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Some Double Generalized Weighted Fractional Integral Inequalities Associated with Monotone Chebyshev Functionals

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Abstract: In this manuscript, we study the unified integrals recently defined by Rahman et al. and present some new double generalized weighted type fractional integral inequalities associated with increasing, positive, monotone and measurable function \mathcal{F} . Also, we establish some new double-weighted inequalities, which are particular cases of the main result and are represented by corollaries. These inequalities are further refinement of all other inequalities associated with increasing, positive, monotone and measurable function existing in literature. The existing inequalities associated with increasing, positive, monotone and measurable function are also restored by applying specific conditions as given in Remarks. Many other types of fractional integral inequalities can be obtained by applying certain conditions on \mathcal{F} and Ψ given in the literature.

Keywords: measurable function; Chebyshev functionals; integral inequalities; monotone; weighted fractional integral

MSC: 26D10; 26D15; 26D53; 05A30



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1. Introduction

In the context of fractional differential equations, integral inequalities are very significant. This field has gained popularity during the last few decades. Various researchers, such as [1–3], have investigated the significant developments in this domain. By employing Riemann-Liouville (R-L) fractional integrals, the authors presented Grüss type and several other new inequalities in [4,5]. Certain inequalities for the generalised (k, ρ) -fractional integral operator are proposed in [6]. In [7], the modified Hermite-Hadamard type inequalities can be found. Dahmani [8] discovered various fractional integral inequalities employing a family of n positive functions. In [9], Srivastava et al. presented the Chebyshev inequality by employing general family of fractional integral operators. Some remarkable inequalities and their applications can be found in the work of [10–15].

In [16,17], the Chebyshev functional for the integrable functions \mathcal{Z}_1 and \mathcal{Z}_2 on $[v_1, v_2]$, is given by

$$\mathcal{H}(\mathcal{Z}_1, \mathcal{Z}_2, \mathfrak{h}_1) = \int_{v_1}^{v_2} \mathfrak{h}_1(\theta) d\theta \int_{v_1}^{v_2} \mathfrak{h}_1(\theta) \mathcal{Z}_1(\theta) \mathcal{Z}_2(\theta) d\theta - \int_{v_1}^{v_2} \mathfrak{h}_1(\theta) \mathcal{Z}_1(\theta) d\theta \int_{v_1}^{v_2} \mathfrak{h}_1(\theta) \mathcal{Z}_2(\theta) d\theta, \quad (1)$$

where the function \mathfrak{h}_1 is a positive and integrable on $[v_1, v_2]$.

The following extended Chebyshev functional for the integrable functions \mathcal{Z}_1 and \mathcal{Z}_2 on $[v_1, v_2]$ can be found in [5,18] by

$$\begin{aligned} \mathcal{H}(\mathcal{Z}_1, \mathcal{Z}_2, \hbar_1, \hbar_2) &= \int_{v_1}^{v_2} \hbar_2(\theta) d\theta \int_{v_1}^{v_2} \hbar_1(\theta) \mathcal{Z}_1(\theta) \mathcal{Z}_2(\theta) d\theta + \int_{v_1}^{v_2} \hbar_1(\theta) d\theta \int_{v_1}^{v_2} \hbar_2(\theta) \mathcal{Z}_1(\theta) \mathcal{Z}_2(\theta) d\theta \\ &\quad - \int_{v_1}^{v_2} \hbar_1(\theta) \mathcal{Z}_1(\theta) d\theta \int_{v_1}^{v_2} \hbar_2(\theta) \mathcal{Z}_2(\theta) d\theta - \int_{v_1}^{v_2} \hbar_2(\theta) \mathcal{Z}_1(\theta) d\theta \int_{v_1}^{v_2} \hbar_1(\theta) \mathcal{Z}_2(\theta) d\theta. \end{aligned} \quad (2)$$

where the two functions \hbar_1 and \hbar_2 are positive and integrable on $[v_1, v_2]$.

Kuang [19] and Mitrinovic [18] proved that $\mathcal{H}(\mathcal{Z}_1, \mathcal{Z}_2, \hbar_1) \geq 0$ and $\mathcal{H}(\mathcal{Z}_1, \mathcal{Z}_2, \hbar_1, \hbar_2) \geq 0$ if the functions \mathcal{Z}_1 and \mathcal{Z}_2 are synchronous on $[v_1, v_2]$.

Remark 1. If we take $\hbar_1(\theta) = \hbar_2(\theta) = 1, \theta \in [v_1, v_2]$, then

$$\mathcal{H}(\mathcal{Z}_1, \mathcal{Z}_2, \hbar_1) = \frac{1}{2} \mathcal{H}(\mathcal{Z}_1, \mathcal{Z}_2, \hbar_1, \hbar_2).$$

Certain remarkable integral inequalities associated with the Chebyshev's functionals (1) and (2) can be found in the work of [20–25].

Awan et al. [26] proposed the following inequality by:

Theorem 1. Let the function g be an absolutely continuous on $[v_1, v_2]$, and \hbar_1 be integrable and positive function on $[v_1, v_2]$ and $(g')^2 \in L_1[v_1, v_2]$, then the following inequality holds;

$$\mathcal{H}(\Phi, \Phi, \hbar_1) \leq \frac{1}{G^2(v_2)} \int_{v_1}^{v_2} \left[\int_{v_1}^{\theta} \hbar_1(\varrho) d\varrho \int_{v_1}^{v_2} \varrho \hbar_1(\varrho) d\varrho - \int_{v_1}^{v_2} \hbar_1(\varrho) d\varrho \int_{v_1}^{\theta} \varrho \hbar_1(\varrho) d\varrho \right] [\Phi'(\theta)]^2 d\theta,$$

where $G(v_2) = \int_{v_1}^{v_2} \hbar_1(\varrho) d\varrho$.

In [27], Bezziou et al. proposed the below result for Riemann-Liouville fractional integral as follows:

Theorem 2. Assume that the function $g : [v_1, v_2] \rightarrow \mathbb{R}$ be an absolutely continuous function, and the function $\hbar_1 : [v_1, v_2] \rightarrow \mathbb{R}^+$ be an integrable, and $(g')^2 \in L_1[v_1, v_2]$. Then the following inequality for $\kappa > 0$ holds:

$$\mathcal{I}_{v_1+}^{\kappa} \hbar_1(v_2) \mathcal{I}_{v_1+}^{\kappa} \hbar_1 g^2(v_2) - (\mathcal{I}_{v_1+}^{\kappa} \hbar_1 g(v_2))^2 \leq \int_{v_1}^{v_2} \Upsilon(\theta) [g'(\theta)]^2 d\theta,$$

where

$$\Upsilon(\theta) = \frac{1}{2} \left[\mathcal{I}_{v_1+v_2}^{\kappa} \hbar_1(v_2) \int_{v_1}^{\theta} (v_2 - \varrho)^{\kappa-1} \hbar_1(\varrho) d\varrho - \mathcal{I}_{v_1+}^{\kappa} \hbar_1(v_2) \int_{v_1}^{\theta} \varrho (v_2 - \varrho)^{\kappa-1} \hbar_1(\varrho) d\varrho \right].$$

Dahmani and Bounoua [28] proposed the following inequality for Riemann-Liouville fractional integral by:

Theorem 3. If the function $g : [v_1, v_2] \rightarrow \mathbb{R}$ be an absolutely continuous and let $\hbar_1 : [v_1, v_2] \rightarrow \mathbb{R}^+$ be an integrable function. If $(\Phi')^2 \in L_1[v_1, v_2]$, then for all $\kappa > 0$, and $\theta \in [v_1, v_2]$, the following inequality holds;

$$\frac{1}{\mathcal{I}_{v_1}^{\kappa} \hbar_1(\theta)} \mathcal{I}_{v_1+}^{\theta} (\hbar_1 g^2)(\theta) - \left[\frac{1}{\mathcal{I}_{v_1}^{\kappa} \hbar_1(\theta)} \mathcal{I}_{v_1+}^{\theta} (\hbar_1 g)(\theta) \right]^2 \leq \frac{1}{[\mathcal{I}_{v_1}^{\kappa} \hbar_1(\theta)]^2} \int_{v_1}^{\theta} P_x(\theta) [g'(\theta)]^2 d\theta,$$

with

$$P_x(\theta) = \frac{1}{\Gamma(\kappa)} \left[\mathcal{I}_{v_1+}^{\kappa} (\hbar_1(x)) \int_{v_1}^{\theta} \hbar_1(\varrho) (x - \varrho)^{\kappa-1} d\varrho - \mathcal{I}_{v_1+}^{\kappa} \hbar_1(x) \int_{v_1}^{\theta} \varrho \hbar_1(\varrho) (x - \varrho)^{\kappa-1} d\varrho \right].$$

Definition 1 ([29]). Suppose that the function $\Psi : [0, \infty) \rightarrow [0, \infty)$ be satisfying the conditions given below:

$$\int_0^1 \frac{\Psi(\varrho)}{\varrho} d\varrho < \infty, \quad (3)$$

$$\frac{1}{P} \leq \frac{\Psi(\hbar_1)}{\Psi(\hbar_2)} \leq P, \frac{1}{2} \leq \frac{\hbar_1}{\hbar_2} \leq 2, \quad (4)$$

$$\frac{\Psi(\hbar_2)}{\hbar_2^2} \leq Q \frac{\Psi(\hbar_1)}{\hbar_1^2}, \hbar_1 \leq \hbar_2, \quad (5)$$

$$\left| \frac{\Psi(\hbar_2)}{\hbar_2^2} - \frac{\Psi(\hbar_1)}{\hbar_1^2} \right| \leq S |\hbar_2 - \hbar_1| \frac{\Psi(\hbar_2)}{\hbar_2^2}, \frac{1}{2} \leq \frac{\hbar_1}{\hbar_2} \leq 2, \quad (6)$$

where $P, Q, S > 0$ and are independent of $\hbar_1, \hbar_2 > 0$. If $\Psi(\hbar_2)\hbar_2^\alpha$ is increasing for some $\alpha > 0$ and $\frac{\Psi(\hbar_2)}{\hbar_2^\beta}$ is decreasing for some $\beta > 0$, then Ψ satisfies (3)–(6).

Next, we recall the following generalized weighted type fractional integral operators recently proposed by Rahman et al. [30].

