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Some Hadamard–Fejér Type Inequalities for LR-Convex Interval-Valued Functions

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Abstract: The purpose of this study is to introduce the new class of Hermite–Hadamard inequality for LR-convex interval-valued functions known as LR-interval Hermite–Hadamard inequality, by means of pseudo-order relation ($\leq_p$). This order relation is defined on interval space. We have proved that if the interval-valued function is LR-convex then the inclusion relation " $\subseteq$ " coincident to pseudo-order relation " $\leq_p$ " under some suitable conditions. Moreover, the interval Hermite–Hadamard–Fejér inequality is also derived for LR-convex interval-valued functions. These inequalities also generalize some new and known results. Useful examples that verify the applicability of the theory developed in this study are presented. The concepts and techniques of this paper may be a starting point for further research in this area.

Keywords: interval-valued function; Riemann integral; LR-convex interval-valued function; interval Hermite–Hadamard inequality; interval Hermite–Hadamard–Fejér inequality

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1. Introduction

In the development of pure and applied mathematics [1,2] convexity has played a key role. Due to their resilience, convex sets and convex functions have been refined and expanded in many mathematical fields; see [3–8]. Convexity theory may be used to generate numerous inequalities in the literature. Integral inequalities [9] have uses in linear programming, combinatory, orthogonal polynomials, quantum theory, number theory, optimization theory, dynamics, and the theory of relativity. Researchers have given this problem a lot of attention [10–14], and it is now regarded an integrative topic involving economics, mathematics, physics, and statistics [15,16]. The Hermite–Hadamard inequality (*HH*-inequality) is, to the best of my knowledge, a well-known, ultimate, and broadly applied inequality. Other classical inequalities, such as the Oslen and Gagliardo–Nirenberg, Oslen, Opial, Hardy, Young, Linger, Ostrowski, levison, Arithmetic's-Geometric, Ky-fan, Minkowski, Beckenbach–Dresher, and Holer inequality, are closely linked to the classical *HH*-inequality [17–20], and it can be put in the following manner.

Let $\mathfrak{S} : K \to \mathbb{R}$ be a convex function on a convex set K and $\mathrm{t}, \upsilon \in K$ with $\mathrm{t} \leq \upsilon$. Then,

$$\mathfrak{S}\left(\frac{\mathrm{t}+\upsilon}{2}\right) \leq \frac{1}{\upsilon-\mathrm{t}} \int_{\mathrm{t}}^{\upsilon} \mathfrak{S}(\omega)d\omega \leq \frac{\mathfrak{S}(\mathrm{t})+\mathfrak{S}(\upsilon)}{2}. \quad (1)$$

In [21], Fejér looked at the key extensions of *HH*-inequality, dubbed Hermite–Hadamard–Fejér inequality (*HH*-Fejér inequality).

Let $\mathfrak{S} : K \to \mathbb{R}$ be a convex function on a convex set K and t, $\upsilon \in K$ with t $\leq \upsilon$. Then,

$$\mathfrak{S}\left(\frac{\mathrm{t}+\upsilon}{2}\right) \leq \frac{1}{\int_{\mathrm{t}}^{\upsilon} \mathfrak{D}(\omega)d\omega} \int_{\mathrm{t}}^{\upsilon} \mathfrak{S}(\omega)\mathfrak{D}(\omega)d\omega \leq \frac{\mathfrak{S}(\mathrm{t}) + \mathfrak{S}(\upsilon)}{2} \int_{\mathrm{t}}^{\upsilon} \mathfrak{D}(\omega))d\omega. \tag{2}$$

If $\mathfrak{D}(\omega) = 1$ then, we obtain (1) from (2). Many classical inequalities may be derived by specific convex functions with the help of inequality (1). Furthermore, in both pure and industrial mathematics, these inequalities play a crucial role for convex functions. We encourage readers to go more into the literature on generalized convex functions and *HH*-integral inequalities, particularly [22–29] and the references therein.

Interval analysis, on the other hand, was mostly forgotten for a long time due to a lack of applicability in other fields. Moore [30] and Kulish and W. Miranker [31] introduced and researched the notion of interval analysis. It is the first time in numerical analysis that it is utilized to calculate the error boundaries of numerical solutions of a finite state machine. Since then, a number of analysts have focused on and studied interval analysis and interval-valued functions (*I.V-Fs*) in both mathematics and applications. As a result, various writers looked into the literature and applications of neural network output optimization, automatic error analysis, computational physics, robotics, computer graphics, and a variety of other well-known scientific and technology fields. We encourage readers to conduct more research into essential aspects and applications in the literature (see [32–40] and the references therein).

The theory of fuzzy sets and systems has progressed in a number of ways from its introduction five decades ago, as seen in [41]. As a result, it is useful in the study of a variety of issues in pure mathematics and applied sciences, such as operation research, computer science, management sciences, artificial intelligence, control engineering, and decision sciences. Convex analysis has contributed significantly to the advancement of several sectors of practical and pure research. Similarly, the concepts of convexity and non-convexity are important in fuzzy optimization because we obtain fuzzy variational inequalities when we characterize the optimality condition of convexity, so variational inequality theory and fuzzy complementary problem theory established powerful mechanisms of mathematical problems and have a friendly relationship. Costa [42], Costa and Roman-Flores [43], Flores-Franulic et al. [44], Roman-Flores et al. [45,46], and Chalco-Cano et al. [47,48] have recently generalized several classical discrete and integral inequalities not only to the environment of the *I.V-Fs* and fuzzy *I.V-Fs*, but also to more general set valued maps by Nikodem et al. Zhang et al. [49] used a pseudo order relation to establish a novel version of Jensen's inequalities for set-valued and fuzzy set-valued functions, proving that these Jensen's inequalities are an expanded form of Costa Jensen's inequalities [42]. Zhao et al. [50], inspired by the literature, introduced $\hbar$-convex *I.V-Fs* and established that the *HH*-inequality for $\hbar$-convex *I.V-Fs*. Yanrong An et al. [51] took a step forward by introducing the class of $(\hbar_1, \hbar_2)$ $\hbar$ -convex *I.V-Fs* and establishing the interval *HH*-inequality for $(\hbar_1, \hbar_2)$ -convex *I.V-Fs*.

This research is structured as follows: preliminary and novel notions and results in interval space and interval-valued convex analysis are presented in Section 2. Section 3 uses LR-convex *I.V-Fs* to generate LR-interval *HH*-inequalities and *HH*-Fejér inequalities. In addition, several intriguing cases are provided to support our findings. Conclusions and future plans are presented in Section 4.

2. Preliminaries

Let $\mathcal{K}_C$ be the collection of all closed and bounded intervals of $\mathbb{R}$ that is $\mathcal{K}_C = \{[\mathcal{Z}_*, \mathcal{Z}^*] : \mathcal{Z}_*, \mathcal{Z}^* \in \mathbb{R}$ and $\mathcal{Z}_* \leq \mathcal{Z}^*\}$. If $\mathcal{Z}_* \geq 0$, then $[\mathcal{Z}_*, \mathcal{Z}^*]$ is named as positive interval. The set of all positive interval is denoted by $\mathcal{K}_C^+$ and defined as $\mathcal{K}_C^+ = \{[\mathcal{Z}_*, \mathcal{Z}^*] : \mathcal{Z}_*, \mathcal{Z}^* \in \mathcal{K}_C$ and $\mathcal{Z}_* \geq 0\}$.

If $[\mathfrak{A}_*, \mathfrak{A}^*]$, $[\mathcal{Z}_*, \mathcal{Z}^*] \in \mathcal{K}_C$ and s $\in \mathbb{R}$, then arithmetic operations are defined by

$$[\mathfrak{A}_*, \mathfrak{A}^*] + [\mathcal{Z}_*, \mathcal{Z}^*] = [\mathfrak{A}_* + \mathcal{Z}_*, \mathfrak{A}^* + \mathcal{Z}^*],$$

$$[\mathfrak{A}_*,\ \mathfrak{A}^*] \times [\mathcal{Z}_*,\ \mathcal{Z}^*] = [\min\{\mathfrak{A}_*\mathcal{Z}_*,\ \mathfrak{A}^*\mathcal{Z}_*,\ \mathfrak{A}_*\mathcal{Z}^*,\ \mathfrak{A}^*\mathcal{Z}^*\},\ \max\{\mathfrak{A}_*\mathcal{Z}_*,\ \mathfrak{A}^*\mathcal{Z}_*,\ \mathfrak{A}_*\mathcal{Z}^*,\ \mathfrak{A}^*\mathcal{Z}^*\}],$$

$$\mathrm{s}.[\mathfrak{A}_*,\ \mathfrak{A}^*] = \begin{cases} [\mathrm{s}\mathfrak{A}_*,\ \mathrm{s}\mathfrak{A}^*] & \text{if } \mathrm{s} > 0 \\ \{0\} & \text{if } \mathrm{s} = 0, \\ [\mathrm{s}\mathfrak{A}^*, \mathrm{s}\mathfrak{A}_*] & \text{if } \mathrm{s} < 0. \end{cases}$$

For $[\mathfrak{A}_*,\ \mathfrak{A}^*],\ [\mathcal{Z}_*,\ \mathcal{Z}^*] \in \mathcal{K}_C$, the inclusion " $\subseteq$ " is defined by

$$[\mathfrak{A}_*,\ \mathfrak{A}^*] \subseteq [\mathcal{Z}_*,\ \mathcal{Z}^*], \text{ if and only if } \mathcal{Z}_* \leq \mathfrak{A}_*,\ \mathfrak{A}^* \leq \mathcal{Z}^*.$$

Remark 1. **[49].** *(i) The relation " $\leq_p$ " defined on $\mathcal{K}_C$ by $[\mathfrak{A}_*,\ \mathfrak{A}^*] \leq_p [\mathcal{Z}_*,\ \mathcal{Z}^*]$ if and only if $\mathfrak{A}_* \leq \mathcal{Z}_*,\ \mathfrak{A}^* \leq \mathcal{Z}^*$, for all $[\mathfrak{A}_*,\ \mathfrak{A}^*],\ [\mathcal{Z}_*,\ \mathcal{Z}^*] \in \mathcal{K}_C$, it is a pseudo-order relation. The relation $[\mathfrak{A}_*,\ \mathfrak{A}^*] \leq_p [\mathcal{Z}_*,\ \mathcal{Z}^*]$ coincident to $[\mathfrak{A}_*,\ \mathfrak{A}^*] \leq [\mathcal{Z}_*,\ \mathcal{Z}^*]$ on $\mathcal{K}_C$.*

(ii) It can be easily seen that " $\leq_p$ " looks similar to "left and right" on the real line $\mathbb{R}$, so we call " $\leq_p$ " is "left and right" (or "LR" order, in short).

