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Nonlocal Boundary Value Problems of Nonlinear Fractional (p,q)-Difference Equations

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Abstract: In this paper, we study nonlinear fractional (p,q)-difference equations equipped with separated nonlocal boundary conditions. The existence of solutions for the given problem is proven by applying Krasnoselskii's fixed-point theorem and the Leray–Schauder alternative. In contrast, the uniqueness of the solutions is established by employing Banach's contraction mapping principle. Examples illustrating the main results are also presented.

Keywords: Caputo fractional (p,q)-difference equations; nonlocal boundary conditions; existence and uniqueness; Leray–Schauder alternative; fixed-point theory

MSC: 05A30; 26A51; 26D10; 26D15

1. Introduction

In recent years, fractional differential equations have been considered a popular field of research and have attracted many researchers' attention. This is mainly because fractional differential equations are found to be effective and more practical than classical differential equations, particularly in the mathematical modeling of dynamical systems, such as fractals and chaos. In the last two decades, the fractional differential equations have developed from theoretical aspects of the existence and uniqueness of solutions to analytic and numerical methods for finding solutions, which can be found in [1–9]. Moreover, in modern mathematical analysis, fractional differential equations have a range of applications, such as engineering and clinical disciplines, including biology, physics, chemistry, economics, signal and image processing, and control theory; see [10–17] for more details.

In 1910, F. H. Jackson initiated a study of the q-difference calculus or quantum calculus (briefly called q-calculus) in a symmetrical manner and introduced the q-derivative and q-integral, which can be found in [18,19]. With these results, q-calculus has arisen in a range of applications, such as combinatorics, orthogonal polynomials, basic hypergeometric functions, number theory, quantum theory, mechanics, and theory of relativity; see [20–40] and the references cited therein. The book by V. Kac and P. Cheung [41] covers the basic theoretical concepts of q-calculus.

Because the exploration has been continued by generalizing the existing results through creative discussions and novel techniques of fractional calculus, fractional q-difference calculus has been introduced by W.A. Al-Salam [42] and R.P. Agarwal [43]. Furthermore, fractional q-difference equations have attracted the attention of researchers;

for instance, B. Ahmad et al. [44] investigated the existence results of nonlocal boundary value problems of nonlinear fractional q-difference equations:

$$\begin{cases} {}^{c}D_q^{\alpha} x(t) = f(t, x(t)), \ t \in [0,1], 1 < \alpha \leq 2, \\ \alpha_1 x(0) + \beta_1 D_q x(0) = \gamma_1 x(\eta_1), \quad \alpha_2 x(1) + \beta_2 D_q x(1) = \gamma_2 x(\eta_2), \end{cases} \tag{1}$$

where $f \in \mathcal{C}([0,1] \times \mathbb{R}, \mathbb{R})$ and $\alpha_i, \beta_i, \gamma_i, \eta_i (i = 1,2)$ are constants with $\eta_i \in (0,1), {}^{c}D_q^{\alpha}$ and D_q are the fractional q-derivative of the Caputo type and the first-order q-difference operator, respectively. For more early and recent works on the existence theory of fractional q-difference equations, we refer to [45–48].

Later on, the subject of (p,q)-calculus was generalized and developed from the q-calculus theory into the two-parameter (p,q)-integer, which is used effectively in many fields, and some results on the study of (p,q)-calculus can be found in [49–74].

Inspired and motivated by some of the above applications, in 2018, N. Kamsrisuk et al. [75] considered the existence and uniqueness of solutions for the first-order quantum (p,q)-difference equation subject to a nonlocal condition:

$$\begin{cases} D_{p,q} x(t) = f(t, x(pt)), \ t \in [0, T/p], \\ x(0) = \alpha_i x(T) + \displaystyle\sum_{i=0}^{m} \beta_i \int_0^{\eta_i} x(s) d_{p_i, q_i} s, \end{cases} \tag{2}$$

where $0 < q < p \leq 1, 0 < q_i < p_i \leq 1 (i = 1,2,3,\ldots,m)$ are quantum numbers, $f \in \mathcal{C}([0, T/p] \times \mathbb{R}, \mathbb{R}), T > 0, D_{p,q}$ is the (p,q)-difference operator, and $\alpha_i, \beta_i, \eta_i$ $(i = 1,2,\ldots m)$ are constants.

Furthermore, C. Promsakon et al. [76] introduced the second-order (p,q)-difference equations with separated boundary conditions:

$$\begin{cases} D_{p,q}^2 x(t) = f\big(t, x(p^2 t)\big), \ t \in [0, T/p^2], \\ \alpha_1 x(0) + \alpha_2 D_{p,q} x(0) = \alpha_3, \quad \beta_1 x(T) + \beta_2 D_{p,q} x(T/p) = \beta_3, \end{cases} \tag{3}$$

where $f \in \mathcal{C}\big([0, T/p^2] \times \mathbb{R}, \mathbb{R}\big)$ and α_i, β_i $(i = 1,2,3)$ are constants. $D_{p,q}^2$ and D_q are the second and first order of the (p,q)-difference operator, respectively.

However, the study of nonlocal boundary value problems of fractional (p,q)-difference equations is at its infancy, and much of the work on the topic is yet to be done.

In this paper, we study the existence and uniqueness of solutions of the nonlocal boundary value problem of nonlinear fractional (p,q)-difference equations given by

$$\begin{cases} {}^{c}D_{p,q}^{\alpha} y(t) = h(t, y(p^{\alpha} t)), \ t \in [0, T/p^{\alpha}], 1 < \alpha \leq 2, \\ \beta_1 y(0) + \gamma_1 D_{p,q} y(0) = \zeta_1 y(\eta_1), \quad \beta_2 y(T) + \gamma_2 D_{p,q} y(T/p) = \zeta_2 y(\eta_2), \end{cases} \tag{4}$$

where $h \in \mathcal{C}([0, T/p^{\alpha}] \times \mathbb{R}, \mathbb{R})$ and $\beta_i, \gamma_i, \eta_i$ $(i = 1,2)$ are constants; ${}^{c}D_{p,q}^{\alpha}$ and $D_{p,q}$ are the fractional (p,q)-derivative of the Caputo type and the first-order (p,q)-difference operator, respectively.

2. Preliminaries

In this section, we give some definitions and fundamental results of the q-calculus and (p,q)-calculus along with the fractional (p,q)-calculus, which can be found in [27,28,56,58]. We also give a lemma that will be used in obtaining the main results of the paper.

Throughout this paper, let $[a,b] \subset \mathbb{R}$ be an interval with $a < b$, and $0 < q < p \leq 1$ be constants,

$$[k]_{p,q} = \frac{p^k - q^k}{p - q}, \quad k \in \mathbb{N}, \tag{5}$$

$$[k]_{p,q}! = \begin{cases} [k]_{p,q}[k-1]_{p,q} \cdots [1]_{p,q} = \prod_{i=1}^{k} \frac{p^i - q^i}{p - q}, & k \in \mathbb{N}, \\ 1, & k = 0. \end{cases} \tag{6}$$

The q-analogue of the power function $(a - b)_q^{(n)}$ with $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ is given by

$$(a - b)_q^{(0)} = 1, \quad (a - b)_q^{(n)} := \prod_{k=0}^{n-1} (a - bq^k), \quad a, b \in \mathbb{R}.$$

The (p, q)-analogue of the power function $(a - b)_{p,q}^{(n)}$ with $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ is given by

$$(a - b)_{p,q}^{(0)} = 1, \quad (a - b)_{p,q}^{(n)} := \prod_{k=0}^{n-1} (ap^k - bq^k), \quad a, b \in \mathbb{R}.$$