Definition 2. The generalized weighted type fractional integral operators both left and right sided are respectively defined by:

$$\left({}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \mathcal{Z}_1 \right) (\theta) = \omega^{-1}(\theta) \int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(\theta) - \mathcal{F}(\varrho))}{\mathcal{F}(\theta) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) \mathcal{Z}_1(\varrho) d\varrho, v_1 < \theta \quad (7)$$

and

$$\left({}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_2-}^{\Psi} \mathcal{Z}_1 \right) (\theta) = \omega^{-1}(\theta) \int_{\theta}^{v_2} \frac{\Psi(\mathcal{F}(\varrho) - \mathcal{F}(\theta))}{\mathcal{F}(\varrho) - \mathcal{F}(\theta)} \omega(\varrho) \mathcal{F}'(\varrho) \mathcal{Z}_1(\varrho) d\varrho, v_2 > \theta. \quad (8)$$

Remark 2. 1. If we consider $\Psi(\mathcal{F}(\theta)) = \mathcal{F}(\theta)$, the fractional integrals (7) and (8) reduce to the following:

$$\left({}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+} \mathcal{Z}_1 \right) (\theta) = \omega^{-1}(\theta) \int_{v_1}^{\theta} \omega(\varrho) \mathcal{F}'(\varrho) \mathcal{Z}_1(\varrho) d\varrho, v_1 < \theta$$

and

$$\left({}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_2-} \mathcal{Z}_1 \right) (\theta) = \omega^{-1}(\theta) \int_{\theta}^{v_2} \omega(\varrho) \mathcal{F}'(\varrho) \mathcal{Z}_1(\varrho) d\varrho, v_2 > \theta,$$

respectively.

2. If we consider $\mathcal{F}(\theta) = \theta$, the fractional integrals (7) and (8) reduce to the following respectively

$$\left({}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+} \mathcal{Z}_1 \right) (\theta) = \omega^{-1}(\theta) \int_{v_1}^{\theta} \frac{\Psi(\theta - \varrho)}{\theta - \varrho} \omega(\varrho) \mathcal{Z}_1(\varrho) d\varrho, v_1 < \theta$$

and

$$\left({}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_2-} \mathcal{Z}_1 \right) (\theta) = \omega^{-1}(\theta) \int_{\theta}^{v_2} \frac{\Psi(\varrho - \theta)}{\varrho - \theta} \omega(\varrho) \mathcal{Z}_1(\varrho) d\varrho, v_2 > \theta.$$

3. If we consider $\Psi(\mathcal{F}(\theta)) = \frac{\mathcal{F}(\theta)^\kappa}{\Gamma(\kappa)}$, the fractional integrals (7) and (8) reduce to the following respectively (see [31]):

$$\left({}_{\omega} \mathcal{I}_{v_1^+}^\kappa \mathcal{Z}_1 \right) (\theta) = \frac{\omega^{-1}(\theta)}{\Gamma(\kappa)} \int_{v_1}^\theta (\mathcal{F}(\theta) - \mathcal{F}(\varrho))^{\kappa-1} \omega(\varrho) \mathcal{F}'(\varrho) \mathcal{Z}_1(\varrho) d\varrho, v_1 < \theta$$

and

$$\left({}_{\omega} \mathcal{I}_{v_2^-}^\kappa \mathcal{Z}_1 \right) (\theta) = \frac{\omega^{-1}(\theta)}{\Gamma(\kappa)} \int_\theta^{v_2} (\mathcal{F}(\varrho) - \mathcal{F}(\theta))^{\kappa-1} \omega(\varrho) \mathcal{F}'(\varrho) \mathcal{Z}_1(\varrho) d\varrho, v_2 > \theta,$$

where $\kappa, \in \mathbb{C}$ with $\Re(\kappa) > 0$.

4. If we consider $\mathcal{F}(\theta) = \theta$ and $\Psi(\mathcal{F}(\theta)) = \frac{\theta^\kappa}{\Gamma(\kappa)}$, the fractional integrals (7) and (8) reduce to the following:

$$\left({}_{\omega} \mathcal{J}_{v_1^+}^\kappa \mathcal{Z}_1 \right) (\theta) = \frac{\omega^{-1}(\theta)}{\Gamma(\kappa)} \int_{v_1}^\theta (\theta - \varrho)^{\kappa-1} \omega(\varrho) \mathcal{Z}_1(\varrho) d\varrho, v_1 < \theta$$

and

$$\left({}_{\omega} \mathcal{J}_{v_2^-}^\kappa \mathcal{Z}_1 \right) (\theta) = \frac{\omega^{-1}(\theta)}{\Gamma(\kappa)} \int_\theta^{v_2} (\varrho - \theta)^{\kappa-1} \omega(\varrho) \mathcal{Z}_1(\varrho) d\varrho, v_2 > \theta,$$

respectively.

5. If we consider $\mathcal{F}(\theta) = \ln \theta$ and $\Psi(\mathcal{F}(\theta)) = \frac{(\ln \theta)^\kappa}{\Gamma(\kappa)}$, the fractional integrals (7) and (8) reduce to the following weighted Hadamard fractional integrals:

$$\left({}_{\omega} \mathcal{I}_{v_1^+}^\kappa \mathcal{Z}_1 \right) (\theta) = \frac{\omega^{-1}(\theta)}{\Gamma(\kappa)} \int_{v_1}^\theta (\ln \theta - \ln \varrho)^{\kappa-1} \omega(\varrho) \mathcal{Z}_1(\varrho) \frac{d\varrho}{\varrho}, v_1 < \theta$$

and

$$\left({}_{\omega} \mathcal{I}_{v_2^-}^\kappa \mathcal{Z}_1 \right) (\theta) = \frac{\omega^{-1}(\theta)}{\Gamma(\kappa)} \int_\theta^{v_2} (\ln \varrho - \ln \theta)^{\kappa-1} \omega(\varrho) \mathcal{Z}_1(\varrho) \frac{d\varrho}{\varrho}, v_2 > \theta.$$

6. If we consider $\mathcal{F}(\theta) = \theta^\eta$ and $\Psi(\mathcal{F}(\theta)) = \frac{\theta^\eta}{\eta}$, $\eta > 0$, the fractional integrals (7) and (8) reduce to the following weighted Katugampola fractional integrals,

$$\left({}_{\omega} \mathcal{I}_{v_1^+}^\kappa \mathcal{Z}_1 \right) (\theta) = \frac{\omega^{-1}(\theta)}{\Gamma(\kappa)} \int_{v_1}^\theta \left(\frac{\theta^\eta - \varrho^\eta}{\eta} \right)^{\kappa-1} \omega(\varrho) \mathcal{Z}_1(\varrho) \frac{d\varrho}{\varrho^{1-\eta}}, v_1 < \theta$$

and

$$\left({}_{\omega} \mathcal{I}_{v_2^-}^\kappa \mathcal{Z}_1 \right) (\theta) = \frac{\omega^{-1}(\theta)}{\Gamma(\kappa)} \int_\theta^{v_2} \left(\frac{\varrho^\eta - \theta^\eta}{\eta} \right)^{\kappa-1} \omega(\varrho) \mathcal{Z}_1(\varrho) \frac{d\varrho}{\varrho^{1-\eta}}, v_2 > \theta.$$

7. If we consider $\mathcal{F}(\theta) = \theta$ and $\Psi(\mathcal{F}(\theta)) = \frac{\theta}{\eta} \exp\left(-\frac{1-\eta}{\eta}\theta\right)$, $\eta \in (0, 1)$, the fractional integrals (7) and (8) reduce to the following weighted fractional integrals,

$$\left({}_{\omega} \mathcal{I}_{v_1^+}^\eta \mathcal{Z}_1 \right) (\theta) = \frac{\omega^{-1}(\theta)}{\eta} \int_{v_1}^\theta \exp\left(-\frac{1-\eta}{\eta}(\theta - \varrho)\right) \omega(\varrho) \mathcal{Z}_1(\varrho), v_1 < \theta$$

and

$$\left({}_{\omega} \mathcal{I}_{v_2^-}^\eta \mathcal{Z}_1 \right) (\theta) = \frac{\omega^{-1}(\theta)}{\eta} \int_\theta^{v_2} \exp\left(-\frac{1-\eta}{\eta}(\varrho - \theta)\right) \omega(\varrho) \mathcal{Z}_1(\varrho) d\varrho, v_2 > \theta.$$

Also, one can derive the weighted form of conformable fractional integrals introduced by [32–35]

The following special cases can be easily obtained by applying the conditions on $\omega(\theta)$ and $\Psi(\mathcal{F}(\theta))$.

Remark 3. 1. If we consider $\omega(\theta) = 1$ and $\Psi(\mathcal{F}(\theta)) = \mathcal{F}(\theta)$, the fractional integrals (7) and (8) reduce to the following:

$$\left({}^{\mathcal{F}}\mathcal{I}_{v_1+}\mathcal{Z}_1\right)(\theta) = \int_{v_1}^{\theta} \mathcal{F}'(\varrho) \mathcal{Z}_1(\varrho) d\varrho, v_1 < \theta$$

and

$$\left({}^{\mathcal{F}}\mathcal{I}_{v_2-}\mathcal{Z}_1\right)(\theta) = \int_{\theta}^{v_2} \mathcal{F}'(\varrho) \mathcal{Z}_1(\varrho) d\varrho, v_2 > \theta,$$

respectively.

2. If we consider $\omega(\theta) = 1$ and $\mathcal{F}(\theta) = \theta$, the fractional integrals (7) and (8) reduce to the following respectively (see [36]) as follows:

$$\left({}^{\mathcal{F}}\mathcal{I}_{v_1+}\mathcal{Z}_1\right)(\theta) = \int_{v_1}^{\theta} \frac{\Psi(\theta - \varrho)}{\theta - \varrho} \mathcal{Z}_1(\varrho) d\varrho, v_1 < \theta$$

and

$$\left({}^{\mathcal{F}}\mathcal{I}_{v_2-}\mathcal{Z}_1\right)(\theta) = \int_{\theta}^{v_2} \frac{\Psi(\varrho - \theta)}{\varrho - \theta} \mathcal{Z}_1(\varrho) d\varrho, v_2 > \theta.$$

3. If we consider $\omega(\theta) = 1$ and $\Psi(\mathcal{F}(\theta)) = \frac{\mathcal{F}(\theta)^\kappa}{\Gamma(\kappa)}$, the fractional integrals (7) and (8) reduce to the following respectively (see [37,38]):

$$\left({}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\kappa}\mathcal{Z}_1\right)(\theta) = \frac{1}{\Gamma(\kappa)} \int_{v_1}^{\theta} (\mathcal{F}(\theta) - \mathcal{F}(\varrho))^{\kappa-1} \mathcal{F}'(\varrho) \mathcal{Z}_1(\varrho) d\varrho, v_1 < \theta$$

and

$$\left({}^{\mathcal{F}}\mathcal{I}_{v_2-}^{\kappa}\mathcal{Z}_1\right)(\theta) = \frac{1}{\Gamma(\kappa)} \int_{\theta}^{v_2} (\mathcal{F}(\varrho) - \mathcal{F}(\theta))^{\kappa-1} \mathcal{F}'(\varrho) \mathcal{Z}_1(\varrho) d\varrho, v_2 > \theta,$$

where $\kappa, \in \mathbb{C}$ with $\Re(\kappa) > 0$.

