\tau=0) = E_0 . \tag{1}$$

Here $\tau = (t+z/c)/\Delta t$ is a dimensionless gain time-scale, where Δt is the interaction time, z is the longitudinal coordinates, t is the current time, and c is the light speed. The dimensionless parameters of the system include the resonance parameter v, which is linked to the laser frequency and scaled by Δt, the small-signal gain coefficient g_0, and the coefficient μ, which relates to a parameter regulating the effects of the gain reduction due to the electron's energy distribution [3].

In the absence of the attenuation of the gained signal, $\mu = 0$, this integro-differential Equation (1) has a solution in the superposition form $E(\tau) = \sum_j E_j \exp(i\Lambda_j\tau)$, where E_j and Λ_j are related to the roots of a cubic equation [10]. Performing the Laplace transform of Equation (1),

$$\tilde{E}(s) = \mathcal{L}[E(\tau)](s) = \int_0^\infty E(\tau)e^{-s\tau}d\tau ,$$

we have for $\mu = 0$

$$\tilde{E}(s) = \frac{E_0(s+iv)^2}{s(s+iv)^2 + i\pi g_0} \equiv \frac{E_0(s+iv)^2}{(s-s_1)(s-s_2)(s-s_3)}\,, \tag{2}$$

where E_0 is the initial condition for Equation (1). The poles s_j are the roots of the cubic equation

$$s^3 + 2ivs^2 - v^2 s + i\pi g_0 = 0 \tag{3}$$

determined by the Cardano rule. This equation defines both Λ_j and E_j.

In the Volterra Equation (1), studied in ref. [3], the integro-differential operator in the r.h.s. was considered by analogy to the Caputo–Fabrizio fractional derivative [11], which is a fractional derivative without a singular kernel [3,11]. This fractional derivative/operator has been treated in the form of an iteration technique, based on an expansion employing a family of two variable Hermite polynomials that eventually leads to the analytical solution [3]. Following this fractional calculus concept of refs. [3,11], it is reasonable to suggest an alternative approach for the kernel of the integral operator, presenting it in the form of the Fox H-functions. In this case, the Laplace transformation of Equation (1) becomes feasible for $\mu \neq 0$.

2. Fox *H*-Function in Laplace Space

Performing the Laplace transformation of Equation (1), one obtains

$$s\tilde{E}(s) - E_0 = -i\pi g_0 \tilde{G}(s)\tilde{E}(s)\,, \tag{4}$$

where $\tilde{G}(s)$ is defined by the integral

$$\tilde{G}(s) = \int_0^\infty \tau e^{-(s+iv)\tau} e^{-\mu\tau^2} d\tau\,. \tag{5}$$

The way of introducing the Fox H-function inside the integrand, based on the representation of the exponential function in the form of the Fox H-function by means of the Mellin–Barnes integration

$$e^{-\mathcal{Z}} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(\xi)\mathcal{Z}^{-\xi} d\xi\,, \tag{6}$$

see, e.g., [2,12,13]. Here $\Gamma(\xi)$ is a gamma function: $\Gamma(\xi+1) = \xi\Gamma(\xi)$. Performing the variable change $y = \tau^2$ in the integral (5), we have

$$\tilde{G}(s) = \frac{1}{2}\int_0^\infty e^{-s_v y^{1/2}} e^{-\mu y} dy\,, \tag{7}$$

where $s_v = s + iv$ is used for brevitys sake. Then taking $\mathcal{Z} = s_v y^{1/2}$ and substituting the Mellin–Barnes integral (6) inside integration (7), we obtain the chain of transformations as follows

$$\begin{aligned}\tilde{G}(s) = \frac{1}{2}\int_0^\infty e^{-\mu y} \cdot \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(\xi) s_v^{-\xi} y^{-\frac{\xi}{2}} d\xi dy \\ = \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(\xi) s_v^{-\xi}\left[\int_0^\infty e^{-\mu y} y^{-\frac{\xi}{2}} dy\right] d\xi \\ = \frac{1}{2\mu}\cdot\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(\xi)\Gamma(1-\xi/2)\left(\mu^{-1/2} s_v\right)^{-\xi} d\xi\,.\end{aligned} \tag{8}$$

