fractal and fractional

Article

Fractional-Order Logistic Differential Equation with Mittag–Leffler-Type Kernel

Iván Area [1,*,†] and Juan J. Nieto [2,†]

[1] CITMAga, Universidade de Vigo, Departamento de Matemática Aplicada II, E.E. Aeronáutica e do Espazo, Campus As Lagoas-Ourense, 32004 Ourense, Spain
[2] CITMAga, Instituto de Matemáticas, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain; juanjose.nieto.roig@usc.es
* Correspondence: area@uvigo.gal
† Both authors contributed equally to this work.

Abstract: In this paper, we consider the Prabhakar fractional logistic differential equation. By using appropriate limit relations, we recover some other logistic differential equations, giving representations of each solution in terms of a formal power series. Some numerical approximations are implemented by using truncated series.

Keywords: logistic differential equation; fractional calculus; Liouville–Caputo fractional derivative; Atangana–Baleanu derivative; Caputo–Fabrizio derivative; Prabhakar derivative; Prabhakar integral

MSC: Primary 34A08; secondary 65Q30

check for updates

Citation: Area, I.; Nieto, J.J. Fractional-Order Logistic Differential Equation with Mittag–Leffler Type Kernel. *Fractal Fract.* **2021**, *5*, 273. https://doi.org/10.3390/fractalfract5040273

Academic Editor: Hari Mohan Srivastava

Received: 10 November 2021
Accepted: 9 December 2021
Published: 14 December 2021

1. Introduction

Let us consider the classical logistic differential equation

$$x'(t) = x(t)(1 - x(t)), \tag{1}$$

which can be explicitly solved. The constant solutions are $x(t) = 0$ and $x(t) = 1$. If another initial condition $x(0) = x_0$ is imposed, the solution is given by

$$x(t) = \frac{x_0}{x_0 + (1 - x_0)\exp(-t)}.$$

The solution can be also obtained in terms of formal power series. Let

$$x(t) = \sum_{n=0}^{\infty} a_n t^n. \tag{2}$$

Then,

$$x'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n,$$

and

$$x^2(t) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} a_j a_{n-j} \right) t^n \tag{3}$$

By substituting into (1), we obtain the following recurrence relation for the coefficients

$$a_{n+1} = \frac{1}{n+1} \left[a_n - \sum_{j=0}^{n} a_j a_{n-j} \right], \quad n \geq 1, \quad a_0 = x(0), \tag{4}$$

which provide a solution in a neighbourhood of $t = 0$ as described in [1].

It is possible to obtain the same recurrence relation if we further apply the Laplace transform to (1). Let $x(t)$ be given in (2), so that (3) holds true. Let $\mathcal{L}$ denote the Laplace transform, and as usual, we shall denote $F(s)$ the Laplace transform of a function $f(t)$. Then,

$$\mathcal{L}[x(t)(1-x(t))] = \mathcal{L}[x(t) - x^2(t)] = \sum_{n=0}^{\infty}(a_n - b_n)\frac{n!}{s^{n+1}}, \quad b_n = \sum_{j=0}^{n} a_j a_{n-j}.$$

Moreover,

$$\mathcal{L}[x'(t)] = sX(s) = \sum_{n=0}^{\infty} a_{n+1}(n+1)\frac{n!}{s^{n+1}}$$

Thus,

$$\sum_{n=0}^{\infty} a_{n+1}(n+1)\frac{n!}{s^{n+1}} = \sum_{n=0}^{\infty}(a_n - b_n)\frac{n!}{s^{n+1}} \tag{5}$$

which implies again the recurrence relation (4) for the coefficients of the power series expansion of the solution. We need to impose that $a_0 = x(0)$ to be able to start the latter recurrence relation.

We have included in Figure 1 some plots by using Mathematica [2] of the logistic function, solution to (1) with $x(0) = 1/2$, and approximations of the function by the corresponding Taylor polynomials.

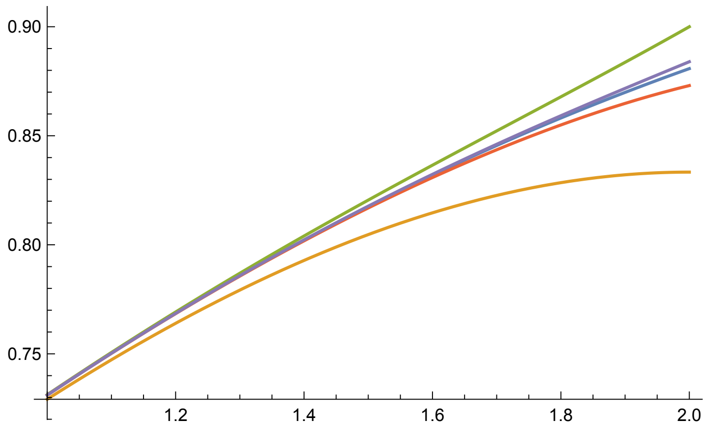

Figure 1. Logistic function solution to Equation (1) with $x(0) = 1/2$, in blue, as well approximations of the function by the corresponding Taylor polynomials in $[1, 2]$: $n = 3$ in orange color, $n = 5$ in green color, $n = 7$ in red color, and $n = 9$ in grey color.

This very classical logistic differential equation has been deeply studied due to its applications in different fields [3]. Recently, it has been used to study the evolution of the COVID-19 pandemic [4,5]. By considering fractional derivatives, the fractional analogue has been analyzed in several works mainly by considering the Liouville–Caputo fractional derivative [1,6–11] (see also [12–14]). The classical logistic ordinary differential equation has been recently studied from the view of fractional calculus and solved in some particular cases [1,8,10]. In this work, we consider the fractional logistic differential equation by using the Prabhakar fractional calculus [15–17].

The main aim of this work is to present the Prabhakar fractional logistic differential equation and, by appropriate limit transitions, recover several logistic differential equations (Liouville–Caputo, Atangana–Baleanu, and Caputo–Fabrizio), providing in each case a representation of the expansion of the solution in formal power series.

For the fractional Prabhakar logistic differential equation, we know the solutions for the Liouville–Caputo fractional derivative [7] (in terms of power series) and for the Caputo–Fabrizio derivative [18] (in implicit form). We emphasize that in this work we present the solution in terms of a power series expansion, as compared with the previous work [18], in which the solution is given in implicit form. It is also important to notice the fact that much more general fractional-calculus operators are available in the literature survey [19].

