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Cauchy Processes, Dissipative Benjamin–Ono Dynamics and Fat-Tail Decaying Solitons

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Abstract: In this paper, a dissipative version of the Benjamin–Ono dynamics is shown to faithfully model the collective evolution of swarms of scalar Cauchy stochastic agents obeying a *follow-the-leader* interaction rule. Due to the Hilbert transform, the swarm dynamic is described by nonlinear and non-local dynamics that can be solved exactly. From the mutual interactions emerges a fat-tail soliton that can be obtained in a closed analytic form. The soliton median evolves nonlinearly with time. This behaviour can be clearly understood from the interaction of mutual agents.

Keywords: Benjamin–Ono dynamics; master equations; jump Markov processes; interacting Cauchy processes; swarm dynamics; fat-tail soliton waves

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1. Introduction

The dynamic of a Markov process $X_t \in \mathbb{R}$ is fully characterised by its transition probability density (TPD) $P(z,t|y,0)$, solving a *differential Chapman–Kolmogorov* (DCK) [1]:

$$\begin{aligned}
&\partial_t P(x,t|y,0) = -\partial_x\{\mathcal{A}(x,t)P(x,t|y,0)\} + \mathcal{B}\partial_{xx}P(x,t|y,0)+ \\
&\qquad\qquad \mathcal{D}\int_{\mathbb{R}}[W(x|z,t)P(z,t|y,0) - W(x|z,t)P(x,t|y,0)]dz, \\
&P(x,t|y,0) \geq 0 \qquad \text{and} \qquad \int_{\mathbb{R}} P(x,t|y,0)dz = 1, \\
&P(x,0|y,0) = \delta(x-y),
\end{aligned} \tag{1}$$

where $t \in \mathbb{R}^+$ and $\delta(x-y)$ is the Dirac probability mass at $x = y$. In Equation (1), the first two terms describe the diffusive component of the stochastic motion (i.e., continuous, yet not differentiable, random trajectories). The integral term takes into account the presence of random jumps. At time t, the kernel $W(x|z,t)$ characterises the distribution of jumps from position z to position x. In close relation with Equation (1), this paper will focus on nonlinear integro-differential PDEs :

$$\begin{aligned}
&\partial_t u(x,t) = -\partial_x\{\mathcal{F}[u(x,t),x,t]u(x,t)\} + \mathcal{B}\partial_{xx}u(x,t)+ \\
&\qquad\qquad \mathcal{D}\int_{\mathbb{R}}[W(x|z,t)u(z,t) - W(z|x,t)u(x,t)]dz, \\
&u(x,t) \geq 0 \qquad \text{and} \qquad \int_{\mathbb{R}} P(x,t)dz = 1, \\
&u(x,0) = u_0(x).
\end{aligned} \tag{2}$$

Equations (1) and (2) describe the evolution of probability densities $P(x,t|y,0)$ and $u(x,t)$, respectively. However, as discussed by T. Frank [2], the nonlinear Equation (2) does not allow the interpretation of $u(x,t)$ as the TPD of an underlying Markov process. The details of the inter-connection between Equations (1) and (2) will be reviewed in Section 3. In Equation (2), the functional $\mathcal{F}[u(x,t),x,t]$ introduces a feedback-type $u(x,t)$ dependence of the drift, and this clearly generates a nonlinear evolution.

For the purely diffusive regimes that appear when $\mathcal{D}=0$, Equation (2) yields well-known specific illustrations of Equation (2). As one of the simplest examples of these, let us mention the Burgers–Kortweg de Vries dynamics (BKDV) (omitting the (x,t) arguments):

$$\begin{cases} \partial_t u = -\partial_x\{\mathcal{F}[u]u\} + \mathcal{B}\partial_{xx}u & u := u(x,t), \\ \mathcal{F}[u] := \mathcal{A}u - \mathcal{C}\frac{\partial_{xx}u}{u}, & u > 0. \end{cases} \tag{3}$$

Equation (3) is known to possess soliton wave solutions. Specifically, for $\mathcal{A}\neq 0$ and $\mathcal{B}=\mathcal{C}=0$, we have the Euler dynamics, which are solved by δ-type solitons [3]. When $\mathcal{A}\neq 0$ and $\mathcal{B}\neq 0$ but $\mathcal{C}=0$, the standard Burgers' dynamics follow, which can be solved by linearisation [4]. With $\mathcal{A}\neq 0$, $\mathcal{B}=0$, and $\mathcal{C}\neq 0$, Equation (3) is the Kortweg de Vries (KDV) and admits symmetric positive definite solitons [4]. Finally, for non-vanishing $\mathcal{A},\mathcal{B},\mathcal{C}$, the BKDV is solved by positive asymmetric solitons [5–8]. All BKDV soliton-type densities exhibit rapidly decreasing tails and arbitrarily high moments. For the case where $\mathcal{D}>0$, the resulting dynamics have been far less explored, and the goal of this paper is to focus on this aspect. Thus, instead of Equation (3), in the following, we will consider the class of dynamics:

$$\partial_t u = -\partial_x\{\mathcal{F}[u]u\} + \mathcal{D}\int_{\mathbb{R}}[W(x|z,t)u(x,t) - W(z|x,t)u(x,t)]dz, \tag{4}$$