For $t \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$, the (p, q)-gamma function is defined by

$$\Gamma_{p,q}(t) = \frac{(p - q)_{p,q}^{(t-1)}}{(p - q)^{t-1}}, \tag{7}$$

and an equivalent definition of (7) is given in [59] as

$$\Gamma_{p,q}(t) = p^{\frac{t(t-1)}{2}} \int_0^\infty x^{t-1} E_{p,q}^{-qx} d_{p,q} x, \tag{8}$$

where

$$E_{p,q}^{-qx} = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{[n]_{p,q}!} (-qx)^n.$$

Obviously, $\Gamma_{p,q}(t + 1) = [t]_{p,q} \Gamma_{p,q}(t)$.
For $s, t > 0$, the definition of the (p, q)-beta function is

$$B_{p,q}(s, t) = \int_0^1 u^{s-1} (1 - qu)_{p,q}^{(t-1)} d_{p,q} u, \tag{9}$$

and (9) can also be written as

$$B_{p,q}(s, t) = p^{(t-1)(2s+t-2)/2} \frac{\Gamma_{p,q}(s) \Gamma_{p,q}(t)}{\Gamma_{p,q}(s + t)}, \tag{10}$$

see [77,78] for more details.

Definition 1 ([56]). *If $f : [0, T] \to \mathbb{R}$ is a continuous function, then the (p, q)-derivative of f is defined by*

$$D_{p,q} f(t) = \frac{f(pt) - f(qt)}{(p - q)t}, \quad t \neq 0 \tag{11}$$

and $D_{p,q} f(0) = \lim_{t \to 0} D_{p,q} f(t)$. Observe that the function $D_{p,q} f(t)$ is defined on $[0, T/p]$ provided that $D_{p,q} f(t)$ exists for all $t \in [0, T/p]$.

Definition 2 ([56]). *If $f : [0, T] \to \mathbb{R}$ is a continuous function, then the (p, q)-integral of f is defined as*

$$\int_0^t f(s)d_{p,q}s = (p-q)t \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}}t\right) \tag{12}$$

provided that the right-hand side converges. Note that the function $\int_0^t f(s)d_{p,q}s$ *is defined on* $[0, pT]$, *which is extended from* $[0, T]$ *of a function* $f(t)$.

Theorem 1 ([56]). *Let* $f, g : [0, T] \to \mathbb{R}$, $g(t) \neq 0$ *for all* $t \in [0, T]$, *be* (p, q)-*differentiable on* $[0, T/p]$ *and* λ *be a constant. Then,*

(i) $D_{p,q}(f(t) + g(t)) = D_{p,q}f(t) + D_{p,q}g(t)$;

(ii) $D_{p,q}\lambda f(s) = \lambda D_{p,q}f(s)$;

(iii) $D_{p,q}(fg)(t) = f(pt)D_{p,q}g(t) + g(qt)D_{p,q}f(t)$;

(iv) $D_{p,q}(f/g)(t) = \dfrac{g(qt)D_{p,q}f(t) - f(qt)D_{p,q}g(t)}{g(pt)g(qt)}$.

Theorem 2 ([79]). *Let* $f : [0, T] \to \mathbb{R}$ *be a continuous function. The following formulas hold for* $t \in [a, b]$:

(i) $D_{p,q} \int_0^t f(s)d_{p,q}s = f(t)$;

(ii) $\int_0^t D_{p,q}f(s)d_{p,q}s = f(t) - f(0)$;

(iii) $\int_c^t D_{p,q}f(s)d_{p,q}s = f(t) - f(c)$, *for* $c \in (0, t)$.

Definition 3 ([77]). *Let* $f : [0, T] \to \mathbb{R}$ *be a continuous function and* $\alpha \geq 0$. *The fractional* (p, q)-*integral of fractional Riemann–Liouville type is given by* $\left(I_{p,q}^0 f\right)(t) = f(t)$ *and*

$$\left(I_{p,q}^\alpha f\right)(t) = \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t - qs)_{p,q}^{(\alpha-1)} f\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s$$

$$= \frac{(p-q)t}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left(t - \frac{q^{n+1}}{p^{n+1}}t\right)_{p,q}^{(\alpha-1)} f\left(\frac{q^n}{p^{\alpha+n}}t\right), \tag{13}$$

where $t \in [0, p^\alpha T]$.

Definition 4 ([77]). *The fractional* (p, q)-*derivative of Riemann-Liouville type of order* $\alpha \geq 0$ *of a continuous function* f *is defined by* $(D_{p,q}^0 f)(t) = f(t)$ *and*

$$(D_{p,q}^\alpha f)(t) = \left(D_{p,q}^{[\alpha]} I_{p,q}^{[\alpha]-\alpha} f\right)(t), \tag{14}$$

where $[\alpha]$ *is the smallest integer greater than or equal to* α.

Definition 5 ([77]). *The fractional* (p, q)-*derivative of Caputo type of order* $\alpha \geq 0$ *of a continuous function* f *is defined by* $\left({}^c D_{p,q}^0 f\right)(t) = f(t)$ *and*

$$({}^c D_{p,q}^\alpha f)(t) = \left(I_{p,q}^{[\alpha]-\alpha} D_{p,q}^{[\alpha]} f\right)(t), \tag{15}$$

where $[\alpha]$ *is the smallest integer greater than or equal to* α.

Lemma 1 ([77]). *Let* f *be a continuous function and* $\alpha, \beta \geq 0$. *Then, the following formulas hold:*

(i) $\left(I_{p,q}^\beta I_{p,q}^\alpha f\right)(t) = \left(I_{p,q}^{\alpha+\beta} f\right)(t)$,

(ii) $\left(D_{p,q}^\alpha I_{p,q}^\alpha f\right)(t) = f(t)$.

Lemma 2 ([77]). *Let f be a continuous function, $\alpha \geq 0$, and $n \in \mathbb{N}$. Then, the following equality holds:*

$$\left(I_{p,q}^{\alpha} D_{p,q}^{n} f\right)(t) = \left(D_{p,q}^{n} I_{p,q}^{\alpha} f\right)(t) - \sum_{k=0}^{[\alpha]-1} \frac{t^{\alpha-n+k}}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha-n+k+1)} \left(D_{p,q}^{k} f\right)(0).$$

Lemma 3 ([77]). *Let f be a continuous function and $\alpha \geq 0$. Then, the following equality holds:*

$$\left(I_{p,q}^{\alpha} {}^{c} D_{p,q}^{\alpha} f\right)(t) = f(t) - \sum_{k=0}^{[\alpha]-1} \frac{t^{k}}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(k+1)} \left(D_{p,q}^{k} f\right)(0).$$

In order to define the solution of the boundary value problem (4), we need the following lemma.