The structure of this work is the following: in Section 2 basic definitions, notations and results are presented. In Section 3 the Prabhakar fractional logistic equation is presented. For specific values of the parameters we recover the Liouville–Caputo, Atangana–Baleanu, and Caputo–Fabrizio logistic differential equations. For each of these cases, the solution is computed in terms of a formal power series. Some numerical experiments are also presented.

2. Basic Definition and Notations

Let $\alpha \in (0,1)$ and $\sigma \in L^1(0,1)$. The (Riemann–Liouville) fractional integral is defined by

$$I^{\alpha}\sigma(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\sigma(s)ds, \quad t \in (0,1)$$

where $\Gamma(z)$ denotes the Euler gamma function [20].

For $z \in \mathbb{C}$, $\alpha, \beta, \gamma \in \mathbb{C}$ with $\Re(\alpha) > 0$, the three-parameter Mittag–Leffler function, introduced by Prabhakar in 1971 [16], is defined by

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(n\alpha+\beta)} \frac{z^n}{n!},$$

which generalizes both the Mittag–Leffler function ($\gamma = 1$) as well as the classical exponential function ($\alpha = \beta = \gamma = 1$). Additionally, $E^0_{\alpha,\beta}(z) = 1/\Gamma(\beta)$. We would like to emphasize that $E^{\gamma}_{\alpha,\beta}(z)$ in an entire function of order $\varrho = 1/\Re(\alpha)$ and type $\sigma = 1$ [21].

Let

$$e^{\gamma}_{\alpha,\beta}(\lambda; t) = t^{\beta-1}E^{\gamma}_{\alpha,\beta}(\lambda t^{\alpha}), \tag{6}$$

be the Prabhakar kernel. In particular,

$$e^1_{1,1}(\lambda; t) = \exp(\lambda t); \quad e^0_{\alpha,\beta}(\lambda; t) = \frac{t^{\beta-1}}{\Gamma(\beta)}.$$

Additionally,

$$e^{\gamma}_{\alpha,\beta}(0; t) = \frac{t^{\beta-1}}{\Gamma(\beta)}.$$

The Prabhakar fractional integral with base point 0 is defined by

$$\mathbb{P}^{\gamma}_{\alpha,\beta,\lambda}\sigma(t) = \int_0^t e^{\gamma}_{\alpha,\beta}(\lambda; t-s)\sigma(s)ds. \tag{7}$$

for $\sigma \in L^1(0,1)$. For $\sigma \in L^1(0,1)$,

$$\mathbb{P}^{\gamma}_{\alpha,\beta,\lambda}\sigma(t) = \sum_{n=0}^{\infty} \frac{(\gamma)_n \lambda^n}{n!} I^{\alpha n+\beta}\sigma(t).$$

Thus, the Prabhakar fractional integral $\mathbb{P}^{\gamma}_{\alpha,\beta,\lambda}$ is linear and bounded from $L^p(0,1)$ into $L^p(0,1)$ for any $1 \leq p \leq \infty$.

Recall that taking $\lambda = -1$

$$e^{\gamma}_{\alpha,\beta}(-1, t) = t^{\beta-1}E^{\gamma}_{\alpha,\beta}(-t^{\alpha})$$

is completely monotone if $0 < \alpha\gamma \leq b \leq 1$ [22]. For example, $e^1_{1,1}(-1, t) = \exp(-t)$.

Moreover, Ref. [15]

$$\mathbb{P}^{\gamma}_{\alpha,\beta,\lambda} e^{\omega}_{\alpha,\mu}(\lambda; t) = e^{\gamma+\omega}_{\alpha,\beta+\mu}(\lambda; t).$$

Let $\sigma \in L^1(0,1)$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$. The Prabhakar fractional derivative in the Riemann–Liouville sense is defined by

$$\mathbb{D}^{\gamma}_{\alpha,\beta,\lambda}\sigma(t) = \frac{d}{dt}\mathbb{P}^{-\gamma}_{\alpha,1-\beta,\lambda}\sigma(t).$$

In doing so, it is required some regularity for σ, for example, that $\sigma \star e^{-\gamma}_{\alpha,1-\beta,\lambda} \in W^{1,1}(0,1)$, where for $\Omega \subset \mathbf{R}^n$ the Sobolev space $W^{m,p}(\Omega)$ is defined by

$$W^{m,p}(\Omega) = \{f \in L^p(\Omega) \,|\, D^\alpha \in L^p(\Omega), \forall \alpha \in \mathbf{N}^n \,:\, |\alpha| \leq m\}.$$

The Prabhakar fractional derivative in the Liouville–Caputo sense is

$$\mathbb{D}^{\gamma}_{\alpha,\beta,\lambda}\sigma(t) = \mathbb{P}^{-\gamma}_{\alpha,1-\beta,\lambda}\sigma'(t).$$

We note that

$$\mathbb{P}^{0}_{\alpha,\beta,1}\sigma(t) = \int_0^t e^{0}_{\alpha,\beta}(\lambda, t-s)\sigma(s)ds = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\sigma(s)ds = I^\beta\sigma(t).$$

Additionally,

$$\mathbb{P}^{\gamma}_{\alpha,\beta,0}\sigma(t) = \int_0^t e^{\gamma}_{\alpha,\beta}(0, t-s)\sigma(s)ds = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\sigma(s)ds = I^\beta\sigma(t).$$

In both cases, the classical Riemann-Liouville integral of order $\beta > 0$ is a particular case of the Prabhakar operator.

The Laplace transform of the Prabhakar fractional derivative in the Liouville–Caputo sense is [15] (page 27, Section 5.1, Equation (5.13))

$$\mathcal{L}[\mathbb{D}^{\gamma}_{\alpha,\beta,\lambda}x(t)](s) = s^{\beta-\alpha\gamma}(s^\alpha - \lambda)^\gamma \left\{ \mathcal{L}[x(t)](s) - \sum_{k=0}^{m-1} s^{-k-1} f^{(k)}(0+) \right\}, \tag{8}$$

where m denotes the integer part of β. In particular, if $m = 0$ or $m = 1$, we have

$$\mathcal{L}[\,\mathbb{D}^{\gamma}_{\alpha,\beta,\lambda}t^n](s) = s^{\beta-\alpha\gamma}(s^\alpha - \lambda)^\gamma \frac{\Gamma(n+1)}{s^{n+1}}. \tag{9}$$

Let us consider the Liouville–Caputo fractional derivative [23] for an absolutely continuous function $f : [0,T] \mapsto \mathbb{R}$

$${}^{\mathrm{C}}\mathbb{D}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} f'(s)ds, \quad t \in [0,T]. \tag{10}$$

We have that

$$\mathcal{L}[\,{}^{\mathrm{C}}\mathbb{D}^\alpha t^n](s) = \frac{\Gamma(n+1)}{s^{n-\alpha+1}}, \quad \alpha > 0. \tag{11}$$

Since

$$\mathcal{L}[\,\mathbb{D}^{\gamma}_{\alpha,\beta,0}t^n](s) = \mathcal{L}[\,\mathbb{D}^{0}_{\alpha,\beta,\lambda}t^n](s) = \frac{\Gamma(n+1)}{s^{n-\beta+1}} = \mathcal{L}[\,{}^{\mathrm{C}}\mathbb{D}^\beta t^n](s).$$

the Liouville–Caputo fractional derivative is a particular case of the Prabhakar fractional derivative.