The concept of Riemann integral for I.V-F first introduced by Moore [30] is defined as follow:

Theorem 1. **[30].** *If $\mathfrak{S} : [\mathrm{t}, v] \subset \mathbb{R} \to \mathcal{K}_C$ is an I.V-F on such that $\mathfrak{S}(\omega) = [\mathfrak{S}_*(\omega),\ \mathfrak{S}^*(\omega)]$. Then $\mathfrak{S}$ is Riemann integrable over $[\mathrm{t}, v]$ if and only if, $\mathfrak{S}_*$ and $\mathfrak{S}^*$ both are Riemann integrable over $[\mathrm{t}, v]$ such that*

$$(IR)\int_{\mathrm{t}}^{v} \mathfrak{S}(\omega)d\omega = [(R)\int_{\mathrm{t}}^{v} \mathfrak{S}_*(\omega)d\omega,\ (R)\int_{\mathrm{t}}^{v} \mathfrak{S}^*(\omega)d\omega].$$

The collection of all Riemann integrable real valued functions and Riemann integrable *I.V-F* is denoted by $\mathcal{R}_{[\mathrm{t},\ v]}$ and $\mathcal{IR}_{[\mathrm{t},\ v]}$, respectively.

Definition 1. *The real mapping $\mathfrak{S} : [\mathrm{t},\ v] \to \mathbb{R}$ is named as convex function if for all $\omega,\ y \in [\mathrm{t},\ v]$ and $\varsigma \in [0,\ 1]$ we have*

$$\mathfrak{S}(\varsigma\omega + (1-\varsigma)y\ ) \leq \varsigma\mathfrak{S}(\omega) + (1-\varsigma)\mathfrak{S}(y), \tag{3}$$

If inequality (3) is reversed, then $\mathfrak{S}$ is named as concave on $[\mathrm{t},\ v]$. A function $\mathfrak{S}$ is named as affine if $\mathfrak{S}$ is both convex and cocave function. The set of all convex (concave) functions is denoted by

$$SX([\mathrm{t},\ v],)(SV([\mathrm{t},\ v],\ \mathbb{R}^+),\ SA([\mathrm{t},\ v],\ \mathbb{R}^+)).$$

Definition 2. **[50].** *The I.V-F $\mathfrak{S} : [\mathrm{t},\ v] \to \mathbb{R}_I^+$ is named as convex I.V-F if for all $\omega,\ y \in [\mathrm{t},\ v]$ and $\varsigma \in [0,\ 1]$, the coming inequality*

$$\mathfrak{S}(\varsigma\omega + (1-\varsigma)y\ ) \supseteq \hbar(\varsigma)\mathfrak{S}(\omega) + \hbar(1-\varsigma)\mathfrak{S}(y), \tag{4}$$

is valid. If inequality (4) is reversed, then $\mathfrak{S}$ is named as concave on $[\mathrm{t},\ v]$. A I.V-F $\mathfrak{S}$ is named as affine if $\mathfrak{S}$ is both convex and cocave I.V-F. The set of all convex (concave, affine) I.V-Fs is denoted by

$$SX([\mathrm{t},\ v],\ \mathcal{K}_C^+)\quad (SV([\mathrm{t},\ v],\ \mathcal{K}_C^+),\ SA([\mathrm{t},\ v],\ \mathcal{K}_C^+)).$$

Definition 3. **[49].** *TheI.V-F $\mathfrak{S} : [\mathrm{t},\ v] \to \mathcal{K}_C^+$ is named as LR-convex I.V-F if for all $\omega,\ y \in [\mathrm{t},\ v]$ and $\varsigma \in [0,\ 1]$, the coming inequality*

$$\mathfrak{S}(\varsigma\omega + (1-\varsigma)y\ ) \leq_p \varsigma\mathfrak{S}(\omega) + (1-\varsigma)\mathfrak{S}(y), \tag{5}$$

is valid. If inequality (5) is reversed, then $\mathfrak{S}$ is named as LR-concave on $[\mathrm{t},\ v]$. A I.V-F $\mathfrak{S}$ is named as LR-affine if $\mathfrak{S}$ is both LR-convex and LR-cocave I.V-F. The set of all LR-convex (LR-concave) I.V-Fs is denoted by

$$LRSX([\mathrm{t},\ v],\ \mathcal{K}_C^+)(LRSV([\mathrm{t},\ v],\ \mathcal{K}_C^+),\ \ LRSA([\mathrm{t},\ v],\ \mathcal{K}_C^+)).$$

Theorem 2. [49]. *Let $\mathfrak{S} : [\mathrm{t},\ v] \to \mathcal{K}_C^+$ be an I.V-F defined by $\mathfrak{S}(\omega) = [\mathfrak{S}_*(\omega),\ \mathfrak{S}^*(\omega)]$, for all $\omega \in [\mathrm{t},\ v]$. Then $\mathfrak{S} \in LRSX\big([\mathrm{t},\ v],\ \mathcal{K}_C^+\big)$ if and only if, $\mathfrak{S}_*,\ \mathfrak{S}^* \in SX([\mathrm{t},\ v])$.*

Example 1. *We consider the I.V-F $\mathfrak{S} : [1,\ 4] \to \mathcal{K}_C^+$ defined by $\mathfrak{S}(\omega) = \left[2\omega, 2\omega^2\right]$. Since end point functions $\mathfrak{S}_*(\omega)$ and $\mathfrak{S}^*(\omega)$ are convex functions. Hence $\mathfrak{S}(\omega)$ is LR-convex I.V-F.*

Remark 2. *By using our Definition 3 and Example 1, it can be easily observed that the concept of set inclusion " $\supseteq$ " coincident to relation " $\leq_p$ " (or " $\leq_p$ " coincident to " $\supseteq$ ") when one of the end point function $\mathfrak{S}_*$ or $\mathfrak{S}^*$ is affine function such that "If $\mathfrak{S} \in SX\big([\mathrm{t},\ v],\ \mathcal{K}_C^+\big)$ then $\mathfrak{S} \in LRSV\big([\mathrm{t},\ v],\ \mathcal{K}_C^+\big)$ if and only if $\mathfrak{S}_* \in SA([\mathrm{t},\ v],\ \mathbb{R}^+)$ and $\mathfrak{S}^* \in SX([\mathrm{t},\ v],\ \mathbb{R}^+)$ ".* Similarly, "If $\mathfrak{S} \in SV\big([\mathrm{t},\ v],\ \mathcal{K}_C^+\big)$ *then $\mathfrak{S} \in LRSX\big([\mathrm{t},\ v],\ \mathcal{K}_C^+\big)$, if and only if $\mathfrak{S}_* \in SV$ $([\mathrm{t},\ v],\ \mathbb{R}^+)$ and $\mathfrak{S}^* \in SA([\mathrm{t},\ v],\ \mathbb{R}^+)$ ".*

Remark 3. *From Theorem 2, it can be easily seen that if $\mathfrak{S}_*(\omega) = \mathfrak{S}^*(\omega)$ then, LR-convex I.V-Fs becomes classical convex functions.*

Example 2. *We consider the I.V-F $\mathfrak{S} : [1,\ 4] \to \mathcal{K}_C^+$ defined by $\mathfrak{S}(\omega) = \left[2\omega^2, 2\omega^2\right]$. Since end point functions $\mathfrak{S}_*(\omega),\ \mathfrak{S}^*(\omega)$, are equal and convex functions. Hence, $\mathfrak{S}(\omega)$ is a convex function.*

3. Interval Inequalities

In this section, we present two classes of *HH*-inequalities and discuss some related results, and verify with the help of use examples. First of all, we derive *HH*-inequality for LR-convex *I.V-F*.

Theorem 3. *Let $\mathfrak{S} : [\mathrm{t},\ v] \to \mathcal{K}_C^+$ be an I.V-F such that $\mathfrak{S}(\omega) = [\mathfrak{S}_*(\omega),\ \mathfrak{S}^*(\omega)]$ for all $\omega \in [\mathrm{t},\ v]$ and $\mathfrak{S} \in \mathcal{IR}_{([\mathrm{t},\ v])}$. If $\mathfrak{S} \in LRSX\big([\mathrm{t},\ v],\ \mathcal{K}_C^+\big)$, then*

$$\mathfrak{S}\left(\frac{\mathrm{t}+v}{2}\right) \leq_p \frac{1}{v-\mathrm{t}}\,(IR)\int_{\mathrm{t}}^{v} \mathfrak{S}(\omega)d\omega \leq_p \frac{\mathfrak{S}(\mathrm{t})+\mathfrak{S}(v)}{2}. \tag{6}$$

If $\mathfrak{S} \in LRSV\big([\mathrm{t},\ v],\ \mathcal{K}_C^+\big)$, then

$$\mathfrak{S}\left(\frac{\mathrm{t}+v}{2}\right) \geq_p \frac{1}{v-\mathrm{t}}\,(IR)\int_{\mathrm{t}}^{v} \mathfrak{S}(\omega)d\omega \geq_p \frac{\mathfrak{S}(\mathrm{t})+\mathfrak{S}(v)}{2}.$$

Proof. Let $\mathfrak{S} \in \mathrm{LRSX}\big([\mathrm{t},\ v],\ \mathcal{K}_C^+\big)$ convex I.V-F. Then, by hypothesis, we have

$$\begin{aligned} 2\mathfrak{S}_*\left(\tfrac{\mathrm{t}+v}{2}\right) &\leq \mathfrak{S}_*(\varsigma\mathrm{t} + (1-\varsigma)v) + \mathfrak{S}_*((1-\varsigma)\mathrm{t} + \varsigma v), \\ 2\mathfrak{S}^*\left(\tfrac{\mathrm{t}+v}{2}\right) &\leq \mathfrak{S}^*(\varsigma\mathrm{t} + (1-\varsigma)v) + \mathfrak{S}^*((1-\varsigma)\mathrm{t} + \varsigma v). \end{aligned}$$

