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Application of the Pick Function in the Lieb Concavity Theorem for Deformed Exponentials

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Abstract: The Lieb concavity theorem, successfully solved in the Wigner–Yanase–Dyson conjecture, is an important application of matrix concave functions. Recently, the Thompson–Golden theorem, a corollary of the Lieb concavity theorem, was extended to deformed exponentials. Hence, it is worthwhile to study the Lieb concavity theorem for deformed exponentials. In this paper, the Pick function is used to obtain a generalization of the Lieb concavity theorem for deformed exponentials, and some corollaries associated with exterior algebra are obtained.

Keywords: Lieb concavity theorem; deformed exponential; Pick function; convexity of matrix

MSC: 15A42; 15A16; 47A56

1. Introduction

Matrix theory is widely used in statistics [1], physics [2], computer science [3] and so on. For convenience, $M(n, \mathbb{C})$ is denoted as the set of all $n \times n$ complex matrices ($\mathbb{C}$ is the set of complex numbers) [4]. A is called a Hermitian matrix when $A \in M(n, \mathbb{C})$ satisfies $A^* = A$ (A^* denotes conjugate transposition of A). The Hermitian matrix is frequently used in quadratic forms and their correlation theory [5]. Let H_n denote the set of $n \times n$ Hermitian matrices and H_n^+ denote the $n \times n$ positive semidefinite Hermitian matrix ($\mathbb{C}^n$ is the n dimensional complex Euclidean space).

Set $u_1, u_2, \cdots, u_n$ to be any orthonormal basis of $\mathbb{C}^n$, and then the trace operator Tr is defined as [4]

$$\mathrm{Tr}[A] = \sum_{i=1}^{n} (u_i, Au_i),$$

where $(\cdot, \cdot)$ is the inner product of $\mathbb{C}^n$. It is well known that for any $A = (a_{ij}) \in M(n, \mathbb{C})$, the following equalities hold [6]

$$\mathrm{Tr}[A] = \sum_{j=1}^{n} \lambda_i = \sum_{j=1}^{n} a_{ii},$$

where λ_i is the eigenvalue of A.

From the spectral theorem [5], $A \in H_n^+$ can be decomposed as

$$A = P^* \Lambda_A P,$$

where P is a unitary matrix and $\Lambda_A := \text{diag}\{\lambda_1, ..., \lambda_n\}$ is a diagonal matrix with eigenvalues $\lambda_1, ..., \lambda_n$. Then, matrix function $f(A)$ is defined as

$$f(A) = P^* f(\Lambda_A) P = \sum_{i=1} f(\lambda_i) P_i, \tag{1}$$

where $f(\Lambda_A) := \text{diag}\{f(\lambda_1), ..., f(\lambda_n)\}$ and $P_i^2 = P_i$.

Based on the above definition, in 1963, the Wigner–Yanase skew information

$$\text{I}_{WY}(\rho) = -\frac{1}{2} \text{Tr}\left[[\sqrt{\rho}, H]^2 \right]$$

was introduced by Wigner and Yanase ([7]), where ρ is a density matrix ($\rho \geq 0, \text{tr}\,\rho = 1$) and H is a Hermitian matrix. Then, an open problem was left

$$\text{Tr}[A^s K A^{1-s} K^*], \tag{2}$$

which is concave for any positive semidefinite matrix A.

In 1973, (2) was proven by Lieb for all $0 < s < 1$ [8], and a more generalized result was obtained from the following fact [9]

$$\text{Tr}[A^s K B^{1-s} K^*] = \langle K, B^{1-s} K^* A^s \rangle_{\mathcal{L}(H)}$$
$$= \langle K, \Psi^{-1}(B^{1-s} \otimes A^s) K^* \rangle_{\mathcal{L}(H)}.$$

where $\Psi^{-1}(A) = \sum_j (Ae_j) \otimes e_j^*$. In fact, the Lieb concavity theorem is equivalent to the concavity of $B^{1-s} \otimes A^s$.

A more elegant proof of the Lieb concavity theorem appeared in [10] using

$$\text{Tr}[K^* A^s K B^{1-s}] = \langle K, (A^s \otimes B^{1-s}) K \rangle_{\mathcal{L}(H)},$$

where

$$[(A \otimes B) K]_{i,j} = \sum_{k,l} A_{i,k} B_{j,l} K_{k,l}.$$

In 2009, Effros gave another proof of the Lieb concavity theorem based on the Hansen–Pedersen–Jensen inequality ([11]). Using

$$L_A(K) = AK, R_B(K) = KB,$$

then one obtains

$$\text{Tr}[K^* A^s K B^{1-s}] = \langle K, L_{A^s} R_{B^{1-s}}(K) \rangle_{\mathcal{L}(H)}$$
$$= \langle K, R_B^{\frac{1}{2}} (R_B^{-\frac{1}{2}} L_A R_B^{-\frac{1}{2}})^s R_B^{\frac{1}{2}}(K) \rangle_{\mathcal{L}(H)}.$$

All the above proof of the Lieb concave theorem is equivalent to the joint concavity of commutative operators. In addition, Epstein also obtained the Lieb concave theorem using the theory of Herglotz functions [12].