Lemma 4. *Let $\alpha_i, \beta_i, \gamma_i$ $(i = 1, 2)$ be constants and ${}^{c} D_{p,q}^{\alpha}$ and $D_{p,q}$ be the fractional (p, q)-derivative of the Caputo type and the first-order (p, q)-difference operator, respectively. For a given $h \in \mathcal{C}([0, T/p^{\alpha}], \mathbb{R})$, a unique solution of the boundary value problem*

$$\begin{cases} {}^{c} D_{p,q}^{\alpha} y(t) = h(t), \ t \in [0, T/p^{\alpha}], \\ \beta_1 y(0) + \gamma_1 D_{p,q} y(0) = \zeta_1 y(\eta_1), \\ \beta_2 y(T) + \gamma_2 D_{p,q} y(T/p) = \zeta_2 y(\eta_2) \end{cases} \tag{16}$$

is given by

$$\begin{aligned} y(t) = &\int_0^t \frac{(t-qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \\ &+ \frac{\zeta_1}{\Delta}[(\beta_2 - \zeta_2)t - (\beta_2 T + \gamma_2 - \zeta_2 \eta_2)] \int_0^{\eta_1} \frac{(\eta_1 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \\ &+ \frac{\zeta_2}{\Delta}[(\gamma_1 - \zeta_1 \eta_1) - (\beta_1 - \zeta_1)t] \int_0^{\eta_2} \frac{(\eta_2 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha-1)} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \\ &+ \frac{\beta_2}{\Delta}[(\beta_1 - \zeta_1)t - (\gamma_1 - \zeta_1 \eta_1)] \int_0^{T} \frac{(T - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha-1)} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \\ &+ \frac{\gamma_2}{\Delta}[(\gamma_1 - \zeta_1)t - (\gamma_1 - \zeta_1 \eta_1)] \int_0^{T/p} \frac{(T/p - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}} \Gamma_{p,q}(\alpha-1)} h\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s, \end{aligned} \tag{17}$$

where

$$\Delta = (\beta_2 - \gamma_2)(\gamma_1 - \zeta_1 \eta_2) - (\beta_1 - \zeta_1)(\beta_2 T + \gamma_2 - \zeta_2 \eta_2) \neq 0.$$

Proof. By taking the fractional (p, q)-integral on (16) and applying Lemma 3, we have

$$y(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s + tc_0 + c_1, \tag{18}$$

where c_0, c_1 are constants and $t \in [0, T]$.

Applying the above boundary condition of (16), we obtain

$$
\begin{cases}
(\gamma_1 - \zeta_1\eta_1)c_1 + (\beta_1 - \zeta_1)c_2 = \zeta_1 \int_0^{\eta_1} \frac{(\eta_1 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s, \\[2mm]
(\beta_2 T + \gamma_2 - \zeta_2\eta_2)c_1 + (\beta_2 - \zeta_2)c_2 = \zeta_2 \int_0^{\eta_2} \frac{(\eta_2 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha-1)} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \\[2mm]
\qquad - \beta_2 \int_0^T \frac{(T - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \\[2mm]
\qquad - \gamma_2 \int_0^{T/p} \frac{(T/p - qs)^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} h\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s.
\end{cases}
\tag{19}
$$

Solving the above system of (19) to find the constants c_1, c_2, we have

$$
\begin{aligned}
c_1 = \frac{1}{\Delta} \Bigg[& \zeta_1(\beta_2 - \zeta_2) \int_0^{\eta_1} \frac{(\eta_1 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \\
& - \zeta_2(\beta_1 - \zeta_1) \int_0^{\eta_2} \frac{(\eta_2 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha-1)} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \\
& + \beta_2(\beta_1 - \zeta_1) \int_0^T \frac{(T - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \\
& + \gamma_2(\beta_1 - \zeta_1) \int_0^{T/p} \frac{(T/p - qs)^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha)} h\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s \Bigg],
\end{aligned}
$$

and

$$
\begin{aligned}
c_2 = \frac{1}{\Delta} \Bigg[& -\zeta_1(\beta_2 T + \gamma_2 - \zeta_2\eta_2) \int_0^{\eta_1} \frac{(\eta_1 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \\
& + \zeta_2(\gamma_1 - \zeta_1\eta_1) \int_0^T \frac{(T - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \\
& - \beta_1\beta_2 \int_0^{T/p} \frac{(T/p - qs)^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} h\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s \Bigg].
\end{aligned}
$$

Substituting the values of c_0, c_1 in (18), we obtain (17). We can prove the converse by direct computation. Therefore, the proof is completed. □

3. Main Results

In view of Lemma 4, we define an operator $\Psi : \mathcal{C}([0,T],\mathbb{R}) \to \mathcal{C}([0,T],\mathbb{R})$ as

$$
\begin{aligned}
(\Psi y)(t) = & \int_0^t \frac{(t - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} h(s, y(p^{\alpha-1}s)) \, d_{p,q}s \\
& + \frac{\zeta_1}{\Delta}\left[(\beta_2 - \zeta_2)t - (\beta_2 T + \gamma_2 - \zeta_2\eta_2)\right] \int_0^{\eta_1} \frac{(\eta_1 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} h(s, y(p^{\alpha-1}s)) \, d_{p,q}s \\
& + \frac{\zeta_2}{\Delta}\left[(\gamma_1 - \zeta_1\eta_1) - (\beta_1 - \zeta_1)t\right] \int_0^{\eta_2} \frac{(\eta_2 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha-1)} h(s, y(p^{\alpha-1}s)) \, d_{p,q}s \\
& + \frac{\beta_2}{\Delta}\left[(\beta_1 - \zeta_1)t - (\gamma_1 - \zeta_1\eta_1)\right] \int_0^T \frac{(T - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha-1)} h(s, y(p^{\alpha-1}s)) \, d_{p,q}s
\end{aligned}
$$

$$+ \frac{\gamma_2}{\Delta}[(\beta_1 - \zeta_1)t - (\gamma_1 - \zeta_1\eta_1)] \int_0^{T/p} \frac{(T/p - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} h(s, y(p^{\alpha-2}s)) \, d_{p,q}s, \quad (20)$$

where

$$\Delta = (\beta_2 - \zeta_2)(\gamma_1 - \zeta_1\eta_2) - (\beta_1 - \zeta_1)(\beta_2 T + \gamma_2 - \zeta_2\eta_2) \neq 0.$$

Let $\Omega := \mathcal{C}([0, T], \mathbb{R})$ denote the Banach space of all continuous functions from $[0, T]$ to $\mathbb{R}$ endowed with norm defined by $\|y\| = \max\{|y(t)| : t \in [0, T]\}$.

Observe that the boundary value problem (4) has a unique solution if the operator equation $\Psi y = y$ has a fixed point, where Ψ is given in (20). For convenience, we let

$$k = \left(I_{p,q}^\alpha \lambda\right)(T) + |\zeta_1|\sigma_1(T)\left(I_{p,q}^\alpha \lambda\right)(\eta_1) + |\zeta_2|\sigma_2(T)\left(I_{p,q}^\alpha \lambda\right)(\eta_2)$$
$$+ |\beta_2|\sigma_2(T)\left(I_{p,q}^\alpha \lambda\right)(T) + |\gamma_2|\sigma_2(T)\left(I_{p,q}^{\alpha-1}\lambda\right)(T/p), \quad (21)$$

where

$$\sigma_1(T) = \frac{(|\beta_2 - \zeta_2|T + |\beta_2 T + \gamma_2 - \zeta_2\eta_2|)}{|\Delta|}$$

and

$$\sigma_2(T) = \frac{(|\beta_1 - \zeta_1|T + |\gamma_1 - \zeta_1\eta_1|)}{|\Delta|}.$$

Theorem 3. *Let $h : [0, T/p^\alpha] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and note that there exists a (p, q)-integrable function $\lambda : [0, T/p^\alpha] \to \mathbb{R}$ such that*
$(A_1) |h(t, y) - h(t, z)| \leq \lambda(t)|y - z|$, for each $t \in [0, T/p^\alpha]$ and $y, z \in \mathbb{R}$.
  If

$$k < 1, \quad (22)$$

where k is given by (21), then the boundary value problem (4) has a unique solution.