In this, the diffusion component of Equation (2) has been removed to focus on pure jump dynamics. Several types of jump kernels $W(z|x,t)$ can be implemented. For example, nonlinear dynamics driven by compound Poisson processes were recently investigated in [9–11]. In the following, we will select Cauchy processes characterised by the jump kernel:

$$W(z|x,t) = W(z|x) = \frac{1}{\pi(z-x)^2}. \tag{5}$$

Using Equation (5), one can rewrite Equation (4) using one of the alternative representations (Appendix A gives details regarding the following notations):

- *Hilbert transform:*

$$\partial_t u = -\partial_x\{\mathcal{F}[u]u\} - \mathcal{D}\partial_x\mathcal{H}[u], \tag{6}$$

  $\mathcal{H}[\cdot]$ standing for the Hilbert transform:

$$\mathcal{H}[f(x)] := \frac{1}{\pi}\,\mathrm{pv}\int_{\mathbb{R}}\frac{f(\xi)}{x-\xi}d\xi, \tag{7}$$

  In this, pv $\int[\cdot]$ means the Cauchy principal part of the integral. For $u>0$, we further focus the discussion on the specific nonlinear functional $\mathcal{F}[u] := \mathcal{A}u - \mathcal{C}\frac{\mathcal{H}\{\partial_x u\}}{u}$, with the constant parameters $\mathcal{A}$ and $\mathcal{C}$. With this, Equation (6) reads:

$$\partial_t u = -\mathcal{A}\partial_x\left[u^2\right] + \mathcal{C}\mathcal{H}[\partial_{xx}u] - \mathcal{D}\partial_x\mathcal{H}[u]. \tag{8}$$

- *Riesz fractional derivative:*
  Here, instead of Equation (8), we can write:

$$\partial_t u(x,t) = -\partial_x\{\mathcal{F}[u]u\} + \sqrt{-\Delta}[u(x,t)] \tag{9}$$

  where the formal root of the Laplacian operator $\Delta[\cdot]$ is a Riesz *fractional derivative* [12].

From now on, we shall only use the Hilbert transform representation. The nonlinear PDE Equation (8) is known as the Landau dissipative Benjamin–Ono (BO) dynamic; it was recently introduced in [13]. With $\mathcal{D} = 0$, Equation (8) is the standard (i.e., conservative) BO dynamic, which has itself been extensively discussed (without attempting to be exhaustive, see, for example, [14–16]).

At first sight, Equations (3) and (8) appear to be drastically different. However, they share a couple of fundamental characteristics: (i) they can be rewritten in the form of continuity equations, namely, $\partial_t u + \partial_x \mathcal{J}_{1,(2)}[u] = 0$, with $\mathcal{J}_{1,(2)}[u]$ being transport currents, and (ii) their drift dynamics:

$$\begin{cases} \partial_t u = -\partial_x \mathcal{J}_1[u] = -\mathcal{A}\partial_x\{u^2\} + \mathcal{C}\partial_{xxx}u, \\ \\ \partial_t u = -\partial_x \mathcal{J}_2[u] = -\mathcal{A}\partial_x\{u^2\} + \mathcal{C}\mathcal{H}\partial_{xx}[u], \end{cases} \tag{10}$$

are both fully integrable nonlinear Hamiltonian field dynamics [14,17] .

Besides the fluid domain, where Equations (3) and (8) play a central role, these also enter into the realm of mean-field (MF) multi-agent (*swarms*) dynamics. This is the aspect that will be discussed here. For Equation (3), the diffusive dynamics are used to model Brownian agents. These enable us to stylise the emergence of *macroscopic* spatio-temporal patterns for *microscopic* agents driven by sources of Gaussian white noise [2,18,19]. In this context, the nonlinearity $\mathcal{A}u\partial_x u$ describes mutual interactions via a *follow-the-leader* algorithm. Along similar lines, our paper explores the swarm dynamics of agents driven by Cauchy processes. With such jump noise sources and a *follow-the-leader* type algorithm, we will show the emergence of fat-tail solitons at the *macroscopic* level.

This paper is organised as follows: In Section 2, we calculate a single decaying soliton solution solving the dissipative BO Equation (8). In Section 3, we expose the multi-agent modelling for agents driven by Cauchy jump processes. Then, we implement *follow-the-leader* mutual interactions and use the MF procedure. Then, we show that the resulting Cauchy swarm behaviour can be faithfully modelled by the dissipative BO dynamics discussed in Section 2. Finally, in Section 4, inspired by Brownian agents we conclude by listing a few open questions that remain to be discussed for Cauchy swarms. Everything in this program can be worked out exactly, and specific technical details are listed in a couple of appendices.

2. Non-Conservative Benjamin–Ono (BO) Dynamics and Fat-Tail Solitons

Consider the dissipative BO dynamics [13] :

$$\begin{cases} \partial_t u(x,t) = -\mathcal{A}\partial_x\{u(x,t)^2\} - \mathcal{C}(t)\partial_{xx}\{\mathcal{H}[u(x,t)]\} - \mathcal{D}(t)\partial_x\mathcal{H}[(u(x,t)], \\ \\ u(x,0) = u_0(x), \qquad \text{and} \qquad u(|\infty|,t) = 0, \end{cases} \tag{11}$$

where $\mathcal{D}(t)$ is a smooth function of time $t \in \mathbb{R}^+$. In [13], several dissipation mechanisms are studied and this specific dissipation is called the Landau dissipation. Equation (11) is a non-conservative extension of the BO equation (dissipation occurs when $\mathcal{D}(t) > 0$). The standard BO equation is obtained when $\mathcal{D} \equiv 0$ and $\mathcal{C}(t) = \text{const}$ [14–16].