Recently, Shi and Hansen [13] generalized the Thompson–Golden theorem

$$\text{Tr}[\exp_q(A + B)] \leq \text{Tr}[(\exp_q(A))^{2-q}(A(q-1) + \exp_q(B))]$$

As the Thompson–Golden theorem can be regarded as a special form of the Lieb concave theorem, it is worthwhile to study the Lieb concavity theorem for deformed exponentials. In this paper, we will use the theory of the Pick function to obtain a generalization of the Lieb concavity theorem and some other corollaries. The rest of the paper is organized as follows. In Section 2, some general definitions and important conclusions are introduced.

With these preparations, we obtain some useful results, such as the Lieb concavity theorem, presented in the final Section 3.

2. Preliminary

In this section, some general definitions and some important properties are introduced.

2.1. The q-Logarithm Function and q-Exponential Function

It is well known that the q-logarithm function $\ln_q(x)$ is defined as [13]

$$\ln_q(x) = \left\{ \begin{array}{ll} \frac{x^{q-1}-1}{q-1}, & q \neq 1 \\ \ln x, & q = 1 \end{array} \right\}$$

for any $x > 0$. The deformed exponential function or the $q-$exponential $\exp_q(x)$ is the inverse function of the $q-$logarithm and is defined as

$$\exp_q(x) = \left\{ \begin{array}{ll} [(q-1)x+1]^{\frac{1}{q-1}}, & x > \frac{1}{q-1}, \ q > 1 \\ [(q-1)x+1]^{\frac{1}{q-1}}, & x < \frac{1}{q-1}, \ q < 1 \\ \exp(x), & x \in \mathbb{R}, \ q = 1 \end{array} \right\}$$

2.2. Tensor Product and Exterior Algebra

The tensor product, denoted by " $\otimes$ ", is also called the Kronecker product. It is a generalization of the outer product from vectors to matrices, and the tensor product of matrices is also referred to as the outer product in certain contexts ([9]). For an $m \times n$ matrix A and a $p \times q$ matrix B, the tensor product of A and B is defined by

$$A \otimes B := \left(\begin{array}{ccc} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{array} \right),$$

where $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$.

The tensor product is different from matrix multiplication, and one of the differences is commutativity

$$(I \otimes B)(A \otimes I) = (A \otimes I)(I \otimes B) = A \otimes B.$$

From the above equations, we obtain

$$\begin{aligned} AC \otimes BD &= (AC \otimes I)(I \otimes BD) \\ &= (A \otimes I)(C \otimes I)(I \otimes B)(I \otimes D) \\ &= (A \otimes I)(I \otimes B)(C \otimes I)(I \otimes D) \\ &= (A \otimes B)(C \otimes D). \end{aligned}$$

For convenience, we denote

$$\otimes_k A = \underbrace{A \otimes A \otimes \cdots \otimes A}_{k}.$$

In addition to the tensor product, there is another common product called exterior algebra [6]. Exterior algebra, denoted by "$\wedge$", is a binary operation for any $A_i \in H_n^+$, and the definition is

$$\underbrace{(A_1 \wedge A_2 \wedge \cdots \wedge A_k)}_{k}(\xi_{i_1} \wedge \xi_{i_2} \cdots \wedge \xi_{i_k})_{1 \leq i_1 < \cdots < i_k \leq n}$$

$$= (A_1 \xi_{i_1} \wedge A_2 \xi_{i_2} \cdots \wedge A_k \xi_{i_k})_{1 \leq i_1 < \cdots < i_k \leq n},$$

where $\{\tilde{\zeta}_j\}_{j=1}^n$ is an orthonormal basis of $\mathbb{C}^n$, and

$$\xi_{i_1} \wedge \xi_{i_2} \cdots \wedge \xi_{i_k} = \frac{1}{\sqrt{n!}} \sum_{\pi \in \sigma_n} (-1)^\pi \xi_{\pi(i_1)} \otimes \xi_{\pi(i_2)} \cdots \otimes \xi_{\pi(i_k)},$$

σ_n is the family of all permutations on $\{1, 2, \cdots, n\}$.