Eventually, we arrive at the definition of the Fox H-function, which is presented in terms of the Mellin–Barnes integral. Its general definition reads [2]

$$H^{m,n}_{p,q}(\mathcal{Z}) = H^{m,n}_{p,q}\left[\mathcal{Z}\left| \begin{array}{c} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{array} \right.\right] = \frac{1}{2\pi i}\int_{\Omega} \Theta(\xi)\mathcal{Z}^{-\xi} d\xi\,, \tag{9}$$

where

$$\Theta(\xi) = \frac{\left\{\prod_{j=1}^{m}\Gamma(b_j + B_j\xi)\right\}\left\{\prod_{j=1}^{n}\Gamma(1 - a_j - A_j\xi)\right\}}{\left\{\prod_{j=m+1}^{q}\Gamma(1 - b_j - B_j\xi)\right\}\left\{\prod_{j=n+1}^{p}\Gamma(a_j + A_j\xi)\right\}}\,, \tag{10}$$

with $0 \le n \le p$, $1 \le m \le q$ and $a_i, b_j \in C$, while $A_i, B_j \in R^+$, for $i = 1, \ldots, p$, and $j = 1, \ldots, q$. The contour Ω starting at $c - i\infty$ and ending at $c + i\infty$, separates the poles of the functions $\Gamma(b_j + B_j\xi)$, $j = 1, \ldots, m$ from those of the function $\Gamma(1 - a_i - A_i\xi)$, $i = 1, \ldots, n$.

In our case of Equation (8), $\mathcal{Z} = \mu^{-1/2}(s + iv)$ and $\Theta(\xi) = \Gamma(\xi)\Gamma(1 - \xi/2)$, while $a_1 = b_1 = 0$ and $2A_1 = B_1 = 1$. Therefore, comparing Equation (8) to Equations (9) and (10) one obtains

$$2\mu\tilde{G}(s) = H^{1,1}_{1,1}\left[\frac{iv}{\mu^{1/2}}(1 - \frac{is}{v})\left| \begin{array}{c} (0, 1/2) \\ (0, 1) \end{array}\right.\right]. \tag{11}$$

Thus, the poles of the Laplace image of the gained signal $\tilde{E}(s)$ are determined by the transcendent equation as follows

$$s - \frac{i\pi g_0}{2\mu} H^{1,1}_{1,1}\left[iv\mu^{-1/2}(1 - is/v)\left| \begin{array}{c} (0,\ 1/2) \\ (0,\ \ 1) \end{array}\right.\right] = 0\,. \tag{12}$$

Limit Case $\mu = 0$

Let us show that for $\mu = 0$, Equation (12) reduces to Equation (6). To that end, the argument of the Fox H-function $\mathcal{Z}(s)$ should be taken as the reciprocal function $1/\mathcal{Z}(s)$ according to the identity [2]

$$H^{m,n}_{p,q}\left[\mathcal{Z}\left| \begin{array}{c} (a_p, A_p) \\ (b_q, B_q) \end{array}\right.\right] = H^{n,m}_{q,p}\left[\frac{1}{\mathcal{Z}}\left| \begin{array}{c} (1 - b_q, B_q) \\ (1 - a_p, A_p) \end{array}\right.\right]. \tag{13}$$

Therefore

$$H^{1,1}_{1,1}\left[iv\mu^{-1/2}(1 - is/v)\left| \begin{array}{c} 0,\ 1/2 \\ 0,\ \ 1 \end{array}\right.\right] = H^{1,1}_{1,1}\left[\frac{\mu^{1/2}}{(s + iv)}\left| \begin{array}{c} (1,\ \ 1) \\ (1,\ 1/2) \end{array}\right.\right]. \tag{14}$$

