fractal and fractional

Article

Some New Versions of Hermite–Hadamard Integral Inequalities in Fuzzy Fractional Calculus for Generalized Pre-Invex Functions via Fuzzy-Interval-Valued Settings

Muhammad Bilal Khan [1,†], Muhammad Aslam Noor [1], Nehad Ali Shah [2,†], Khadijah M. Abualnaja [3] and Thongchai Botmart [4,*]

1 Department of Mathematics, COMSATS University Islamabad, Islamabad 44000, Pakistan; bilal42742@gmail.com (M.B.K.); aslamnoor@comsats.edu.pk (M.A.N.)
2 Department of Mechanical Engineering, Sejong University, Seoul 05006, Korea; nehadali199@yahoo.com
3 Department of Mathematics and Statistics, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia; kh.abualnaja@tu.edu.sa
4 Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand
* Correspondence: thongbo@kku.ac.th
† These authors contributed equally to this work and co-first authors.

Abstract: The purpose of this study is to prove the existence of fractional integral inclusions that are connected to the Hermite–Hadamard and Hermite–Hadamard–Fejér type inequalities for χ-pre-invex fuzzy-interval-valued functions. Some of the related fractional integral inequalities are also proved via Riemann–Liouville fractional integral operator, where integrands are fuzzy-interval-valued functions. To prove the validity of our main results, some of the nontrivial examples are also provided. As specific situations, our findings can provide a variety of new and well-known outcomes which can be viewed as applications of our main results. The results in this paper can be seen as refinements and improvements to previously published findings.

Keywords: χ-pre-invex fuzzy-interval-valued function; fuzzy-interval Riemann–Liouville fractional integral operator; Hermite–Hadamard type inequality; Hermite–Hadamard–Fejér type inequality

MSC: 26A33; 26A51; 26D10

Citation: Khan, M.B.; Noor, M.A.; Shah, N.A.; Abualnaja, K.M.; Botmart, T. Some New Versions of Hermite–Hadamard Integral Inequalities in Fuzzy Fractional Calculus for Generalized Pre-Invex Functions via Fuzzy-Interval-Valued Settings. *Fractal Fract.* **2022**, *6*, 83. https://doi.org/10.3390/fractalfract6020083

Academic Editor: Paul Eloe

Received: 11 December 2021
Accepted: 31 January 2022
Published: 2 February 2022

1. Introduction

The Hermite–Hadamard inequality is one of the most well-known inequalities in the theory of convex functions with geometrical meaning, and it has a wide range of applications. This disparity might be seen as a refinement of the idea of convexity. In recent years, the Hermite–Hadamard inequality ($H.H$ inequality) for convex functions has gotten a lot of attention, and there have been some impressive improvements and generalizations, see [1,2].

The relevance of set-valued analysis research from both a theoretical and practical standpoint is well understood. Control theory and dynamical games have fueled several developments in set-valued analysis. Since the dawn of the 1960s, optimal control theory and mathematical programming have been pushing these disciplines. Interval analysis is a special instance that was developed to deal with interval uncertainty that may be found in many mathematical or computer models of deterministic real-world processes. In recent years, a few major inequalities for interval-valued functions, such as $H.H$ and Ostrowski type inequalities have been developed. Chalco-Cano et al. used Hukuhara derivatives for interval-valued functions to develop Ostrowski type inequalities for interval-valued functions in [3,4]. For interval-valued functions, Roman-Flores et al. established Minkowski and Beckenbach's inequalities in [5]. Readers [6–13] are referred to for further relevant findings.

Novel inequalities for various forms of convexities, pre-invexities, statistical theory, and other fields have lately been developed by a number of authors. Several debates illustrate that inequality theory and convex functions are inextricably linked. For the first time in 1981, Hanson looked at the invex function in the setting of bi-function $\Lambda(.,.)$ (see [14]). Following in the footsteps of Hanson, Ben-Israel and Mond sought to explore deeper into linked invexity, for the first time presenting the ideas of invex sets and pre-invex functions (see [15]). According to Mohan and Neogy [16], the pre-invex and invex functions in the form of differentiability are equivalent under certain situations. In 2005, Antczak [17] was the first to uncover and study the characteristics of pre-invex functions. Fuzzy mappings (*F-Ms*) are functions with a fuzzy interval value. Nanda and Kar [18], Syau [19], and Furukawa [20], introduced the notion of convex *F-Ms* from $\mathbb{R}^n$ to the set of fuzzy numbers. They also constructed various forms of convex *F-Ms*, such as logarithmic convex *F-Ms* and quasi-convex *F-Ms*, and studied Lipschitz continuity of fuzzy-valued mappings. Based on Goetschel and Voxman's idea of ordering [21], Yan and Xu [22] presented the concepts of epigraphs and convexity of *F-Ms*, as well as the features of convex *F-Ms* and quasi-convex *F-Ms*,. Noor [23] proposed and investigated the notion of fuzzy pre-invex mapping on the invex set. He also showed how to express the fuzzy optimality conditions of differentiable pre-invex fuzzy mappings using variational inequalities.