Let $\bigwedge^k \mathbb{C}^n$ be the span of the $\{\xi_{i_1} \wedge \xi_{i_2} \cdots \wedge \xi_{i_k}\}_{1 \le i_1 < \cdots < i_k \le n}$, and then a simple calculation shows that

$$\wedge_n A = (\underbrace{A \wedge A \wedge \cdots \wedge A}_{k}) = \det(A)$$

2.3. Pick Function

Let $z = x + iy$ be a complex number where i is the imaginary unit and $f(z) = U(z) + iV(z)$ is analytic where $U(z), V(z)$ are all real functions. $\operatorname{Re} z = x$ denotes the real part of z, and $\operatorname{Im} z = y$ is the imaginary part of z. If $\operatorname{Im} f(z) > 0$ for any $\operatorname{Im} z > 0$, then we call the analytic function $f(z)$ a Pick function [14]. It is equivalent that $f(z)$ is analytic in the upper half-plane with the positive imaginary part.

The Pick functions evidently form a convex cone—for instance, if α and β are positive numbers and $f(z)$ and $g(z)$ are two Pick functions, then the function $\alpha f(z) + \beta g(z)$ is also a Pick function. A simple example is that $\tan(z)$ is a Pick function.

$$\tan(x + iy) = \frac{\tan(x) + \tan(iy)}{1 - \tan(x)\tan(iy)}$$
$$= \frac{\tan(x) + i\tanh(y)}{1 - i\tan(x)\tanh(y)}.$$

Hence, $\operatorname{Im} \tan(z) = \frac{(1+\tan^2(x))\tanh(y)}{1+\tan^2(x)\tanh^2(y)}$, and this implies that $\operatorname{Im} \tan(z) > 0$ when $y > 0$.

It is well known that the Pick function has a integral representation, such as the following lemma [14].

Lemma 1. *Let $f(z)$ be a Pick function. Then, $f(z)$ has a unique canonical representation of the form*

$$f(z) = \alpha + \beta z + \int_R \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\mu(\lambda),$$

where α is real, $\beta \ge 0$ and $d\mu(\lambda)$ is a positive Borel measure on the real λ-axis that $\int_R (1 + \lambda^2)^{-1} d\mu(\lambda)$ is finite. Conversely, any function of this form is also a Pick function.

Lemma 1 is frequently used for functions that are positive and harmonic in the half-plane.

2.4. The Matrix-Monotone Function

A matrix function f is said to be matrix-monotonic if it satisfies

$$f(A) \ge f(B) \quad \text{for all} \quad A \ge B > 0. \tag{3}$$

where $A \ge B >$ is equivalent to $A - B$ is a positive semidefinite Hermitian matrix.

Since the matrix-monotone function is a special kind of operator monotone function, we have the following general conclusions [14].

Lemma 2. *The following statements for a real valued continuous function f on $(0, +\infty)$ are equivalent:*

(1) *$f(z)$ is matrix-monotone;*
(2) *$f(z)$ admits an analytic continuation to the whole domain $\operatorname{Im} z \ne 0$ and $\operatorname{Im}(z) \operatorname{Im} f(z) \ge 0$.*

(3) *f admits an integral representation:*

$$f(\lambda) = \alpha + \beta\lambda + \int_{-\infty}^{0} (1 + \lambda t)(t - \lambda)^{-1} \mathrm{d}\mu(t), \text{ for any } \lambda > 0, \tag{4}$$

where α is a real number, β is non-negative and μ is a finite positive measure on $(-\infty, 0)$.

From Lemmas 1 and 2, we know that a Pick function must be a matrix-monotone function.

2.5. Convexity of Matrix

Suppose that X is a convex set in $\mathbb{R}^n$ and f is a function defined on X. Then, we call f a convex function if

$$f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2), \forall x_1, x_2 \in X, \ \forall t \in [0, 1],$$

for all $x_1, x_2 \in X$ and $t \in [0, 1]$.

A matrix function f is called convex if [15–17]

$$f(tA + (1-t)B) \le tf(A) + (1-t)f(B), \tag{5}$$

for any $A, B \in H_n^+$ and any $t \in [0, 1]$. Replacing $\le$ by $<$ in (5), this gives the definition of a strictly matrix convex function. A matrix function f is called (strictly) concave if $-f$ is (strictly) convex. More details can be found in [18].

A matrix convex function must be a convex function; however, the inverse claim is not always true. For instance, the function $f : [0, +\infty) \to \mathbb{R}$ given by $f(x) = x^3$ is a convex function. However, the matrix function $f(A) = A^3$ for any $A \in H_n^+$ is not convex.

Let $f(\cdot, \cdot)$ be a bivariate function defined on $H_n^+ \times H_n^+$. We call $f(\cdot, \cdot)$ jointly convex if

$$f(tA_1 + (1-t)A_2, tB_1 + (1-t)B_2) \le tf(A_1, B_1) + (1-t)f(A_2, B_2),$$

for all $A_1, A_2, B_1, B_2 \in \mathrm{H}_n^+$ and all $t \in [0, 1]$.

2.6. Brunn–Minkowski Inequality

Finally, let us review the Brunn–Minkowski inequality [19].