Proof. Let us fix $\sup_{t \in [0, T/p^\alpha]} |h(t, 0)| = M$ and choose

$$r \geq \frac{MA}{1 - k},$$

where

$$A = \frac{T^\alpha}{\Gamma_{p,q}(\alpha+1)} + |\beta_2|\sigma_2(T)\frac{T^\alpha}{\Gamma_{p,q}(\alpha+1)} + |\gamma_2|\sigma_2(T)\frac{T^{\alpha-1}}{\Gamma_{p,q}(\alpha)}$$
$$+ |\zeta_1|\sigma_1(T)\frac{\eta_1^\alpha}{\Gamma_{p,q}(\alpha+1)} + |\zeta_2|\sigma_2(T)\frac{\eta_2^\alpha}{\Gamma_{p,q}(\alpha+1)}. \quad (23)$$

We define $B_r = \{y \in \mathbb{C} : \|y\| \leq r\}$ and we shall show that $\Psi B_r \subset B_r$, where Ψ is defined (20). For any $y \in B_r$, observe that

$$|h(t, y(t))| \leq |h(t, y(t)) - h(t, 0)| + |h(t, 0)| \leq \lambda(t)r + M \quad (24)$$

and

$$|(\Psi y)(t)| \leq \int_0^t \frac{(t - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} |h(s, y(p^{\alpha-1}s))| \, d_{p,q}s$$
$$+ |\zeta_1|\sigma_1(T)\int_0^{\eta_1} \frac{(\eta_1 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} |h(s, y(p^{\alpha-1}s))| \, d_{p,q}s$$

$$+ |\zeta_2|\sigma_2(T) \int_0^{\eta_2} \frac{(\eta_2 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha-1)} |h(s,y(p^{\alpha-1}s))| \, d_{p,q}s$$

$$+ |\beta_2|\sigma_2(T) \int_0^{T} \frac{(T - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha-1)} |h(s,y(p^{\alpha-1}s))| \, d_{p,q}s$$

$$+ |\gamma_2|\sigma_2(T) \int_0^{T/p} \frac{(T/p - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} |h(s,y(p^{\alpha-2}s))| \, d_{p,q}s$$

$$\leq \int_0^{T} \frac{(T - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} |\lambda\left(\frac{s}{p^{\alpha-1}}\right)r + M| \, d_{p,q}s$$

$$+ |\zeta_1|\sigma_1(T) \int_0^{\eta_1} \frac{(\eta_1 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} |\lambda\left(\frac{s}{p^{\alpha-1}}\right)r + M| \, d_{p,q}s$$

$$+ |\zeta_2|\sigma_2(T) \int_0^{\eta_2} \frac{(\eta_2 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha-1)} |\lambda\left(\frac{s}{p^{\alpha-1}}\right)r + M| \, d_{p,q}s$$

$$+ |\beta_2|\sigma_2(T) \int_0^{T} \frac{(T - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha-1)} |\lambda\left(\frac{s}{p^{\alpha-2}}\right)r + M| \, d_{p,q}s$$

$$+ |\gamma_2|\sigma_2(T) \int_0^{T/p} \frac{(T/p - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} |\lambda\left(\frac{s}{p^{\alpha-2}}\right)r + M| \, d_{p,q}s$$

$$\leq M\left\{ \frac{T^\alpha}{\Gamma_{p,q}(\alpha+1)} + |\beta_2|\sigma_1(T)\frac{T^\alpha}{\Gamma_{p,q}(\alpha+1)} + |\gamma_2|\sigma_2(T)\frac{T^{\alpha-1}}{\Gamma_{p,q}(\alpha)} \right.$$

$$\left. + |\zeta_1|\sigma_2(T)\frac{\eta_1^\alpha}{\Gamma_{p,q}(\alpha+1)} + |\zeta_2|\sigma_2(T)\frac{\eta_1^\alpha}{\Gamma_{p,q}(\alpha+1)} \right\}$$

$$+ r\left\{ \int_0^{T} \frac{(T - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \lambda\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \right.$$

$$+ |\zeta_1|\sigma_1(T) \int_0^{\eta_1} \frac{(\eta_1 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \lambda\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s$$

$$+ |\zeta_2|\sigma_2(T) \int_0^{\eta_2} \frac{(\eta_2 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \lambda\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s$$

$$+ |\beta_2|\sigma_2(T) \int_0^{T} \frac{(T - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \lambda\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s$$

$$+ |\gamma_2|\sigma_2(T) \int_0^{T/p} \frac{(T/p - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} \lambda\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s \right\}, \tag{25}$$

which, in view of (21) and (23), implies that

$$\|\Psi y\| \leq MA + rk \leq r.$$

This implies that $\Psi y \subset B_r$.
Now, for $y, z \in \mathcal{C}$, we obtain

$$\|\Psi y - \Psi z\|$$

$$\leq \sup_{t \in [0, T/p^\alpha]} \left\{ \int_0^t \frac{(t - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left| h(s, y(p^{\alpha-1}s)) - h(s, z(p^{\alpha-1}s)) \right| d_{p,q}s \right.$$

$$+ |\zeta_1|\sigma_1(T) \int_0^{\eta_1} \frac{(\eta_1 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left| h(s, y(p^{\alpha-1}s)) - h(s, z(p^{\alpha-1}s)) \right| d_{p,q}s$$

$$+ |\zeta_2|\sigma_2(T) \int_0^{\eta_2} \frac{(\eta_2 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left| h(s, y(p^{\alpha-1}s)) - h(s, z(p^{\alpha-1}s)) \right| d_{p,q}s$$

$$+ |\beta_2|\sigma_2(T) \int_0^{T} \frac{(T - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left| h(s, y(p^{\alpha-1}s)) - h(s, z(p^{\alpha-1}s)) \right| d_{p,q}s$$

$$\left. + |\gamma_2|\sigma_2(T) \int_0^{T/p} \frac{(T/p - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} |h(s, y(p^\alpha s)) - h(s, z(p^\alpha s))| d_{p,q}s \right\}$$

$$\leq \|y - z\| \left\{ \int_0^{T} \frac{(T - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \lambda\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \right.$$

$$+ |\zeta_1|\sigma_1(T) \int_0^{\eta_1} \frac{(\eta_1 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \lambda\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s$$

$$+ |\zeta_2|\sigma_2(T) \int_0^{\eta_2} \frac{(\eta_2 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \lambda\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s$$

$$+ \sigma_1(T)|\beta_2|\sigma_2(T) \int_0^{T} \frac{(T - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \lambda\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s$$

$$\left. + \sigma_2(T)|\gamma_2|\sigma_2(T) \int_0^{T/p} \frac{(T/p - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} \lambda\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s \right\},$$

from which, in view of (21), we obtain

$$\|\Psi y - \Psi z\| \leq k\|y - z\|.$$

Because $k \in (0,1)$ by assumptions (22), therefore, Ψ is a contraction. Thus, it follows by Banach's contraction principle that the boundary value problem (4) has a unique solution. $\square$

In case $\lambda(t) = \lambda$, where λ is a constant, the condition (22) becomes $\lambda A < 1$ and Theorem 3 takes the form of the following results.

Remark 1. *If* $h : [0, T/p^\alpha] \times \mathbb{R} \to \mathbb{R}$ *is a contraction function and there exists a constant* $\lambda > 0$ *with*

$$|h(t,y) - h(t,z)| \leq \lambda|y - z|,$$

for each $t \in [0, T/p^\alpha]$ *and* $y, z \in \mathbb{R}$, *then the boundary value problem* (4) *has a unique solution, provided that* $k < 1$.

Our next existence results are based on Krasnoselskii's fixed-point theorem.

Lemma 5 ([80]). *(Krasnoselskii's fixed-point theorem) Let M be a closed, bounded, convex, and non-empty subset of a Banach space X. Let A, B be two operators such that:*

(i) $Ay + Bz \in M$, *whenever* $y, z \in M$;

(ii) *A is compact and continuous;*

(iii)　B is a contraction mapping.