Khan et al. [24–27] extended the class of convex *F·Ms* and defined *h*-convex and (h_1, h_2)-convex *F-IV-Fs* using fuzzy partial order relation. Moreover, they introduced $H.H$, Hermite–Hadamard–Fejér ($H.H$ Fejér), fractional $H.H$ and $H.H$ Fejér for *h*-convex and (h_1, h_2)-convex *F·I·V·Fs* via fuzzy Riemannian and fuzzy Riemann–Liouville fractional integrals. We suggest readers to [28–63] and the references therein for more study of literature on the applications and properties of fuzzy-interval, inequalities, and generalized convex *F-Ms*.

In Section 2, motivated by ongoing research work some preliminary notions, definitions and some new results are introduced. In the Riemann–Liouville interval-valued fractional operator settings, Section 3 develops $H.H$ and $H.H$ Fejér type inequalities and some related inequalities for χ-pre-invex fuzzy-interval-valued functions (χ-pre-invex *F-IV-Fs*). Moreover, some non-trivial examples are also given to prove the validity of our main results. We end Section 4 with conclusions and future plans.

2. Preliminaries

Let $\mathcal{K}_C$ and $\mathbb{F}_0$ be the collection of all closed and bounded intervals, and fuzzy intervals of $\mathbb{R}$. We use $\mathcal{K}_C^+$ to represent the set of all positive intervals. The collection of all Riemann integrable real valued functions, Riemann integrable *IV-Fs* and fuzzy Riemann integrable *F-IV-Fs* over $[t, v]$ is denoted by $\mathcal{R}_{[t,\, v]}$, $\mathcal{IR}_{[t,\, v]}$, and $\mathcal{FR}_{([t,\, v])}$, respectively. For more conceptions on interval-valued functions and fuzzy-interval-valued functions, see [28,34,35]. Moreover, some more Preliminaries notions are as follows:

The inclusion "$\subseteq$" means that

$$\xi \subseteq \eta \text{ if and only if, } [\xi_*,\ \xi^*] \subseteq [\eta_*,\ \eta^*], \text{ if and only if } \eta_* \leq \xi_*,\ \xi^* \leq \eta^*, \tag{1}$$

for all $[\mathscr{r}_*,\ \mathscr{r}^*],\ [\eta_*,\ \eta^*] \in \mathcal{K}_C$.

Remark 1. [31] *The relation "$\leq_I$" defined on $\mathcal{K}_C$ by*

$$[\mathscr{r}_*,\ \mathscr{r}^*] \leq_I [\eta_*,\ \eta^*], \text{ if and only if } \mathscr{r}_* \leq \eta_*,\ \mathscr{r}^* \leq \eta^*, \tag{2}$$

for all $[\mathscr{r}_*,\ \mathscr{r}^*],\ [\eta_*,\ \eta^*] \in \mathcal{K}_C$, it is an order relation.

Proposition 1. [39] *If $\xi, \varpi \in \mathbb{F}_0$, then relation "$\preccurlyeq$" defined on $\mathbb{F}_0$ by*

$$\xi \preccurlyeq \varpi \text{ if and only if, } [\xi]^\varsigma \leq_I [\varpi]^\varsigma, \text{ for all } \varsigma \in [0,\ 1], \tag{3}$$

this relation is known as the partial order relation.