Then

$$\begin{aligned} 2\int_0^1 \mathfrak{S}_*\left(\tfrac{\mathrm{t}+v}{2}\right)d\varsigma &\leq \int_0^1 \mathfrak{S}_*(\varsigma\mathrm{t} + (1-\varsigma)v)d\varsigma + \int_0^1 \mathfrak{S}_*((1-\varsigma)\mathrm{t} + \varsigma v)d\varsigma, \\ 2\int_0^1 \mathfrak{S}^*\left(\tfrac{\mathrm{t}+v}{2}\right)d\varsigma &\leq \int_0^1 \mathfrak{S}^*(\varsigma\mathrm{t} + (1-\varsigma)v)d\varsigma + \int_0^1 \mathfrak{S}^*((1-\varsigma)\mathrm{t} + \varsigma v)d\varsigma. \end{aligned}$$

It follows that

$$\begin{aligned} \mathfrak{S}_*\left(\tfrac{\mathrm{t}+v}{2}\right) &\leq \tfrac{1}{v-\mathrm{t}}\int_{\mathrm{t}}^{v} \mathfrak{S}_*(\omega)d\omega, \\ \mathfrak{S}^*\left(\tfrac{\mathrm{t}+v}{2}\right) &\leq \tfrac{1}{v-\mathrm{t}}\int_{\mathrm{t}}^{v} \mathfrak{S}^*(\omega)d\omega. \end{aligned}$$

That is

$$\left[\mathfrak{S}_*\left(\frac{\mathrm{t}+v}{2}\right),\ \mathfrak{S}^*\left(\frac{\mathrm{t}+v}{2}\right)\right] \leq_p \frac{1}{v-\mathrm{t}}\left[\int_{\mathrm{t}}^{v} \mathfrak{S}_*(\omega)d\omega,\ \int_{\mathrm{t}}^{v} \mathfrak{S}^*(\omega)d\omega\right].$$

Thus,

$$\mathfrak{S}\left(\frac{\mathrm{t}+v}{2}\right) \leq_p \frac{1}{v-\mathrm{t}}\,(IR)\int_{\mathrm{t}}^{v} \mathfrak{S}(\omega)d\omega. \tag{7}$$

In a similar way as above, we have

$$\frac{1}{\upsilon - \mathrm{t}}\,(IR)\int_{\mathrm{t}}^{\upsilon} \mathfrak{S}(\omega)d\omega \leq_p \frac{\mathfrak{S}(\mathrm{t}) + \mathfrak{S}(\upsilon)}{2}. \tag{8}$$

Combining (7) and (8), we have

$$\mathfrak{S}\left(\frac{\mathrm{t}+\upsilon}{2}\right) \leq_p \frac{1}{\upsilon - \mathrm{t}}\,(IR)\int_{\mathrm{t}}^{\upsilon} \mathfrak{S}(\omega)d\omega \leq_p \frac{\mathfrak{S}(\mathrm{t}) + \mathfrak{S}(\upsilon)}{2}.$$

Hence, the required result. □

Remark 4. *If $\mathfrak{S}_*(\omega) = \mathfrak{S}^*(\omega)$, then Theorem 3, reduces to the result for convex function:*

$$\mathfrak{S}\left(\frac{\mathrm{t}+\upsilon}{2}\right) \leq \frac{1}{\upsilon - \mathrm{t}}\,(R)\int_{\mathrm{t}}^{\upsilon} \mathfrak{S}(\omega)d\omega \leq \frac{\mathfrak{S}(\mathrm{t}) + \mathfrak{S}(\upsilon)}{2}.$$

It is easy to see that due to the convexity of end point functions $\mathfrak{S}_*(\omega)$ and $\mathfrak{S}^*(\omega)$ have following two possibilities to satisfy (1) either both are convex or affine convex functions. However, in the case of interval inclusion both functions $\mathfrak{S}_*(\omega)$ and $\mathfrak{S}^*(\omega)$ has only one possibility to satisfy (1) such that both end point functions should be affine convex because in interval inclusion $\mathfrak{S}_*(\omega)$ is convex and $\mathfrak{S}^*(\omega)$ is concave, see [50].

Example 3. *We consider the function $\mathfrak{S} : [\mathrm{t},\ \upsilon] = [0,\ 2] \to \mathcal{K}_C^+$ defined by, $\mathfrak{S}(\omega) = \left[\omega^2, 2\omega^2\right]$. Since end point functions $\mathfrak{S}_*(\omega) = \omega^2$, $\mathfrak{S}^*(\omega) = 2\omega^2$ LR-convex functions. Hence $\mathfrak{S}(\omega)$ is LR-convex I.V-F. We now compute the following*

$$\mathfrak{S}_*\left(\frac{\mathrm{t}+\upsilon}{2}\right) \leq \frac{1}{\upsilon - \mathrm{t}}\int_{\mathrm{t}}^{\upsilon} \mathfrak{S}_*(\omega)d\omega \leq \frac{\mathfrak{S}_*(\mathrm{t}) + \mathfrak{S}_*(\upsilon)}{2}.$$

$$\mathfrak{S}_*\left(\frac{\mathrm{t}+\upsilon}{2}\right) = \mathfrak{S}_*(1) = 1,$$

$$\frac{1}{\upsilon - \mathrm{t}}\int_{\mathrm{t}}^{\upsilon} \mathfrak{S}_*(\omega)d\omega = \frac{1}{2}\int_0^2 \omega^2 d\omega = \frac{4}{3},$$

$$\frac{\mathfrak{S}_*(\mathrm{t}) + \mathfrak{S}_*(\upsilon)}{2} = 2.$$

That means

$$1 \leq \frac{4}{3} \leq 2.$$

Similarly, it can be easily show that

$$\mathfrak{S}^*\left(\frac{\mathrm{t}+\upsilon}{2}\right) \leq \frac{1}{\upsilon - \mathrm{t}}\int_{\mathrm{t}}^{\upsilon} \mathfrak{S}^*(\omega)d\omega \leq \frac{\mathfrak{S}^*(\mathrm{t}) + \mathfrak{S}^*(\upsilon)}{2}.$$

such that

$$\mathfrak{S}^*\left(\tfrac{\mathrm{t}+\upsilon}{2}\right) = \mathfrak{S}_*(1) = 2,$$

$$\tfrac{1}{\upsilon-\mathrm{t}}\int_{\mathrm{t}}^{\upsilon} \mathfrak{S}^*(\omega)d\omega = \tfrac{1}{2}\int_0^2 2\omega^2 d\omega = \tfrac{8}{3},$$

$$\tfrac{\mathfrak{S}^*(\mathrm{t})+\mathfrak{S}^*(\upsilon)}{2} = 4,$$

from which, it follows that

$$2 \leq \frac{8}{3} \leq 4,$$

that is

$$[1,\ 2] \leq \left[\frac{4}{3},\ \frac{8}{3}\right] \leq [2,\ 4].$$

Hence,

$$\mathfrak{S}\left(\frac{\mathsf{t}+v}{2}\right) \leq_p \frac{1}{v-\mathsf{t}}\,(IR)\int_{\mathsf{t}}^{v}\mathfrak{S}(\omega)d\omega \leq_p \frac{\mathfrak{S}(\mathsf{t})+\mathfrak{S}(v)}{2}.$$

Theorem 4. *Let $\mathfrak{S}:[\mathsf{t},\ v] \to \mathcal{K}_C^+$ be an I.V-F such that $\mathfrak{S}(\omega) = [\mathfrak{S}_*(\omega),\ \mathfrak{S}^*(\omega)]$ for all $\omega \in [\mathsf{t},\ v]$ and $\mathfrak{S} \in \mathcal{IR}_{([\mathsf{t},\ v])}$. If $\mathfrak{S} \in LRSX([\mathsf{t},\ v],\ \mathcal{K}_C{}^+)$, then*

$$\mathfrak{S}\left(\frac{\mathsf{t}+v}{2}\right) \leq_p \rhd_2 \leq_p \frac{1}{v-\mathsf{t}}\,(IR)\int_{\mathsf{t}}^{v}\mathfrak{S}(\omega)d\omega \leq_p \rhd_1 \leq_p \frac{\mathfrak{S}(\mathsf{t})+\mathfrak{S}(v)}{2},$$

where

$$\rhd_1 = \frac{\frac{\mathfrak{S}(\mathsf{t})+\mathfrak{S}(v)}{2}+\mathfrak{S}\left(\frac{\mathsf{t}+v}{2}\right)}{2},\ \rhd_2 = \frac{\mathfrak{S}\left(\frac{3\mathsf{t}+v}{4}\right)+\mathfrak{S}\left(\frac{\mathsf{t}+3v}{4}\right)}{2}$$

and $\rhd_1 = [\rhd_{1*}, \rhd_1{}^*]$, $\rhd_2 = [\rhd_{2*}, \rhd_2{}^*]$.

Proof. Take $\left[\mathsf{t},\ \frac{\mathsf{t}+v}{2}\right]$, we have

$$2\mathfrak{S}\left(\frac{\varsigma\mathsf{t}+(1-\varsigma)\frac{\mathsf{t}+v}{2}}{2}+\frac{(1-\varsigma)\mathsf{t}+\varsigma\frac{\mathsf{t}+v}{2}}{2}\right) \leq_p \mathfrak{S}\left(\varsigma\mathsf{t}+(1-\varsigma)\frac{\mathsf{t}+v}{2}\right)+\mathfrak{S}\left((1-\varsigma)\mathsf{t}+\varsigma\frac{\mathsf{t}+v}{2}\right).$$

From which, we have

$$2\mathfrak{S}_*\left(\frac{\varsigma\mathsf{t}+(1-\varsigma)\frac{\mathsf{t}+v}{2}}{2}+\frac{(1-\varsigma)\mathsf{t}+\varsigma\frac{\mathsf{t}+v}{2}}{2}\right) \leq \mathfrak{S}_*\left(\varsigma\mathsf{t}+(1-\varsigma)\frac{\mathsf{t}+v}{2}\right)+\mathfrak{S}_*\left((1-\varsigma)\mathsf{t}+\varsigma\frac{\mathsf{t}+v}{2}\right),$$
$$2\mathfrak{S}^*\left(\frac{\varsigma\mathsf{t}+(1-\varsigma)\frac{\mathsf{t}+v}{2}}{2}+\frac{(1-\varsigma)\mathsf{t}+\varsigma\frac{\mathsf{t}+v}{2}}{2}\right) \leq \mathfrak{S}^*\left(\varsigma\mathsf{t}+(1-\varsigma)\frac{\mathsf{t}+v}{2}\right)+\mathfrak{S}^*\left((1-\varsigma)\mathsf{t}+\varsigma\frac{\mathsf{t}+v}{2}\right).$$