　　Then, there exists $x \in M$ such that $x = Ax + Bx$.

Theorem 4. *Let $h : [0, T/p^\alpha] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (A_1). In addition, we assume that*

(A_2) there exists a function $\mu \in \mathcal{C}([0, T/p^\alpha], \mathbb{R}^+)$ and a non-decreasing function $\phi \in \mathcal{C}([0, T/p^\alpha], \mathbb{R}^+)$ with

$$|h(t,y)| \leq \mu(t)\phi(|y|),$$

where $(t,y) \in [0, T/p^\alpha] \times \mathbb{R}$.

If

$$\int_0^T \frac{(T - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \lambda\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s$$

$$+ |\zeta_1|\sigma_1(T) \int_0^{\eta_1} \frac{(\eta_1 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \lambda\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s$$

$$+ |\zeta_2|\sigma_2(T) \int_0^{\eta_2} \frac{(\eta_2 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \lambda\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s$$

$$+ \sigma_1(T)|\beta_2|\sigma_2(T) \int_0^T \frac{(T - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \lambda\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s$$

$$+ \sigma_2(T)|\gamma_2|\sigma_2(T) \int_0^{T/p} \frac{(T/p - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha - 1)} \lambda\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s < 1,$$

then the boundary value problem (4) *has at least one solution on $[0, T]$.*

Proof. Let $\mathcal{B}_{\bar{r}} := \{y \in \mathcal{C} : \|y\| \leq \bar{r}\}$, where

$$\bar{r} \geq \phi(\bar{r})\|\mu\|\left\{\frac{T^\alpha}{\Gamma_{p,q}(\alpha + 1)} + |\beta_2|\sigma_1(T)\frac{T^\alpha}{\Gamma_{p,q}(\alpha + 1)} + |\gamma_2|\sigma_2(T)\frac{T^{\alpha-1}}{\Gamma_{p,q}(\alpha)}\right.$$

$$\left. + |\zeta_1|\sigma_2(T)\frac{\eta_1^\alpha}{\Gamma_{p,q}(\alpha + 1)} + |\zeta_2|\sigma_2(T)\frac{\eta_1^\alpha}{\Gamma_{p,q}(\alpha + 1)}\right\},$$

where $\|\mu\| = \sup_{[0, T/p^\alpha]} |\mu(t)|$ and define the operators $\mathcal{P}$ and $\mathcal{Q}$ on $\mathcal{B}_{\bar{r}}$ as

$$(\mathcal{P}y)(t) = \int_0^t \frac{(t - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} h(s, y(p^{\alpha-1}s)) \, d_{p,q}s$$

and

$$(\mathcal{Q}y)(t) = \frac{\zeta_1}{\Delta}[(\beta_2 - \zeta_2)t - (\beta_2 T + \gamma_2 - \zeta_2\eta_2)] \int_0^{\eta_1} \frac{(\eta_1 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} h(s, y(p^{\alpha-1}s)) \, d_{p,q}s$$

$$+ \frac{\zeta_2}{\Delta}[(\gamma_1 - \zeta_1\eta_1) - (\beta_1 - \zeta_1)t] \int_0^{\eta_2} \frac{(\eta_2 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha - 1)} h(s, y(p^{\alpha-1}s)) \, d_{p,q}s$$

$$+ \frac{\beta_2}{\Delta}[(\beta_1 - \zeta_1)t - (\gamma_1 - \zeta_1\eta_1)] \int_0^T \frac{(T - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha - 1)} h(s, y(p^{\alpha-1}s)) \, d_{p,q}s$$

$$+ \frac{\gamma_2}{\Delta}[(\beta_1 - \zeta_1)t - (\gamma_1 - \zeta_1\eta_1)] \int_0^{T/p} \frac{(T/p - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha - 1)} h(s, y(p^{\alpha-2}s)) \, d_{p,q}s, \tag{26}$$

where

$$\Delta = (\beta_2 - \zeta_2)(\gamma_1 - \zeta_1\eta_2) - (\beta_1 - \zeta_1)(\beta_2 T + \gamma_2 - \zeta_2\eta_2) \neq 0.$$

For $y, z \in \mathcal{B}_{\bar{r}}$, we have

$$|(\mathcal{P}y + \mathcal{Q}z)(t)| \leq \int_0^t \frac{(t - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left| h(s, y(p^{\alpha-1}s)) \right| d_{p,q}s$$

$$+ \frac{\zeta_1}{\Delta}[(\beta_2 - \zeta_2)t - (\beta_2 T + \gamma_2 - \zeta_2\eta_2)] \int_0^{\eta_1} \frac{(\eta_1 - qs)^{(\alpha-1)}_{p,q}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} h(s, z(p^{\alpha-1}s)) d_{p,q}s$$

$$+ \frac{\zeta_2}{\Delta}[(\gamma_1 - \zeta_1\eta_1) - (\beta_1 - \zeta_1)t] \int_0^{\eta_2} \frac{(\eta_2 - qs)^{(\alpha-1)}_{p,q}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha - 1)} h(s, z(p^{\alpha-1}s)) d_{p,q}s$$

$$+ \frac{\beta_2}{\Delta}[(\beta_1 - \zeta_1)t - (\gamma_1 - \zeta_1\eta_1)] \int_0^{T} \frac{(T - qs)^{(\alpha-1)}_{p,q}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha - 1)} h(s, z(p^{\alpha-1}s)) d_{p,q}s$$

$$+ \frac{\gamma_2}{\Delta}[(\beta_1 - \zeta_1)t - (\gamma_1 - \zeta_1\eta_1)] \int_0^{T/p} \frac{(T/p - qs)^{(\alpha-2)}_{p,q}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha - 1)} h(s, z(p^{\alpha-2}s)) d_{p,q}s$$

$$\leq \int_0^t \frac{(t - qs)^{(\alpha-1)}_{p,q}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \mu(t)\phi(|y|) d_{p,q}s$$

$$+ |\zeta_1|\sigma_1(T) \int_0^{\eta_1} \frac{(\eta_1 - qs)^{(\alpha-1)}_{p,q}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \mu(t)\phi(|z|) d_{p,q}s$$

$$+ |\zeta_2|\sigma_2(T) \int_0^{\eta_2} \frac{(\eta_2 - qs)^{(\alpha-1)}_{p,q}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha - 1)} \mu(t)\phi(|z|) d_{p,q}s$$

$$+ |\beta_2|\sigma_2(T) \int_0^{T} \frac{(T - qs)^{(\alpha-1)}_{p,q}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha - 1)} \mu(t)\phi(|z|) d_{p,q}s$$

$$+ |\gamma_2|\sigma_2(T) \int_0^{T/p} \frac{(T/p - qs)^{(\alpha-2)}_{p,q}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha - 1)} \mu(t)\phi(|z|) d_{p,q}s$$

$$\leq \phi(\bar{r})\|\mu\| \left\{ \frac{T^\alpha}{\Gamma_{p,q}(\alpha + 1)} + |\beta_2|\sigma_1(T)\frac{T^\alpha}{\Gamma_{p,q}(\alpha + 1)} + |\gamma_2|\sigma_2(T)\frac{T^{\alpha-1}}{\Gamma_{p,q}(\alpha)} \right.$$

$$\left. + |\zeta_1|\sigma_2(T)\frac{\eta_1^\alpha}{\Gamma_{p,q}(\alpha + 1)} + |\zeta_2|\sigma_2(T)\frac{\eta_1^\alpha}{\Gamma_{p,q}(\alpha + 1)} \right\}$$

$$\leq \bar{r}.$$

Thus, $\mathcal{P}y + \mathcal{Q}z \in \mathcal{B}_{\bar{r}}$. From (A_1) and (26), it follows that $\mathcal{Q}$ is a contraction mapping. By continuity of h, we obtain that the operator $\mathcal{P}$ is continuous. It is not difficult to verify that

$$\|\mathcal{P}\| \leq \frac{\phi(\bar{r})\|\mu\|T^\alpha}{\Gamma_{p,q}(\alpha + 1)}.$$