4. If we consider $\omega(\theta) = 1$, $\mathcal{F}(\theta) = \theta$ and $\Psi(\mathcal{F}(\theta)) = \frac{\theta^\kappa}{\Gamma(\kappa)}$, the fractional integrals (7) and (8) reduce to the following (see [37,38]):

$$\left(\mathcal{I}_{v_1+}^{\kappa}\mathcal{Z}_1\right)(\theta) = \frac{1}{\Gamma(\kappa)} \int_{v_1}^{\theta} (\theta - \varrho)^{\kappa-1} \mathcal{Z}_1(\varrho) d\varrho, v_1 < \theta$$

and

$$\left(\mathcal{I}_{v_2-}^{\kappa}\mathcal{Z}_1\right)(\theta) = \frac{1}{\Gamma(\kappa)} \int_{\theta}^{v_2} (\varrho - \theta)^{\kappa-1} \mathcal{Z}_1(\varrho) d\varrho, v_2 > \theta,$$

respectively.

5. If we consider $\omega(\theta) = 1$, $\mathcal{F}(\theta) = \ln \theta$ and $\Psi(\mathcal{F}(\theta)) = \frac{(\ln \theta)^\kappa}{\Gamma(\kappa)}$, the fractional integrals (7) and (8) reduce to the following weighted Hadamard fractional integrals (see [37,38]):

$$\left(\mathcal{I}_{v_1+}^{\kappa}\mathcal{Z}_1\right)(\theta) = \frac{1}{\Gamma(\kappa)} \int_{v_1}^{\theta} (\ln \theta - \ln \varrho)^{\kappa-1} \mathcal{Z}_1(\varrho) \frac{d\varrho}{\varrho}, v_1 < \theta$$

and

$$(\mathcal{I}_{v_2-}^{\kappa} \mathcal{Z}_1)(\theta) = \frac{1}{\Gamma(\kappa)} \int_{\theta}^{v_2} (\ln \varrho - \ln \theta)^{\kappa-1} \mathcal{Z}_1(\varrho) \frac{d\varrho}{\varrho}, \quad v_2 > \theta.$$

6. If we consider $\omega(\theta) = 1$, $\mathcal{F}(\theta) = \theta^{\eta}$ and $\Psi(\mathcal{F}(\theta)) = \frac{\theta^{\eta}}{\eta}$, $\eta > 0$, the fractional integrals (7) and (8) reduce to the following Katugampola [39] fractional integrals respectively,

$$(\mathcal{I}_{v_1}^{\kappa} \mathcal{Z}_1)(\theta) = \frac{1}{\Gamma(\kappa)} \int_{v_1}^{\theta} \left(\frac{\theta^{\eta} - \varrho^{\eta}}{\eta} \right)^{\kappa-1} \mathcal{Z}_1(\varrho) \frac{d\varrho}{\varrho^{1-\eta}}, \quad v_1 < \theta$$

and

$$(\mathcal{I}_{v_2}^{\kappa} \mathcal{Z}_1)(\theta) = \frac{1}{\Gamma(\kappa)} \int_{\theta}^{v_2} \left(\frac{\varrho^{\eta} - \theta^{\eta}}{\eta} \right)^{\kappa-1} \mathcal{Z}_1(\varrho) \frac{d\varrho}{\varrho^{1-\eta}}, \quad v_2 > \theta.$$

7. If we consider $\omega(\theta) = 1$, $\mathcal{F}(\theta) = \theta$ and $\Psi(\mathcal{F}(\theta)) = \frac{\theta}{\eta} \exp\left(-\frac{1-\eta}{\eta}\theta\right)$, $\eta \in (0, 1)$, the fractional integrals (7) and (8) reduce to the following weighted fractional integrals,

$$(\mathcal{I}_{v_1+}^{\eta} \mathcal{Z}_1)(\theta) = \frac{1}{\eta} \int_{v_1}^{\theta} \exp\left(-\frac{1-\eta}{\eta}(\theta - \varrho)\right) \mathcal{Z}_1(\varrho), \quad v_1 < \theta$$

and

$$(\mathcal{I}_{v_2-}^{\eta} \mathcal{Z}_1)(\theta) = \frac{1}{\eta} \int_{\theta}^{v_2} \exp\left(-\frac{1-\eta}{\eta}(\varrho - \theta)\right) \mathcal{Z}_1(\varrho) d\varrho, \quad v_2 > \theta.$$

Similarly, (7) and (8) will lead to the fractional integrals defined by [32–35].

2. Some Double-Weighted Generalized Fractional Integral Inequalities

In this section, some double-weighted generalized fractional integral inequalities are presented. To this end, we begin by proving the following Lemma.

Lemma 1. Let the function \mathcal{F} be measurable, increasing, positive and monotone function on (v_1, v_2) , and has a continuous derivative \mathcal{F}' on $[v_1, v_2]$. If $\Phi : [v_1, v_2] \rightarrow \mathbb{R}$ is continuous on $[v_1, v_2]$, and $\mathfrak{h}_1, \mathfrak{h}_2 : [u, v] \rightarrow \mathbb{R}^+$ are positive integrable. Then, we have

$$\begin{aligned} & \left[{}_{\omega}^{\mathcal{F}} \mathcal{I}_{v_1+}^{\Psi} \mathfrak{h}_1(v_2) \right] \left[{}_{\omega}^{\mathcal{F}} \mathcal{I}_{v_1+}^{\Psi} v_2(\mathfrak{h}_2 \Phi)(v_2) \right] + \left[{}_{\omega}^{\mathcal{F}} \mathcal{I}_{v_1+}^{\Psi} \Phi(v_2) \right] \left[{}_{\omega}^{\mathcal{F}} \mathcal{I}_{v_1+}^{\Psi} v_2(\mathfrak{h}_1 \Phi)(v_2) \right] \\ & - \left[{}_{\omega}^{\mathcal{F}} \mathcal{I}_{v_1+}^{\Psi} v_2 \mathfrak{h}_1(v_2) \right] \left[{}_{\omega}^{\mathcal{F}} \mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_2 \Phi)(v_2) \right] - \left[{}_{\omega}^{\mathcal{F}} \mathcal{I}_{v_1+}^{\Psi} v_2 \mathfrak{h}_2(v_2) \right] \left[{}_{\omega}^{\mathcal{F}} \mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_1 \Phi)(v_2) \right] \\ & = \frac{1}{\omega(\theta)} \int_{v_1}^{v_2} \left[{}_{\omega}^{\mathcal{F}} \mathcal{I}_{v_1+}^{\Psi} v_2 \mathfrak{h}_1(v_2) \int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) \mathfrak{h}_2(\varrho) d\varrho \right. \\ & \left. - {}_{\omega}^{\mathcal{F}} \mathcal{I}_{v_1+}^{\Psi} \mathfrak{h}_1(v_2) \int_{v_1}^{\theta} \varrho \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) \mathfrak{h}_2(\varrho) d\varrho \right] (\Phi'(\theta)) d\theta. \end{aligned} \quad (9)$$

Proof. Assume that $\mathcal{Z}_1 : [v_1, v_2] \rightarrow \mathbb{R}$ is a continuous function on $[v_1, v_2]$. Then, one may get

$$\begin{aligned} & \left[{}_{\omega}^{\mathcal{F}} \mathcal{I}_{v_1+}^{\Psi} \mathfrak{h}_2(v_2) \right] \left[{}_{\omega}^{\mathcal{F}} \mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_2 \mathcal{Z}_1 \Phi)(v_2) \right] + \left[{}_{\omega}^{\mathcal{F}} \mathcal{I}_{v_1+}^{\Psi} \Phi(v_2) \right] \left[{}_{\omega}^{\mathcal{F}} \mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_1 \mathcal{Z}_1 \Phi)(v_2) \right] \\ & - \left[{}_{\omega}^{\mathcal{F}} \mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_1 \mathcal{Z}_1)(v_2) \right] \left[{}_{\omega}^{\mathcal{F}} \mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_2 \Phi)(v_2) \right] - \left[{}_{\omega}^{\mathcal{F}} \mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_2 \mathcal{Z}_1)(v_2) \right] \left[{}_{\omega}^{\mathcal{F}} \mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_1 \Phi)(v_2) \right] \\ & = \frac{1}{\omega^2(\theta)} \int_{v_1}^{v_2} \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \omega(\xi) \mathcal{F}'(\xi) \\ & \times \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) \mathfrak{h}_1(\xi) \mathfrak{h}_2(\varrho) [(\mathcal{Z}_1(\xi) - \mathcal{Z}_1(\varrho))(\Phi(\xi) - \Phi(\varrho))] d\xi d\varrho. \end{aligned}$$

Consequently, it follows

$$\begin{aligned} & \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \mathfrak{h}_1(v_2) \right] \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_2 \mathcal{Z}_1 \Phi)(v_2) \right] + \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \Phi(v_2) \right] \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_1 \mathcal{Z}_1 \Phi)(v_2) \right] \\ & - \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_1 \mathcal{Z}_1)(v_2) \right] \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_2 \Phi)(v_2) \right] - \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_2 \mathcal{Z}_1)(v_2) \right] \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_1 \Phi)(v_2) \right] \\ & = \frac{1}{\omega^2(\theta)} \int_{v_1}^{v_2} \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \omega(\xi) \mathcal{F}'(\xi) \\ & \times \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\rho))}{\mathcal{F}(v_2) - \mathcal{F}(\rho)} \omega(\rho) \mathcal{F}'(\rho) \mathfrak{h}_1(\xi) \mathfrak{h}_2(\rho) (\mathcal{Z}_1(\xi) - \mathcal{Z}_1(\rho)) \left(\int_{\rho}^{\xi} \Phi'(\vartheta) d\vartheta \right) d\xi d\rho. \quad (10) \end{aligned}$$