Theorem 1. [29] *Let* $\widetilde{\mathfrak{U}} : [t,\ v] \subset \mathbb{R} \to \mathbb{F}_0$ *be a F-IV-F, whose* ς*-levels define the family of IV-Fs* $\mathfrak{U}_\varsigma : [t,\ v] \subset \mathbb{R} \to \mathcal{K}_C$ *are given by* $\mathfrak{U}_\varsigma(\varkappa) = [\mathfrak{U}_*(\varkappa,\varsigma),\ \mathfrak{U}^*(\varkappa,\varsigma)]$ *for all* $\varkappa \in [t,\ v]$ *and for all* $\varsigma \in (0,\ 1]$. *Then* $\widetilde{\mathfrak{U}}$ *is fuzzy Riemann integrable over* $[t,\ v]$ *if and only if,* $\mathfrak{U}_*(\varkappa,\varsigma)$ *and* $\mathfrak{U}^*(\varkappa,\varsigma)$ *both are Riemann integrable over* $[t,\ v]$. *Moreover, if* $\widetilde{\mathfrak{U}}$ *is fuzzy Riemann integrable over* $[t,\ v]$, *then*

$$\left[(FR)\int_t^v \widetilde{\mathfrak{U}}(\varkappa)d\right]^\varsigma = \left[(R)\int_t^v \mathfrak{U}_*(\varkappa,\varsigma)d,\ (R)\int_t^v \mathfrak{U}^*(\varkappa,\varsigma)d\varkappa\right] = (IR)\int_t^v \mathfrak{U}_\varsigma(\varkappa)d\varkappa \quad (4)$$

for all $\varsigma \in (0,\ 1]$.

The following FI Riemann–Liouville fractional integral operators were introduced by Allahviranloo et al. [33].

Definition 1. *Let* $\alpha > 0$ *and* $L([t,\ v],\mathbb{F}_0)$ *be the collection of all Lebesgue measurable F·I·V·Fs on* $[t,v]$. *Then the fuzzy left and right Riemann–Liouville fractional integral of* $\widetilde{\mathfrak{U}} \in L([t,\ v],\mathbb{F}_0)$ *with order* $\alpha > 0$ *are defined by*

$$\mathcal{I}_{t^+}^\alpha\ \widetilde{\mathfrak{U}}(\varkappa) = \frac{1}{\Gamma(\alpha)}\int_t^\varkappa (\varkappa - s)^{\alpha-1}\widetilde{\mathfrak{U}}(s)ds,\ (\varkappa > t), \quad (5)$$

and

$$\mathcal{I}_{v^-}^\alpha\ \widetilde{\mathfrak{U}}(\varkappa) = \frac{1}{\Gamma(\alpha)}\int_\varkappa^v (s - \varkappa)^{\alpha-1}\widetilde{\mathfrak{U}}(s)ds,\ (\varkappa < v) \quad (6)$$

respectively, where $\Gamma(\varkappa) = \int_0^\infty s^{-1}e^{-s}ds$ *is the Euler gamma function. The fuzzy left and right Riemann–Liouville fractional integral based on left and right end point functions can be defined, that is*

$$\begin{aligned}\left[\mathcal{I}_{t^+}^\alpha\ \widetilde{\mathfrak{U}}(\varkappa)\right]^\varsigma &= \tfrac{1}{\Gamma(\alpha)}\int_t^\varkappa (\varkappa - s)^{\alpha-1}\mathfrak{U}_\varsigma(s)ds \\ &= \tfrac{1}{\Gamma(\alpha)}\int_t^\varkappa (\varkappa - s)^{\alpha-1}[\mathfrak{U}_*(s,\ \varsigma),\mathfrak{U}^*(s,\ \varsigma)]ds,\ (\varkappa > t),\end{aligned} \quad (7)$$

where

$$\mathcal{I}_{t^+}^\alpha\ \mathfrak{U}_*(\varkappa,\ \varsigma) = \frac{1}{\Gamma(\alpha)}\int_t^\varkappa (\varkappa - s)^{\alpha-1}\mathfrak{U}_*(s,\ \varsigma)ds,\ (\varkappa > t), \quad (8)$$

and

$$\mathcal{I}_{t^+}^\alpha\ \mathfrak{U}^*(\varkappa,\ \varsigma) = \frac{1}{\Gamma(\alpha)}\int_t^\varkappa (\varkappa - s)^{\alpha-1}\mathfrak{U}^*(s,\ \varsigma)ds,\ (\varkappa > t), \quad (9)$$

Similarly, the left and right end-point functions can be used to define the right Riemann–Liouville fractional integral $\widetilde{\mathfrak{U}}$ *of* $\varkappa$.