Therefore, the set $\mathcal{P}(\mathcal{B}_{\bar{r}})$ is uniformly bounded. Next, we shall prove the compactness of the operator $\mathcal{P}$. Now, for any $y \in B_{\bar{r}}$ $t_1, t_2 \in [0, T]$ with $t_1 < t_2$. Then, we obtain

$$|(\mathcal{P}y)(t_2) - (\mathcal{Q}y)(t_1)|$$

$$= \left| \int_0^{t_2} \frac{(t - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} f(s, y(p^{\alpha-1}s)) \, d_{p,q}s - \int_0^{t_1} \frac{(t - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} f(s, y(p^{\alpha-1}s)) \, d_{p,q}s \right|$$

$$\leq \left| \int_0^{t_1} \frac{(t_2 - qs)^{(\alpha-1)} - (t_1 - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} f(s, y(p^{\alpha-1}s)) \, d_{p,q}s \right|$$

$$+ \left| \int_{t_1}^{t_2} \frac{(t_2 - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} f(s, y(p^{\alpha-1}s)) \, d_{p,q}s \right|$$

$$\leq \phi(\bar{r})\|\mu\| \left[\int_0^{t_1} \frac{(t_2 - qs)^{(\alpha-1)} - (t_1 - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} f(s, y(p^{\alpha-1}s)) \, d_{p,q}s \right.$$

$$\left. + \int_{t_1}^{t_2} \frac{(t_2 - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} f(s, y(p^{\alpha-1}s)). \, d_{p,q}s. \right]$$

which is independent of x and tends to zero as $t_1 \to t_2$. Thus, the set $\mathcal{P}(B_{\bar{r}})$ is equicontinuous. By the Arzelá–Ascoli theorem, $\mathcal{P}$ is compact on $B_{\bar{r}}$. Therefore, the boundary value problem (4) has at least one solution on $[0, T]$. This completes the proof. $\square$

Remark 2. *Let $h : [0, T/p^\alpha] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (A_1). In addition, we assume that*

$$|h(t,y)| \leq \mu(t), \ \forall (t,y) \in [0, T/p^\alpha] \times \mathbb{R} \ \text{and} \ \mu \in \mathbb{C}([0, T/p^\alpha], \mathbb{R}^+).$$

If (26) holds, then the boundary value problem (4) has at least one solution on $[0, T]$.

The next existence results are based on the Leray–Schauder nonlinear alternative.

Lemma 6 ([81]). *(Nonlinear alternative for single value maps) Let E be a Banach space, $\mathbb{C}$ a closed, convex subset of E, and $\mathbb{U}$ an open subset of $\mathbb{C}$ with $0 \in \mathbb{U}$. Suppose that $\Psi : \overline{\mathbb{U}} \to \mathcal{C}$ is a continuous, compact function; that is, $\Psi\left(\overline{\mathbb{U}}\right)$ is a relatively compact subset of $\mathcal{C}$ map. Then, either*

(i) Ψ has a fixed point in $\overline{\mathbb{U}}$, or
(ii) there is a $u \in \partial\mathbb{U}$ (the boundary of $\mathbb{U}$ in $\mathbb{C}$) and $\lambda \in (0,1)$ with $u = \lambda\Psi(u)$.

Theorem 5. *Let $h : [0, T/p^\alpha] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that:*
(A_3) *there exist functions $p_1, p_2 \in \mathbb{C}([0, T/p^\alpha], \mathbb{R}^+)$, and a non-decreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that*
$$|h(t,y)| \leq p_1(t)\psi(|y|) + p_2(t), t,y \in [0, T/p^\alpha] \times \mathbb{R};$$
(A_4) *there exists a number $M > 0$ such that*

$$\frac{M}{\psi(M)\omega_1 + \omega_2} > 1, \tag{27}$$

where

$$\omega_i := (1 + |\beta_2|\sigma_2)\left(I_{p,q}^\alpha p_i\right)(T) + |\zeta_1|\sigma_1(T)\left(I_{p,q}^\alpha p_i\right)(\eta_1) + |\zeta_2|\sigma_2(T)\left(I_{p,q}^\alpha p_i\right)(\eta_2)$$

$$+ |\gamma_2|\sigma_2(T)\left(I_{p,q}^{\alpha-1} p_i\right)(T/p), \ i = 1, 2.$$

Then, the boundary value problem (4) has at least one solution on $[0, T]$.

Proof. Consider the operator $\Psi : \mathcal{C} \to \mathcal{C}$ defined by (20). We first show that Ψ is continuous. Let $\{y_n\}$ be a sequence of function such that $y_n \to y$ on $[0, T/p^\alpha]$. Given that h is a continuous function on $[0, T/p^\alpha]$, we have

$$h(t, y_n(p^\alpha t)) \to h(t, y(p^\alpha t)).$$

Therefore, we obtain

$$|(\Psi y_n)(t) - (\Psi y)(t)| \leq \int_0^t \frac{(t - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} |h(s, y_n(p^\alpha s)) - h(s, y(p^\alpha s))| \, d_{p,q}s$$

$$+ |\zeta_1|\sigma_1(T) \int_0^{\eta_1} \frac{(\eta_1 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} |h(s, y_n(p^\alpha s)) - h(s, y(p^\alpha s))| \, d_{p,q}s$$

$$+ |\zeta_2|\sigma_2(T) \int_0^{\eta_2} \frac{(\eta_2 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha - 1)} |h(s, y_n(p^\alpha s)) - h(s, y(p^\alpha s))| \, d_{p,q}s$$

$$+ |\beta_2|\sigma_2(T) \int_0^{T} \frac{(T - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha - 1)} |h(s, y_n(p^\alpha s)) - h(s, y(p^\alpha s))| \, d_{p,q}s$$

$$+ |\gamma_2|\sigma_2(T) \int_0^{T/p} \frac{(T/p - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha - 1)} |h(s, y_n(p^\alpha s)) - h(s, y(p^\alpha s))| \, d_{p,q}s,$$

which implies that

$$\|\Psi y_n - \Psi y\| \to 0 \quad \text{as } n \to \infty.$$

Thus, the operator $\mathbb{F}$ is continuous.