By utilizing the given condition $v_1 \leq \rho \leq \theta \leq \xi \leq v_2$, we get

$$\begin{aligned} & \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \mathfrak{h}_1(v_2) \right] \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_2 \mathcal{Z}_1 \Phi)(v_2) \right] + \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \Phi(v_2) \right] \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_1 \mathcal{Z}_1 \Phi)(v_2) \right] \\ & - \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_1 \mathcal{Z}_1)(v_2) \right] \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_2 \Phi)(v_2) \right] - \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_2 \mathcal{Z}_1)(x_2) \right] \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_1 \Phi)(x_2) \right] \\ & = \frac{1}{\omega^2(\theta)} \int_{v_1}^{v_2} \left[\int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\rho))}{\mathcal{F}(v_2) - \mathcal{F}(\rho)} \omega(\rho) \mathcal{F}'(\rho) \mathfrak{h}_2(\rho) \right. \\ & \left. \times \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} (\mathcal{Z}_1(\xi) - \mathcal{Z}_1(\rho)) \omega(\xi) \mathcal{F}'(\xi) \mathfrak{h}_1(\xi) d\xi d\rho \right] (\Phi'(\theta)) d\theta. \quad (11) \end{aligned}$$

Applying (11) for the particular case when $\mathcal{Z}_1(v) = v$, then we can write

$$\begin{aligned} & \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \mathfrak{h}_1(v_2) \right] \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} v_2 (\mathfrak{h}_2 \Phi)(v_2) \right] + \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \Phi(v_2) \right] \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} v_2 (\mathfrak{h}_1 \Phi)(v_2) \right] \\ & - \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} v_2 \mathfrak{h}_1(v_2) \right] \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_2 \Phi)(v_2) \right] - \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} v_2 \mathfrak{h}_2(v_2) \right] \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_1 \Phi)(v_2) \right] \\ & = \frac{1}{\omega^2(\theta)} \int_{v_1}^{v_2} \left[\int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\rho))}{\mathcal{F}(v_2) - \mathcal{F}(\rho)} \omega(\rho) \mathcal{F}'(\rho) \mathfrak{h}_2(\rho) \right. \\ & \left. \times \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} (\xi - \rho) \omega(\xi) \mathcal{F}'(\xi) \mathfrak{h}_1(\xi) d\xi d\rho \right] (\Phi'(\theta)) d\theta \\ & = \frac{1}{\omega(\theta)} \int_{v_1}^{v_2} \left[\frac{1}{\omega(\theta)} \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \omega(\xi) \mathcal{F}'(\xi) \xi \mathfrak{h}_1(\xi) d\xi \right. \\ & \left. \times \int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\rho))}{\mathcal{F}(v_2) - \mathcal{F}(\rho)} \omega(\rho) \mathcal{F}'(\rho) \mathfrak{h}_2(\rho) d\rho - \frac{1}{\omega(\theta)} \right. \\ & \left. \times \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \omega(\xi) \mathcal{F}'(\xi) \mathfrak{h}_1(\xi) d\xi \times \int_{v_1}^{\theta} \right. \\ & \left. \times \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\rho))}{\mathcal{F}(v_2) - \mathcal{F}(\rho)} \omega(\rho) \mathcal{F}'(\rho) \rho \mathfrak{h}_2(\rho) d\rho \right] (\Phi'(\theta)) d\theta. \end{aligned}$$

Thus with the aid of (7), the above equation gives,

$$\begin{aligned} & \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \mathfrak{h}_1(v_2) \right] \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} v_2 (\mathfrak{h}_2 \Phi)(v_2) \right] + \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \Phi(v_2) \right] \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} v_2 (\mathfrak{h}_1 \Phi)(v_2) \right] \\ & - \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} v_2 \mathfrak{h}_1(v_2) \right] \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_2 \Phi)(v_2) \right] - \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} v_2 \mathfrak{h}_2(v_2) \right] \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_1 \Phi)(v_2) \right] \\ & = \frac{1}{\omega(\theta)} \int_{v_1}^{v_2} \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} v_2 \mathfrak{h}_1(v_2) \int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\rho))}{\mathcal{F}(v_2) - \mathcal{F}(\rho)} \omega(\rho) \mathcal{F}'(\rho) \rho \mathfrak{h}_2(\rho) d\rho \right. \\ & \left. - {}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \mathfrak{h}_1(v_2) \int_{x_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\rho))}{\mathcal{F}(v_2) - \mathcal{F}(\rho)} \omega(\rho) \mathcal{F}'(\rho) \rho \mathfrak{h}_2(\rho) d\rho \right] (\Phi'(\theta)) d\theta, \end{aligned}$$

which completes the proof. \square

Based on Lemma 1, we prove the following theorem.

Theorem 4. Suppose that the function \mathcal{F} be measurable, increasing, positive and monotone function on (v_1, v_2) , and has a continuous derivative \mathcal{F}' on (v_1, v_2) . Assume that $\Phi : [v_1, v_2] \rightarrow \mathbb{R}$ is absolutely continuous on $[v_1, v_2]$, and $\mathfrak{h}_1, \mathfrak{h}_2 : [v_1, v_2] \rightarrow \mathbb{R}^+$ are positive integrable functions. If $(\Phi')^2 \in L_1[v_1, v_2]$. Then, we have

$$\begin{aligned} & \left[{}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \mathfrak{h}_1(v_2) \right] \left[{}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_2\Phi^2)(v_2) \right] + \left[{}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \mathfrak{h}_2(v_2) \right] \left[{}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_1\Phi^2)(v_2) \right] \\ & - 2 \left[{}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_1\Phi)(v_2) \right] \left[{}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_2\Phi)(v_2) \right] \\ & \leq \frac{1}{\omega(\theta)} \int_{v_1}^{v_2} \left[{}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} v_2 \mathfrak{h}_1(v_2) \int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) \mathfrak{h}_2(\varrho) d\varrho \right. \\ & \left. - {}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \mathfrak{h}_1(v_2) \int_{v_1}^{\theta} \varrho \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) \mathfrak{h}_2(\varrho) d\varrho \right] (\Phi'(\theta))^2 d\theta. \end{aligned} \quad (12)$$

Proof. By employing the definition (7) and Lemma 1, we obtain

$$\begin{aligned} & \left[{}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \mathfrak{h}_1(v_2) \right] \left[{}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_2\Phi^2)(v_2) \right] + \left[{}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \mathfrak{h}_2(v_2) \right] \left[{}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_1\Phi^2)(v_2) \right] \\ & - 2 \left[{}_{x_1}^{\Psi}\mathcal{I}_{\kappa}^{\theta} (\mathfrak{h}_1\Phi)(x_2) \right] \left[{}_{x_1}^{\Psi}\mathcal{I}_{\kappa}^{\theta} (\mathfrak{h}_2\Phi)(x_2) \right] \\ & = \frac{1}{\omega^2(\theta)} \int_{v_1}^{v_2} \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \\ & \times \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\xi) \mathcal{F}'(\xi) \mathfrak{h}_1(\xi) \omega(\varrho) \mathcal{F}'(\varrho) \mathfrak{h}_2(\varrho) (\Phi(\xi) - \Phi(\varrho))^2 d\xi d\varrho \\ & = \frac{1}{\omega^2(\theta)} \int_{v_1}^{v_2} \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \\ & \times \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\xi) \mathcal{F}'(\xi) \mathfrak{h}_1(\xi) \omega(\varrho) \mathcal{F}'(\varrho) \mathfrak{h}_2(\varrho) (\xi - \varrho)^2 \left(\frac{\Phi(\xi) - \Phi(\varrho)}{\xi - \varrho} \right)^2 d\xi d\varrho. \end{aligned}$$

Consequently, it follows that

$$\begin{aligned} & \left[{}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \mathfrak{h}_1(v_2) \right] \left[{}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_2\Phi^2)(v_2) \right] + \left[{}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \mathfrak{h}_2(v_2) \right] \left[{}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_1\Phi^2)(v_2) \right] \\ & - 2 \left[{}_{x_1}^{\Psi}\mathcal{I}_{\kappa}^{\theta} (\mathfrak{h}_1\Phi)(x_2) \right] \left[{}_{x_1}^{\Psi}\mathcal{I}_{\kappa}^{\theta} (\mathfrak{h}_2\Phi)(x_2) \right] \\ & = \frac{1}{\omega^2(\theta)} \int_{v_1}^{v_2} \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \\ & \times \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\xi) \mathcal{F}'(\xi) \mathfrak{h}_1(\xi) \omega(\varrho) \mathcal{F}'(\varrho) \mathfrak{h}_2(\varrho) (\xi - \varrho)^2 \left(\frac{\int_{\varrho}^{\xi} \Phi'(\theta) d\theta}{\xi - \varrho} \right)^2 d\xi d\varrho. \end{aligned}$$

By applying Cauchy-Schwartz inequality [40], we get

$$\begin{aligned}
& \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \mathfrak{h}_1(v_2) \right] \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_2 \Phi^2)(v_2) \right] + \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \mathfrak{h}_2(v_2) \right] \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_1 \Phi^2)(v_2) \right] \\
& - 2 \left[{}^{\Psi}\mathcal{I}_{x_1}^{\theta} (\mathfrak{h}_1 \Phi)(x_2) \right] \left[{}^{\Psi}\mathcal{I}_{x_1}^{\theta} (\mathfrak{h}_2 \Phi)(x_2) \right] \\
& = \frac{1}{\omega^2(\theta)} \int_{v_1}^{v_2} \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \\
& \times \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\rho))}{\mathcal{F}(v_2) - \mathcal{F}(\rho)} \omega(\xi) \mathcal{F}'(\xi) \mathfrak{h}_1(\xi) \omega(\rho) \mathcal{F}'(\rho) \mathfrak{h}_2(\rho) (\xi - \rho)^2 \left(\frac{\int_{\rho}^{\xi} \Phi'(\theta) d\theta}{\xi - \rho} \right)^2 d\xi d\rho \\
& \leq \frac{1}{\omega^2(\theta)} \int_{v_1}^{v_2} \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \\
& \times \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\rho))}{\mathcal{F}(v_2) - \mathcal{F}(\rho)} \omega(\xi) \mathcal{F}'(\xi) \mathfrak{h}_1(\xi) \omega(\rho) \mathcal{F}'(\rho) \mathfrak{h}_2(\rho) (\xi - \rho)^2 \left(\frac{\left(\int_{\rho}^{\xi} d\theta \right)^{\frac{1}{2}} \left(\int_{\rho}^{\xi} (\Phi'(\theta))^2 d\theta \right)^{\frac{1}{2}}}{\xi - \rho} \right)^2 d\xi d\rho \\
& \leq \frac{1}{\omega^2(\theta)} \int_{v_1}^{v_2} \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \\
& \times \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\rho))}{\mathcal{F}(v_2) - \mathcal{F}(\rho)} \omega(\xi) \mathcal{F}'(\xi) \mathfrak{h}_1(\xi) \omega(\rho) \mathcal{F}'(\rho) \mathfrak{h}_2(\rho) (\xi - \rho)^2 \left(\frac{\int_{\rho}^{\xi} (\Phi'(\theta))^2 d\theta}{\xi - \rho} \right) d\xi d\rho \\
& \leq \frac{1}{\omega^2(\theta)} \int_{v_1}^{v_2} \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \\
& \times \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\rho))}{\mathcal{F}(v_2) - \mathcal{F}(\rho)} \omega(\xi) \mathcal{F}'(\xi) \mathfrak{h}_1(\xi) \omega(\rho) \mathcal{F}'(\rho) \mathfrak{h}_2(\rho) (\xi - \rho) \left(\int_{\rho}^{\xi} (\Phi'(\theta))^2 d\theta \right) d\xi d\rho
\end{aligned} \tag{13}$$

Hence, using (11) and (13) concludes the proof. \square

The following new particular results of Theorem 4 can be easily obtained.