Proposition 1 (Cauchy-type soliton)**.** *For constant $\mathcal{C} = \frac{\mathcal{A}}{2\pi}$, the nonlinear PDE Equation (11) is solved by a fat-tail decaying soliton:*

$$u(x,t) = \frac{\varphi(t)}{\pi\Big[\varphi^2(t) + [x+\phi(t)]^2\Big]}, \qquad t \in \mathbb{R}^+, \tag{12}$$

where $\varphi(t)$ and $\phi(t)$ solve the set of coupled odes:

$$\begin{cases} \dot{\varphi}(t) = \mathcal{D}(t), \\ \dot{\phi} = -\frac{3\mathcal{A}}{2\pi\varphi(t)}. \end{cases} \tag{13}$$

Proof. By direct verification using the list of identities in Appendix B. □

For $\mathcal{D}(t) > 0$, Equation (13) forces $\varphi(t)$ to strictly increase, and $u(x,t)$ is therefore a time-evanescent soliton that travels with a non-constant velocity $\phi(t)$. The case $\mathcal{D} =$ constant is also discussed in [13]. We emphasise that the soliton amplitude depends exclusively on the parameter $\mathcal{D}(t)$. However, both $\mathcal{D}(t)$ and $\mathcal{C} = \frac{\mathcal{A}}{(2\pi)}$ determine the soliton velocity $\phi(t)$. For $\mathcal{D} = 0 \Rightarrow \mathcal{D}(t) = \varphi_0$, $u(x,t)$ is a constant φ_0-amplitude soliton travelling with a ϕ_0-constant velocity where $\phi_0 = -3t[\varphi_0]^{-1}$ showing that the amplitude and velocity of the soliton are interdependent.

Remark 1. *The solution of Equation (13) involves two integration constants: φ_0 and ϕ_0. These can be fixed by the initial profile $u_0(x)$, which itself reads as*

$$u_0(x) = \frac{\varphi_0}{\pi[\varphi_0^2 + (x+\phi_0)^2]}. \tag{14}$$

Mixed-Canonic Dissipative Dynamics

From Appendix B, it can be verified that for the u given by Equation (12), we have $\partial_x u + 2\pi(u\,\mathcal{H}[u]) = 0$. This immediately implies:

Corollary 1. *The generalised dissipative BO equation is as follows:*

$$\begin{cases} \partial_t u(x,t) = -\mathcal{A}\partial_x\{u(x,t)^2, x, t\} - \frac{\mathcal{A}}{2\pi}\partial_{xx}\{\mathcal{H}[u(x,t)]\} - \mathcal{D}(u,x,t)\partial_x\mathcal{H}[(u(x,t)], \\ \mathcal{D}(u,x,t) := \mathcal{D}(t) + [\partial_x u(x,t) + 2\pi(u(x,t)\,\mathcal{H}[u(x,t)])], \end{cases} \tag{15}$$

is solved by $u(x,t)$ given by Equations (12) and (13).

It is important to point out that Equation (15) cannot be written as a continuity equation. Hence, Equation (15) fundamentally differs from Equation (11). Along the same lines of Corollary 1, we can now construct soliton amplitude selection mechanisms. For this, recall that for $\mathcal{D}(t) = 0$ with $\mathcal{C}$ as a constant, the BO evolution Equation (11), is a fully integrable Hamiltonian field dynamic [17]. The Hamiltonian $\mathbf{H}$ and an infinite set of constants of motion $\{\mathbf{I}_1, \mathbf{I}_2, \cdots\}$ are explicitly known [15,17]. In particular, we know that

$$\mathbf{H}[u(x,t)] = \mathcal{A}\int_{\mathbb{R}}\left\{\frac{1}{3}u^3(x,t) + \frac{1}{\pi}u(x,t)\mathcal{H}\partial_x u(x,t)\right\}dx. \tag{16}$$

is a constant of the motion. For $u_0(x)$, as in Equation (14) and with $\mathcal{D} = 0$, Proposition 1 implies that $u(x,t) = u_0(x - \frac{3\mathcal{A}}{2\pi\varphi_0}t)$ and $\mathcal{E}(\varphi_0) := \mathbf{H}[u_0(x - \frac{3\mathcal{A}}{2\pi\varphi_0}t)]$ is time-independent. Now, consider the non-conservative dynamics:

$$\begin{cases} \partial_t u(x,t) = -\mathcal{A}\partial_x\{u(x,t)^2, x, t\} - \frac{\mathcal{A}}{2\pi}\partial_{xx}\{\mathcal{H}[u(x,t)]\} - \mathcal{D}_{\mathbf{H}}(u,x,t)\{\mathcal{H}[u(x,t)]\} \\ \mathcal{D}_{\mathbf{H}}(u,x,t) := -\kappa\{\mathbf{H}[u(x,t)] - \mathcal{E}(\varphi_0)\}\partial_x\mathcal{H}[(u(x,t)], \end{cases} \tag{17}$$

where $\kappa \in \mathbb{R}^+$ is a constant. As long as $\mathcal{D}_{\mathbf{H}}(u,x,t) > 0$, Equation (17) describes a dissipative dynamic that persists as long as the system's energy exceeds the level $\mathcal{E}(\varphi_0)$. Conversely, when the energy is less than $\mathcal{E}(\varphi_0)$, we have $\mathcal{D} < 0$, implying that energy is fed into the system. Once the system's $\mathcal{E}(\varphi_0)$ is reached, the evolution is Hamiltonian, and we have $u(x,t) = u_0(x - \frac{3\mathcal{A}}{2\pi\varphi_0}t)$. Accordingly, $\mathcal{D}_{\mathbf{H}}(u,x,t)$ creates an attractor that drives the dynamics towards a conservative orbit with energy $\mathcal{E}(\varphi_0)$. Along the same lines, more general regulators based on the other constants of the motion $\{\mathbf{I}_1, \mathbf{I}_2, \cdots\}$ can be similarly constructed. Note that for finite dimensional systems, a similar type of regulator was implemented for diffusive dynamics in [20].