The Atangana–Baleanu operator in the sense of Caputo for $u \in \mathrm{AC}(0,1) = W^{1,1}(0,1)$ is defined by [24]

$${}^{\mathrm{AB}}\mathbb{D}^\alpha u(t) = \frac{1}{1-\alpha}\int_0^t E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-s)^\alpha\right) u'(s)ds. \tag{12}$$

It yields

$$\mathcal{L}[{}^{\mathrm{AB}}\mathbb{D}^{\alpha}t^{n}](s) = \frac{B(\alpha)}{1-\alpha}\frac{1}{s^{\alpha}+\frac{\alpha}{1-\alpha}}\frac{\Gamma(n+1)}{s^{n-\alpha+1}}. \tag{13}$$

where $B(\alpha)$ is a normalizing function satisfying $B(0) = B(1) = 1$. Let $\beta = 0$, $\gamma = -1$, $\lambda = \alpha/(\alpha-1)$, so that

$$\mathcal{L}[\mathbb{D}^{-1}_{\alpha,0,\alpha/(\alpha-1)}t^{n}](s) = \frac{\Gamma(n+1)s^{\alpha-n-1}}{s^{\alpha}+\frac{\alpha}{1-\alpha}} = \frac{1-\alpha}{B(\alpha)}\mathcal{L}[{}^{\mathrm{AB}}\mathbb{D}^{\alpha}t^{n}](s).$$

As a consequence, the Atangana–Baleanu derivative is a particular case of the Prabhakar fractional derivative.

For $u \in \mathrm{AC}(0,1) = W^{1,1}(0,1)$, the Caputo–Fabrizio fractional derivative is defined by [25] (see also [26])

$${}^{\mathrm{CF}}\mathbb{D}^{\alpha}u(t) = \frac{1}{1-\alpha}\int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right)u'(s)ds. \tag{14}$$

We have that

$$\mathcal{L}[{}^{\mathrm{CF}}\mathbb{D}^{\alpha}t^{n}](s) = \frac{\Gamma(n+1)}{\alpha+(1-\alpha)s}\frac{1}{s^{n}}. \tag{15}$$

Let $\alpha = 1$, $\beta = 0$, $\gamma = -1$, $\lambda = \alpha/(\alpha-1)$, so that

$$\mathcal{L}[\mathbb{D}^{1}_{1,0,\alpha/(\alpha-1)}t^{n}](s) = \frac{s^{-n}\Gamma(n+1)}{s-\frac{\alpha}{\alpha-1}} = (1-\alpha)\mathcal{L}[{}^{\mathrm{CF}}\mathbb{D}^{\alpha}t^{n}](s),$$

revealing that the Caputo–Fabrizio derivative [25] is also a particular case of the Prabhakar fractional derivative.

3. Prabhakar Fractional Logistic Equation and Its Limiting Cases

Let $\mathbb{D}^{\gamma}_{\alpha,\beta,\lambda}x(t)$ be the Prabhakar fractional derivative of a given function $x(t)$. Let us now consider the Prabhakar fractional logistic differential equation

$$\Lambda(\alpha,\beta,\gamma,\lambda)\mathbb{D}^{\gamma}_{\alpha,\beta,\lambda}x(t) = x(t)(1-x(t)), \tag{16}$$

where the constant $\Lambda(\alpha,\beta,\gamma,\lambda)$ is defined by

$$\Lambda(\alpha,\beta,\gamma,\lambda) = \begin{cases} 1-\dfrac{\lambda}{\lambda-1}, & \alpha = 1, \\ \left(\dfrac{B(\alpha)}{1-\alpha}\right)^{\frac{(1-\alpha)\gamma\lambda}{\alpha}}, & \alpha \neq 1. \end{cases}$$

Let

$$x(t) = \sum_{n=0}^{\infty} a_n t^{n\xi}, \tag{17}$$

so that

$$x(t)(1-x(t)) = \sum_{n=0}^{\infty}(a_n - b_n)t^{n\xi}, \quad b_n = \sum_{j=0}^{n} a_j a_{n-j}. \tag{18}$$

If we apply the Laplace transform, we obtain

$$\Lambda(\alpha,\beta,\gamma,\lambda)s^{\beta-\alpha\gamma}(s^{\alpha}-\lambda)^{\gamma}\sum_{n=0}^{\infty}a_n\frac{\Gamma(\xi n+1)}{s^{\xi n+1}} = \sum_{n=0}^{\infty}(a_n-b_n)\frac{\Gamma(\xi n+1)}{s^{\xi n+1}},$$

or equivalently

$$\Lambda(\alpha,\beta,\gamma,\lambda)(s^{\alpha}-\lambda)^{\gamma}\sum_{n=0}^{\infty}a_n\frac{\Gamma(\xi n+1)}{s^{\xi n+1-\beta+\alpha\gamma}}=\sum_{n=0}^{\infty}(a_n-b_n)\frac{\Gamma(\xi n+1)}{s^{\xi n+1}}. \tag{19}$$

By using the binomial theorem,

$$\Lambda(\alpha,\beta,\gamma,\lambda)(s^{\alpha}-\lambda)^{\gamma}=\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+1)}{\Gamma(k+1)\Gamma(\gamma-n+1)}s^{\alpha n}(-\lambda)^{\gamma-n}=\sum_{n=0}^{\infty}c_n s^{\alpha n},$$

where

$$c_n=\frac{\Gamma(\gamma+1)}{\Gamma(n+1)\Gamma(\gamma-n+1)}(-\lambda)^{\gamma-n}$$

and then

$$\Lambda(\alpha,\beta,\gamma,\lambda)\sum_{n=0}^{\infty}\frac{c_n}{s^{-\alpha n}}\sum_{n=0}^{\infty}a_n\frac{\Gamma(\xi n+1)}{s^{\xi n+1-\beta+\alpha\gamma}}=\sum_{n=0}^{\infty}(a_n-b_n)\frac{\Gamma(\xi n+1)}{s^{\xi n+1}}.$$