Lemma 3. *for any $A, B > 0$, and then*

$$\{\mathrm{Tr}[\wedge^k(A+B)]\}^{\frac{1}{k}} \ge \{\mathrm{Tr}[\wedge^k A]\}^{\frac{1}{k}} + \{\mathrm{Tr}[\wedge^k B]\}^{\frac{1}{k}}.$$

Proof. Let $\{\xi_i\}_{i=1}^n$ be the eigenvectors of $A + B$ with the eigenvalue $\{\lambda_i\}_{i=1}^n$, then

$$\begin{aligned}
\{\mathrm{Tr}[\wedge^k(A+B)]\}^{\frac{1}{k}} &= \left[\sum_{1 \le \xi_{i_1} < \cdots < \xi_{i_k} \le n} \lambda_{i_1} \cdots \lambda_{i_k}\right]^{\frac{1}{k}} \\
&= \left[\sum_{1 \le \xi_{i_1} < \cdots < \xi_{i_k} \le n} \left(\det\left|P_{i_1,\cdots,i_k}^* (A+B) P_{i_1,\cdots,i_k}\right|\right)\right]^{\frac{1}{k}} \\
&\ge \left[\sum_{1 \le \xi_{i_1} < \cdots < \xi_{i_k} \le n} \left(\det\left|P_{i_1,\cdots,i_k}^* A P_{i_1,\cdots,i_k}\right| + \det\left|P_{i_1,\cdots,i_k}^* B P_{i_1,\cdots,i_k}\right|\right)\right]^{\frac{1}{k}}
\end{aligned}$$

where $P_{i_1,\cdots,i_k} = (\xi_{i_1}, \cdots, \xi_{i_k})$ and $\ge$ holds due to $\det(A+B) \ge \det(A) + \det(B)$.

As $S_k = \left[\sum\limits_{1 \le \xi_{i_1} < \cdots < \xi_{i_k} \le n} x_{i_1} \cdots x_{i_k} \right]^{\frac{1}{k}}$ is concave [20], we have

$$
\{\mathrm{Tr}[\wedge^k(A+B)]\}^{\frac{1}{k}} \ge \left[\sum\limits_{1 \le \xi_{i_1} < \cdots < \xi_{i_k} \le n} \det\left| P^*_{i_1,\cdots,i_k} A P_{i_1,\cdots,i_k} \right| \right]^{\frac{1}{k}}
$$

$$
+ \left[\sum\limits_{1 \le \xi_{i_1} < \cdots < \xi_{i_k} \le n} \det\left| P^*_{i_1,\cdots,i_k} B P_{i_1,\cdots,i_k} \right| \right]^{\frac{1}{k}}
$$

$$
= \left[\sum\limits_{1 \le \xi_{i_1} < \cdots < \xi_{i_k} \le n} \left(\xi_{i_1} \wedge \cdots \wedge \xi_{i_k}, A\xi_{i_1} \wedge \cdots \wedge A\xi_{i_k} \right) \right]^{\frac{1}{k}}
$$

$$
+ \left[\sum\limits_{1 \le \xi_{i_1} < \cdots < \xi_{i_k} \le n} \left(\xi_{i_1} \wedge \cdots \wedge \xi_{i_k}, B\xi_{i_1} \wedge \cdots \wedge B\xi_{i_k} \right) \right]^{\frac{1}{k}}
$$

$$
= \{\mathrm{Tr}[\wedge^k A]\}^{\frac{1}{k}} + \{\mathrm{Tr}[\wedge^k B]\}^{\frac{1}{k}}.
$$

□

3. Lieb Concavity Theorem for Deformed Exponential

In this section, we obtain some useful conclusions, and some simple and straightforward computations are omitted. Recently, by using the Young inequality,

$$
\mathrm{Tr}[Y] = \max_{X \ge 0}\{\mathrm{Tr}[X] - \mathrm{Tr}[X^{2-q}(\ln_q X - \ln_q Y)]\},
$$

Shi and Hansen obtained that $F(A) = \mathrm{Tr}\left[\exp_q^{\frac{1}{p}}(K^* \ln_q(A^p)K) \right]$ is concave for any $1 \le q \le 2$ where $K^*K \le I$ (I is the identity matrix of $M(n,\mathbb{C})$) [13], namely, the following theorem.

Theorem 1. *For $0 < p \le 1$, $1 < q \le 2$ and $K^*K \le I$, the function*

$$
F(A) = \mathrm{Tr}\left[\exp_q^{\frac{1}{p}}(K^* \ln_q(A^p)K) \right] \tag{6}
$$

is concave for the strictly positive $A \in H_n^+$.