In consequence, we obtain

$$\frac{\mathfrak{S}_*\left(\frac{3\mathsf{t}+v}{4}\right)}{2} \leq \frac{1}{v-\mathsf{t}}\int_{\mathsf{t}}^{\frac{\mathsf{t}+v}{2}}\mathfrak{S}_*(\omega)d\omega,$$
$$\frac{\mathfrak{S}^*\left(\frac{3\mathsf{t}+v}{4}\right)}{2} \leq \frac{1}{v-\mathsf{t}}\int_{\mathsf{t}}^{\frac{\mathsf{t}+v}{2}}\mathfrak{S}^*(\omega)d\omega.$$

That is

$$\frac{\left[\ \mathfrak{S}_*\left(\frac{3\mathsf{t}+v}{4}\right),\ \mathfrak{S}^*\left(\frac{3\mathsf{t}+v}{4}\right)\ \right]}{2} \leq \frac{1}{v-\mathsf{t}}\left[\int_{\mathsf{t}}^{\frac{\mathsf{t}+v}{2}}\mathfrak{S}_*(\omega)d\omega,\ \int_{\mathsf{t}}^{\frac{\mathsf{t}+v}{2}}\mathfrak{S}^*(\omega)d\omega\right].$$

It follows that

$$\frac{\mathfrak{S}\left(\frac{3\mathsf{t}+v}{4}\right)}{2} \leq_p \frac{1}{v-\mathsf{t}}\,(IR)\int_{\mathsf{t}}^{\frac{\mathsf{t}+v}{2}}\mathfrak{S}(\omega)d\omega. \tag{9}$$

In a similar way as above, we have

$$\frac{\mathfrak{S}\left(\frac{\mathsf{t}+3v}{4}\right)}{2} \leq_p \frac{1}{v-\mathsf{t}}\,(IR)\int_{\frac{\mathsf{t}+v}{2}}^{v}\mathfrak{S}(\omega)d\omega. \tag{10}$$

Combining (9) and (10), we have

$$\frac{\left[\mathfrak{S}\left(\frac{3\mathsf{t}+v}{4}\right)+\mathfrak{S}\left(\frac{\mathsf{t}+3v}{4}\right)\right]}{2} \leq_p \frac{1}{v-\mathsf{t}}\,(IR)\int_{\mathsf{t}}^{v}\mathfrak{S}(\omega)d\omega.$$

By using Theorem 3, we have

$$\mathfrak{S}\left(\frac{\mathsf{t}+v}{2}\right) = \mathfrak{S}\left(\frac{1}{2}.\frac{3\mathsf{t}+v}{4}+\frac{1}{2}.\frac{\mathsf{t}+3v}{4}\right).$$

From which, we have

$$
\begin{aligned}
\mathfrak{S}_*\left(\tfrac{\mathrm{t}+v}{2}\right) &= \mathfrak{S}_*\left(\tfrac{1}{2}.\tfrac{3\mathrm{t}+v}{4}+\tfrac{1}{2}.\tfrac{\mathrm{t}+3v}{4}\right),\\
\mathfrak{S}^*\left(\tfrac{\mathrm{t}+v}{2}\right) &= \mathfrak{S}^*\left(\tfrac{1}{2}.\tfrac{3\mathrm{t}+v}{4}+\tfrac{1}{2}.\tfrac{\mathrm{t}+3v}{4}\right),\\
&\leq \left[\tfrac{1}{2}\mathfrak{S}_*\left(\tfrac{3\mathrm{t}+v}{4}\right)+\tfrac{1}{2}\mathfrak{S}_*\left(\tfrac{\mathrm{t}+3v}{4}\right)\right],\\
&\leq \left[\tfrac{1}{2}\mathfrak{S}^*\left(\tfrac{3\mathrm{t}+v}{4}\right)+\tfrac{1}{2}\mathfrak{S}^*\left(\tfrac{\mathrm{t}+3v}{4}\right)\right],\\
&= \rhd_{2*},\\
&= \rhd_2{}^*,\\
&\leq \tfrac{1}{v-\mathrm{t}}\int_{\mathrm{t}}^{v}\mathfrak{S}_*(\omega)d\omega,\\
&\leq \tfrac{1}{v-\mathrm{t}}\int_{\mathrm{t}}^{v}\mathfrak{S}^*(\omega)d\omega,\\
&\leq \tfrac{1}{2}\left[\tfrac{\mathfrak{S}_*(\mathrm{t})+\mathfrak{S}_*(v)}{2}+\mathfrak{S}_*\left(\tfrac{\mathrm{t}+v}{2}\right)\right],\\
&\leq \tfrac{1}{2}\left[\tfrac{\mathfrak{S}^*(\mathrm{t})+\mathfrak{S}^*(v)}{2}+\mathfrak{S}^*\left(\tfrac{\mathrm{t}+v}{2}\right)\right],\\
&= \rhd_{1*},\\
&= \rhd_1{}^*,\\
&\leq \tfrac{1}{2}\left[\tfrac{\mathfrak{S}_*(\mathrm{t})+\mathfrak{S}_*(v)}{2}+\tfrac{\mathfrak{S}_*(\mathrm{t})+\mathfrak{S}_*(v)}{2}\right],\\
&\leq \tfrac{1}{2}\left[\tfrac{\mathfrak{S}^*(\mathrm{t})+\mathfrak{S}^*(v)}{2}+\tfrac{\mathfrak{S}_*(\mathrm{t})+\mathfrak{S}_*(v)}{2}\right],\\
&= \tfrac{\mathfrak{S}_*(\mathrm{t})+\mathfrak{S}_*(v)}{2},\\
&= \tfrac{\mathfrak{S}^*(\mathrm{t})+\mathfrak{S}^*(v)}{2},
\end{aligned}
$$

that is

$$\mathfrak{S}\left(\frac{\mathrm{t}+v}{2}\right) \leq_p \rhd_2 \leq_p \frac{1}{v-\mathrm{t}}\,(IR)\int_{\mathrm{t}}^{v}\mathfrak{S}(\omega)d\omega \leq_p \rhd_1 \leq_p \frac{\mathfrak{S}(\mathrm{t})+\mathfrak{S}(v)}{2},$$

hence, the result follows. □

Example 4. *We consider the function* $\mathfrak{S}:[\mathrm{t},\ v]=[0,\ 2]\to\mathcal{K}_C^+$ *defined by,* $\mathfrak{S}(\omega)=\left[\omega^2,2\omega^2\right]$, *as in Example 3, then* $\mathfrak{S}(\omega)$ *is LR-convex I.V-F and satisfying (10). We have* $\mathfrak{S}_*(\omega)=\omega^2$ *and* $\mathfrak{S}^*(\omega)=2\omega^2$ *. We now compute the following*

$$
\begin{aligned}
\tfrac{\mathfrak{S}_*(\mathrm{t})+\mathfrak{S}_*(v)}{2} &= 2,\\
\tfrac{\mathfrak{S}^*(\mathrm{t})+\mathfrak{S}^*(v)}{2} &= 4,\\
\rhd_{1*} = \frac{\frac{\mathfrak{S}_*(\mathrm{t})+\mathfrak{S}_*(v)}{2}+\mathfrak{S}_*\left(\frac{\mathrm{t}+v}{2}\right)}{2} &= \tfrac{3}{2},\\
\rhd_1{}^* = \frac{\frac{\mathfrak{S}^*(\mathrm{t})+\mathfrak{S}^*(v)}{2}+\mathfrak{S}^*\left(\frac{\mathrm{t}+v}{2}\right)}{2} &= 3,\\
\rhd_{2*} = \frac{\mathfrak{S}_*\left(\frac{3\mathrm{t}+v}{4}\right)+\mathfrak{S}_*\left(\frac{\mathrm{t}+3v}{4}\right)}{2} &= \tfrac{5}{4},\\
\rhd_2{}^* = \frac{\mathfrak{S}^*\left(\frac{3\mathrm{t}+v}{4}\right)+\mathfrak{S}^*\left(\frac{\mathrm{t}+3v}{4}\right)}{2} &= \tfrac{5}{2},
\end{aligned}
$$

Then we obtain that

$$1 \leq \tfrac{5}{4} \leq \tfrac{4}{3} \leq \tfrac{3}{2} \leq 2,$$
$$2 \leq \tfrac{5}{2} \leq \tfrac{8}{3} \leq 3 \leq 4,$$

Hence, Theorem 4 is verified.

Theorem 5. *Let* $\mathfrak{S}$, $g : [\mathsf{t}, v] \to \mathcal{K}_C^+$ *be two I.V-F such that* $\mathfrak{S}(\omega) = [\mathfrak{S}_*(\omega), \mathfrak{S}^*(\omega)]$ *and* $g(\omega) = [g_*(\omega), g^*(\omega)]$ *for all* $\omega \in [\mathsf{t}, v]$ *and* $\mathfrak{S}g \in \mathcal{IR}_{([\mathsf{t}, v])}$. *If* $\mathfrak{S}$, $g \in LRSX([\mathsf{t}, v], \mathcal{K}_C^+)$, *then*

$$\frac{1}{v-\mathsf{t}}\,(IR)\int_{\mathsf{t}}^{v}\mathfrak{S}(\omega)g(\omega)d\omega \leq_p \frac{\mathfrak{B}(\mathsf{t},v)}{3}+\frac{\mathfrak{C}(\mathsf{t},v)}{6},$$

where $\mathfrak{B}(\mathsf{t},v) = \mathfrak{S}(\mathsf{t})g(\mathsf{t}) + \mathfrak{S}(v)g(v)$, $\mathfrak{C}(\mathsf{t},v) = \mathfrak{S}(\mathsf{t})g(v) + \mathfrak{S}(v)g(\mathsf{t})$, *and* $\mathfrak{B}(\mathsf{t},v) = [\mathfrak{B}_*((\mathsf{t},v)), \mathfrak{B}^*((\mathsf{t},v))]$ *and* $\mathfrak{C}(\mathsf{t},v) = [\mathfrak{C}_*((\mathsf{t},v)), \mathfrak{C}^*((\mathsf{t},v))]$.