Corollary 1. Suppose that the function \mathcal{F} be measurable, increasing, positive and monotone function on (v_1, v_2) , and has a continuous derivative \mathcal{F}' on (v_1, v_2) . Assume that $\Phi : [v_1, v_2] \rightarrow \mathbb{R}$ is absolutely continuous on $[v_1, v_2]$, and $\mathfrak{h}_1, \mathfrak{h}_2 : [v_1, v_2] \rightarrow \mathbb{R}^+$ are positive integrable functions. If $(\Phi')^2 \in L_1[v_1, v_2]$. Then, we have

$$\begin{aligned}
& \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} 1 \right] \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_2 \Phi^2)(v_2) \right] + \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \mathfrak{h}_2(v_2) \right] \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\Phi^2)(v_2) \right] \\
& - 2 \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\Phi)(v_2) \right] \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_2 \Phi)(v_2) \right] \\
& \leq \frac{1}{\omega(\theta)} \int_{v_1}^{v_2} \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} v_2 \int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\rho))}{\mathcal{F}(v_2) - \mathcal{F}(\rho)} \omega(\rho) \mathcal{F}'(\rho) \mathfrak{h}_2(\rho) d\rho \right. \\
& \left. - {}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} 1 \int_{v_1}^{\theta} \rho \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\rho))}{\mathcal{F}(v_2) - \mathcal{F}(\rho)} \omega(\rho) \mathcal{F}'(\rho) \mathfrak{h}_2(\rho) d\rho \right] (\Phi'(\theta))^2 d\theta.
\end{aligned}$$

Proof. By considering $\mathfrak{h}_1(\theta) = 1, \theta \in [v_1, v_2]$ in Theorem 4, the desired result is obtained. \square

Corollary 2. Suppose that the function \mathcal{F} be measurable, increasing, positive and monotone function on (v_1, v_2) , and has a continuous derivative \mathcal{F}' on (v_1, v_2) . Assume that $\Phi : [v_1, v_2] \rightarrow \mathbb{R}$ is absolutely continuous on $[v_1, v_2]$, and $\mathfrak{h}_1, \mathfrak{h}_2 : [v_1, v_2] \rightarrow \mathbb{R}^+$ are positive integrable functions. If $(\Phi')^2 \in L_1[v_1, v_2]$. Then, we have

$$\begin{aligned}
& \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \mathfrak{h}_1(v_2) \right] \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\Phi^2)(v_2) \right] + \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} 1 \right] \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_1 \Phi^2)(v_2) \right] \\
& - 2 \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_1 \Phi)(v_2) \right] \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\Phi)(v_2) \right] \\
& \leq \frac{1}{\omega(\theta)} \int_{v_1}^{v_2} \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} v_2 \mathfrak{h}_1(v_2) \int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) d\varrho \right. \\
& \left. - {}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \mathfrak{h}_1(v_2) \int_{v_1}^{\theta} \varrho \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) d\varrho \right] (\Phi'(\theta))^2 d\theta.
\end{aligned}$$

Proof. By considering $\mathfrak{h}_2(\theta) = 1$, $\theta \in [v_1, v_2]$ in Theorem 4, desired corollary is proven. \square

Corollary 3. Suppose that the function \mathcal{F} be measurable, increasing, positive and monotone function on (v_1, v_2) , and has a continuous derivative \mathcal{F}' on (v_1, v_2) . Assume that $\Phi : [v_1, v_2] \rightarrow \mathbb{R}$ is absolutely continuous on $[v_1, v_2]$, and $\mathfrak{h}_1, \mathfrak{h}_2 : [v_1, v_2] \rightarrow \mathbb{R}^+$ are positive integrable functions. If $(\Phi')^2 \in L_1[v_1, v_2]$. Then, we have

$$\begin{aligned}
& \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} 1 \right] \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\Phi^2)(v_2) \right] - \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\Phi)(v_2) \right]^2 \\
& \leq \frac{1}{\omega(\theta)} \int_{v_1}^{v_2} \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} v_2 \int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) d\varrho \right. \\
& \left. - {}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} 1 \int_{v_1}^{\theta} \varrho \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) d\varrho \right] (\Phi'(\theta))^2 d\theta.
\end{aligned}$$

Proof. Taking $\mathfrak{h}_1(\theta) = \mathfrak{h}_2(\theta) = 1$, $\theta \in [v_1, v_2]$ in Theorem 4, the desired result is obtained. \square

Remark 4. If we consider $\omega(\theta) = 1$ and $\Psi(\mathcal{F}(\theta)) = \frac{\mathcal{F}(\theta)^{\kappa}}{\Gamma(\kappa)}$, then Theorem 4 and Corollaries 1–3 will reduce to the work of Bezziou et al. [27].

Remark 5. If we consider $\omega(\theta) = 1$ and $\Psi(\mathcal{F}(\theta)) = \frac{\mathcal{F}(\theta)^{\frac{\kappa}{\eta}}}{\Gamma_{\eta}(\kappa)}$, then Theorem 4 and Corollaries 1–3 will reduce to the work of Rahman et al. [41].

Theorem 5. Let \mathcal{F} be measurable, increasing, positive and monotone function on (v_1, v_2) , and having a continuous derivative \mathcal{F}' on (v_1, v_2) . Assume that $f_1, f_2 : [v_1, v_2] \rightarrow \mathbb{R}$ are absolutely continuous on $[v_1, v_2]$, and $\mathfrak{h}_1, \mathfrak{h}_2 : [v_1, v_2] \rightarrow \mathbb{R}^+$ are positive integrable. If $(f_1')^2 \in L_1[v_1, v_2]$ and $(f_2')^2 \in L_1[v_1, v_2]$. Then, we have

$$\begin{aligned}
& \left| {}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \mathfrak{h}_1(v_2) {}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_2 f_1 f_2)(v_2) + {}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \mathfrak{h}_2(v_2) {}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_1 f_1 f_2)(v_2) \right. \\
& \left. - {}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_2 f_1)(v_2) {}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_1 f_2)(v_2) - {}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_1 f_1)(v_2) {}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\mathfrak{h}_2 f_2)(v_2) \right| \\
& \leq \frac{1}{\omega(\theta)} \left(\int_{v_1}^{v_2} \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} v_2 \mathfrak{h}_1(v_2) \int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) \mathfrak{h}_2(\varrho) d\varrho \right. \right. \\
& \left. \left. - {}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \mathfrak{h}_1(v_2) \int_{v_1}^{\theta} \varrho \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) \mathfrak{h}_2(\varrho) d\varrho \right] (f_1'(\theta))^2 d\theta \right)^{\frac{1}{2}} \\
& \times \left(\int_{v_1}^{v_2} \left[{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} v_2 \mathfrak{h}_1(v_2) \int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) \mathfrak{h}_2(\varrho) d\varrho \right. \right. \\
& \left. \left. - {}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \mathfrak{h}_1(v_2) \int_{v_1}^{\theta} \varrho \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) \mathfrak{h}_2(\varrho) d\varrho \right] (f_2'(\theta))^2 d\theta \right)^{\frac{1}{2}}
\end{aligned} \tag{14}$$

Proof. Consider the left-hand side of (14), we have

$$\begin{aligned}
& \left| \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \hbar_2(v_2) {}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_2 f_1 f_2)(v_2) + {}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \hbar_2(v_2) {}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_1 f_1 f_2)(v_2)}{\omega^2(\theta)} \right. \\
& \left. - \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_2 f_1)(v_2) {}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_1 f_2)(v_2) - {}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_1 f_1)(v_2) {}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_2 f_2)(v_2)}{\omega^2(\theta)} \right| \\
& \leq \frac{1}{\omega^2(\theta)} \left(\int_{v_1}^{v_2} \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\rho))}{\mathcal{F}(v_2) - \mathcal{F}(\rho)} \right. \\
& \times \omega(\xi) \mathcal{F}'(\xi) \omega(\rho) \mathcal{F}'(\rho) \hbar_1(\xi) \hbar_2(\rho) (f_1(\xi) - f_1(\rho))^2 d\xi d\rho \Big)^{\frac{1}{2}} \times \left(\int_{v_1}^{v_2} \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \right. \\
& \times \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\rho))}{\mathcal{F}(v_2) - \mathcal{F}(\rho)} \omega(\xi) \mathcal{F}'(\xi) \omega(\rho) \mathcal{F}'(\rho) \hbar_1(\xi) \hbar_2(\rho) (f_2(\xi) - f_2(\rho))^2 d\xi d\rho \Big)^{\frac{1}{2}} \\
& \leq \frac{1}{\omega^2(\theta)} \left(\int_{v_1}^{v_2} \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\rho))}{\mathcal{F}(v_2) - \mathcal{F}(\rho)} \omega(\xi) \mathcal{F}'(\xi) \omega(\rho) \mathcal{F}'(\rho) \hbar_1(\xi) \hbar_2(\rho) \right. \\
& \times \left(\int_{\rho}^{\xi} f_1'(\theta) d\theta \right)^2 d\xi d\rho \Big)^{\frac{1}{2}} \times \left(\int_{v_1}^{v_2} \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \right. \\
& \times \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\rho))}{\mathcal{F}(v_2) - \mathcal{F}(\rho)} \omega(\xi) \mathcal{F}'(\xi) \omega(\rho) \mathcal{F}'(\rho) \hbar_1(\xi) \hbar_2(\rho) \left(\int_{\rho}^{\xi} f_2'(\theta) d\theta \right)^2 d\xi d\rho \Big)^{\frac{1}{2}}
\end{aligned}$$