Proof. (The first proof of Theorem 1)
　　Since [21]
$$
\mathrm{D}\, f(A)(B) = \sum_i \sum_j \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_i} P_i B P_j,
$$
we obtain

$$
\frac{\mathrm{d}(\mathrm{Tr}[f(A+tB) - f(A)])}{\mathrm{d}\, t} = \mathrm{Tr}\left[\sum_i \sum_j \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_i} P_i B P_j \right]
$$

$$
= \mathrm{Tr}\left[\sum_i P_j \sum_j \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_i} P_i B \right]
$$

$$
= \mathrm{Tr}\left[\sum_i f'(\lambda_i) P_i B \right] = \mathrm{Tr}[f'(A)B],
$$

where λ_i are eigenvalues of A. When $f(x)$ is a convex function, we obtain

$$\text{Tr}[f(A + tB) - f(A)] \geq \text{Tr}[f'(A)tB]$$

for any t. This implies that

$$\text{Tr}[f(C)] = \max\{\text{Tr}[f(D) + f'(D)(C - D)] : D > 0\}.$$

Therefore, we obtain

$$\text{Tr}[(K^* A^{pq-p} K + I - K^* K)^{\frac{1}{pq-p}}]$$

$$= \max\{\text{Tr}[D^{\frac{1}{pq-p}} + \frac{D^{\frac{1}{pq-p}-1}(K^* A^{pq-p} K + I - K^* K - D)}{pq - p}] : D > 0\}$$

$$= \max\{\text{Tr}[C + \frac{C^{1-pq+p}(K^* A^{pq-p} K + I - K^* K - C^{pq-p})}{pq - p}] : C = D^{\frac{1}{pq-p}} > 0\}$$

$$= \max\{\text{Tr}[C(1 - \frac{1}{pq - p}) + \frac{C^{1-pq+p} K^* A^{pq-p} K}{pq - p} + C^{1-pq+p}(I - K^* K)] : C > 0\}$$

Thus, the concavity of $F(A)$ is equivalent to the jointly concavity of $\text{Tr}[\frac{C^{1-pq+p} K^* A^{pq-p} K}{pq - p}]$ for the strictly positive A and C, which is the Lieb concavity theorem [22,23]. $\square$

Unfortunately, Theorem 1 cannot be obtained using Epstein's theorem. Hence, we require a more general generalization of Epstein's theorem. First, for any $\text{Im}(z) > 0$, we know that $A + zB$ is invertible and $x^*(A + zB)x$ is a Pick function for any $x \in \mathbb{C}^n$ [14]. For any $A \in M(n, \mathbb{C})$, we know $f(A)$ is defined as [12]

$$f(A) = \frac{1}{2\pi} \oint_C \frac{f(z)}{z - A} \, dz,$$

where $f(z)$ is a complex holomorphic function in an open set of the complex plane containing $\text{Sp}(A)$ (the set of all eigenvalues of A). Then, we have the following lemma.

Lemma 4. *Let $A, B \in H_n^+$ and $0 < \alpha \leq 1$, then*

$$x^*(A + zB)^\alpha x$$

is a Pick function for any $x \in \mathbb{C}^n$ and $0 < \arg(x^*(A + zB)^\alpha x) < \alpha\pi$ if $0 < \arg(z) = \theta < \pi$, such as $\text{Sp}((A + zB)^\alpha) \subseteq (\text{Sp}(A + zB)^\alpha)$. Generally, we can find that

$$x^* f(A + zB)x$$

is a Pick function when f is a Pick function.

Proof. Setting $z = \rho e^{i\theta}$, we have

$$(A + zB)^\alpha = \int_0^{+\infty} (\frac{A + zB}{t + A + zB}) \, d\mu(t)$$

$$= \int_0^{+\infty} (\frac{1}{\frac{t}{A+zB} + 1}) \, d\mu(t),$$

where $d\mu(t) = \frac{t^{\alpha-1}\pi}{\sin\alpha\pi}$.

Since $\text{Im } z > 0$, we see that $A + zB$ is invertible. Hence, we have

$$x^*(A + zB)^\alpha x = \int_0^{+\infty} x^* \left(\frac{1}{\frac{t}{A+zB} + 1} \right) x \, d \, \mu(t)$$

$$= \int_0^{+\infty} y^* \left(\frac{t}{A + z^*B} + 1 \right) y \, d \, \mu(t), \quad y = \left(\frac{t}{A+zB} + 1 \right)^{-1} x$$

$$= \int_0^{+\infty} y^* y + t w^* (A + zB) w \, d \, \mu(t), \quad w = (A + z^*B)^{-1} y$$

$$= \int_0^{+\infty} y^* y + t w^* A w \, d \, \mu(t) + z \int_0^{+\infty} t w^* B w \, d \, \mu(t).$$

This implies that

$$\text{Im } x^*(A + zB)^\alpha x = \text{Im}(z) \cdot \int_0^{+\infty} t w^* B w \, d \, \mu(t) > 0;$$

hence, $0 < \arg(x^*(A + zB)^\alpha x)$ when $0 < \arg(z) = \theta < \pi$.