3. Interactive Multi-Agent Dynamics with Fat-Tail Solitons

Consider a swarm of N indistinguishable agents $\{a_j\}_{j=1,2,\cdots,N}$ with individual dynamics described by N independent scalar Cauchy process $\mathbf{X}(t) := (X_1(t), X_2(t), \cdots, X_N(t))$ solutions of stochastic differential equations:

$$dX_k(t) = \mathcal{D}(t)dC_k(t), \qquad k = 1,2,\cdots,N, \tag{18}$$

where $C_k(t)$ stands for the Cauchy processes, and $\mathcal{D}(t) \in \mathbb{R}^+$ is an amplitude modulation. At time $t = 0$, we assume the initial positions $\{X_j(0)\}$ to be drawn at random from a probability law $u_0(x)$. Let us define the empirical distribution $\tilde{u}(\vec{X}(t), x, t)$:

$$\tilde{u}(\vec{X}(t), x, t) := \frac{1}{N}\sum_{j=1}^{N} \delta[x - X_j(t)], \qquad \tilde{u}(\vec{X}(0), x, 0) = u_0(x), \tag{19}$$

where $\vec{X}(t) := (X_1(t), X_2(t), \cdots, X_N(t))$. In Equation (18), let us now include mutual agents interactions by rewriting the dynamics as

$$dX_k(t) = \mathcal{J}[X_k(t), \tilde{u}(\vec{X}(t), x, t)]dt + \mathcal{D}(t)dC_k(t), \qquad k = 1,2,\cdots,N. \tag{20}$$

The interaction kernel $\mathcal{J}[X_k(t), \tilde{u}(\vec{X}(t), x, t)]$ indicates, via $\tilde{u}(\vec{X}(t), x, t)$ dependence, that an arbitrary agent a_k interacts with all other fellows within the swarm. For a large population N, we can adopt a mean-field (MF) description of the dynamics. In a nutshell, the MF approach consists of assuming that (i) the empirical distribution $\tilde{u}(x,t)$ can be approximated by a smooth function $u(x,t)$ and that (ii) the swarm behaviour can be characterised by single representative agents interacting with their fellows via an external effective field (i.e., *the mean field*). Following the procedure exposed for Brownian agents in [2], the MF approach adapted to the Cauchy dynamics in Equation (20) can be summarised as

$$\begin{cases} \partial_t P(x,t|y,t_0) = -\partial_x\{u(x,t)P(x,t|y,t_0)\} - \mathcal{D}(t)\mathcal{H}[\partial_x P(x,t|y,t_0) \\ \partial_t u(x,t) = -\partial_x\{\mathcal{J}[u(x,t),x,t]u\} - \mathcal{D}(t)\mathcal{H}[\partial_x u(x,t)] \\ P(x,t|y,t_0) \geq 0 \quad \text{and} \quad u(x,t) \geq 0, \\ P(x,t_0|y,t_0) = \delta(x-y), \\ \int_{\mathbb{R}} P(x,t|y,t_0)dx = \int_{\mathbb{R}} u(x,t)dx = 1, \end{cases} \tag{21}$$

In Equation (21), we have the following:

- $P(x,t|y,t_0)$ is the TPD of the MF representative agent. This agent evolves as a Markov process with Cauchy jumps and effective MF drift $u(x,t)$. This drift encapsulates the effective influence of the whole swarm on a randomly chosen representative agent (it is necessary to remember that the agents are indistinguishable). As is required for master-type equations, the evolution of $P(x,t|y,t_0)$ is linear.
- The swarm collective behaviour is itself described by $u(x,t)$. The functional $\mathcal{J}[u(x,t),x,t]$ encapsulates the effective result of mutual interactions. Here, we point out that $u(x,t)$ obeys a nonlinear PDE. Note that $u(x,t)$ is a density but itself is not the TPD of a Markov process.

- $P(x,t|y,t_0)$ and $u(x,t)$ are connected via a self-consistent constraint:

$$\begin{cases} u(x,t) = \int_{\mathbb{R}} P(x,t|y,t_0)u_0(y)dy, \\ u_0(x) = \delta(x) \quad \Rightarrow \quad P(x,t|0,0) = u(x,t). \end{cases} \tag{22}$$

There are very few multi-agents models for which Equations (21) and (22) can be analytically solved. An example of where this procedure is feasible is the Shimizu–Yamada model, which describes Brownian agents. In this case, the solution is based on the Gaussian properties of the Ornstein–Uhlenbeck process [2]. In Section 3.1 below, we show that this MF procedure can also be analytically solved for Cauchy jump processes.