Proof. Since $\mathfrak{S}$, $g \in \mathcal{IR}_{([\mathsf{t}, v])}$, then we have

$$\mathfrak{S}_*(\varsigma\mathsf{t}+(1-\varsigma)v) \leq \varsigma\mathfrak{S}_*(\mathsf{t})+(1-\varsigma)\mathfrak{S}_*(v),$$
$$\mathfrak{S}^*(\varsigma\mathsf{t}+(1-\varsigma)v) \leq \varsigma\mathfrak{S}^*(\mathsf{t})+(1-\varsigma)\mathfrak{S}^*(v).$$

And

$$g_*(\varsigma\mathsf{t}+(1-\varsigma)v) \leq \varsigma g_*(\mathsf{t})+(1-\varsigma)g_*(v),$$
$$g^*(\varsigma\mathsf{t}+(1-\varsigma)v) \leq \varsigma g^*(\mathsf{t})+(1-\varsigma)g^*(v).$$

From the definition of LR-convex *I.V-Fs* it follows that $0 \leq_p \mathfrak{S}(\omega)$ and $0 \leq_p g(\omega)$, so

$$\begin{aligned}
&\mathfrak{S}_*(\varsigma\mathsf{t}+(1-\varsigma)v)g_*(\varsigma\mathsf{t}+(1-\varsigma)v)\\
&\quad\leq \left(\varsigma\mathfrak{S}_*(\mathsf{t})+(1-\varsigma)\mathfrak{S}_*(v)\right)\left(\varsigma g_*(\mathsf{t})+(1-\varsigma)g_*(v)\right)\\
&\quad= \mathfrak{S}_*(\mathsf{t})g_*(\mathsf{t})\varsigma^2+\mathfrak{S}_*(v)g_*(v)\varsigma^2+\mathfrak{S}_*(\mathsf{t})g_*(v)\varsigma(1-\varsigma)+\mathfrak{S}_*(v)g_*(\mathsf{t})\varsigma(1-\varsigma)\\
&\mathfrak{S}^*(\varsigma\mathsf{t}+(1-\varsigma)v)g^*(\varsigma\mathsf{t}+(1-\varsigma)v)\\
&\quad\leq \left(\varsigma\mathfrak{S}^*(\mathsf{t})+(1-\varsigma)\mathfrak{S}^*(v)\right)\left(\varsigma g^*(\mathsf{t})+(1-\varsigma)g^*(v)\right)\\
&\quad= \mathfrak{S}^*(\mathsf{t})g^*(\mathsf{t})\varsigma^2+\mathfrak{S}^*(v)g^*(v)\varsigma^2+\mathfrak{S}^*(\mathsf{t})g^*(v)\varsigma(1-\varsigma)+\mathfrak{S}^*(v)g^*(\mathsf{t})\varsigma(1-\varsigma),
\end{aligned}$$

Integrating both sides of above inequality over [0, 1] we obtain

$$\begin{aligned}
\int_0^1\mathfrak{S}_*(\varsigma\mathsf{t}+(1-\varsigma)v)g_*(\varsigma\mathsf{t}+(1-\varsigma)v) &= \tfrac{1}{v-\mathsf{t}}\int_{\mathsf{t}}^{v}\mathfrak{S}_*(\omega)g_*(\omega)d\omega\\
&\leq (\mathfrak{S}_*(\mathsf{t})g_*(\mathsf{t})+\mathfrak{S}_*(v)g_*(v))\int_0^1\varsigma^2 d\varsigma\\
&\quad+(\mathfrak{S}_*(\mathsf{t})g_*(v)+\mathfrak{S}_*(v)g_*(\mathsf{t}))\int_0^1\varsigma(1-\varsigma)d\varsigma,\\
\int_0^1\mathfrak{S}^*(\varsigma\mathsf{t}+(1-\varsigma)v)g^*(\varsigma\mathsf{t}+(1-\varsigma)v) &= \tfrac{1}{v-\mathsf{t}}\int_{\mathsf{t}}^{v}\mathfrak{S}^*(\omega)g^*(\omega)d\omega\\
&\leq (\mathfrak{S}^*(\mathsf{t})g^*(\mathsf{t})+\mathfrak{S}^*(v)g^*(v))\int_0^1\varsigma^2 d\varsigma\\
&\quad+(\mathfrak{S}^*(\mathsf{t})g^*(v)+\mathfrak{S}^*(v)g^*(\mathsf{t}))\int_0^1\varsigma(1-\varsigma)d\varsigma.
\end{aligned}$$

It follows that,

$$\tfrac{1}{v-\mathsf{t}}\int_{\mathsf{t}}^{v}\mathfrak{S}_*(\omega)g_*(\omega)d\omega \leq \mathfrak{B}_*((\mathsf{t},v))\int_0^1\varsigma^2 d\varsigma+\mathfrak{C}_*((\mathsf{t},v))\int_0^1\varsigma(1-\varsigma)d\varsigma,$$
$$\tfrac{1}{v-\mathsf{t}}\int_{\mathsf{t}}^{v}\mathfrak{S}^*(\omega)g^*(\omega)d\omega \leq \mathfrak{B}^*((\mathsf{t},v))\int_0^1\varsigma^2 d\varsigma+\mathfrak{C}^*((\mathsf{t},v))\int_0^1\varsigma(1-\varsigma)d\varsigma,$$

that is

$$\frac{1}{v-\mathsf{t}}\left[\int_{\mathsf{t}}^{v}\mathfrak{S}_*(\omega)g_*(\omega)d\omega,\ \int_{\mathsf{t}}^{v}\mathfrak{S}^*(\omega)g^*(\omega)d\omega\right] \leq_p \left[\frac{\mathfrak{B}_*((\mathsf{t},v))}{3},\ \frac{\mathfrak{B}^*((\mathsf{t},v))}{3}\right]+\left[\frac{\mathfrak{C}_*((\mathsf{t},v))}{6},\ \frac{\mathfrak{C}^*((\mathsf{t},v))}{6}\right].$$

Thus,

$$\frac{1}{v-\mathsf{t}}\,(IR)\int_{\mathsf{t}}^{v}\mathfrak{S}(\omega)g(\omega)d\omega \leq_p \frac{\mathfrak{B}(\mathsf{t},v)}{3}+\frac{\mathfrak{C}(\mathsf{t},v)}{6},$$

and the theorem has been established. □

Theorem 6. *Let* $\mathfrak{S}, g : [\mathsf{t}, v] \to \mathcal{K}_C^+$ *be two I.V-Fs such that* $\mathfrak{S}(\omega) = [\mathfrak{S}_*(\omega), \mathfrak{S}^*(\omega)]$ *and* $g(\omega) = [g_*(\omega), g^*(\omega)]$ *for all* $\omega \in [\mathsf{t}, v]$ *and* $\mathfrak{S}g \in \mathcal{IR}_{([\mathsf{t}, v])}$. *If* $\mathfrak{S}, g \in LRSX([\mathsf{t}, v], \mathcal{K}_C^+)$, *then*

$$2\,\mathfrak{S}\left(\frac{\mathsf{t}+v}{2}\right)g\left(\frac{\mathsf{t}+v}{2}\right) \leq_p \frac{1}{v-\mathsf{t}}\,(IR)\int_{\mathsf{t}}^{v}\mathfrak{S}(\omega)g(\omega)d\omega + \frac{\mathfrak{B}(\mathsf{t},v)}{6} + \frac{\mathfrak{C}(\mathsf{t},v)}{3},$$

where $\mathfrak{B}(\mathsf{t},v) = \mathfrak{S}(\mathsf{t})g(\mathsf{t}) + \mathfrak{S}(v)g(v)$, $\mathfrak{C}(\mathsf{t},v) = \mathfrak{S}(\mathsf{t})g(v) + \mathfrak{S}(v)g(\mathsf{t})$, *and* $\mathfrak{B}(\mathsf{t},v) = [\mathfrak{B}_*((\mathsf{t},v)), \mathfrak{B}^*((\mathsf{t},v))]$ *and* $\mathfrak{C}(\mathsf{t},v) = [\mathfrak{C}_*((\mathsf{t},v)), \mathfrak{C}^*((\mathsf{t},v))]$.

Proof. By hypothesis, we have

$$\begin{aligned}
&\mathfrak{S}_*\left(\tfrac{\mathsf{t}+v}{2}\right)g_*\left(\tfrac{\mathsf{t}+v}{2}\right)\\
&\mathfrak{S}^*\left(\tfrac{\mathsf{t}+v}{2}\right)g^*\left(\tfrac{\mathsf{t}+v}{2}\right)\\
&\leq \tfrac{1}{4}\left[\begin{array}{l}\mathfrak{S}_*(\varsigma\mathsf{t}+(1-\varsigma)v)g_*(\varsigma\mathsf{t}+(1-\varsigma)v)\\ +\mathfrak{S}_*(\varsigma\mathsf{t}+(1-\varsigma)v)g_*((1-\varsigma)\mathsf{t}+\varsigma v)\end{array}\right]\\
&+\tfrac{1}{4}\left[\begin{array}{l}\mathfrak{S}_*((1-\varsigma)\mathsf{t}+\varsigma v)g_*(\varsigma\mathsf{t}+(1-\varsigma)v)\\ +\mathfrak{S}_*((1-\varsigma)\mathsf{t}+\varsigma v)g_*((1-\varsigma)\mathsf{t}+\varsigma v)\end{array}\right],\\
&\leq \tfrac{1}{4}\left[\begin{array}{l}\mathfrak{S}^*(\varsigma\mathsf{t}+(1-\varsigma)v)g^*(\varsigma\mathsf{t}+(1-\varsigma)v)\\ +\mathfrak{S}^*(\varsigma\mathsf{t}+(1-\varsigma)v)g^*((1-\varsigma)\mathsf{t}+\varsigma v)\end{array}\right]\\
&+\tfrac{1}{4}\left[\begin{array}{l}\mathfrak{S}^*((1-\varsigma)\mathsf{t}+\varsigma v)g^*(\varsigma\mathsf{t}+(1-\varsigma)v)\\ +\mathfrak{S}^*((1-\varsigma)\mathsf{t}+\varsigma v)g^*((1-\varsigma)\mathsf{t}+\varsigma v)\end{array}\right],\\
&\leq \tfrac{1}{4}\left[\begin{array}{l}\mathfrak{S}_*(\varsigma\mathsf{t}+(1-\varsigma)v)g_*(\varsigma\mathsf{t}+(1-\varsigma)v)\\ +\mathfrak{S}_*((1-\varsigma)\mathsf{t}+\varsigma v)g_*((1-\varsigma)\mathsf{t}+\varsigma v)\end{array}\right]\\
&+\tfrac{1}{4}\left[\begin{array}{l}(\varsigma\mathfrak{S}_*(\mathsf{t})+(1-\varsigma)\mathfrak{S}_*(v))\\ ((1-\varsigma)g_*(\mathsf{t})+\varsigma g_*(v))\\ +((1-\varsigma)\mathfrak{S}_*(\mathsf{t})+\varsigma\mathfrak{S}_*(v))\\ (\varsigma g_*(\mathsf{t})+(1-\varsigma)g_*(v))\end{array}\right],\\
&\leq \tfrac{1}{4}\left[\begin{array}{l}\mathfrak{S}_*(\varsigma\mathsf{t}+(1-\varsigma)v)g_*(\varsigma\mathsf{t}+(1-\varsigma)v)\\ +\mathfrak{S}_*((1-\varsigma)\mathsf{t}+\varsigma v)g_*((1-\varsigma)\mathsf{t}+\varsigma v)\end{array}\right]\\
&+\tfrac{1}{4}\left[\begin{array}{l}(\varsigma\mathfrak{S}^*(\mathsf{t})+(1-\varsigma)\mathfrak{S}^*(v))\\ ((1-\varsigma)g^*(\mathsf{t})+\varsigma g^*(v))\\ +((1-\varsigma)\mathfrak{S}^*(\mathsf{t})+\varsigma\mathfrak{S}^*(v))\\ (\varsigma g^*(\mathsf{t})+(1-\varsigma)g^*(v))\end{array}\right],\\
&= \tfrac{1}{4}\left[\begin{array}{l}\mathfrak{S}_*(\varsigma\mathsf{t}+(1-\varsigma)v)g_*(\varsigma\mathsf{t}+(1-\varsigma)v)\\ +\mathfrak{S}_*((1-\varsigma)\mathsf{t}+\varsigma v)g_*((1-\varsigma)\mathsf{t}+\varsigma v)\end{array}\right]\\
&+\tfrac{1}{2}\left[\begin{array}{l}\left\{\varsigma^2+(1-\varsigma)^2\right\}\mathfrak{C}_*((\mathsf{t},v))\\ +\{\varsigma(1-\varsigma)+(1-\varsigma)\varsigma\}\mathfrak{B}_*((\mathsf{t},v))\end{array}\right],\\
&= \tfrac{1}{4}\left[\begin{array}{l}\mathfrak{S}^*(\varsigma\mathsf{t}+(1-\varsigma)v)g^*(\varsigma\mathsf{t}+(1-\varsigma)v)\\ +\mathfrak{S}^*((1-\varsigma)\mathsf{t}+\varsigma v)g^*((1-\varsigma)\mathsf{t}+\varsigma v)\end{array}\right]\\
&+\tfrac{1}{2}\left[\begin{array}{l}\left\{\varsigma^2+(1-\varsigma)^2\right\}\mathfrak{C}^*((\mathsf{t},v))\\ +\{\varsigma(1-\varsigma)+(1-\varsigma)\varsigma\}\mathfrak{B}^*((\mathsf{t},v))\end{array}\right].
\end{aligned}$$