In the same way, we can obtain

$$\text{Im } w^* [(-A - z^*B)^{-\alpha}] w = \text{Im}(e^{-i\alpha\pi} z^*) \cdot \int_0^{+\infty} t v^* B v \, d \, \mu(t) < 0, \quad v = (t(A + z^*B) + 1)^{-1} w.$$

In particular, letting $w = (A + z^*B)^\alpha x$, we have

$$\text{Im}(e^{-i\alpha\pi} x^*(A + zB)^\alpha x) < 0.$$

This is equivalent to $\arg(x^*(A + zB)^\alpha x) < \alpha\pi$.

To prove $\text{Sp}((A + zB)^\alpha) \subseteq (\text{Sp}(A + zB)^\alpha)$, let $(A + zB)\xi = \lambda\xi$, we find

$$\xi^*(A + zB)^\alpha \xi = [\xi^*(A + zB)\xi]^\alpha = [\xi^* A \xi + z\xi^* B \xi]^\alpha = \rho^\alpha e^{i\alpha\theta},$$

where $\tan\theta = \frac{\xi^* B \xi \, \text{Im}(z)}{\xi^* A \xi + \xi^* B \xi \, \text{Re}(z)} \leq \tan\arg(z)$.

When $f(z)$ is a Pick function, using the integral represented of $f(z)$, in a similar way, we can obtain that

$$x^* f(A + zB) x$$

is a Pick function for any $x \in \mathbb{C}^n$. $\quad \square$

Using Lemma 4, another proof of Theorem 1 can be obtained.

Theorem 2. *For $0 < p \leq 1$, $1 < q \leq 2$ and $K^*K \leq I$, the function*

$$F(A) = \text{Tr} \left[\exp_q^{\frac{1}{p}} (K^* \ln_q(A^p) K) \right]$$

is concave for the strictly positive $A \in H_n^+$.

Proof. (The second proof of Theorem 1)

First, setting $f(z) = \text{Tr}[(A(z) + iB(z))^{\frac{1}{pq-p}}]$ where $A(z) = \text{Re}(K^*(A + zB)^{pq-p}K + I - K^*K)$ and $B(z) = \text{Im}(K^*(A + zB)^{pq-p}K + I - K^*K) \in H_n^+$. As

$$
\text{Im}\left[\text{Tr}[(A(z) + iB(z))^{\frac{1}{pq-p}}]\right] = \text{Im}\left[\text{Tr}[\int_0^{+\infty}(\frac{A(z) + iB(z)}{t + A(z) + iB(z)})\,d\,\mu(t)]\right]
$$

$$
= \text{Im}\left[\int_0^{+\infty}\text{Tr}[(\frac{\Lambda_{A(z)+iB(z)}}{t + \Lambda_{A(z)+iB(z)}})\,d\,\mu(t)]\right]
$$

$$
= \text{Im}\left[\int_0^{+\infty}\sum_{i=1}^{n}[(\frac{\lambda_i(A(z) + iB(z))}{t + \lambda_i(A(z) + iB(z))})\,d\,\mu(t)]\right]
$$

$$
= \text{Im}\left[\sum_{i=1}^{n}[(\lambda_i(A(z) + iB(z))^{\frac{1}{pq-p}}]\right],
$$

when $\arg(z) \in (0, \pi)$ and $K^*K \le I$, then

$$
\arg(\lambda_i(A(z) + iB(z)))
$$
$$
= \arg(x_i^*(A(z) + iB(z))x_i)
$$
$$
= \arg(x_i^*K^*(A + zB)^{pq-p}Kx_i + x_i^*(I - K^*K)x_i) \in (0, (pq - p)\pi),
$$

where $x_i \in \mathbb{C}^n$ are the eigenvectors of $K^*(A + zB)^{pq-p}K + I - K^*K$.

Hence,

$$
\text{Im}\left[\text{Tr}[(A(z) + iB(z))^{\frac{1}{pq-p}}]\right] = \text{Im}\left[\sum_{i=1}^{n}z_i\right],
$$

where z_i is the i eigenvalue of $(A(z) + iB(z))^{\frac{1}{pq-p}}$ and $\arg(z_i) \in (0, \pi)$.

Thus, $f(z) = \text{Tr}[(A(z) + iB(z))^{\frac{1}{pq-p}}]$ is a Pick function, and this implies that $F(A)$ is concave. $\square$

Using a similar method, we can obtain the following corollary.

Corollary 1. *For $0 < p \le 1$ and $1 < q \le 2$, the function*

$$
E(A) = \text{Tr}\left[\exp_q^{\frac{1}{p}}[B + \ln_q(A^p)]\right] \tag{7}
$$

is concave for the strictly positive $A \in H_n^+$.