Applying Cauchy-Schwartz inequality [40] to the above equation yields,

$$\begin{aligned}
& \left| \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \hbar_2(v_2) {}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_2 f_1 f_2)(v_2) + {}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \hbar_2(v_2) {}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_1 f_1 f_2)(v_2)}{\omega^2(\theta)} \right. \\
& \left. - \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_2 f_1)(v_2) {}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_1 f_2)(v_2) - {}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_1 f_1)(v_2) {}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_2 f_2)(v_2)}{\omega^2(\theta)} \right| \\
& \leq \frac{1}{\omega^2(\theta)} \left\{ \int_{v_1}^{v_2} \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\rho))}{\mathcal{F}(v_2) - \mathcal{F}(\rho)} \omega(\xi) \mathcal{F}'(\xi) \omega(\rho) \mathcal{F}'(\rho) \hbar_1(\xi) \hbar_2(\rho) \right. \\
& \times \left(\left(\int_{\rho}^{\xi} d\theta \right)^{\frac{1}{2}} \left(\int_{\rho}^{\xi} (f_1'(\theta))^2 d\theta \right)^{\frac{1}{2}} \right)^2 d\xi d\rho \Big\}^{\frac{1}{2}} \times \left\{ \int_{v_1}^{v_2} \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \right. \\
& \times \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\rho))}{\mathcal{F}(v_2) - \mathcal{F}(\rho)} \omega(\xi) \mathcal{F}'(\xi) \omega(\rho) \mathcal{F}'(\rho) \hbar_1(\xi) \hbar_2(\rho) \left(\left(\int_{\rho}^{\xi} d\theta \right)^{\frac{1}{2}} \left(\int_{\rho}^{\xi} (f_2'(\theta))^2 d\theta \right)^{\frac{1}{2}} \right)^2 d\xi d\rho \Big\}^{\frac{1}{2}} \\
& \leq \frac{1}{\omega^2(\theta)} \left\{ \int_{v_1}^{v_2} \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\rho))}{\mathcal{F}(v_2) - \mathcal{F}(\rho)} \omega(\xi) \mathcal{F}'(\xi) \omega(\rho) \mathcal{F}'(\rho) \hbar_1(\xi) \hbar_2(\rho) \right. \\
& \times \omega(\rho) \mathcal{F}'(\rho) \hbar_1(\xi) \hbar_2(\rho) (\xi - \rho) \left(\int_{\rho}^{\xi} (f_1'(\theta))^2 d\theta \right) d\xi d\rho \Big\}^{\frac{1}{2}} \times \left\{ \int_{v_1}^{v_2} \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \right. \\
& \times \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\rho))}{\mathcal{F}(v_2) - \mathcal{F}(\rho)} \omega(\xi) \mathcal{F}'(\xi) \omega(\rho) \mathcal{F}'(\rho) \hbar_1(\xi) \hbar_2(\rho) (\xi - \rho) \left(\int_{\rho}^{\xi} (f_2'(\theta))^2 d\theta \right) d\xi d\rho \Big\}^{\frac{1}{2}} \\
& \leq \frac{1}{\omega^2(\theta)} \left\{ \int_{v_1}^{v_2} \left(\int_{v_1}^{v_2} \xi \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \omega(\xi) \mathcal{F}'(\xi) \hbar_1(\xi) d\xi \right. \right. \\
& \times \int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\rho))}{\mathcal{F}(v_2) - \mathcal{F}(\rho)} \omega(\rho) \mathcal{F}'(\rho) \hbar_2(\rho) d\rho \\
& \left. \left. - \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \omega(\xi) \mathcal{F}'(\xi) \hbar_1(\xi) d\xi \right. \right. \\
& \times \left. \int_{x_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\rho))}{\mathcal{F}(v_2) - \mathcal{F}(\rho)} \omega(\rho) \mathcal{F}'(\rho) \hbar_2(\rho) d\rho \right) (f_1'(\theta))^2 \Big\}^{\frac{1}{2}} \\
& - \left\{ \int_{v_1}^{v_2} \left(\int_{v_1}^{v_2} \xi \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \omega(\xi) \mathcal{F}'(\xi) \hbar_1(\xi) d\xi \right. \right. \\
& \times \left. \int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\rho))}{\mathcal{F}(v_2) - \mathcal{F}(\rho)} \omega(\rho) \mathcal{F}'(\rho) \hbar_2(\rho) d\rho \right. \\
& \left. \left. - \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \omega(\xi) \mathcal{F}'(\xi) \hbar_1(\xi) d\xi \right. \right. \\
& \times \left. \int_{x_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\rho))}{\mathcal{F}(v_2) - \mathcal{F}(\rho)} \omega(\rho) \mathcal{F}'(\rho) \hbar_2(\rho) d\rho \right) (f_2'(\theta))^2 \Big\}^{\frac{1}{2}}.
\end{aligned}$$

In view of (7), we get the desired proof of (14). \square

Corollary 4. Let \mathcal{F} be measurable, increasing, positive and monotone function on (v_1, v_2) , and having a continuous derivative \mathcal{F}' on (v_1, v_2) . Assume that $f_1, f_2 : [v_1, v_2] \rightarrow \mathbb{R}$ are absolutely continuous on $[v_1, v_2]$, and $\mathfrak{h}_1, \mathfrak{h}_2 : [v_1, v_2] \rightarrow \mathbb{R}^+$ are positive integrable. If $(f_1')^2 \in L_1[v_1, v_2]$ and $(f_2')^2 \in L_1[v_1, v_2]$. Then, we have

$$\begin{aligned} & \left| \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(1) \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(\mathfrak{h}_2 f_1 f_2)(v_2) + \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(\mathfrak{h}_2(v_2)) \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(f_1 f_2)(v_2) \right. \\ & \left. - \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(\mathfrak{h}_2 f_1)(v_2) \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(f_2)(v_2) - \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(f_1)(v_2) \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(\mathfrak{h}_2 f_2)(v_2) \right| \\ & \leq \frac{1}{\omega(\theta)} \left(\int_{v_1}^{v_2} \left[\frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} v_2 \int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) \mathfrak{h}_2(\varrho) d\varrho \right. \right. \\ & \left. \left. - \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(1) \int_{v_1}^{\theta} \varrho \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) \mathfrak{h}_2(\varrho) d\varrho \right] (f_1'(\theta))^2 d\theta \right)^{\frac{1}{2}} \\ & \times \left(\int_{v_1}^{v_2} \left[\frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} v_2 \int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) \mathfrak{h}_2(\varrho) d\varrho \right. \right. \\ & \left. \left. - \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(1) \int_{v_1}^{\theta} \varrho \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) \mathfrak{h}_2(\varrho) d\varrho \right] (f_2'(\theta))^2 d\theta \right)^{\frac{1}{2}} \end{aligned}$$

Proof. Applying Theorem 5 for $\mathfrak{h}_1(\theta) = 1, \theta \in [v_1, v_2]$, the desired result is obtained. \square

Corollary 5. Let \mathcal{F} be measurable, increasing, positive and monotone function on (v_1, v_2) , and having a continuous derivative \mathcal{F}' on (v_1, v_2) . Assume that $f_1, f_2 : [v_1, v_2] \rightarrow \mathbb{R}$ are absolutely continuous on $[v_1, v_2]$, and $\mathfrak{h}_1, \mathfrak{h}_2 : [v_1, v_2] \rightarrow \mathbb{R}^+$ are positive integrable. If $(f_1')^2 \in L_1[v_1, v_2]$ and $(f_2')^2 \in L_1[v_1, v_2]$. Then, we have

$$\begin{aligned} & \left| \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(\mathfrak{h}_1(v_2)) \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(f_1 f_2)(v_2) + \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(1) \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(\mathfrak{h}_1 f_1 f_2)(v_2) \right. \\ & \left. - \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(f_1)(v_2) \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(\mathfrak{h}_1 f_2)(v_2) - \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(\mathfrak{h}_1 f_1)(v_2) \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(f_2)(v_2) \right| \\ & \leq \frac{1}{\omega(\theta)} \left(\int_{v_1}^{v_2} \left[\frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} v_2 \mathfrak{h}_1(v_2) \int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) d\varrho \right. \right. \\ & \left. \left. - \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(\mathfrak{h}_1(v_2)) \int_{v_1}^{\theta} \varrho \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) d\varrho \right] (f_1'(\theta))^2 d\theta \right)^{\frac{1}{2}} \\ & \times \left(\int_{v_1}^{v_2} \left[\frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} v_2 \mathfrak{h}_1(v_2) \int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) d\varrho \right. \right. \\ & \left. \left. - \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(\mathfrak{h}_1(v_2)) \int_{v_1}^{\theta} \varrho \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) d\varrho \right] (f_2'(\theta))^2 d\theta \right)^{\frac{1}{2}} \end{aligned}$$

Proof. Applying Theorem 5 for $\mathfrak{h}_2(\theta) = 1, \theta \in [v_1, v_2]$, the desired result is obtained. \square

Corollary 6. Let \mathcal{F} be measurable, increasing, positive and monotone function on (v_1, v_2) , and having a continuous derivative \mathcal{F}' on (v_1, v_2) . Assume that $f_1, f_2 : [v_1, v_2] \rightarrow \mathbb{R}$ are absolutely continuous on $[v_1, v_2]$, and $\mathfrak{h}_1, \mathfrak{h}_2 : [v_1, v_2] \rightarrow \mathbb{R}^+$ are positive integrable. If $(f_1')^2 \in L_1[v_1, v_2]$ and $(f_2')^2 \in L_1[v_1, v_2]$. Then, we have

$$\begin{aligned} & \left| \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(1) \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(f_1 f_2)(v_2) - \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(f_1)(v_2) \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(f_2)(v_2) \right| \\ & \leq \frac{1}{\omega(\theta)} \left(\int_{v_1}^{v_2} \left[\frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} v_2 \int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) d\varrho \right. \right. \\ & \left. \left. - \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(1) \int_{v_1}^{\theta} \varrho \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) d\varrho \right] (f_1'(\theta))^2 d\theta \right)^{\frac{1}{2}} \\ & \times \left(\int_{v_1}^{v_2} \left[\frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} v_2 \int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) d\varrho \right. \right. \\ & \left. \left. - \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(1) \int_{v_1}^{\theta} \varrho \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) d\varrho \right] (f_2'(\theta))^2 d\theta \right)^{\frac{1}{2}} \end{aligned}$$

Proof. Applying Theorem 5 for $\hbar_1(\theta) = \hbar_2(\theta) = 1, \theta \in [v_1, v_2]$, the desired result is obtained. \square