Thus, we have

$$\begin{aligned}\Lambda(\beta,\alpha,\gamma,0)\mathbb{D}^{\gamma}_{\beta,\alpha,0}&={}^{C}\mathbb{D}^{\alpha} &&\text{(Liouville–Caputo)},\\ \Lambda(\beta,\alpha,0,\lambda)\mathbb{D}^{0}_{\beta,\alpha,\lambda}&={}^{C}\mathbb{D}^{\alpha} &&\text{(Liouville–Caputo)},\\ \Lambda(\alpha,0,-1,\alpha/(\alpha-1))\mathbb{D}^{-1}_{\alpha,0,\alpha/(\alpha-1)}&={}^{AB}\mathbb{D}^{\alpha} &&\text{(Atangana–Baleanu)},\\ \Lambda(1,0,-1,\alpha/(\alpha-1))\mathbb{D}^{-1}_{\alpha,0,\alpha/(\alpha-1)}&={}^{CF}\mathbb{D}^{\alpha} &&\text{(Caputo–Fabrizio)}.\end{aligned}$$

3.1. Fractional Liouville–Caputo Logistic Differential Equation

Let us consider the Liouville–Caputo fractional logistic differential equation

$${}^{C}\mathbb{D}^{\alpha}x(t)=x(t)(1-x(t)), \tag{20}$$

where ${}^{C}\mathbb{D}^{\alpha}$ is defined in (10). Since

$$\begin{aligned}\Lambda(\beta,\alpha,\gamma,0)\mathbb{D}^{\gamma}_{\beta,\alpha,0}&={}^{C}\mathbb{D}^{\alpha},\\ \Lambda(\beta,\alpha,0,\lambda)\mathbb{D}^{0}_{\beta,\alpha,\lambda}&={}^{C}\mathbb{D}^{\alpha},\end{aligned}$$

by applying the Laplace transform, taking into account (19) and (18), we obtain

$$\sum_{n=1}^{\infty}\frac{\Gamma(\xi n+1)}{s^{n\xi-\alpha+1}}a_n=\sum_{n=0}^{\infty}(a_n-b_n)\frac{\Gamma(n\xi+1)}{s^{n\xi+1}}.$$

For $\xi=\alpha$, equating coefficients we have the following recurrence relation for the coefficients in the power series expansion (19)

$$a_1=\frac{a_0-b_0}{\Gamma(1+\alpha)},\quad a_n=\frac{\Gamma((n-1)\alpha+1)}{\Gamma(n\alpha+1)}(a_{n-1}-b_{n-1}). \tag{21}$$

We have included in Figures 2 and 3 some plots of the logistic function, solution to (1) with $x(0)=1/2$, as well as some approximations of the solution to the Liouville–Caputo fractional logistic differential Equation (20).

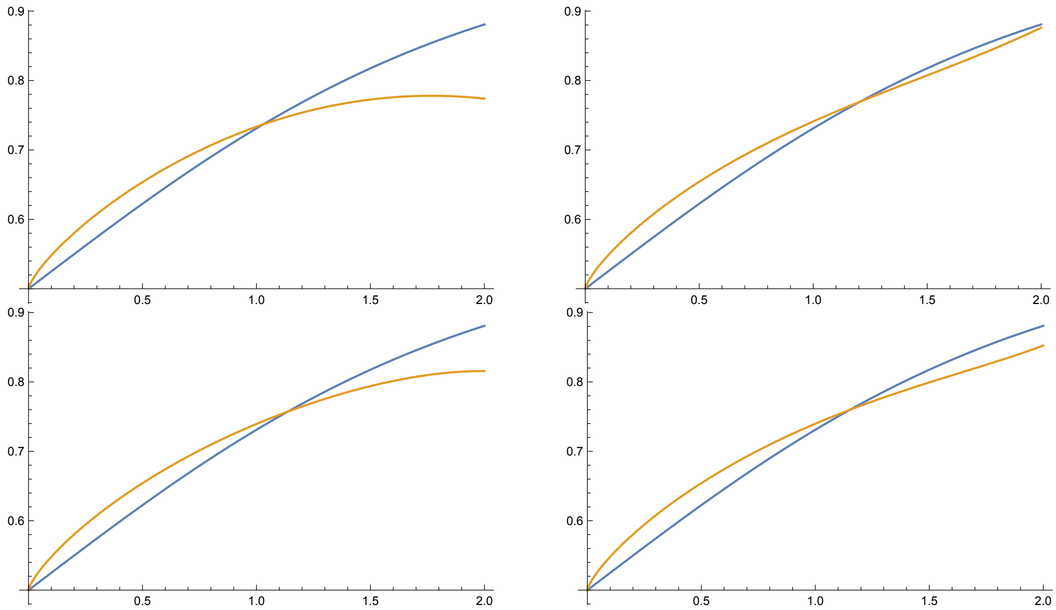

Figure 2. Logistic function solution to Equation (1) with $x(0) = 1/2$, in blue, as well as some approximations of the solution to the Caputo fractional logistic differential Equation (20) in $[0, 2]$ for $\alpha = 0.75$, in orange. From left to right and top to bottom the approximations are shown for $n = 3$, $n = 5$, $n = 7$, and $n = 9$. From these figures, one must use α closer to one as shown in Figure 3 in order to approximate the classical solution.