Since the Thompson–Golden theorem can be seen as a corollary of the Lieb concavity theorem, we discuss the Lieb concavity theorem for deformed exponentials. Setting $\text{SP}(A) \subset \{z = \rho e^{i\theta} : 0 < \rho, 0 < \theta < \alpha\}$ and $\text{SP}(B) \subset \{z = \rho e^{i\theta} : 0 < \rho, 0 < \theta < \beta\}$, then for any $A_1, B_1 \in H_n$, $A_2, B_2 \in H_n^+$ and $A = A_1 + iA_2$, $B = B_1 + iB_2$, we have [12]

$$
\text{SP}(AB) \subset \{z = \rho e^{i\theta} : 0 < \rho, 0 < \theta < \alpha + \beta\}. \tag{8}
$$

and then the following theorem can be obtained.

Theorem 3. *For $0 < p \le 1$, $1 < q \le 2$ and $P^*P \le I$, the following function*

$$
L(A) = \text{Tr}[\exp_q(P^* \ln_q(K^* A^p K)P) \exp_q(P^* \ln_q A^{1-p}P)] \tag{9}
$$

is concave for any $A \in H_n^+$.

Proof. Set $L_{A,B}(z) = \text{Tr}[\exp_q(P^* \ln_q(K^*(A + zB)^p K)P) \exp_q(P^* \ln_q(A + zB)^{1-p}P)]$.

When $x_i \in \mathbb{C}^n$ is a eigenvector of $P^*(A + zB)^{pq-p}P + I - P^*P$ and $P^*P \leq I$,

$$\arg(x_i^* P^* K^*(A + zB)^{pq-p}KPx_i + x_i^*(I - P^*P)x_i) \in (0, (pq - p)\pi),$$

if $\arg(z) \in (0, \pi)$. This implies

$$\mathrm{SP}(P^*K^*(A + zB)^{pq-p}KP + I - P^*P) \subset \{z = \rho e^{i\theta} : 0 < \rho, 0 < \theta < (pq - p)\pi\},$$

such as

$$\mathrm{SP}(\exp_q(P^* \ln_q(K^*(A + zB)^p K)P)) \subset \{z = \rho e^{i\theta} : 0 < \rho, 0 < \theta < p\pi\}.$$

Similarly, we can also obtain

$$\mathrm{SP}(\exp_q(P^* \ln_q(A + zB)^{1-p}P)) \subset \{z = \rho e^{i\theta} : 0 < \rho, 0 < \theta < (1 - p)\pi\}.$$

Hence, using (8), we see that

$$\mathrm{SP}[\exp_q(P^* \ln_q(K^*(A + zB)^p K)P) \exp_q(P^* \ln_q(A + zB)^{1-p}P)]$$
$$\subset \{z = \rho e^{i\theta} : 0 < \rho, 0 < \theta < \pi\}.$$

Thus, we know $\arg(L_{A,B}(z)) \in (0, \pi)$, which implies that $L_{A,B}(z)$ is a Pick function. Hence, L(A) is concave. $\square$

In fact, Theorem 3 is a generalization of the Lieb concavity theorem setting $P = I$, $K = \begin{pmatrix} 0 & 0 \\ H & 0 \end{pmatrix}$ and $A = \begin{pmatrix} Z & 0 \\ 0 & B \end{pmatrix}$. Moreover, we can obtain the following theorem.

Theorem 4. *For $0 < p, s \leq 1, 1 < q \leq 2$ and $P^*P \leq I$, the functions*

$$\mathrm{Tr}\left[\left[\exp_q(P^* \ln_q A^{\frac{ps}{2}} P) \exp_q(P^* \ln_q(K^* A^{s-sp}K)P) \exp_q(P^* \ln_q A^{\frac{ps}{2}} P)\right]^{\frac{1}{s}}\right] \quad (10)$$

and

$$\left[\mathrm{Tr}\exp_q(P^* \ln_q A^{\frac{ps}{2}} P) \exp_q(P^* \ln_q(K^* A^{s-sp}K)P) \exp_q(P^* \ln_q A^{\frac{ps}{2}} P)\right]^{\frac{1}{s}} \quad (11)$$

are jointly concave for any $A \in H_n^+$.

The proof of Theorem 4 is similar to Theorem 3; here, we do not repeat the proof. In [19], Huang used exterior algebra to find that

$$\left[\mathrm{Tr} \wedge^k [\exp(K^* \ln(A)K)]\right]^{\frac{1}{k}}$$

is a concave function for any $A \in H_n^+$, $K^*K \leq I$ and $k \leq n$. Associated with Theorem 1, we can obtain a generalization as the following theorem.

Theorem 5. *For $0 < p \leq 1, 1 < q \leq 2$ and $K^*K \leq I$, the function*

$$\left[\mathrm{Tr} \wedge^k \left[\exp_q^{\frac{1}{p}}(K^* \ln_q(A^p)K)\right]\right]^{\frac{1}{k}} \quad (12)$$

is concave for the strictly positive $A \in H_n^+$ and $k \leq n$.