Next, we show that Ψ maps a bounded set into a bounded set in $\mathbb{C}([0, T], \mathbb{R})$. For a positive number $r > 0$, let $B_r = \{y \in \mathbb{C}([0, T]) : \|y\| \leq r\}$. Then, for any $y \in B_r$, we have

$$(\Psi y)(t) = \int_0^t \frac{(t - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} h(s, y(p^{\alpha-1}s)) \, d_{p,q}s$$

$$+ \frac{\zeta_1}{\Delta}[(\beta_2 - \zeta_2)t - (\beta_2 T + \gamma_2 - \zeta_2\eta_2)] \int_0^{\eta_1} \frac{(\eta_1 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} h(s, y(p^{\alpha-1}s)) \, d_{p,q}s$$

$$+ \frac{\zeta_2}{\Delta}[(\gamma_1 - \zeta_1\eta_1) - (\beta_1 - \zeta_1)t] \int_0^{\eta_2} \frac{(\eta_2 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha - 1)} h(s, y(p^{\alpha-1}s)) \, d_{p,q}s$$

$$+ \frac{\beta_2}{\Delta}[(\beta_1 - \zeta_1)t - (\gamma_1 - \zeta_1\eta_1)] \int_0^{T} \frac{(T - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha - 1)} f(s, y(p^{\alpha-1}s)) \, d_{p,q}s$$

$$+ \frac{\gamma_2}{\Delta}[(\beta_1 - \zeta_1)t - (\gamma_1 - \zeta_1\eta_1)] \int_0^{T/p} \frac{(T/p - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha - 1)} f(s, y(p^{\alpha-2}s)) \, d_{p,q}s$$

$$+ \frac{1}{|\Delta|}[\alpha_1\beta_2 t + |\beta_1||\beta_2|] \int_0^{T/p} \frac{(T/p - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha - 1)} |h(s, y(p^\alpha s))| \, d_{p,q}s$$

$$\leq \int_0^T \frac{(T - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left[p_1 \left(\frac{s}{p^{\alpha-1}} \right) \psi(\|y\|) + p_2 \left(\frac{s}{p^{\alpha-1}} \right) \right] d_{p,q}s$$

$$+ |\zeta_1|\sigma_1(T) \int_0^{\eta_1} \frac{(\eta_1 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left[p_1 \left(\frac{s}{p^{\alpha-1}} \right) \psi(\|y\|) + p_2 \left(\frac{s}{p^{\alpha-1}} \right) \right] d_{p,q}s$$

$$+ |\zeta_2|\sigma_1(T) \int_0^{\eta_2} \frac{(\eta_2 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left[p_1 \left(\frac{s}{p^{\alpha-1}} \right) \psi(\|y\|) + p_2 \left(\frac{s}{p^{\alpha-1}} \right) \right] d_{p,q}s$$

$$+ |\beta_2|\sigma_2(T) \int_0^T \frac{(T - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left[p_1 \left(\frac{s}{p^{\alpha-1}} \right) \psi(\|y\|) + p_2 \left(\frac{s}{p^{\alpha-1}} \right) \right] d_{p,q}s$$

$$+ |\gamma_2|\sigma_2(T) \int_0^{T/p} \frac{(T/p - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} \left[p_1 \left(\frac{s}{p^{\alpha-1}} \right) \psi(\|y\|) + p_2 \left(\frac{s}{p^{\alpha-1}} \right) \right] d_{p,q}s$$

$$\leq \psi(r) \left\{ \int_0^T \frac{(T - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} p_1 \left(\frac{s}{p^{\alpha-1}} \right) d_{p,q}s \right.$$

$$+ |\zeta_1|\sigma_1(T) \int_0^{\eta_1} \frac{(T - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} p_1 \left(\frac{s}{p^{\alpha-1}} \right) d_{p,q}s$$

$$+ |\zeta_2|\sigma_2(T) \int_0^{\eta_2} \frac{(\eta_2 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} p_1 \left(\frac{s}{p^{\alpha-1}} \right) d_{p,q}s$$

$$+ |\beta_2|\sigma_2(T) \int_0^T \frac{(T - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} p_1 \left(\frac{s}{p^{\alpha-1}} \right) d_{p,q}s$$

$$+ |\gamma_2|\sigma_2(T) \int_0^{T/p} \frac{(T/p - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} p_1 \left(\frac{s}{p^{\alpha-1}} \right) d_{p,q}s \right\}$$

$$+ \left\{ \int_0^T \frac{(T - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} p_1 \left(\frac{s}{p^{\alpha-1}} \right) d_{p,q}s \right.$$

$$+ |\zeta_1|\sigma_1(T) \int_0^{\eta_1} \frac{(\eta_1 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} p_1 \left(\frac{s}{p^{\alpha-1}} \right) d_{p,q}s$$

$$+ |\zeta_2|\sigma_2(T) \int_0^{\eta_2} \frac{(\eta_2 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} p_1 \left(\frac{s}{p^{\alpha-1}} \right) d_{p,q}s$$

$$+ |\beta_2|\sigma_2(T) \int_0^T \frac{(T - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} p_1 \left(\frac{s}{p^{\alpha-1}} \right) d_{p,q}s$$

$$+ |\gamma_2|\sigma_2(T) \int_0^{T/p} \frac{(T/p - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} p_1 \left(\frac{s}{p^{\alpha-1}} \right) d_{p,q}s \right\}.$$

We have

$$\|\Psi y\| \leq \psi(r)\omega_1 + \omega_2 \leq M.$$

Therefore, the set ΨB_r is uniformly bounded.

Now, we show that Ψ maps bounded sets into equicontinuous sets of $\mathbb{C}([0, T], \mathbb{R})$. Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$ be two points and B_r be a bounded ball in Ψ. Then, for any $y \in B_r$, we obtain

$$|(\Psi y)(t_2) - (\Psi y)(t_1)|$$

$$\leq \left| \int_0^{t_2} \frac{(t_2 - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left| h(s, y(p^{\alpha-1}s)) \right| d_{p,q}s \right.$$

$$\left. - \int_0^{t_1} \frac{(t_1 - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left| h(s, y(p^{\alpha-1}s)) \right| d_{p,q}s \right|$$

$$+ \frac{1}{|\Delta|} |\zeta_1(\beta_2 - \zeta_2)|(t_2 - t_1) \int_0^{\eta_1} \frac{(\eta_1 - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left| h(s, y(p^{\alpha-1}s)) \right| d_{p,q}s$$

$$+ \frac{1}{|\Delta|} |\alpha_1||\alpha_2|(t_2 - t_1) \int_0^{T} \frac{(T - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left| h(s, y(p^{\alpha-1}s)) \right| d_{p,q}s$$

$$+ \frac{1}{|\Delta|} |\alpha_1||\alpha_2|(t_2 - t_1) \int_0^{T} \frac{(T - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left| h(s, y(p^{\alpha-1}s)) \right| d_{p,q}s$$

$$+ \frac{1}{|\Delta|} |\alpha_1\beta_2|(t_2 - t_1) \int_0^{T/p} \frac{(T/p - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha - 1)} |h(s, y(p^{\alpha}s))| d_{p,q}s$$

$$\leq \left| \int_0^{t_1} \left[\frac{(t_2 - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} - \frac{(t_1 - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \right] \left[p_1\left(\frac{s}{p^{\alpha-1}}\right)\psi(r) + p_2\left(\frac{s}{p^{\alpha-1}}\right) \right] d_{p,q}s \right.$$

$$\left. + \int_{t_1}^{t_2} \frac{(t_1 - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left[p_1\left(\frac{s}{p^{\alpha-1}}\right)\psi(r) + p_2\left(\frac{s}{p^{\alpha-1}}\right) \right] d_{p,q}s \right|$$

$$+ \frac{1}{|\Delta|} |\alpha_1||\alpha_2|(t_2 - t_1) \int_0^{T} \frac{(T - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left[p_1\left(\frac{s}{p^{\alpha-1}}\right)\psi(r) + p_2\left(\frac{s}{p^{\alpha-1}}\right) \right] d_{p,q}s$$

$$+ \frac{1}{|\Delta|} |\alpha_1||\alpha_2|(t_2 - t_1) \int_0^{T} \frac{(T - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left[p_1\left(\frac{s}{p^{\alpha-1}}\right)\psi(r) + p_2\left(\frac{s}{p^{\alpha-1}}\right) \right] d_{p,q}s$$

$$+ \frac{1}{|\Delta|} |\alpha_1||\alpha_2|(t_2 - t_1) \int_0^{T} \frac{(T - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left[p_1\left(\frac{s}{p^{\alpha-1}}\right)\psi(r) + p_2\left(\frac{s}{p^{\alpha-1}}\right) \right] d_{p,q}s$$

$$+ \frac{1}{|\Delta|} |\alpha_1\beta_2|(t_2 - t_1) \int_0^{T/p} \frac{(T/p - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha - 1)} \left[p_1\left(\frac{s}{p^{\alpha-1}}\right)\psi(r) + p_2\left(\frac{s}{p^{\alpha-1}}\right) \right] d_{p,q}s.$$

Obviously, the right-hand side of the above inequality tends to zero independently of $x \in B_r$ as $t_2 \to t_1$. Thus, it follows by the Arzelá–Ascoli theorem that $\Psi : \mathbb{C}([0, T], \mathbb{R}) \to \mathbb{C}([0, T], \mathbb{R})$ is completely continuous.