IR -Integrating over $[0,\ 1]$, we have

$$2\ \mathfrak{S}_*\left(\frac{t+v}{2}\right)g_*\left(\frac{t+v}{2}\right) \leq \frac{1}{v-t}\ \int_t^v \mathfrak{S}_*(\omega)g_*(\omega)d\omega + \frac{\mathfrak{B}_*((t,v))}{6} + \frac{\mathfrak{C}_*((t,v))}{3},$$
$$2\ \mathfrak{S}^*\left(\frac{t+v}{2}\right)g^*\left(\frac{t+v}{2}\right) \leq \frac{1}{v-t}\ \int_t^v \mathfrak{S}^*(\omega)g^*(\omega)d\omega + \frac{\mathfrak{B}^*((t,v))}{6} + \frac{\mathfrak{C}^*((t,v))}{3},$$

that is

$$2\ \mathfrak{S}\left(\frac{t+v}{2}\right)g\left(\frac{t+v}{2}\right) \leq_p \frac{1}{v-t}\ (IR)\int_t^v \mathfrak{S}(\omega)g(\omega)d\omega + \frac{\mathfrak{B}(t,v)}{6} + \frac{\mathfrak{C}(t,v)}{3}.$$

Hence, the required result. □

Example 5. *We consider the I.V-Fs* $\mathfrak{S},\ g : [t,\ v] = [0,\ 1] \to \mathcal{K}_C^+$ *defined by* $\mathfrak{S}(\omega) = \left[2\omega^2, 4\omega^2\right]$ *and* $g(\omega) = [\omega, 2\omega]$. *Since end point functions* $\mathfrak{S}_*(\omega) = 2\omega^2$, $\mathfrak{S}^*(\omega) = 4\omega^2$ *and* $g_*(\omega) = \omega$, $g^*(\omega) = 2\omega$ *are convex functions. Hence* $\mathfrak{S}$, g *both are LR-convex I.V-Fs. We now compute the following*

$$\frac{1}{v-t}\ \int_t^v \mathfrak{S}_*(\omega)g_*(\omega)d\omega = \frac{1}{2},$$
$$\frac{1}{v-t}\ \int_t^v \mathfrak{S}^*(\omega)g^*(\omega)d\omega = 2,$$
$$\frac{\mathfrak{B}_*((t,v))}{3} = \frac{2}{3},$$
$$\frac{\mathfrak{B}^*((t,v))}{3} = \frac{8}{3},$$
$$\frac{\mathfrak{C}_*((t,v))}{6} = 0,$$
$$\frac{\mathfrak{C}^*((t,v))}{6} = 0,$$

that means

$$\frac{1}{2} \leq \frac{2}{3} + 0 = \frac{2}{3},$$
$$2 \leq \frac{8}{3} + 0 = \frac{8}{3},$$

Consequently, Theorem 5 is verified.
For Theorem 6, we have

$$2\ \mathfrak{S}_*\left(\frac{t+v}{2}\right)g_*\left(\frac{t+v}{2}\right) = \frac{1}{2},$$
$$2\ \mathfrak{S}^*\left(\frac{t+v}{2}\right)g^*\left(\frac{t+v}{2}\right) = 2,$$
$$\frac{1}{v-t}\ \int_t^v \mathfrak{S}_*(\omega)g_*(\omega)d\omega = \frac{1}{2},$$
$$\frac{1}{v-t}\ \int_t^v \mathfrak{S}^*(\omega)g^*(\omega)d\omega = 2,$$
$$\frac{\mathfrak{B}_*((t,v))}{6} = \frac{1}{3},$$
$$\frac{\mathfrak{B}^*((t,v))}{6} = \frac{4}{3},$$
$$\frac{\mathfrak{C}_*((t,v))}{3} = 0,$$
$$\frac{\mathfrak{C}^*((t,v))}{3} = 0,$$

From which, we have

$$\frac{1}{2} \leq \frac{1}{2} + 0 + \frac{1}{3} = \frac{5}{6},$$
$$2 \leq 2 + 0 + \frac{4}{3} = \frac{10}{3},$$

Consequently, Theorem 6 is demonstrated.

We now give *HH*-Fejér inequalities for LR-convex *I.V-Fs*. Firstly, we obtain the second *HH*-Fejér inequality for LR-convex *I.V-F*.

Theorem 7. *Let* $\mathfrak{S} : [\mathrm{t},\ v] \to \mathcal{K}_C^+$ *be an I.V-F with* $\mathrm{t} < v$*, such that* $\mathfrak{S}(\omega) = [\mathfrak{S}_*(\omega),\ \mathfrak{S}^*(\omega)]$ *for all* $\omega \in [\mathrm{t},\ v]$ *and* $\mathfrak{S} \in \mathcal{IR}_{([\mathrm{t},\ v])}$. *If* $\mathfrak{S} \in LRSX([\mathrm{t},\ v],\ \mathcal{K}_C^+)$*, then* $\mathfrak{D} : [\mathrm{t},\ v] \to \mathbb{R}$, $\mathfrak{D}(\omega) \geq 0$, *symmetric with respect to* $\frac{\mathrm{t}+v}{2}$*, then*

$$\frac{1}{v-\mathrm{t}}\ (IR)\int_{\mathrm{t}}^{v} \mathfrak{S}(\omega)\mathfrak{D}(\omega)d\omega \leq_p [\mathfrak{S}(\mathrm{t}) + \mathfrak{S}(v)]\int_0^1 \varsigma\mathfrak{D}((1-\varsigma)\mathrm{t} + \varsigma v)d\varsigma. \tag{11}$$

Proof. Let $\mathfrak{S} \in LRSX([\mathrm{t},\ v],\ \mathcal{K}_C^+)$. Then we have

$$\begin{aligned} &\mathfrak{S}_*(\varsigma\mathrm{t} + (1-\varsigma)v)D(\varsigma\mathrm{t} + (1-\varsigma)v) \\ &\qquad \leq (\varsigma\mathfrak{S}_*(\mathrm{t}) + (1-\varsigma)\mathfrak{S}_*(v))D(\varsigma\mathrm{t} + (1-\varsigma)v), \\ &\mathfrak{S}^*(\varsigma\mathrm{t} + (1-\varsigma)v)D(\varsigma\mathrm{t} + (1-\varsigma)v) \\ &\qquad \leq (\varsigma\mathfrak{S}^*(\mathrm{t}) + (1-\varsigma)\mathfrak{S}^*(v))D(\varsigma\mathrm{t} + (1-\varsigma)v). \end{aligned} \tag{12}$$

And

$$\begin{aligned} &\mathfrak{S}_*((1-\varsigma)\mathrm{t} + \varsigma v)D((1-\varsigma)\mathrm{t} + \varsigma v) \leq ((1-\varsigma)\mathfrak{S}_*(\mathrm{t}) + \varsigma\mathfrak{S}_*(v))D((1-\varsigma)\mathrm{t} + \varsigma v), \\ &\mathfrak{S}^*((1-\varsigma)\mathrm{t} + \varsigma v)D((1-\varsigma)\mathrm{t} + \varsigma v) \leq ((1-\varsigma)\mathfrak{S}^*(\mathrm{t}) + \varsigma\mathfrak{S}^*(v))D((1-\varsigma)\mathrm{t} + \varsigma v). \end{aligned} \tag{13}$$