3.1. Follow-the-Leader Interactions

Inspired by [19], let us now introduce the *"follow-the-leader"* (FLA) interaction kernel and discuss the corresponding Equation (20). The FLA algorithm assumes that agents permanently observe the relative positions of $\mathbb{R}$ of their fellows. Based on these real-time observations, each agent dynamically updates their drift. The update depends on the number of leaders $\hat{L}(\lambda; x,t)$ detected within a range λ. For agent a_k, this is achieved by an empirical counting operation:

$$\hat{L}_{k,\lambda}(x,t) = \frac{1}{N}\sum_{j\neq k} \delta(x - X_j(t))\mathbb{I}[0 \leq (X_j(t) - X_k(t)) \leq \lambda], \tag{23}$$

where $\mathbb{I}[\cdot]$ is the indicator function. For large swarms, (i.e., $N \to \infty$), one uses the density function $u(x,t)$ to rewrite Equation (23) as:

$$\lim_{N\to\infty} \hat{L}_{k,\lambda}(x,t) \simeq \int_x^{x+\lambda} u(\xi,t)d\xi := L_{k,\lambda}(u,x,t) \in [0,1]. \tag{24}$$

With Equation (24), the FLA algorithm can now be defined by the rules:

$\mathbf{A_1}$: each agent implements an extra drift $\Gamma L_\lambda(u,x,t)$ (for the MF description, we drop the index k of the representative agent), thus describing a *follow-the-leader* tendency. This introduces nonlinearity into the dynamic.
$\mathbf{A_2}$: in the absence of jumps—i.e., when $\mathcal{D}(t) = 0$—we impose the relative ranking of agents to be unchanged. To this end, we introduce an extra operator $\mathcal{R}[u]$, for which an explicit form will be given shortly.

In terms of the rules $\mathbf{A_1}$ and $\mathbf{A_2}$, Equation (20) becomes:

$$\begin{cases} \partial_t X_k(t) = \{\mathcal{J}[L_{k,\lambda}(u,x,t)]u\}dt - \mathcal{D}(t)dC_k(t), \\ \mathcal{J}[L_{k,\lambda}(x,t)] = \Gamma L_{k,\lambda}(u,x,t) + \frac{\mathcal{R}[u]}{[u]}, \qquad u > 0. \end{cases} \tag{25}$$

Since agents are assumed to be indistinguishable, we drop index k. Hence, in the MF limit with Equation (25), the swarm dynamics $u(x,t)$ given by Equation (21) take the form:

$$\partial_t u = \{\Gamma\partial_x[uL_\lambda(u,x,t)] + \mathcal{R}[u]\} - \mathcal{D}(t)\mathcal{H}[\partial_x u], \tag{26}$$

For an arbitrary observation range λ and with the presence of the Hilbert transform, Equation (26) is a non-local and nonlinear PDE for which analytic solutions are generally not known. To progress further, let us focus on limiting situations:

$$\begin{aligned} &\text{short observation range}: && L_\lambda(u,x,t) \simeq \lambda u(x,t) + \mathcal{O}(\lambda^2), \\ &\text{infinite observation range}: && \lim_{\lambda\to\infty} L_\lambda(u,x,t) = \int_x^\infty u(\xi,t)d\xi := U(x,t). \end{aligned} \tag{27}$$

Since u is normalised to unity and, by definition, we have $u = -\partial_x U$, Equation (26) becomes:

$$\text{short range}: \quad \begin{cases} \partial_t u(x,t) = -\Gamma\lambda\partial_x\left[u^2\right] + \mathcal{R}[u] - \mathcal{D}(t)\mathcal{H}[\partial_x u], \\ u(-\infty,t) = u(+\infty) = 0. \end{cases} \tag{28}$$

$$\text{infinite range}: \quad \begin{cases} \partial_t U(x,t) = -\Gamma\partial_x\left[U^2\right] + \mathcal{R}[U] - \mathcal{D}(t)\mathcal{H}[\partial_x U], \\ U(-\infty,t) = 1, \quad \text{and} \quad U(+\infty,t) = 0. \end{cases} \tag{29}$$

For the short-range regime Equation (28), we have

Proposition 2. *In the short range regime Equation (28) and with $\mathcal{D}(t) = \mathcal{D}$, the specific choice $\mathcal{R}[u] := -\frac{\Gamma\lambda}{2\pi}\mathcal{H}\partial_x[u]$ is sufficient to ensure that the rules $\mathbf{A}_1$ and $\mathbf{A}_2$ hold and that the swarm density $u(x,t)$ reads:*

$$\begin{cases} u(x,t) = \frac{\varphi(t)}{\pi\left[\varphi^2(t)+[x+\phi(t)]^2\right]}, \qquad t \in \mathbb{R}^+, \\ \\ \varphi(t) = \mathcal{D}t + \varphi_0 \quad \text{and} \quad \phi(t) = -\frac{3\Gamma\lambda}{2\pi\mathcal{D}}\ln\left[\frac{t+\varphi_0}{\varphi_0}\right]. \end{cases} \tag{30}$$

Proof. (i) By construction, the rule $\mathbf{A}_1$ implies the drift nonlinearity in Equation (28). The choice $\mathcal{R}[u] = -\frac{\Gamma\lambda}{2\pi}\mathcal{H}\partial_x[u]$ ensures that Proposition 1 holds. Hence, the resulting swarm's evolution is given by Equation (30). Note that for $\mathcal{D} = 0$, Equation (13) directly implies:

$$\begin{cases} u(x,t) = \frac{\varphi_0}{\pi\left[\varphi_0^2+[x+\phi(t)]^2\right]}, \qquad t \in \mathbb{R}^+, \\ \\ \phi(t) = -\frac{\Gamma\lambda}{2\pi\varphi_0}t. \end{cases} \tag{31}$$