When $\mu \to 0$, the argument $1/\mathcal{Z}(s) = \mu^{1/2}/(s + iv) \to 0$, as well. Then the asymptotic behavior of the Fox H-function for the small argument, limiting to zero, reads [2]

$$H^{p,q}_{m,n}(\mathcal{Z}) \sim a\mathcal{Z}^c, \quad c = \min\left[\frac{\operatorname{Re} b_j}{B_j}\right].$$

Taking into account that $b_1 = 1$ and $B_1 = 1/2$, this yields for the r.h.s. of Equation(14)

$$H^{1,1}_{1,1}\left[\mu^{1/2}(s + iv)^{-1}\left| \begin{array}{c} (1,\ \ 1) \\ (1,\ 1/2) \end{array}\right.\right] \sim a\left(\frac{\mu}{(s + iv)^2}\right). \tag{15}$$

Taking $a = 2$ and substituting Equation (15) into Equation (11), we obtain that the latter reduces to the cubic Equation (6).

3. Series Expansion and Asymptotics

The Fox H-function can be presented in the form of series expansion [2]. Then we have the l.h.s. of Equation (14)

$$H^{1,1}_{1,1}\left[iv\mu^{-1/2}(1-is/v)\middle|\begin{matrix}(0,\ 1/2)\\(0,\quad 1)\end{matrix}\right]=\sum_{r=0}^{\infty}\frac{(is/v)^r}{r!}H^{1,1}_{1,1}\left[iv\mu^{-1/2}\middle|\begin{matrix}(0,\ 1/2)\\(r,\quad 1)\end{matrix}\right]. \tag{16}$$

Let us consider asymptotic behavior, when $|s/v|\ll 1$ and $|v\mu^{-1/2}|\gg 1$. Then to obtain a gained signal at least the first four terms in the expansion should be accounted for. Then we have

$$H^{1,1}_{1,1}\left[iv\mu^{-1/2}(1-is/v)\middle|\begin{matrix}(0,\ 1/2)\\(0,\quad 1)\end{matrix}\right]\approx C_0+iC_1s/v-C_2s^2/2v^2-iC_3s^3/6v^3)\,, \tag{17}$$

where

$$C_r=C_r\left(iv\mu^{1/2}\right)=H^{1,1}_{1,1}\left[iv\mu^{-1/2}\middle|\begin{matrix}(0,\ 1/2)\\(r,\quad 1)\end{matrix}\right]. \tag{18}$$

Then, Equation (12) reduces to the cubic equation with roots $s_j=s_j(\mu,v)$.

The coefficients of the expansion in the form of the Fox H-functions of large arguments behave as follows [2]

$$H^{1,1}_{1,1}(\mathcal{Z})\approx\mathcal{Z}^d,\qquad d=\min\frac{\operatorname{Re}a_1-1}{A_1}=-2\,.$$

Then $C_r=\mu/v^2$ for $r=0,1,2,3$, and we have

$$H^{1,1}_{1,1}\left[iv\mu^{-1/2}(1-is/v)\middle|\begin{matrix}(0,\ 1/2)\\(0,\quad 1)\end{matrix}\right]\approx-\mu/v^2(1+is/v-s^2/2v^2-is^3/6v^3)\,. \tag{19}$$

Thus, by analogy with Equation (12), the poles of $\tilde{E}(s)$ are determined by a cubic equation, which now reads

$$s^3-3vis^2-\frac{12v^5}{\pi g_0}\left(1+\frac{\pi g_0}{2v^2}\right)s+6v^3i=0\,. \tag{20}$$

Note that this expression is independent of μ, which results from the asymptotic consideration for both $|s/v|\ll 1$ and $|v\mu^{-1/2}|\gg 1$.

In any case of the cubic equation, the solution is the superposition of three waves

$$E(\tau)=\sum_{j=1}^{3}E_je^{s_j\tau}\,, \tag{21}$$

where s_j are roots of the cubic equation, which is an approximation of Equation (12) that also determine E_j with the initial condition $\sum_{j=1}^3 E_j=E_0$.