Definition 2. *A non-negative real mapping* $\mathfrak{U} : [t,\ v] \to \mathbb{R}^+$ *is named as a convex function if*

$$\mathfrak{U}(s\mu + (1 - s\varkappa)) \le s\mathfrak{U}(\mu) + (1 - s)\mathfrak{U}(\varkappa), \quad (10)$$

for all $\mu,\ \varkappa \in [t,\ v]$, $s \in [0,\ 1]$. *If (10) is reversed, then* $\mathfrak{U}$ *is named as a concave.*

Definition 3. [18] *The F-IV-F* $\widetilde{\mathfrak{U}} : [t,\ v] \to \mathbb{F}_0$ *is named as a convex F-IV-F on* $[t,\ v]$ *if*

$$\widetilde{\mathfrak{U}}(s\mu + (1 - s)\ ) \preccurlyeq s\widetilde{\mathfrak{U}}(\mu)\widetilde{+}(1 - s\varkappa)\widetilde{\mathfrak{U}}(\varkappa), \quad (11)$$

for all $\mu,\ \varkappa \in [t,\ v]$, $s \in [0,\ 1]$, *where* $\widetilde{\mathfrak{U}}(\varkappa) \succcurlyeq \widetilde{0}$ *for all* $\varkappa \in [t,\ v]$. *If (11) is reversed, then* $\widetilde{\mathfrak{U}}$ *is named as a concave F-IV-F on* $[t,\ v]$. $\widetilde{\mathfrak{U}}$ *is affine if and only if it is both convex and concave F-IV-F.*

Remark 2. *If* $\mathfrak{U}_*(\varkappa,\varsigma) = \mathfrak{U}^*(\varkappa,\varsigma)$ *and* $\varsigma = 1$, *then we obtain the classical convex function.*

Definition 4. [23] *The F·I·V·F* $\widetilde{\mathfrak{U}} : [t,\ v] \to \mathbb{F}_0$ *is named as a pre-invex F·I·V·F on invex interval* $[t,\ v]$ *if*

$$\widetilde{\mathfrak{U}}(\mu + (1-s)\Lambda(\mu,\varkappa)) \preccurlyeq s\widetilde{\mathfrak{U}}(\mu)\widetilde{+}(1-s)\widetilde{\mathfrak{U}}(\varkappa), \tag{12}$$

for all $\mu,\ \varkappa \in [t,\ v]$, $s \in [0,\ 1]$, *where for all* $\widetilde{\mathfrak{U}}(\mu) \succcurlyeq \widetilde{0}$ *for all* $\mu \in [t,\ v]$. *If (12) is reversed, then* $\widetilde{\mathfrak{U}}$ *is named as a pre-incave F·I·V·F on* $[t,\ v]$. $\widetilde{\mathfrak{U}}$ *is affine if and only if it is both pre-invex and pre-incave F·I·V·Fs.*

Definition 5. [41] *Let* $\chi_1,\ \chi_2 : [0,\ 1] \subseteq [t,v] \to \mathbb{R}^+$ *such that* $\chi_1,\ \chi_2 \not\equiv 0$. *Then, F-IV-F* $\widetilde{\mathfrak{U}} : [t,v] \to \mathbb{F}_0$ *is said to be* $(\chi_1,\ \chi_2)$*-pre-invex F-IV-F on* $[t,v]$ *if*

$$\widetilde{\mathfrak{U}}(\mu + (1-s)\Lambda(\mu,\varkappa)) \preccurlyeq \chi_1(s)\chi_2(1-s)\widetilde{\mathfrak{U}}(\mu)\widetilde{+}\chi_1(1-s)\chi_2(s)\widetilde{\mathfrak{U}}(\varkappa), \tag{13}$$

for all $\mu,\ \varkappa \in [t,v]$, $s \in [0,\ 1]$, *where* $\widetilde{\mathfrak{U}}(\mu) \succcurlyeq \widetilde{0}$. *If* $\widetilde{\mathfrak{U}}$ *is* $(\chi_1,\ \chi_2)$*-pre-incave on* $[t,v]$, *then inequality (13) is reversed.*