Theorem 6. Let \mathcal{F} be measurable, increasing, positive and monotone function on (v_1, v_2) , and having a continuous derivative \mathcal{F}' on (v_1, v_2) . Assume that $f_1 : [v_1, v_2] \rightarrow \mathbb{R}$ is absolutely continuous function on $[v_1, v_2]$, and $f_2 : [v_1, v_2] \rightarrow \mathbb{R}$ is non-decreasing on $[v_1, v_2]$. Moreover, suppose that both $\hbar_1, \hbar_2 : [v_1, v_2] \rightarrow \mathbb{R}^+$ are positive integrable. If $(f'_1) \in L^\infty[v_1, v_2]$, then we have

$$\begin{aligned} & \left| \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} \hbar_1(v_2) \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} (\hbar_2 f_1 f_2)(v_2) + \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} \hbar_2(v_2) \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} (\hbar_1 f_1 f_2)(v_2) \right. \\ & \left. - \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} (\hbar_1 f_1)(v_2) \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} (\hbar_2 f_2)(v_2) - \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} (\hbar_2 f_1)(v_2) \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} (\hbar_1 f_2)(v_2) \right| \\ & \leq \frac{\|f'_1\|_\infty}{\omega(\theta)} \left[\frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} v_2 \hbar_1(v_2) \int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) \hbar_2(\varrho) d\varrho \right. \\ & \left. - \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} \hbar_1(v_2) \int_{v_1}^{\theta} \varrho \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) \hbar_2(\varrho) d\varrho \right] \int_{v_1}^{v_2} f'_2(\theta) d\theta. \end{aligned} \tag{15}$$

Proof. Consider the left-hand side of (15), we have

$$\begin{aligned} & \left| \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} \hbar_1(v_2) \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} (\hbar_2 f_1 f_2)(v_2) + \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} \hbar_2(v_2) \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} (\hbar_1 f_1 f_2)(v_2) \right. \\ & \left. - \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} (\hbar_2 f_1)(v_2) \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} (\hbar_1 f_2)(v_2) - \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} (\hbar_1 f_1)(v_2) \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} (\hbar_2 f_2)(v_2) \right| \\ & \leq \frac{1}{\omega^2(\theta)} \left| \int_{v_1}^{v_2} \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\xi) \mathcal{F}'(\xi) \omega(\varrho) \mathcal{F}'(\varrho) \hbar_1(\xi) \hbar_2(\varrho) \right. \\ & \left. \times [(f_1(\xi) - f_1(\varrho))(f_2(\xi) - f_2(\varrho))] d\xi d\varrho \right| \\ & \leq \frac{1}{\omega^2(\theta)} \left| \int_{v_1}^{v_2} \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\xi) \mathcal{F}'(\xi) \omega(\varrho) \mathcal{F}'(\varrho) \hbar_1(\xi) \hbar_2(\varrho) \right. \\ & \left. \times \left| \frac{(f_1(\xi) - f_1(\varrho))}{\xi - \varrho} \right| |(\xi - \varrho)(f_2(\xi) - f_2(\varrho))| d\xi d\varrho \right| \\ & \leq \frac{\|f'_1\|_\infty}{\omega^2(\theta)} \left| \int_{v_1}^{v_2} \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\xi) \mathcal{F}'(\xi) \omega(\varrho) \mathcal{F}'(\varrho) \hbar_1(\xi) \hbar_2(\varrho) \right. \\ & \left. \times (\xi - \varrho) \left(\int_{v_1}^{v_2} f'_2(\theta) d\theta \right) d\xi d\varrho \right| \\ & \leq \frac{\|f'_1\|_\infty}{\omega^2(\theta)} \left[\int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \xi \mathcal{F}'(\xi) \omega(\xi) \hbar_2(\xi) d\xi \int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \mathcal{F}'(\varrho) \omega(\varrho) \hbar_2(\varrho) d\varrho \right. \\ & \left. - \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \mathcal{F}'(\xi) \omega(\varrho) \hbar_2(\xi) d\xi \int_{v_1}^{\theta} \varrho \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \mathcal{F}'(\varrho) \omega(\varrho) \hbar_2(\varrho) d\varrho \right] \int_{v_1}^{v_2} f'_2(\theta) d\theta \end{aligned}$$

Hence taking (7) into account, the proof of (15) is completed. \square

Corollary 7. Let \mathcal{F} be measurable, increasing, positive and monotone function on (v_1, v_2) , and having a continuous derivative \mathcal{F}' on (v_1, v_2) . Assume that $f_1 : [v_1, v_2] \rightarrow \mathbb{R}$ is absolutely continuous function on $[v_1, v_2]$, and $f_2 : [v_1, v_2] \rightarrow \mathbb{R}$ is non-decreasing on $[v_1, v_2]$. Moreover, suppose that both $\hbar_1, \hbar_2 : [v_1, v_2] \rightarrow \mathbb{R}^+$ are positive integrable. If $(f'_1) \in L^\infty[v_1, v_2]$, then we have

$$\begin{aligned} & \left| \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(1) \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(\hbar_2 f_1 f_2)(v_2) + \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} \hbar_2(v_2) \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(f_1 f_2)(v_2) \right. \\ & \left. - \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(f_1)(v_2) \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(\hbar_2 f_2)(v_2) - \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(\hbar_2 f_1)(v_2) \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(f_2)(v_2) \right| \\ & \leq \frac{\|f_1'\|_{\infty}}{\omega(\theta)} \left[\frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} v_2 \int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) \hbar_2(\varrho) d\varrho \right. \\ & \left. - \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(1) \int_{v_1}^{\theta} \varrho \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) \hbar_2(\varrho) d\varrho \right] \int_{v_1}^{v_2} f_2'(\theta) d\theta. \end{aligned}$$

Proof. Applying Theorem 6 for $\hbar_1(\theta) = 1, \theta \in [v_1, v_2]$, the desired result is obtained. \square

Corollary 8. Let \mathcal{F} be measurable, increasing, positive and monotone function on (v_1, v_2) , and having a continuous derivative \mathcal{F}' on (v_1, v_2) . Assume that $f_1 : [v_1, v_2] \rightarrow \mathbb{R}$ is absolutely continuous function on $[v_1, v_2]$, and $f_2 : [v_1, v_2] \rightarrow \mathbb{R}$ is non-decreasing on $[v_1, v_2]$. Moreover, suppose that both $\hbar_1, \hbar_2 : [v_1, v_2] \rightarrow \mathbb{R}^+$ are positive integrable. If $(f_1') \in L^{\infty}[v_1, v_2]$, then we have

$$\begin{aligned} & \left| \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} \hbar_1(v_2) \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(f_1 f_2)(v_2) + \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(1) \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(f_1 f_2)(v_2) \right. \\ & \left. - \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(\hbar_1 f_1)(v_2) \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(f_2)(v_2) - \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(f_1)(v_2) \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(\hbar_1 f_2)(v_2) \right| \\ & \leq \frac{\|f_1'\|_{\infty}}{\omega(\theta)} \left[\frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} v_2 \hbar_1(v_2) \int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) d\varrho \right. \\ & \left. - \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} \hbar_1(v_2) \int_{v_1}^{\theta} \varrho \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) d\varrho \right] \int_{v_1}^{v_2} f_2'(\theta) d\theta. \end{aligned}$$

Proof. Applying Theorem 6 for $\hbar_2(\theta) = 1, \theta \in [v_1, v_2]$, the desired result is obtained. \square

Corollary 9. Let \mathcal{F} be measurable, increasing, positive and monotone function on (v_1, v_2) , and having a continuous derivative \mathcal{F}' on (v_1, v_2) . Assume that $f_1 : [v_1, v_2] \rightarrow \mathbb{R}$ is absolutely continuous function on $[v_1, v_2]$, and $f_2 : [v_1, v_2] \rightarrow \mathbb{R}$ is non-decreasing on $[v_1, v_2]$. Moreover, suppose that both $\hbar_1, \hbar_2 : [v_1, v_2] \rightarrow \mathbb{R}^+$ are positive integrable. If $(f_1') \in L^{\infty}[v_1, v_2]$, then we have

$$\begin{aligned} & \left| \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(1) \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(f_1 f_2)(v_2) - \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(f_1)(v_2) \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(f_2)(v_2) \right| \\ & \leq \frac{\|f_1'\|_{\infty}}{2\omega(\theta)} \left[\frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi} v_2 \int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) d\varrho \right. \\ & \left. - \frac{\mathcal{F}}{\omega} \mathcal{I}_{v_1+}^{\Psi}(1) \int_{v_1}^{\theta} \varrho \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) d\varrho \right] \int_{v_1}^{v_2} f_2'(\theta) d\theta. \end{aligned}$$

Proof. Applying Theorem 6 for $\hbar_1(\theta) = \hbar_2(\theta) = 1, \theta \in [v_1, v_2]$, the desired result is obtained. \square

Theorem 7. Let \mathcal{F} be measurable, increasing, positive and monotone function on (v_1, v_2) and having continuous derivative \mathcal{F}' on (v_1, v_2) . Assume that $f_1, f_2 : [v_1, v_2] \rightarrow \mathbb{R}$ are absolutely continuous on $[v_1, v_2]$ and $f_2 : [v_1, v_2] \rightarrow \mathbb{R}$ is nondecreasing on $[v_1, v_2]$. Suppose that $\hbar_1, \hbar_2 : [v_1, v_2] \rightarrow \mathbb{R}^+$ are positive integrable. If $f_1', f_2' \in L^{\infty}[v_1, v_2]$, then we have

$$\begin{aligned}
& \left| \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \hbar_1(v_2)}{\omega} \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_2 f_1 f_2)(v_2)}{\omega} + \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \hbar_2(v_2)}{\omega} \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_1 f_1 f_2)(v_2)}{\omega} \right. \\
& \left. - \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_2 f_1)(v_2)}{\omega} \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_1 f_2)(x_2)}{\omega} - \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_1 f_1)(v_2)}{\omega} \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_2 f_2)(v_2)}{\omega} \right| \\
& \leq \frac{\|f'_1\|_{\infty} \|f'_2\|_{\infty}}{\omega(\theta)} \left[\frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} v_2 \hbar_1(v_2)}{\omega} \int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) \hbar_2(\varrho) d\varrho \right. \\
& \left. - 2 \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} v_2 \hbar_1(v_2)}{\omega} \int_{v_1}^{\theta} \varrho \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) \hbar_2(\varrho) d\varrho \right. \\
& \left. + \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \hbar_1(v_2)}{\omega} \int_{v_1}^{\theta} \varrho^2 \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) \hbar_2(\varrho) d\varrho \right]. \quad (16)
\end{aligned}$$