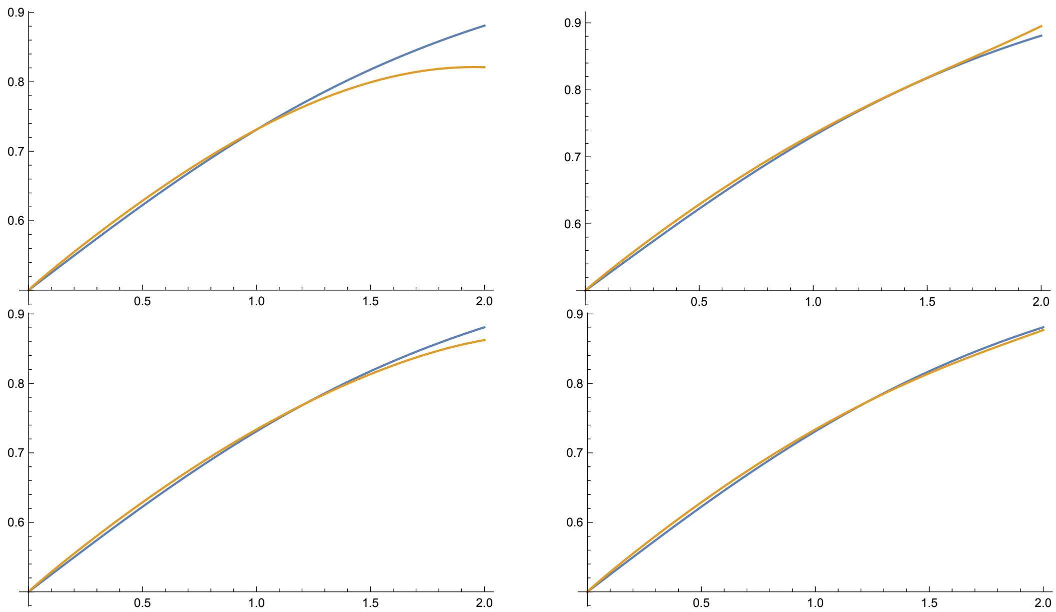

Figure 3. Logistic function solution to (1) with $x(0) = 1/2$, in blue, as well as some approximations of the solution to the Caputo fractional logistic differential Equation (20) in $[0, 2]$ for $\alpha = 0.95$, in orange. From left to right and top to bottom the approximations are shown for $n = 3$, $n = 5$, $n = 7$, and $n = 9$.

3.2. Atangana–Baleanu Logistic Differential Equation

Let us consider

$$ {}^{AB}\mathbb{D}^{\alpha} x(t) = x(t)(1 - x(t)), \tag{22} $$

where ${}^{AB}\mathbb{D}^{\alpha} f(t)$ is the Atangana–Baleanu derivative defined in (12).

Since

$$ \Lambda(\alpha, 0, -1, \alpha/(\alpha-1))\mathbb{D}^{-1}_{\alpha,0,\alpha/(\alpha-1)} = {}^{AB}\mathbb{D}^{\alpha}, $$

if we apply the Laplace transform to (22), by using (19) and (18) we obtain

$$\mathcal{L}[\,{}^{AB}\mathbb{D}^{\alpha}x(t)](s) = \sum_{n=0}^{\infty}(a_n - b_n)\frac{\Gamma(n\xi+1)}{s^{n\xi+1}}. \tag{23}$$

Thus, for $\xi = \alpha$,

$$\frac{B(\alpha)}{(1-\alpha)(s^{\alpha}+\frac{\alpha}{1-\alpha})}\sum_{n=1}^{\infty}a_n\frac{\Gamma(n\alpha+1)}{s^{(n-1)\alpha+1}} = \sum_{n=0}^{\infty}(a_n - b_n)\frac{\Gamma(n\alpha+1)}{s^{n\alpha+1}}. \tag{24}$$

Hence,

$$\begin{aligned}&\frac{B(\alpha)}{(1-\alpha)}\sum_{n=1}^{\infty}a_n\frac{\Gamma(n\alpha+1)}{s^{(n-1)\alpha+1}}\\ &= \sum_{n=0}^{\infty}(a_n - b_n)\frac{\Gamma(n\alpha+1)}{s^{\alpha(n-1)+1}} + \frac{\alpha}{1-\alpha}\sum_{n=0}^{\infty}(a_n - b_n)\frac{\Gamma(n\alpha+1)}{s^{n\alpha+1}}.\end{aligned} \tag{25}$$

Equating the coefficients, we obtain

$$\begin{aligned}a_1 &= \frac{a_0 - b_0 + b_1(\alpha-1)\Gamma(\alpha)}{(B(\alpha)+\alpha-1)\Gamma(\alpha)},\\ a_n &= \frac{(\alpha-1)b_n + \frac{\alpha(a_{n-1}-b_{n-1})\Gamma((n-1)\alpha+1)}{\Gamma(n\alpha+1)}}{B(\alpha)+\alpha-1}.\end{aligned}$$

By using

$$b_n = 2a_0a_n + \sum_{j=1}^{n-1}a_ja_{n-j}, \tag{26}$$

we finally obtain the initial step in terms of the initial condition

$$a_1 = \frac{(a_0-1)a_0}{\Gamma(\alpha)((2a_0-1)(\alpha-1)-B(\alpha))}, \tag{27}$$

as well the recurrence relation for the coefficients

$$a_n = \frac{(1-\alpha)\sum_{j=1}^{n-1}a_ja_{n-j} + \frac{\alpha\Gamma(\alpha(n-1)+1)\left(\sum_{j=1}^{n-1}a_ja_{n-j}+(2a_0-1)a_{n-1}\right)}{\Gamma(\alpha n+1)}}{(2a_0-1)(\alpha-1)-B(\alpha)}, \tag{28}$$

which in the limit as $\alpha \to 1$ converge to (21).

We have included in Figures 4 and 5 some plots of the logistic function, solution to (1) with $x(0) = 1/2$, as well as some approximations of the solution to the Atangana–Baleanu logistic differential Equation (22).