After adding (12) and (13), and integrating over $[0,\ 1]$, we obtain

$$\begin{aligned} &\int_0^1 \mathfrak{S}_*(\varsigma\mathrm{t} + (1-\varsigma)v)\mathfrak{D}(\varsigma\mathrm{t} + (1-\varsigma)v)d\varsigma + \int_0^1 \mathfrak{S}_*((1-\varsigma)\mathrm{t} + \varsigma v)\mathfrak{D}((1-\varsigma)\mathrm{t} + \varsigma v)d\varsigma \\ &\qquad \leq \int_0^1 \left[\begin{array}{l} \mathfrak{S}_*(\mathrm{t})\{\varsigma\mathfrak{D}(\varsigma\mathrm{t} + (1-\varsigma)v) + (1-\varsigma)\mathfrak{D}((1-\varsigma)\mathrm{t} + \varsigma v)\} \\ +\mathfrak{S}_*(v)\{(1-\varsigma)\mathfrak{D}(\varsigma\mathrm{t} + (1-\varsigma)v) + \varsigma\mathfrak{D}((1-\varsigma)\mathrm{t} + \varsigma v)\} \end{array}\right] d\varsigma, \\ &\int_0^1 \mathfrak{S}^*((1-\varsigma)\mathrm{t} + \varsigma v)\mathfrak{D}((1-\varsigma)\mathrm{t} + \varsigma v)d\varsigma + \int_0^1 \mathfrak{S}^*(\varsigma\mathrm{t} + (1-\varsigma)v)\mathfrak{D}(\varsigma\mathrm{t} + (1-\varsigma)v)d\varsigma \\ &\qquad \leq \int_0^1 \left[\begin{array}{l} \mathfrak{S}^*(\mathrm{t})\{\varsigma\mathfrak{D}(\varsigma\mathrm{t} + (1-\varsigma)v) + (1-\varsigma)\mathfrak{D}((1-\varsigma)\mathrm{t} + \varsigma v)\} \\ +\mathfrak{S}^*(v)\{(1-\varsigma)\mathfrak{D}(\varsigma\mathrm{t} + (1-\varsigma)v) + \varsigma\mathfrak{D}((1-\varsigma)\mathrm{t} + \varsigma v)\} \end{array}\right] d\varsigma. \\ &\qquad = 2\mathfrak{S}_*(\mathrm{t})\int_0^1 \ \varsigma D(\varsigma\mathrm{t} + (1-\varsigma)v)\ d\varsigma + 2\mathfrak{S}_*(v)\int_0^1 \ \varsigma D((1-\varsigma)\mathrm{t} + \varsigma v)\ d\varsigma, \\ &\qquad = 2\mathfrak{S}^*(\mathrm{t})\int_0^1 \ \varsigma D(\varsigma\mathrm{t} + (1-\varsigma)v)\ d\varsigma + 2\mathfrak{S}^*(v)\int_0^1 \ \varsigma D((1-\varsigma)\mathrm{t} + \varsigma v)\ d\varsigma. \end{aligned}$$

Since $\mathfrak{D}$ is symmetric, then

$$\begin{aligned} &= 2[\mathfrak{S}_*(\mathrm{t}) + \mathfrak{S}_*(v)]\int_0^1 \ \varsigma D((1-\varsigma)\mathrm{t} + \varsigma v)\ d\varsigma, \\ &= 2[\mathfrak{S}^*(\mathrm{t}) + \mathfrak{S}^*(v)]\int_0^1 \ \varsigma D((1-\varsigma)\mathrm{t} + \varsigma v)\ d\varsigma. \end{aligned} \tag{14}$$

Since

$$\begin{aligned} &\int_0^1 \mathfrak{S}_*(\varsigma\mathrm{t} + (1-\varsigma)v)\mathfrak{D}(\varsigma\mathrm{t} + (1-\varsigma)v)d\varsigma \\ &\qquad = \int_0^1 \mathfrak{S}_*((1-\varsigma)\mathrm{t} + \varsigma v)\mathfrak{D}((1-\varsigma)\mathrm{t} + \varsigma v)d\varsigma = \frac{1}{v-\mathrm{t}}\ \int_{\mathrm{t}}^{v} \mathfrak{S}_*(\omega)\mathfrak{D}(\omega)d\omega \\ &\int_0^1 \mathfrak{S}^*((1-\varsigma)\mathrm{t} + \varsigma v)\mathfrak{D}((1-\varsigma)\mathrm{t} + \varsigma v)d\varsigma \\ &\qquad = \int_0^1 \mathfrak{S}^*(\varsigma\mathrm{t} + (1-\varsigma)v)\mathfrak{D}(\varsigma\mathrm{t} + (1-\varsigma)v)d\varsigma = \frac{1}{v-\mathrm{t}}\ \int_{\mathrm{t}}^{v} \mathfrak{S}^*(\omega)\mathfrak{D}(\omega)d\omega \end{aligned} \tag{15}$$

From (15), we have

$$\begin{aligned} &\frac{1}{v-\mathrm{t}}\ \int_{\mathrm{t}}^{v} \mathfrak{S}_*(\omega)\mathfrak{D}(\omega)d\omega \leq [\mathfrak{S}_*(\mathrm{t}) + \mathfrak{S}_*(v)]\int_0^1 \varsigma D((1-\varsigma)\mathrm{t} + \varsigma v)d\varsigma, \\ &\frac{1}{v-\mathrm{t}}\ \int_{\mathrm{t}}^{v} \mathfrak{S}^*(\omega)\mathfrak{D}(\omega)d\omega\ \leq [\mathfrak{S}^*(\mathrm{t}) + \mathfrak{S}^*(v)]\int_0^1 \ \varsigma D((1-\varsigma)\mathrm{t} + \varsigma v)\ d\varsigma, \end{aligned}$$

that is

$$\begin{aligned} &\left[\frac{1}{v-\mathrm{t}}\ \int_{\mathrm{t}}^{v} \mathfrak{S}_*(\omega)\mathfrak{D}(\omega)d\omega,\ \frac{1}{v-\mathrm{t}}\ \int_{\mathrm{t}}^{v} \mathfrak{S}^*(\omega)\mathfrak{D}(\omega)d\omega\right] \\ &\leq_p [\mathfrak{S}_*(\mathrm{t}) + \mathfrak{S}_*(v),\ \mathfrak{S}^*(\mathrm{t}) + \mathfrak{S}^*(v)]\int_0^1 \ \varsigma D((1-\varsigma)\mathrm{t} + \varsigma v)\ d\varsigma \end{aligned}$$

hence

$$\frac{1}{v-\mathrm{t}}\ (IR)\int_{\mathrm{t}}^{v}\mathfrak{S}(\omega)\mathfrak{D}(\omega)d\omega \leq_p [\mathfrak{S}(\mathrm{t})+\mathfrak{S}(v)]\int_0^1 \varsigma\mathfrak{D}((1-\varsigma)\mathrm{t}+\varsigma v)d\varsigma.$$

Next, we construct first *HH*-Fejér inequality for LR-convex *I.V-F*, which generalizes first *HH*-Fejér inequalities for convex function, see [21]. □

Theorem 8. *Let* $\mathfrak{S}:[\mathrm{t},\ v]\to\mathcal{K}_C^+$ *be an I.V-F with* $\mathrm{t}<v$, *such that* $\mathfrak{S}(\omega)=[\mathfrak{S}_*(\omega),\ \mathfrak{S}^*(\omega)]$ *for all* $\omega\in[\mathrm{t},\ v]$ *and* $\mathfrak{S}\in\mathcal{IR}_{([\mathrm{t},\ v])}$ *. If* $\mathfrak{S}\in LRSX([\mathrm{t},\ v],\ \mathcal{K}_C^+)$ *and* $\mathfrak{D}:[\mathrm{t},\ v]\to\mathbb{R}$, $\mathfrak{D}(\omega)\geq 0$, *symmetric with respect to* $\frac{\mathrm{t}+v}{2}$, *and* $\int_{\mathrm{t}}^{v}\mathfrak{D}(\omega)d\omega>0$, *then*

$$\mathfrak{S}\left(\frac{\mathrm{t}+v}{2}\right)\leq_p \frac{1}{\int_{\mathrm{t}}^{v}\mathfrak{D}(\omega)d\omega}\ (IR)\int_{\mathrm{t}}^{v}\mathfrak{S}(\omega)\mathfrak{D}(\omega)d\omega. \tag{16}$$

Proof. Since $\mathfrak{S}\in LRSX([\mathrm{t},\ v],\ \mathcal{K}_C^+)$, then we have

$$\begin{aligned}\mathfrak{S}_*\left(\tfrac{\mathrm{t}+v}{2}\right) &\leq \tfrac{1}{2}(\mathfrak{S}_*(\varsigma\mathrm{t}+(1-\varsigma)v)+\mathfrak{S}_*((1-\varsigma)\mathrm{t}+\varsigma v)),\\ \mathfrak{S}^*\left(\tfrac{\mathrm{t}+v}{2}\right) &\leq \tfrac{1}{2}(\mathfrak{S}^*(\varsigma\mathrm{t}+(1-\varsigma)v)+\mathfrak{S}^*((1-\varsigma)\mathrm{t}+\varsigma v)),\end{aligned} \tag{17}$$

By multiplying (17) by $\mathfrak{D}(\varsigma\mathrm{t}+(1-\varsigma)v)=\mathfrak{D}((1-\varsigma)\mathrm{t}+\varsigma v)$ and integrate it by ς over $[0,\ 1]$, we obtain

$$\begin{aligned}&\mathfrak{S}_*\left(\tfrac{\mathrm{t}+v}{2}\right)\int_0^1\mathfrak{D}((1-\varsigma)\mathrm{t}+\varsigma v)d\varsigma\\ &\qquad\leq\tfrac{1}{2}\left(\begin{array}{c}\int_0^1\mathfrak{S}_*(\varsigma\mathrm{t}+(1-\varsigma)v)\mathfrak{D}(\varsigma\mathrm{t}+(1-\varsigma)v)d\varsigma\\ +\int_0^1\mathfrak{S}_*((1-\varsigma)\mathrm{t}+\varsigma v)\mathfrak{D}((1-\varsigma)\mathrm{t}+\varsigma v)d\varsigma\end{array}\right),\\ &\mathfrak{S}^*\left(\tfrac{\mathrm{t}+v}{2}\right)\int_0^1\mathfrak{D}((1-\varsigma)\mathrm{t}+\varsigma v)d\varsigma\\ &\qquad\leq\tfrac{1}{2}\left(\begin{array}{c}\int_0^1\mathfrak{S}^*(\varsigma\mathrm{t}+(1-\varsigma)v)\mathfrak{D}(\varsigma\mathrm{t}+(1-\varsigma)v)d\varsigma\\ +\int_0^1\mathfrak{S}^*((1-\varsigma)\mathrm{t}+\varsigma v)\mathfrak{D}((1-\varsigma)\mathrm{t}+\varsigma v)d\varsigma\end{array}\right),\end{aligned} \tag{18}$$