Proof. In fact, we can prove that

$$\left[\mathrm{Tr} \wedge^k \left[(H^* A^s H + B)^{\frac{1}{s}}\right]\right]^{\frac{1}{k}}$$

is a concave function for any $A \in H_n^+$ where $0 < s \leq 1$ and $B \in H_n^+$.

Using Theorem 1, we know that

$$\mathrm{Tr}\left[(H^* A^p H + C)^{\frac{1}{p}} \right] \tag{13}$$

is a concave function for any $A \in H_n^+$ where $0 < p \leq 1$ and $C \in H_n^+$.

Then, for any $A_1, A_2 \in H_n^+$, we have

$$\left[\mathrm{Tr} \wedge^k \left[(H^*(\frac{A_1 + A_2}{2})^s H + B)^{\frac{1}{s}} \right] \right]^{\frac{1}{k}}$$

$$= \left[\mathrm{Tr}\left[(\bar{H}^*(\frac{A_1 \wedge^{k-1} I + A_2 \wedge^k I}{2})^s \bar{H} + \bar{B})^{\frac{1}{s}} \right] \right]^{\frac{1}{k}}$$

$$\geq \left[\mathrm{Tr}\left[\frac{(\bar{H}^*(A_1 \wedge^{k-1} I)^s \bar{H} + \bar{B})^{\frac{1}{s}} + (\bar{H}^*(A_2 \wedge^{k-1} I)^s \bar{H} + \bar{B})^{\frac{1}{s}}}{2} \right] \right]^{\frac{1}{k}}$$

$$= \left[\mathrm{Tr}\left[\left(\frac{(H^* A_1^s H + B)^{\frac{1}{s}} + (H^* A_2^s H + B)^{\frac{1}{s}}}{2} \right) \wedge^{k-1} (H^*(\frac{A_1 + A_2}{2})^s H + B)^{\frac{1}{s}} \right] \right]^{\frac{1}{k}},$$

where $\bar{H} = H \wedge^{k-1} (H^*(\frac{A_1+A_2}{2})^s H + B)^{\frac{1}{s}}$ and $\bar{B} = B \wedge^{k-1} (H^*(\frac{A_1+A_2}{2})^s H + B)^{\frac{1}{s}}$. Analogously, we can obtain

$$\left[\mathrm{Tr} \wedge^k \left[(H^*(\frac{A_1 + A_2}{2})^s H + B)^{\frac{1}{s}} \right] \right]^{\frac{1}{k}}$$

$$\geq \left[\mathrm{Tr}\left[\wedge^k \left(\frac{(H^* A_1^s H + B)^{\frac{1}{s}} + (H^* A_2^s H + B)^{\frac{1}{s}}}{2} \right) \right] \right]^{\frac{1}{k}}.$$

Using lemma 3, we obtain

$$\left[\mathrm{Tr} \wedge^k \left[(H^*(\frac{A_1 + A_2}{2})^s H + B)^{\frac{1}{s}} \right] \right]^{\frac{1}{k}}$$

$$\geq \frac{\left[\mathrm{Tr}\left[\wedge^k \left((H^* A_1^s H + B)^{\frac{1}{s}} \right) \right] \right]^{\frac{1}{k}} + \left[\mathrm{Tr}\left[\wedge^k \left((H^* A_2^s H + B)^{\frac{1}{s}} \right) \right] \right]^{\frac{1}{k}}}{2}.$$

□

Clearly, the proof of Theorem 5 is in the application of exterior algebra and the Brunn–Minkowski inequality. Hence, other theorems, such as the Thompson–Golden theorem in a deformed exponential, can be generalized to a more general form, but we do not discuss this here.

4. Conclusions

In this paper, we used the Pick function to obtain a generalization of the Lieb concavity theorem and some corollaries. The advantage of using the Pick function is that it avoids discussing the commutativity of the matrix and variational method. Generally, we obtain that the following two functions are concave for $0 < p, s \leq 1, 1 < q \leq 2$ and $P^* P \leq I$

$$\left[\mathrm{Tr} \wedge^k \left[\exp_q(P^* \ln_q A^{\frac{ps}{2}} P) \exp_q(P^* \ln_q(K^* A^{s-sp} K)P) \exp_q(P^* \ln_q A^{\frac{ps}{2}} P) \right]^{\frac{1}{s}} \right]^{\frac{1}{k}} \tag{14}$$

and

$$\left[\operatorname{Tr} \wedge^k \left[\exp_q(P^* \ln_q A^{\frac{ps}{2}} P) \exp_q(P^* \ln_q(K^* A^{s-sp} K) P) \exp_q(P^* \ln_q A^{\frac{ps}{2}} P) \right] \right]^{\frac{1}{ks}}, \quad (15)$$

where $A \in H_n^+$ and $k \le n$, and this provides work for the future.

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