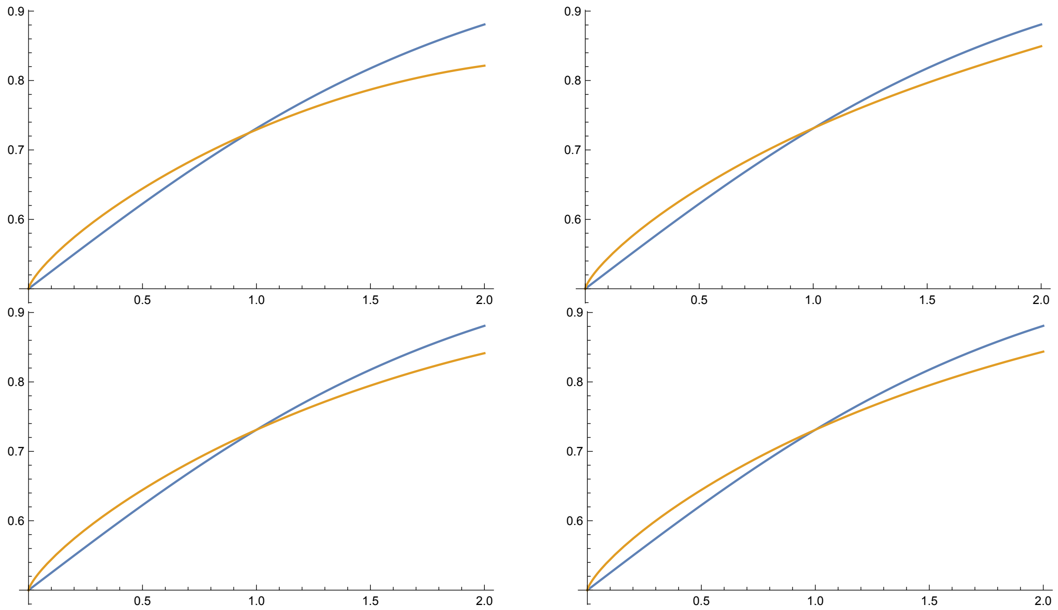

Figure 4. Logistic function solution to (1) with $x(0) = 1/2$, in blue, as well as some approximations of the solution to the Atangana–Baleanu logistic differential equation in $[0, 2]$ for $\alpha = 0.75$, in orange. From left to right and top to bottom the approximations are shown for $n = 3$, $n = 5$, $n = 7$, and $n = 9$. From these figures, one must use α closer to one as shown in Figure 5 in order to approximate the classical solution.

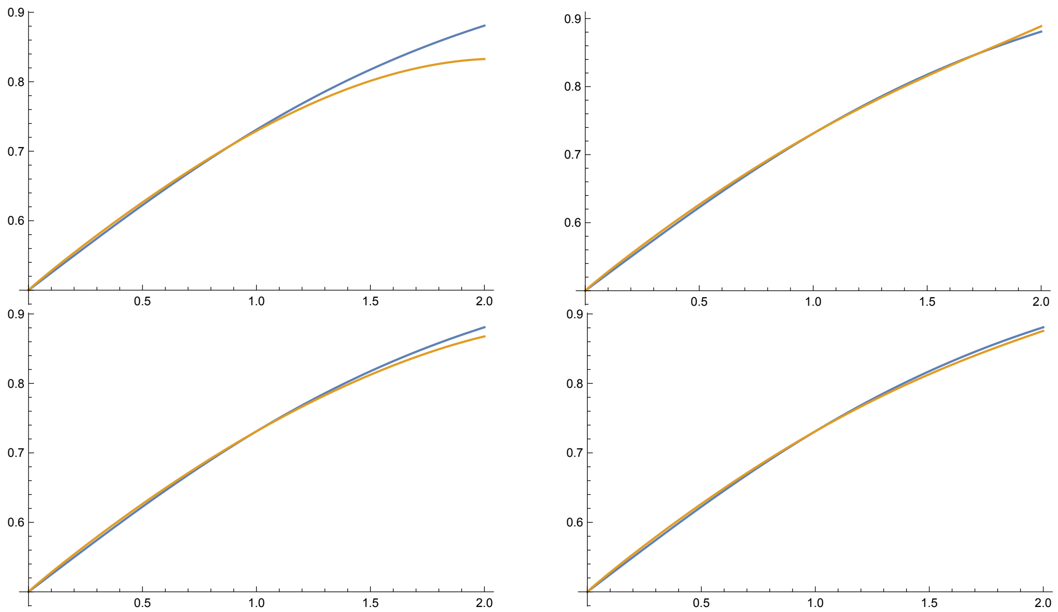

Figure 5. Logistic function solution to (1) with $x(0) = 1/2$, in blue, as well as some approximations of the solution to the Atangana–Baleanu logistic differential equation in $[0, 2]$ for $\alpha = 0.95$, in orange. From left to right and top to bottom the approximations are shown for $n = 3$, $n = 5$, $n = 7$, and $n = 9$.

3.3. Caputo–Fabrizio Logistic Differential Equation

Let us consider

$$ {}^{\mathrm{CF}}\mathbb{D}^{\alpha} x(t) = x(t)(1 - x(t)), \tag{29} $$

where ${}^{\mathrm{CF}}\mathbb{D}^{\alpha}$ is the Caputo–Fabrizio derivative introduced in (14).

Since

$$ \Lambda(1, 0, -1, \alpha/(\alpha - 1))\mathbb{D}^{-1}_{\alpha,0,\alpha/(\alpha-1)} = {}^{\mathrm{CF}}\mathbb{D}^{\alpha}, $$

if we apply the Laplace transform to (29), we obtain

$$\sum_{n=1}^{\infty} \frac{\Gamma(n\xi+1)}{(\alpha(1-s)+s)} \frac{a_n}{s^{n\xi}} = \sum_{n=0}^{\infty} (a_n - b_n) \frac{\Gamma(n\xi+1)}{s^{n\xi+1}},$$

which, if we fix $\xi = \alpha$, in the limit as $\alpha \to 1^-$, gives (5). Equivalently,

$$\sum_{n=1}^{\infty} \Gamma(n\xi+1) \frac{a_n}{s^{n\xi}} = (\alpha + (1-\alpha)s) \sum_{n=0}^{\infty} (a_n - b_n) \frac{\Gamma(n\xi+1)}{s^{n\xi+1}},$$

which can be rewritten as

$$\sum_{n=1}^{\infty} \Gamma(n\xi+1) \frac{a_n}{s^{n\xi}} = \alpha \sum_{n=0}^{\infty} (a_n - b_n) \frac{\Gamma(n\xi+1)}{s^{n\xi+1}} + (1-\alpha) \sum_{n=0}^{\infty} (a_n - b_n) \frac{\Gamma(n\xi+1)}{s^{n\xi}}.$$

Let $\xi = 1$. If we equate the coefficients, we obtain

$$a_n = \frac{1}{n} \left(a_{n-1} - b_{n-1} + n b_n \frac{\alpha - 1}{\alpha} \right)$$

which in the limit as $\alpha \to 1$ converges to (4). This relation can be obtained from [18] (Equation (8)).

We have included in Figures 6 and 7 some plots of the logistic function, solution to (1) with $x(0) = 1/2$, as well as some approximations of the solution to the Caputo–Fabrizio logistic differential Equation (29).