Asymptotics of Small τ and Series Expansion

Disregarding the fractional concept, related to the Fox H-function and looking for the asymptotic solution of Equation (1) for the small gain time-scale $\tau\ll 1$, a simplified consideration can be suggested as follows. Noting that the Laplace image $\tilde{G}(s)$ in Equation (5) is a table integral [14], which reads

$$\tilde{G}(s)=-\frac{d}{d\,s}\sqrt{\frac{\pi}{4\mu}}e^{\frac{(s+iv)^2}{4\mu}}\operatorname{Erfc}\left(\frac{s+iv}{\mu^{1/2}}\right). \tag{22}$$

Taking into account the asymptotic behavior of the Erfc-function for the large values of $|s+iv| \gg 1$ [15], we have

$$\tilde{G}(s) \approx -\frac{d}{ds}\sqrt{\frac{\pi}{4\mu}}\frac{\mu^{1/2}}{2(s+iv)}\left[1+\sum_{n=1}^{\infty}(2n-1)!!\left(\frac{-2(s+iv)^2}{\mu}\right)^{-n}\right] \approx \frac{\pi^{1/2}}{4(s+iv)^2}\,. \tag{23}$$

Therefore, the initial gain solution of Equation (1) is $G(\tau) = \sum_{j=1}^{3} E_j e^{s_j \tau}$, where s_j is the root of the cubic equation.

Taking into account that Equations (5), (11) and (22) describe the same image $\tilde{G}(s)$, we obtain the asymptotic series expansion for $|s+iv| \gg 1$, which reads

$$H_{1,1}^{1,1}\left[(s+iv)\mu^{-1/2}\middle|\begin{array}{c}(0,1/2)\\(0,1)\end{array}\right] \approx \frac{\pi^{1/2}\mu}{2(s+iv)^2}\left[1+\sum_{n=1}^{\infty}(2n-1)!!\left(\frac{-2(s+iv)^2}{\mu}\right)^{-n}\right]. \tag{24}$$

4. Conclusions

A fractional calculus concept was considered in the framework of a Volterra type integro-differential equation, which is known is employed for the self-consistent description of the high-gain free-electron laser (FEL). We have shown that the Fox H-function can be employed for the Laplace image of the kernel of the integro-differential equation. The analysis was performed in the framework of the Laplace transformation with respect to the gain time-scale τ. Note that the FEL geometry can be chosen in such a way that $\tau > 0$. This approach makes it possible to obtain an exact analytical expression for the Laplace image of the gained signal $\tilde{E}(s)$, and its singular behavior is determined by the roots of the transcendent Equation (12). Further analytical analysis is possible (and presented) in the asymptotic approximation for both large $\tau \gg 1$ and small $\tau \ll 1$. In either case, these solutions are described by cubic equations with coefficients depending on μ. It is worth mentioning that an alternating approach to the electron energy distribution has been considered as well [16] in the framework of the fractional generalization of the FEL Equation (1) with $\mu = 0$.

Discussing a mathematical aspects related to fractional calculus, it should be admitted that, the r.h.s. of Equation (1) can be considered as an FEL fractional derivative by analogy with the Caputo–Fabrizio fractional derivative [11,17], which reads as follows

$$\mathcal{D}_{\mathrm{CF}}f(\tau) = \frac{m(\alpha)}{1-\alpha}\int_a^{\tau} f(\xi)e^{-\frac{\tau-\xi}{1-\alpha}}d\xi\,, \tag{25}$$

where $m(\alpha)$ is a normalization term with constant α. Therefore the FEL fractional derivative reads

$$\mathcal{D}_{\mathrm{FEL}}f(\tau) = \int_0^{\tau} e^{-iv(\tau-\xi)}e^{-\mu(\tau-\xi)^2}(\tau-\xi)f(\xi)d\xi\,. \tag{26}$$

Therefore, the Laplace image of the kernel of the FEL fractional derivative, $\tilde{G}(s)$ obtained in Equation (11) is the Fox H-function in Laplace space. We also note in passing that $\mathcal{D}_{FEL}$ in Equation (26) differs from those introduced in ref. [3] by the term $(\tau-\xi)f(\xi) \to \xi f(\xi)$.

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