Since

$$\begin{aligned}&\int_0^1\mathfrak{S}_*(\varsigma\mathrm{t}+(1-\varsigma)v)\mathfrak{D}(\varsigma\mathrm{t}+(1-\varsigma)v)d\varsigma\\ &\qquad=\int_0^1\mathfrak{S}_*((1-\varsigma)\mathrm{t}+\varsigma v)\mathfrak{D}((1-\varsigma)\mathrm{t}+\varsigma v)d\varsigma\\ &\qquad=\tfrac{1}{v-\mathrm{t}}\int_{\mathrm{t}}^{v}\mathfrak{S}_*(\omega)\mathfrak{D}(\omega)d\omega\\ &\int_0^1\mathfrak{S}^*((1-\varsigma)\mathrm{t}+\varsigma v)\mathfrak{D}((1-\varsigma)\mathrm{t}+\varsigma v)d\varsigma\\ &\qquad=\int_0^1\mathfrak{S}^*(\varsigma\mathrm{t}+(1-\varsigma)v)\mathfrak{D}(\varsigma\mathrm{t}+(1-\varsigma)v)d\varsigma\\ &\qquad=\tfrac{1}{v-\mathrm{t}}\int_{\mathrm{t}}^{v}\mathfrak{S}^*(\omega)\mathfrak{D}(\omega)d\omega\end{aligned} \tag{19}$$

From (19), we have

$$\begin{aligned}\mathfrak{S}_*\left(\tfrac{\mathrm{t}+v}{2}\right) &\leq \tfrac{1}{\int_{\mathrm{t}}^{v}\mathfrak{D}(\omega)d\omega}\int_{\mathrm{t}}^{v}\mathfrak{S}_*(\omega)\mathfrak{D}(\omega)d\omega,\\ \mathfrak{S}^*\left(\tfrac{\mathrm{t}+v}{2}\right) &\leq \tfrac{1}{\int_{\mathrm{t}}^{v}\mathfrak{D}(\omega)d\omega}\int_{\mathrm{t}}^{v}\mathfrak{S}^*(\omega)\mathfrak{D}(\omega)d\omega,\end{aligned}$$

From which, we have

$$\begin{aligned}&\left[\mathfrak{S}_*\left(\tfrac{\mathrm{t}+v}{2}\right),\ \mathfrak{S}^*\left(\tfrac{\mathrm{t}+v}{2}\right)\right]\\ &\qquad\leq_p\tfrac{1}{\int_{\mathrm{t}}^{v}\mathfrak{D}(\omega)d\omega}\left[\int_{\mathrm{t}}^{v}\mathfrak{S}_*(\omega)\mathfrak{D}(\omega)d\omega,\ \ \int_{\mathrm{t}}^{v}\mathfrak{S}^*(\omega)\mathfrak{D}(\omega)d\omega\right],\end{aligned}$$

that is

$$\mathfrak{S}\left(\frac{\mathrm{t}+v}{2}\right)\leq_p\frac{1}{\int_{\mathrm{t}}^{v}\mathfrak{D}(\omega)d\omega}\ (IR)\int_{\mathrm{t}}^{v}\mathfrak{S}(\omega)\mathfrak{D}(\omega)d\omega.$$

This completes the proof. □

Remark 5. *If $\mathfrak{D}(\omega) = 1$ then, combining Theorems 7 and 8, we obtain Theorem 3.*

If $\mathfrak{S}_*(\mathrm{t}) = \mathfrak{S}^*(\mathrm{t})$ then, Theorems 7 and 8 reduces to classical first and second HH-*Fejér inequality for convex function, see [21].*

If $\mathfrak{S}_*(\mathrm{t}) = \mathfrak{S}^*(\mathrm{t})$ with $\mathfrak{D}(\omega) = 1$ then, Theorems 7 and 8 reduces to classical first and second HH-*Fejér inequality for convex function, see [17,18].*

Example 6. *We consider the I.V-F $\mathfrak{S} : [\mathrm{t}, v] = \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \to \mathcal{K}_C^+$ defined by,*

$$\mathfrak{S}(\omega) = [\exp(\sin(\omega)), 2\exp(\sin(\omega))]$$

Since end point functions $\mathfrak{S}_*(\omega) = \exp(\sin(\omega))$, $\mathfrak{S}^*(\omega) = 2\exp(\sin(\omega))$ convex functions then, by Theorem 2, $\mathfrak{S}(\omega)$ is LR-convex I.V-F. If

$$\mathfrak{D}(\omega) = \begin{cases} \omega - \frac{\pi}{4}, & S \in \left[\frac{\pi}{4}, \frac{3\pi}{8}\right], \\ \frac{\pi}{2} - \omega, & S \in \left(\frac{3\pi}{8}, \frac{\pi}{2}\right]. \end{cases}$$

then, we have

$$\frac{1}{v-\mathrm{t}} \int_{\mathrm{t}}^{v} [\mathfrak{S}_*(\omega)]\mathfrak{D}(\omega)d\omega = \frac{4}{\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} [\mathfrak{S}_*(\omega)]\mathfrak{D}(\omega)d\omega = \frac{4}{\pi} \int_{\frac{\pi}{4}}^{\frac{3\pi}{8}} [\mathfrak{S}_*(\omega)]\mathfrak{D}(\omega)d\omega + \frac{4}{\pi} \int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \mathfrak{S}_*(\omega)\mathfrak{D}(\omega)d\omega,$$

$$\frac{1}{v-\mathrm{t}} \int_{\mathrm{t}}^{v} [\mathfrak{S}^*(\omega)]\mathfrak{D}(\omega)d\omega = \frac{4}{\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} [\mathfrak{S}^*(\omega)]\mathfrak{D}(\omega)d\omega = \frac{4}{\pi} \int_{\frac{\pi}{4}}^{\frac{3\pi}{8}} [\mathfrak{S}^*(\omega)]\mathfrak{D}(\omega)d\omega + \frac{4}{\pi} \int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \mathfrak{S}^*(\omega)\mathfrak{D}(\omega)d\omega,$$

$$\begin{aligned} &= \frac{4}{\pi} \int_{\frac{\pi}{4}}^{\frac{3\pi}{8}} [\exp(\sin(\omega))]\left(\omega - \frac{\pi}{4}\right)d\omega + \frac{4}{\pi} \int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \exp(\sin(\omega))\left(\frac{\pi}{2} - \omega\right)d\omega \approx \frac{63}{100\pi}, \\ &= \frac{8}{\pi} \int_{\frac{\pi}{4}}^{\frac{3\pi}{8}} \exp(\sin(\omega))\left(\omega - \frac{\pi}{4}\right)d\omega + \frac{8}{\pi} \int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \exp(\sin(\omega))\left(\frac{\pi}{2} - \omega\right)d\omega \approx \frac{63}{50\pi}, \end{aligned} \tag{20}$$

and

$$\begin{aligned} &[\mathfrak{S}_*(\mathrm{t}) + \mathfrak{S}_*(v)] \int_0^1 \varsigma D(\mathrm{t} + \varsigma\partial(v, \mathrm{t}))\ d\varsigma \\ &[\mathfrak{S}^*(\mathrm{t}) + \mathfrak{S}^*(v)] \int_0^1 \varsigma D(\mathrm{t} + \varsigma\partial(v, \mathrm{t}))d\varsigma \\ &= \frac{\pi}{2}\left[\int_0^{\frac{1}{2}} \varsigma^2 d\varsigma + \int_{\frac{1}{2}}^1 \varsigma(1+\varsigma)d\varsigma\right] = \frac{17\pi}{48}, \\ &= \pi\left[\int_0^{\frac{1}{2}} \varsigma^2 d\varsigma + \int_{\frac{1}{2}}^1 \varsigma(1+\varsigma)d\varsigma\right] = \frac{17\pi}{24}. \end{aligned} \tag{21}$$

From (20) and (21), we have

$$\left[\frac{63}{100\pi}, \frac{63}{50\pi}\right] \quad \leq_p \left[\frac{17\pi}{48}, \frac{17\pi}{24}\right].$$

Hence, Theorem 7 is verified.

For Theorem 8, we have

$$\begin{aligned} \mathfrak{S}_*\left(\frac{\mathrm{t}+v}{2}\right) &= \mathfrak{S}_*\left(\frac{3\pi}{8}\right) \approx 1\,, \\ \mathfrak{S}^*\left(\frac{\mathrm{t}+v}{2}\right) &= \mathfrak{S}^*\left(\frac{3\pi}{8}\right) \approx 2\,, \end{aligned} \tag{22}$$

$$\int_{\mathrm{t}}^{v} \mathfrak{D}(\omega)d\omega = \int_{\frac{\pi}{4}}^{\frac{3\pi}{8}} \left(\omega - \frac{\pi}{4}\right)d\omega + \int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \left(\frac{\pi}{2} - \omega\right)d\omega \approx \frac{4}{25},$$

$$\begin{aligned} \frac{1}{\int_{\mathrm{t}}^{v} \mathfrak{D}(\omega)d\omega} \int_{\mathrm{t}}^{v} \mathfrak{S}_*(\omega)\mathfrak{D}(\omega)d\omega &\approx 1.1 \\ \frac{1}{\int_{\mathrm{t}}^{v} \mathfrak{D}(\omega)d\omega} \int_{\mathrm{t}}^{v} \mathfrak{S}^*(\omega)\mathfrak{D}(\omega)d\omega &\approx 2.1. \end{aligned} \tag{23}$$

From (22) and (23), we have

$$[1,\ 2] \quad \leq \quad {}_p[1.1,\ 2.1].$$

Hence, Theorem 8 is verified.

4. Results and Discussion

For LR-convex I.V-Fs, we find Hermite–Hadamard type inequalities. Our findings not only improve on Zhao's work, but they also investigate some of the findings of Sarikaya et al. We have not looked into inequalities using interval derivatives since there are not any "interval derivatives" with desirable characteristics.

5. Conclusions

In this paper, *HH*-inequalities have been investigated for the concept of LR-convex I.V-Fs. The most important thing in this study is that we have proved that both concepts LR-convex I.V-F and convex I.V-Fs coincide under some mild conditions when these conditions are defined on the endpoint functions. As for future research, we try to explore this concept for generalized LR-convex I.V-Fs and some applications in interval nonlinear programing. This is an open problem for the readers and anyone can investigate this concept, "the optimality conditions of LR-convex I.V-Fs can be obtained through variational inequalities". We hope that this concept will be helpful for other authors to play their roles in different fields of sciences. Moreover, in future, we will also start exploring this concept and their generalizations by using different fractional integral operators.

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