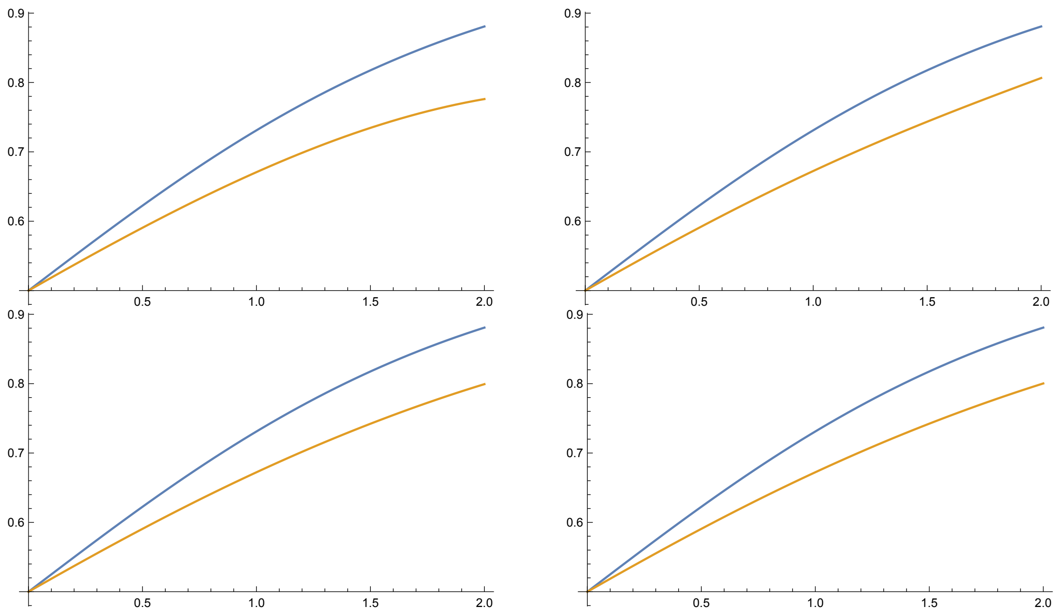

Figure 6. Logistic function solution to (1) with $x(0) = 1/2$, in blue, as well as some approximations of the solution to the Caputo–Fabrizio logistic differential Equation (29) in $[0, 2]$ for $\alpha = 0.75$, in orange. From left to right and top to bottom the approximations are shown for $n = 3$, $n = 5$, $n = 7$, and $n = 9$. As in the previous cases, from these figures one must use α closer to one as shown in Figure 7 in order to approximate the classical solution.

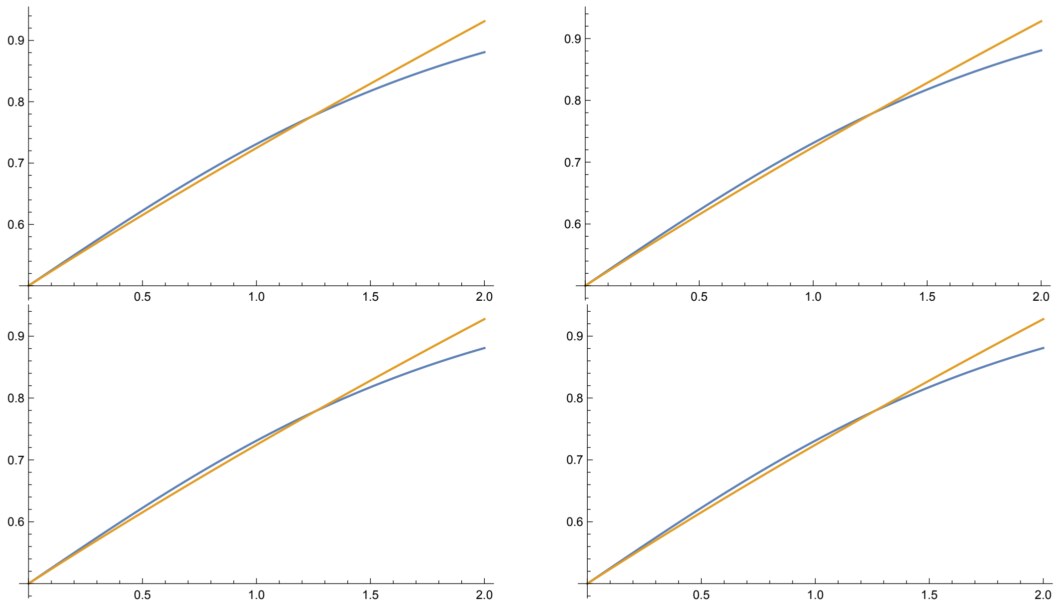

Figure 7. Logistic function solution to (1) with $x(0) = 1/2$, in blue, as well as some approximations of the solution to the Caputo–Fabrizio logistic differential Equation (29) in $[0, 2]$ for $\alpha = 0.95$, in orange. From left to right and top to bottom the approximations are shown for $n = 3$, $n = 5$, $n = 7$, and $n = 9$.

Moreover, in Figure 8, we show a comparison between the results in [18] in implicit form, and the results presented here in terms of recurrence relation for the coefficients in the power series expansion. For this last comparison, we have chosen $\alpha = 0.9$.

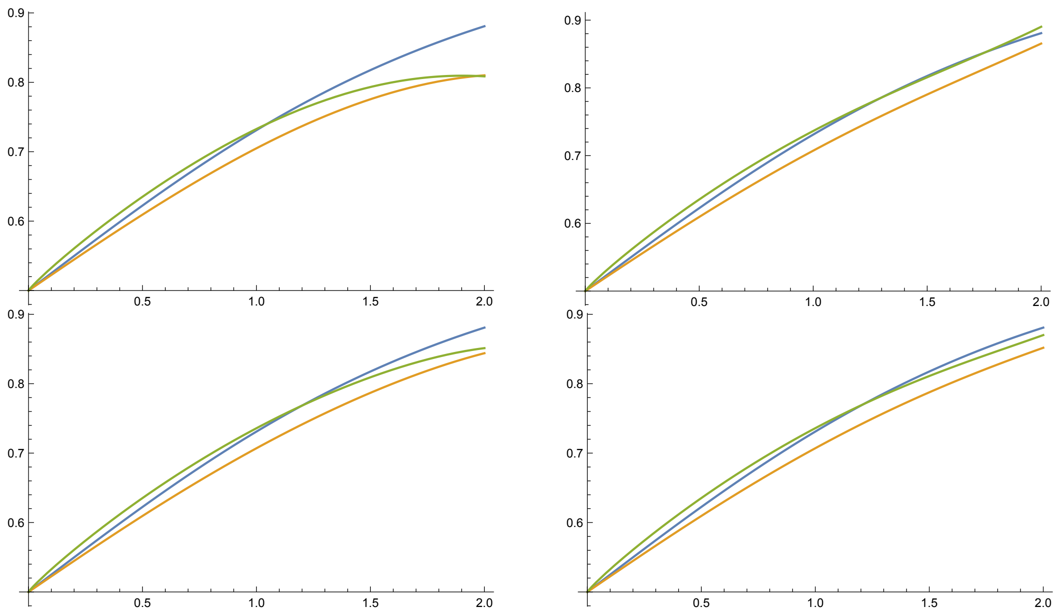

Figure 8. In $[0, 2]$ for $\alpha = 0.9$, logistic function solution to (1) with $x(0) = 1/2$, in blue, some approximations of the solution to the Caputo–Fabrizio logistic differential Equation (29), as well as the solution given in [18] in orange. From left to right and top to bottom the approximations are shown for $n = 3$, $n = 5$, $n = 7$, and $n = 9$.