Equation (31) shows that $u(x,t) = u(x - \frac{\Gamma\lambda}{2\pi\varphi_0}t)$. Such pure translation behaviour excludes a shock formation during the evolution. Hence, the operator $\mathcal{R}[u]$ is sufficient to guarantee that $\mathbf{A}_2$ is fulfilled. Indeed, in the absence of jumps, the pure translation behaviour indicates that agents do not overtake; thus, $\mathbf{A}_2$ is satisfied. □

From Equation (31), we have

$$\lim_{t\to 0} u(x,t) = \frac{\varphi_0}{\pi\left[\varphi_0^2 + x^2\right]} \qquad \text{so that} \qquad \lim_{\varphi_0 \to 0} \frac{\varphi_0}{\pi\left[\varphi_0^2 + x^2\right]} = \delta(x) \tag{32}$$

where $u(x,0) = P(x,t|0,0) = \delta(x)$, as imposed by Equation (22). Note that with $u(x,t)$ given by Equation (31), we are unable to explicitly solve the linear Equation (21) for $P(x,t|y,0)$ for arbitrary y (only $P(x,t|0,0) = u(x,t)$ is explicitly known). Proposition 2 therefore shows that the dissipative BO dynamic faithfully describes the dynamics of a swarm of Cauchy agents with mutual interactions.

Equation (30) explicitly shows how from microscopic interactions a fat-tail decaying soliton emerges. The median of the soliton evolves logarithmically over time. This nonlinear behaviour with time can be heuristically understood directly from rule $\mathbf{A}_1$. The normalised decaying soliton density Equation (31) generates a dispersion enhancement. Hence, for a fixed λ, the short-range regime implies a decreasing number of observed leaders. As a consequence, the amplitude of the nonlinear drift component decreases, which produces the logarithmic behaviour.

Let us emphasise that it is truly remarkable that the nonlinearity together with the nonlocal character (Hilbert transform) of the swarm dynamics still allow exact results to be obtained in the short-range interaction regime. Hence, the dissipative BO enriches the very short list of nonlinear dynamics of the Equation (21) for which explicit solutions can be expressed. Despite the fact that Equations (29) and (28) differ only by their boundary conditions, solving Equation (29) remains an open challenge.

4. Conclusions and Perspectives

Besides the fundamental relevance of the BO in fluid dynamics, it is remarkable that adding an extra dissipative term enables us to describe the collective motion of swarms of Cauchy processes. This statistical mechanics point of view offers a heuristic interpretation of the role played by Hilbert transform in the BO dynamics. This contributes to building a complementary understanding of the physical meaning of BO. Inspired by the rich corpus of results already derived for Brownian agents, the new interpretation of the dissipative Benjamin–Ono dynamics leaves several open questions that still need to be addressed:

- **Long-range agent interactions**. For short-range observations (λ infinitesimal), a decaying soliton Equation (12) emerges. Short-range interactions are not sufficient to sustain a constant-amplitude soliton. For long-range interactions $\lambda \to \infty$, the solution $U(x,t)$ in Equation (29) is yet unknown. One might infer whether the $\lambda \to \infty$ regime sustains a steady-amplitude soliton, which does actually occur for Brownian agents [19].
- **Optimal control and mean-field games**. Does a utility function $\mathcal{L}$ for which the dissipative BO dynamics can be seen as the Hamilton–Jacobi–Bellman of the resulting optimal control problem exist? This happens to be the case for Brownian and two-state Markov chains (i.e., random telegraph) agents [21].
- **Multi-solitons and agent clustering**. The standard BO (i.e., conservative) dynamics are well known to possess multi-soliton solutions. Similar decaying multi-solitons also exist for the dissipative BO. Do such multi-soliton evolutions describe non-overlapping clusters of Cauchy agents? Such a possibility does not exist for Brownian clusters [22].

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Appendix A. Master Equation and Hilbert Transformation

Let us recall some basic properties of the Cauchy process $P_c(x,t)$. This is a pure jump Markov process that is fully characterised by the master equation:

$$\begin{aligned} &\partial_t P_c(x,t|y,0) = \text{pv} \int_{\mathbb{R}} \left\{ \frac{P_c(z,t|y,0)}{\pi(x-z)^2} - \frac{P_c(x,t|y,0)}{\pi(z-x)^2} \right\} dz, \\ &P_c(x,0|y,0) = \delta(x-y), \end{aligned} \tag{A1}$$

with pv $\int_{\mathbb{R}}$ standing for the Cauchy principal value integral. Equation (A1) is solved by the Cauchy probability density:

$$P_c(z,t|y,0) = \frac{t}{\pi[t^2 + (z-y)^2]} \quad \Rightarrow \quad \lim_{t\to 0} \frac{t}{\pi[t^2 + (z-y)^2]} = \delta(x-y). \tag{A2}$$

Lemma A1. *The master equation Equation (A1) can alternatively be written as:*

$$\partial_t P_c(x,t|y,0) = -\partial_x \mathcal{H}[P_c(x,t|y,0)], \tag{A3}$$

where $\mathcal{H}[\cdot]$ stands for the Hilbert transform Equation (7).