Proof. Consider the left-hand side of (16), we have

$$\begin{aligned}
& \left| \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \hbar_2(v_2)}{\omega} \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_2 f_1 f_2)(v_2)}{\omega} + \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \hbar_2(v_2)}{\omega} \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_1 f_1 f_2)(v_2)}{\omega} \right. \\
& \left. - \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_2 f_1)(v_2)}{\omega} \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_1 f_2)(x_2)}{\omega} - \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_1 f_1)(v_2)}{\omega} \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_2 f_2)(v_2)}{\omega} \right| \\
& \leq \frac{1}{\omega^2(\theta)} \left| \int_{v_1}^{v_2} \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \right. \\
& \times \omega(\xi) \mathcal{F}'(\xi) \hbar_1(\xi) \omega(\varrho) \mathcal{F}'(\varrho) \hbar_2(\varrho) [(f_1(\xi) - f_1(\varrho))(f_2(\xi) - f_2(\varrho))] d\xi d\varrho \left| \right. \\
& \leq \frac{1}{\omega^2(\theta)} \int_{v_1}^{v_2} \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\xi) \mathcal{F}'(\xi) \hbar_1(\xi) \omega(\varrho) \mathcal{F}'(\varrho) \hbar_2(\varrho) \\
& \times \left| \frac{(f_1(\xi) - f_1(\varrho))}{\xi - \varrho} \right| \left| \frac{(f_2(\xi) - f_2(\varrho))}{\xi - \varrho} \right| (\xi - \varrho)^2 d\xi d\varrho \\
& \leq \frac{\|f'_1\|_{\infty} \|f'_2\|_{\infty}}{\omega^2(\theta)} \int_{v_1}^{v_2} \int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\xi) \mathcal{F}'(\xi) \hbar_1(\xi) \omega(\varrho) \mathcal{F}'(\varrho) \hbar_2(\varrho) \\
& \times (\xi^2 - 2\xi\varrho + \varrho^2) d\xi d\varrho \\
& \leq \frac{\|f'_1\|_{\infty} \|f'_2\|_{\infty}}{\omega^2(\theta)} \left[\int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \xi^2 \omega(\xi) \mathcal{F}'(\xi) \hbar_1(\xi) d\xi \int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) \hbar_2(\varrho) d\varrho \right. \\
& \left. - 2 \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \xi \omega(\xi) \mathcal{F}'(\xi) \hbar_1(\xi) d\xi \int_{v_1}^{\theta} \varrho \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) \hbar_2(\varrho) d\varrho \right. \\
& \left. + \int_{v_1}^{v_2} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\xi))}{\mathcal{F}(v_2) - \mathcal{F}(\xi)} \omega(\xi) \mathcal{F}'(\xi) \hbar_1(\xi) d\xi \int_{v_1}^{\theta} \varrho^2 \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) \hbar_2(\varrho) d\varrho \right].
\end{aligned}$$

Hence, by using (7), the proof of the theorem is completed. \square

Corollary 10. Let \mathcal{F} be measurable, increasing, positive and monotone function on (v_1, v_2) and having continuous derivative \mathcal{F}' on (v_1, v_2) . Assume that $f_1, f_2 : [v_1, v_2] \rightarrow \mathbb{R}$ are absolutely continuous on $[v_1, v_2]$ and $f_2 : [v_1, v_2] \rightarrow \mathbb{R}$ is nondecreasing on $[v_1, v_2]$. Suppose that $\hbar_1, \hbar_2 : [v_1, v_2] \rightarrow \mathbb{R}^+$ are positive integrable. If $f'_1, f'_2 \in L^{\infty}[v_1, v_2]$, then we have

$$\begin{aligned}
& \left| \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (1)}{\omega} \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_2 f_1 f_2)(v_2)}{\omega} + \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \hbar_2(v_2)}{\omega} \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (f_1 f_2)(v_2)}{\omega} \right. \\
& \left. - \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_2 f_1)(v_2)}{\omega} \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (f_2)(x_2)}{\omega} - \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (f_1)(v_2)}{\omega} \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_2 f_2)(v_2)}{\omega} \right| \\
& \leq \frac{\|f'_1\|_{\infty} \|f'_2\|_{\infty}}{\omega(\theta)} \left[\frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} v_2}{\omega} \int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) \hbar_2(\varrho) d\varrho \right. \\
& \left. - 2 \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} v_2}{\omega} \int_{v_1}^{\theta} \varrho \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) \hbar_2(\varrho) d\varrho \right. \\
& \left. + \frac{{}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (1)}{\omega} \int_{v_1}^{\theta} \varrho^2 \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) \hbar_2(\varrho) d\varrho \right].
\end{aligned}$$

Proof. Setting $\hbar_1(\theta) = 1, \theta \in [v_1, v_2]$ in Theorem 7, then the desired result is obtained. \square

Corollary 11. Let \mathcal{F} be measurable, increasing, positive and monotone function on (v_1, v_2) and having continuous derivative \mathcal{F}' on (v_1, v_2) . Assume that $f_1, f_2 : [v_1, v_2] \rightarrow \mathbb{R}$ are absolutely continuous on $[v_1, v_2]$ and $f_2 : [v_1, v_2] \rightarrow \mathbb{R}$ is nondecreasing on $[v_1, v_2]$. Suppose that $\hbar_1, \hbar_2 : [v_1, v_2] \rightarrow \mathbb{R}^+$ are positive integrable. If $f'_1, f'_2 \in L^\infty[v_1, v_2]$, then we have

$$\begin{aligned} & \left| {}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \hbar_1(v_2) {}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (f_1 f_2)(v_2) + {}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (1) {}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_1 f_1 f_2)(v_2) \right. \\ & \left. - {}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (f_1)(v_2) {}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_1 f_2)(x_2) - {}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (\hbar_1 f_1)(v_2) {}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (f_2)(v_2) \right| \\ & \leq \frac{\|f'_1\|_{\infty} \|f'_2\|_{\infty}}{\omega(\theta)} \left[{}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} v_2 \hbar_1(v_2) \int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) d\varrho \right. \\ & \left. - 2 {}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} v_2 \hbar_1(v_2) \int_{v_1}^{\theta} \varrho \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) d\varrho \right. \\ & \left. + {}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \hbar_1(v_2) \int_{v_1}^{\theta} \varrho^2 \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) d\varrho \right]. \end{aligned}$$

Proof. Setting $\hbar_2(\theta) = 1, \theta \in [v_1, v_2]$ in Theorem 7, then the desired result is proven. \square

Corollary 12. Let \mathcal{F} be measurable, increasing, positive and monotone function on (v_1, v_2) and having continuous derivative \mathcal{F}' on (v_1, v_2) . Assume that $f_1, f_2 : [v_1, v_2] \rightarrow \mathbb{R}$ are absolutely continuous on $[v_1, v_2]$ and $f_2 : [v_1, v_2] \rightarrow \mathbb{R}$ is nondecreasing on $[v_1, v_2]$. Suppose that $\hbar_1, \hbar_2 : [v_1, v_2] \rightarrow \mathbb{R}^+$ are positive integrable. If $f'_1, f'_2 \in L^\infty[v_1, v_2]$, then we have

$$\begin{aligned} & \left| {}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (1) {}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (f_1 f_2)(v_2) + {}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (1) {}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (f_1 f_2)(v_2) \right. \\ & \left. - {}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (f_1)(v_2) {}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (f_2)(x_2) - {}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (f_1)(v_2) {}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} (f_2)(v_2) \right| \\ & \leq \frac{\|f'_1\|_{\infty} \|f'_2\|_{\infty}}{\omega(\theta)} \left[{}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} v_2 \int_{v_1}^{\theta} \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) d\varrho \right. \\ & \left. - 2 {}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} v_2 \int_{v_1}^{\theta} \varrho \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) d\varrho \right. \\ & \left. + {}_{\omega}^{\mathcal{F}}\mathcal{I}_{v_1+}^{\Psi} \int_{v_1}^{\theta} \varrho^2 \frac{\Psi(\mathcal{F}(v_2) - \mathcal{F}(\varrho))}{\mathcal{F}(v_2) - \mathcal{F}(\varrho)} \omega(\varrho) \mathcal{F}'(\varrho) d\varrho \right]. \end{aligned}$$

Proof. Setting $\hbar_1(\theta) = \hbar_2(\theta) = 1, \theta \in [v_1, v_2]$ in Theorem 7, then the desired result is obtained. \square

Remark 6. One can easily derive some new inequalities by applying the following conditions.

- i. Setting $\hbar_1(\theta) = \hbar_2(\theta), \mathcal{F}(\theta) = \theta$ and $\Psi(\mathcal{F}(\theta)) = \theta$ throughout in the paper.
- ii. Setting $\hbar_1(\theta) = \hbar_2(\theta) = 1$ and $\mathcal{F}(\theta) = \theta$ and $\Psi(\mathcal{F}(\theta)) = \theta$ throughout in the paper.

Remark 7. Throughout in this article, if we put $\omega(\theta) = 1$ and $\Psi(\mathcal{F}(\theta)) = \frac{\mathcal{F}(\theta)^\kappa}{\Gamma(\kappa)}$, then all the inequalities will reduce to the work of Bezziou et al. [27].

Remark 8. Throughout in this article, if we consider $\omega(\theta) = 1$ and $\Psi(\mathcal{F}(\theta)) = \frac{\mathcal{F}(\theta)^\kappa}{\Gamma_\eta(\kappa)}$, then all the inequalities will reduce to the work of Rahman et al. [41].

Remark 9. Taking $\hbar_1(\theta) = \hbar_2(\theta), \omega(\theta) = 1, \Psi(\mathcal{F}(\theta)) = \theta$ and $\Psi(\mathcal{F}(\theta)) = \frac{(\theta)^\kappa}{\Gamma(\kappa)}$ in Theorems 4–7, the results of Bezziou et al. [42] are restored.

3. Concluding Remarks

In the study of mathematics and related subjects, mathematical inequalities are extremely important. Fractional integral inequalities are now useful in determining the uniqueness of fractional partial differential equation solutions. They also guarantee the boundedness of fractional boundary value problem solutions. These suggestions have promoted the future research in the subject of integral inequalities to investigate the extensions of integral inequalities using fractional calculus operators. In the present investigation, we have proposed some double-weighted generalized fractional integral inequalities by utilizing more generalized class of fractional integrals associated with integrable, measurable, positive and monotone function \mathcal{F} in its kernel. The derived inequalities are more general than the existing inequalities cited therein. All the classical inequalities can be easily restored by applying specific conditions on \mathcal{F} and $\Psi(\theta)$ given in Remark 3. Also, we can derive some new weighted type double fractional integral inequalities by applying specific conditions on \mathcal{F} and $\Psi(\theta)$ given in Remark 2. In future research, some new other type of inequalities will be derived by employing the proposed operator. The special cases of the obtained result can be found in [24,25,27,41,42].

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