4. Conclusions

The Prabhakar fractional calculus, based on the three-parameter generalization of the Mittag–Leffler function, provides some physical examples such as anomalous phenomena showing the need for an extension of ordinary calculus based on the Prabhakar function [15].

We contribute to the study of the fractional logistic differential equation in the setting of Prabhakar fractional calculus by solving that logistic equation in some cases.

Author Contributions: All authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: This work has been partially supported by the Agencia Estatal de Investigación (AEI) of Spain under Grant PID2020-113275GB-I00, cofinanced by the European Community fund FEDER, and Xunta de Galicia, grant ED431C 2019/02 for Competitive Reference Research Groups (2019–2022).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors thank the three anonymous referees and editor for their valuable comments which improved a preliminary version of this material.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Area, I.; Nieto, J.J. Power series solution of the fractional logistic equation. *Phys. A Stat. Mech. Its Appl.* **2021**, *573*, 125947. [CrossRef]
2. Wolfram Research, Inc. *Mathematica, Version 12.1*; Wolfram Research, Inc.: Champaign, IL, USA, 2020.
3. Thieme, H.R. *Mathematics in Population Biology, Princeton series in Theoretical and Computational Biology*; Princeton University Press: Princeton, NJ, USA, 2003.
4. Saito, T. A Logistic Curve in the SIR Model and Its Application to Deaths by COVID-19 in Japan. *MedRxiv* **2020**. [CrossRef]
5. Pelinovsky, E.; Kurkin, A.; Kurkina, O.; Kokoulina, M.; Epifanova, A. Logistic equation and COVID-19. *Chaos Solitons Fractals* **2020**, *140*, 110241. [CrossRef] [PubMed]
6. Ortigueira, M.; Bengochea, G. A new look at the fractionalization of the logistic equation. *Phys. A Stat. Mech. Its Appl.* **2017**, *467*, 554–561. [CrossRef]
7. Area, I.; Losada, J.; Nieto, J.J. A note on the fractional logistic equation. *Physica A* **2016**, *444*, 182–187. [CrossRef]
8. D'Ovidio, M.; Loreti, P.; Sarv Ahrabi, S. Modified fractional logistic equation, *Phys. A Stat. Mech. Its Appl.* **2018**, *505*, 818–824. [CrossRef]
9. El-Sayed, A.M.A.; El-Mesiry, A.E.M.; El-Saka, H.A.A. On the fractional-order logistic equations. *Appl. Math. Lett.* **2007**, *20*, 817–823. [CrossRef]
10. Kaharuddin, L.N.; Phang, C.; Jamaian, S.S. Solution to the fractional logistic equation by modified Eulerian numbers. *Eur. Phys. J. Plus* **2020**, *135*, 229. [CrossRef]
11. West, B.J. Exact solution to fractional logistic equation. *Physica A* **2015**, *429*, 103–108. [CrossRef]
12. Izadi, M.; Srivastava, H.M. A discretization approach for the nonlinear fractional logistic equation. *Entropy* **2020**, *22*, 1328. [CrossRef] [PubMed]
13. Izadi, M.; Srivastava, H.M. Numerical approximations to the nonlinear fractional-order logistic population model with fractional-order Bessel and Legendre bases. *Chaos Solitons Fractals* **2021**, *145*, 110779. [CrossRef]
14. Oldham, K.B.; Spanier, J. *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*; Academic Press: Cambridge, MA, USA, 1974.
15. Giusti, A.; Colombaro, I.; Garra, R.; Garrappa, R.; Polito, F.; Popolizio, M.; Mainardi, F. A practical guide to Prabhakar fractional calculus. *Frac. Calcul. Appl. Anal.* **2020**, *23*, 9–54. [CrossRef]
16. Prabhakar, T.R. A singular integral equation with a generalized Mittag- Leffler function in the kernel. *Yokohama Math. J.* **1971**, *19*, 7–15.
17. Samko, S.; Kilbas, A.A.; Marichev, O. *Fractional Integrals and Derivatives*; Taylor & Francis: Abingdon, UK, 1993.
18. Nieto, J. Solution of a fractional logistic ordinary differential equation. *Appl. Math. Lett.* **2022**, *123*, 107568. [CrossRef]
19. Srivastava, H.M. An introductory overview of fractional-calculus operators based upon the Fox-Wright and related higher transcendental functions. *J. Adv. Eng. Comput.* **2021**, *5*, 135–166.
20. Abramowitz, M.; Stegun, I.A. *Handbook of Mathematical Functions*, 9th ed.; Dover: New York, NY, USA, 1972.
21. Gorenflo, R.; Kilbas, A.A.; Mainardi, F.; Rogosin, S. Mittag-Leffler functions. In *Theory and Applications*; Springer Monographs in Mathematics; Springer: Berlin, Germany, 2014.
22. Garra, R.; Garrappa, R. The Prabhakar or three parameter Mittag-Leffler function: Theory and application. *Commun. Nonlin. Sci. Numer. Simul.* **2018**, *56*, 314–329. [CrossRef]
23. Caputo, M. Linear model of dissipation whose Q is almost frequency independent. II. *Geophys. J. Int.* **1967**, *13*, 529–539. [CrossRef]
24. Atangana, A.; Baleanu, D. New fractional derivatives with non-local and non-singular kernel. Theory and application to heat transfer model. *Ther. Sci.* **2016**, *20*, 763–769. [CrossRef]

25. Caputo, M.; Fabrizio, M. On the singular kernels for fractional derivatives. Some applications to partial differential equations. *Progr. Fract. Differ. Appl.* **2021**, *7*, 1–21.
26. Losada, J.; Nieto, J.J. Fractional integral associated to fractional derivatives with nonsingular kernels. *Progr. Fract. Differ. Appl.* **2021**, *7*, 79–82.