Proof of Lemma A1. The right-hand-side of the master Equation (A1) is the difference between two integrals:

$$\begin{cases} I_{\text{in}}(x,t) := \int_{\mathbb{R}} \frac{P(z,t|y,0)}{\pi(x-z)^2} d\xi = \int_{\mathbb{R}} \frac{P(\eta,t|0,0)}{\pi(z-\eta)^2} d\eta, & (\eta := \xi - y), \\ I_{\text{out}}(x,t) := -P(x,t|y,0) \int_{\mathbb{R}} \frac{1}{\pi(\xi-z)^2} d\xi. \end{cases}$$

We have

$$I_{\text{in}}(z,t) := -\partial_z \left\{ \frac{1}{\pi} \int_{\mathbb{R}} \frac{P(\eta,t|0,0)}{(z-\eta)} d\eta \right\} = -\partial_z \mathcal{H}\{P(z,t \mid 0,0)\}$$

and

$$\mathcal{H}\{P(z,t \mid 0,0)\} := \frac{1}{\pi} \int_{\mathbb{R}} \frac{P(\eta,t|0,0)}{(z-\eta)} d\eta = \frac{1}{\pi^2} \int_{\mathbb{R}} \left[\frac{t}{(t^2+\eta^2)(z-\eta)} \right] d\eta =$$

$$\frac{1}{\pi^2} \left[\int_{\mathbb{R}} \frac{t}{(z-\eta)(\eta+it)(\eta-it)} \right\} d\eta = \frac{1}{\pi^2} \left\{ 2\pi i \left[\frac{t}{(z-it)(2it)} \right] - i\pi \left[\frac{t}{(z+it)(z-it)} \right] \right] = \tag{A4}$$

$$\frac{1}{\pi} \left(\frac{1}{(z-it)} - \frac{it}{t^2+z^2} \right) = \frac{1}{\pi} \left(\frac{z}{t^2+z^2} \right),$$

where in Equation (A4) the integration contour is taken in the upper complex plane (for the pole on the real axis, only an $i\pi$ contribution is subtracted). Hence, we have

$$-\partial_z \mathcal{H}\{P(z,t \mid 0,0)\} = \frac{1}{\pi} \frac{z^2 - t^2}{[t^2 + z^2]^2} = \partial_t P(z,t \mid 0,0).$$

Finally, we have $I_{\text{out}}(z,t) = \frac{P(z,t|0,0)}{\pi} \partial_z \left\{ \int_{\mathbb{R}} \frac{d\xi}{\xi - z} \right\} = \partial_z[-i\pi] = 0.$ □

Remark A1 (Riesz fractional derivative)**.** *In terms of the spatial Fourier transform:*

$$P_c(k,t) := \int_{\mathbb{R}} P_c(z,t|0,0) e^{ikz} dz, \qquad z := (x-y), \tag{A5}$$

Equations (A1) and (A3) can also be formally rewritten as

$$\partial_t P_c(k,t) = -|k| P_c(k,t) \quad \Rightarrow \quad P_c(k,t) = e^{-|k|t}, \tag{A6}$$

showing that we formally have

$$\partial_t P_c(k,t) = \sqrt{-\Delta}[P_c(k,t)] \tag{A7}$$

with the operator $\sqrt{-\Delta}[\cdot]$ being a Riesz *fractional derivative* [12].

Remark A2 (Lévy processes)**.** *The jumps of the Cauchy processes are fully characterised by a Lévy measure [23]:*

$$W_{a,b}(z) = \begin{cases} \frac{a+b}{z^2} & \text{when} \quad z < 0, \\ \frac{a-b}{z^2} & \text{when} \quad z > 0, \end{cases} \tag{A8}$$

where $a > b \in \mathbb{R}^+$ are two constants. For the jump Lévy measure Equation (A8), the resulting Lévy Khinchine triplet $\Psi(k) = [0, 0, \psi_{a,b}(k)]$ with [24]:

$$\begin{cases} \psi_{a,b}(k) := \int_{\mathbb{R}} \left\{ 1 - e^{ikx} + z\mathbb{I}(|z| < 1) \right\} W_{a,b}(z) dz = -a|k| - i\beta(a,b) k \ln(|k|), \\ \psi_{a,0}(k) = -a|k|. \end{cases} \tag{A9}$$

In this paper, we focus exclusively on symmetric jumps, for which $\psi_{a,0}(k) = -a|k|$.

Appendix B. Basic Identities

We define $D(x,t) := \varphi^2(t) + [x+\phi(t)]^2 := \varphi^2(t) + z^2$ and $\frac{d}{dt}f(t) := \dot{f}$. Omitting the (x,t) arguments, we have

$$u(x,t) = \frac{\varphi}{\pi[(x+\phi)^2+\varphi^2]} = \frac{\varphi}{\pi[z^2+\varphi^2]} = \frac{\varphi}{\pi D}, \qquad (z := x+\phi),$$

$$D := \varphi^2 + z^2 \qquad \Rightarrow \qquad \dot{D} = 2\varphi\dot{\varphi} + 2z\dot{\phi},$$

$$\partial_t u(x,t) = \frac{\dot{\varphi}D - \varphi(2z\dot{\phi}) - 2\varphi^2\dot{\varphi}}{\pi D^2} = \frac{\dot{\varphi}z^2 - \varphi^2\dot{\varphi} - 2z\varphi\dot{\phi}}{\pi D^2} = \frac{\dot{\varphi}[z^4-\varphi^4] - 2\dot{\phi}[\varphi z^3 + z\varphi^3]}{\pi D^3}$$