Now, the operator Ψ satisfies all the conditions of Lemma 6 and, therefore, by its conclusion, either condition (i) or condition (ii) holds.

Now, we show that the conclusion (ii) is not possible. Let

$$\mathcal{U} = \{ y \in \mathbb{C}([0, T], \mathbb{R}) : \|y\| \leq M \}$$

with $\psi(M)\omega_1 + \omega_2 < M$. Then, it can be shown that

$$|\Psi y| \leq \psi(\|y\|) \left\{ \int_0^T \frac{(T-qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} p_1\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \right.$$

$$+ |\zeta_1|\sigma_1(T) \int_0^{\eta_1} \frac{(\eta_1-qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} p_1\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s$$

$$+ |\zeta_2|\sigma_2(T) \int_0^{\eta_2} \frac{(\eta_2-qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} p_1\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s$$

$$+ |\beta_2|\sigma_2(T) \int_0^T \frac{(T-qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} p_1\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s$$

$$\left. + |\gamma_2|\sigma_2(T) \int_0^{T/p} \frac{(T/p-qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} p_2\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \right\}$$

$$+ \left\{ \int_0^T \frac{(T-qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} p_2\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \right.$$

$$+ |\zeta_1|\sigma_1(T) \int_0^{\eta_1} \frac{(\eta_1-qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} p_2\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s$$

$$+ |\zeta_2|\sigma_2(T) \int_0^{\eta_2} \frac{(\eta_2-qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} p_2\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s$$

$$+ |\beta_2|\sigma_2(T) \int_0^T \frac{(T-qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} p_2\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s$$

$$\left. + |\gamma_2|\sigma_2|(T) \int_0^{T/p} \frac{(T/p-qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} p_2\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \right\}$$

$$\leq \psi(M)\omega_1 + \omega_2 \leq M.$$

Suppose there exists $y \in \partial \mathcal{U}$ and $\lambda \in (0,1)$ such that $y = \lambda \Psi x$. Then, for such choices of x and λ, we have

$$M = \|y\| = \lambda\|\Psi y\| < \psi(\|y\|)\omega_1 + \omega_2 < M.$$

This leads to a contradiction. Accordingly, by Lemma 6, we have that Ψ has a fixed point $y \in \overline{\mathcal{U}}$, which is a solution of the boundary value problem (4). Therefore, the proof is completed. $\square$

Remark 3. *If p_1, p_2 in (A_4) are continuous, then $\omega_i \leq A\|p_i\|$, $i = 1, 2$, where A is defined by (23).*

4. Examples

In this section, we give some examples to illustrate the investigated results.

Example 1. *Consider the following fractional (p,q)-difference equation with the boundary value problem*

$$\begin{cases} {}^cD_{1/4,1/5}^{3/2}y(t) = \frac{\lambda}{2}\left(\sin(t) + \tan^{-1}\left(y(t/8)\right) + y(t/8)\right), \ t \in [0,8], \\ y(0) + \frac{1}{2}D_{p,q}y(0) = y(1/3), \quad \frac{1}{2}y(1) + \frac{3}{2}D_{p,q}y(4) = y(2/3). \end{cases} \tag{28}$$

By applying Theorem 3 with $p = 1/4, q = 1/5, \alpha = 3/2, \beta_1 = 1, \gamma_1 = 1/2, \zeta_1 = 1, \beta_2 = 1/2, \gamma_2 = 3/2, \zeta_2 = 1, \eta_1 = 1/3, \eta_2 = 2/3, T = 1$, and λ is a constant to be fixed later on. By simple calculation, we have $\Delta = 1/12, \sigma_1(1) = 4/3$, and $\sigma_2(1) = 2$. Let

$$\lambda < \left[\frac{1}{\Gamma_{1/4,1/5}(3/2)} \left(\frac{171\sqrt{30} + 8\sqrt{15} + 48\sqrt{5} - 216\sqrt{6}}{45\sqrt{30} - 72\sqrt{6}} \right) \right]^{-1}.$$

It is not difficult to show that

$$|f(t,y) - f(t,z)| \le \lambda |y - z|,$$

for each $t \in [0,8]$, and

$$k = \frac{\lambda}{\Gamma_{1/4,1/5}(3/2)} \left(\frac{171\sqrt{30} + 8\sqrt{15} + 48\sqrt{5} - 216\sqrt{6}}{45\sqrt{30} - 72\sqrt{6}} \right) < 1.$$

Then, all the assumptions of Theorem 3 are satisfied. Accordingly, by Theorem 3, the boundary value problem (28) has a unique solution.

Example 2. *Consider the following fractional (p,q)-difference equation with the boundary value problem*

$$\begin{cases} {}^{c}D^{3/2}_{1/4,1/5}y(t) = \frac{1}{16}\cos t^2 \sin(\frac{|y(t/8)|}{2}) + \frac{e^{-y(t/8)}(t^2+1)}{2+t^2} + \frac{1}{3},\ t \in [0,8], \\ y(0) + \frac{1}{2}D_{p,q}y(0) = y(1/3), \quad \frac{1}{2}y(1) + \frac{3}{2}D_{p,q}y(4) = y(2/3). \end{cases} \tag{29}$$

By applying Theorem 5 with $p = 1/4, q = 1/5, \alpha = 3/2, \beta_1 = 1, \gamma_1 = 1/2, \zeta_1 = 1, \beta_2 = 1/2, \gamma_2 = 3/2, \zeta_2 = 1, \eta_1 = 1/3, \eta_2 = 2/3, T = 1$. By simple calculation, we have $\Delta = 1/12, \sigma_1(1) = 4/3$, and $\sigma_2(1) = 2$.

On the other hand, we have

$$|f(t,y)| = \left| \frac{1}{16}\cos t^2 |\sin(|y|/2)| + \frac{e^{-y}(t^2+1)}{2+t^2} + \frac{1}{3} \right|$$

$$\le \frac{1}{32}|y| + 1.$$

Obviously, $p_1 = 1/16, p_2 = 1, \psi(M) = M$. By simple computation, we obtain $\omega_1 \approx 0.9149007474, \omega_2 \approx 29.27682392$. Additionally, from the condition (27), it follows that $M > 344.0315046$. Thus, all assumptions of Theorem 5 are satisfied. Therefore, the boundary value problem (29) has at least one solution.

5. Conclusions

In this paper, we study nonlocal fractional (p,q)-difference equations with separated nonlocal boundary conditions. The existence of solutions for the problem is given by applying some well-known tools in fixed-point theory, such as Banach's contraction mapping principle, Krasnoselskii's fixed-point theorem, and the Leray–Schauder nonlinear alternative. Some illustrating examples are also presented. We hope that the paper will inspire interested readers working in this field to draw upon these ideas and techniques.

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