$$\partial_x u(x,t) = -\frac{2z\varphi}{\pi D^2},$$

$$u(x,t)\partial_x u(x,t) = -\frac{2z\varphi^2}{\pi^2 D^3},$$

$$\mathcal{H}[u(x,t)] = \frac{1}{\pi}\int_{\mathbb{R}} \frac{u(\xi,t)}{x-\xi}d\xi = \frac{1}{\pi^2}\int_{\mathbb{R}} \frac{\varphi d\xi}{(x-\xi)(\xi+\phi-i\varphi)(\xi+\phi+i\varphi)} =$$

$$\frac{\varphi}{\pi^2}\left\{\frac{2\pi i}{2i\varphi(z-i\phi)}\right\} - \frac{\varphi}{\pi^2}\left\{\frac{\pi i}{z^2+\phi^2}\right\} = \frac{z}{\pi D},$$

$$\partial_x\{\mathcal{H}[u(x,t)]\} = \frac{D-2z^2}{\pi D^2} = \frac{\varphi^2-z^2}{\pi D^2} = \frac{\varphi^4-z^4}{\pi D^3},$$

$$\partial_{xx}\{\mathcal{H}[u(x,t)]\} = \frac{-2zD^2 + [\varphi^2-z^2]4Dz}{\pi D^4} = \frac{-6z^3+2z\varphi^2}{\pi D^3}.$$

References

1. Gardiner, C.W. *Stochastic Methods: A Handbook for the Natural and Social Sciences*; Springer: Berlin, Germany, 2009.
2. Frank, T.D. *Nonlinear Fokker-Planck Equations*; Springer: Berlin, Germany 2005.
3. Sarrico, C.O.R. Distributional products and global solutions for the nonconservative inviscid Burgers equation. *J. Math. Anal.* **2003**, *281*, 641–656. [CrossRef]
4. Debnath, L. *Nonlinear Partial Differential Equations for Scientists and Engineers*; Birkhäuser/Springer: New York, NY, USA, 2012.
5. Hu, Y.; Liu, Q. On traveling wave solutions of a class of KDV-Burgers-Kuramoto. *AIMS Math.* **2019**, *4*, 1450–1465. [CrossRef]
6. Feng, Z.; Chen, G. Solitary wave solutions of the compound Burgers-Korteweg-de Vries equation. *Phys. A Stat. Mech. Appl.* **2005**, *352*, 419–435. [CrossRef]
7. Feng, Z.S.; Meng, Q.G. Burgers-Korteweg-de Vries equation and its traveling solitary waves. *Sci. China Ser. A* **2007**, *50*, 412–422. [CrossRef]
8. Hongler, M.O. Brownian swarm dynamics and Burgers equations with higher order dispersion. *Symmetry* **2021**, *13*, 57. [CrossRef]
9. Balazs, M.; Racz, M.Z.; Toth, B. Modeling flocks and prices: Jumping particles with an attractive interactions. *Ann. Inst. H. Poincare (B) Prob. Stat.* **2014**, *50*, 425–454. [CrossRef]
10. Filiger, R.; Hongler, M.O. On Jump-Diffusive Driving Noise Sources. *Methodol. Comput. Appl. Probab.* **2019**, *21*, 753–764.
11. Hongler, M.O. Exact soliton-like probability measures for interacting jump processes. *Math. Sci.* **2015**, *40*, 62–66.
12. Bayin, S.S. Definition of the Riesz derivative and its applications to fractional quantum mechanics. *J. Math. Phys.* **2016**, *57*, 123501. [CrossRef]
13. Grimshaw, R.H.; Smyth, N.F.; Stepanyants, Y.A. Decay of Benjamin-Ono solitons under the influence of dissipation. *Wave Motion* **2018**, *78*, 98–115. [CrossRef]
14. Matsuno, Y. *Bilinear Transformation Method*; Academic Press: Cambridge, MA, USA, 1984.
15. Case, K. Benjamin-Ono and related equations. *Proc. Natl. Acad. Sci. USA* **1979**, *76*, 1–3. [CrossRef] [PubMed]
16. Saut, J.C. *Benjamin-Ono and Intermediate Long Wave Equations: Modeling, IST and PDE*; Springer: New York, NY, USA, 2019; Volume 83, pp. 95–160.
17. Lakshmanan, M.; Rajasekar, S. *Nonlinear Dynamics. Integrability, Chaos and Patterns*; Springer: Berlin, Germany, 2003.
18. Schweitzer, F. *Brownian Agents and Active Particles*; Springer Series in Synergetics; Springer: Berlin, Germany, 2003.
19. Gallay, O.; Hashemi, F.; Hongler, M. Imitation, Proximity, and Growth—A Collective Swarm Dynamics Approach. *Adv. Complex Syst.* **2019**, *22*. [CrossRef]
20. Hongler, M.O.; Ryter, D.M. Hard mode stationary states generated by fluctuations. *Z. Für Phys. B Condens. Matter Quanta* **1978**, *31*, 333–337. [CrossRef]
21. Hongler, M.O. Mean-field games and swarms dynamics in Gaussian and non-Gaussian environments. *J. Dyn. Games* **2020**, *7*, 1–20. [CrossRef]

22. Sartoretti, G.; Hongler, M.O. Interacting Brownian swarms: Some analytical results. *Entropy* **2016**, *18*, 27. [CrossRef]
23. Kyprianou, A.E. *Fluctuations of Levy Processes with Applications*; Springer: Berlin, Germany, 2010.
24. Kovalenko, I.N.; Kutznetsov, N.Y.; Shurenkov, V.M. *Models of Random Processes—A Handbook for Mathematicians and Engineers*; CRC Press: Boca Raton, FL, USA, 1996.