Remark 3. [41] *If one attempt to take* $\chi_2(s) \equiv 1$, *then from* $(\chi_1,\ \chi_2)$*-pre-invex F-IV-F one achieves* χ*-pre-invex F-IV-F, that is*

$$\widetilde{\mathfrak{U}}(\mu + (1-s)\Lambda(\mu,\varkappa)) \preccurlyeq \chi_1(s)\widetilde{\mathfrak{U}}(\mu)\widetilde{+}\chi_1(1-s)\widetilde{\mathfrak{U}}(\varkappa),\ \forall\ \mu,\varkappa\ \in [t,v],\ \in [0,\ 1]. \tag{14}$$

If one attempt to take $\chi_1(s) = s$, $\chi_2(s) \equiv 1$, *then from* $(\chi_1,\ \chi_2)$*-pre-invex F-IV-F one achieves pre-invex F-IV-F, that is*

$$\widetilde{\mathfrak{U}}(\mu + (1-s)\Lambda(\mu,\varkappa)) \preccurlyeq s\widetilde{\mathfrak{U}}(\mu)\widetilde{+}(1-s)\widetilde{\mathfrak{U}}(\varkappa),\ \forall\ \mu,\varkappa\ \in [t,v],\ s \in [0,\ 1]. \tag{15}$$

If one attempts to take $\chi_1(s) = \chi_2(s) \equiv 1$, *then from* $(\chi_1,\ \chi_2)$*-pre-invex F-IV-F one achieves P-pre-invex F-IV-F, that is*

$$\widetilde{\mathfrak{U}}(\mu + (1-s)\Lambda(\mu,\varkappa)) \preccurlyeq \widetilde{\mathfrak{U}}(\mu)\widetilde{+}\widetilde{\mathfrak{U}}(\varkappa),\ \forall\ \mu,\varkappa\ \in [t,v],\ s \in [0,\ 1]. \tag{16}$$

Theorem 2. [41] *Let* $\chi : [0,\ 1] \subseteq [t,v] \to \mathbb{R}$ *be a non-negative real-valued function such that* $\chi 0$ *and let* $\widetilde{\mathfrak{U}} : [t,v] \to \mathbb{F}_0$ *be a F-IV-F, whose* ς*-cuts define the family of IV-Fs* $\mathfrak{U}_\varsigma : [t,v] \to \mathcal{K}_C{}^+ \subset \mathcal{K}_C$ *are given by*

$$\mathfrak{U}_\varsigma(\varkappa) = [\mathfrak{U}_*(\varkappa,\varsigma),\ \mathfrak{U}^*(\varkappa,\varsigma)], \tag{17}$$

for all $\varkappa \in [t,v]$ *and for all* $\varsigma \in [0,\ 1]$. *Then,* $\widetilde{\mathfrak{U}}$ *is* χ*-pre-invex F-IV-F on* $[t,v]$, *if and only if, for all* $\varsigma \in [0,\ 1]$, $\mathfrak{U}_*(\varkappa,\ \varsigma)$ *and* $\mathfrak{U}^*(\varkappa,\ \varsigma)$ *are* χ*-pre-invex function.*

Example 1. *If we attempt to take* $\chi(s) = s$, *for* $s \in [0,\ 1]$ *and the F-IV-F* $\widetilde{\mathfrak{U}} : [0,\ 4] \to \mathbb{F}_0$ *defined by*

$$\widetilde{\mathfrak{U}}(\varkappa)(\theta) = \begin{cases} \frac{\theta}{2e^{\varkappa^2}}\ \theta \in \left[0,\ 2e^{\varkappa^2}\right] \\ \frac{4e^{\varkappa^2}-\theta}{2e^{\varkappa^2}}\ \theta \in \left(2e^{\varkappa^2},\ 4e^{\varkappa^2}\right] \\ 0 \text{ otherwise}, \end{cases}$$

then, for each $\varsigma \in [0,\ 1]$, *we have* $\mathfrak{U}_\varsigma(\varkappa) = \left[2\varsigma e^{\varkappa^2}, 2(2-\varsigma)e^{\varkappa^2}\right]$. *Since end point functions* $\mathfrak{U}_*(\varkappa,\varsigma)$, $\mathfrak{U}^*(\varkappa,\varsigma)$ *are* χ*-pre-invex functions with respect to* $\Lambda(v,t) = v - t$, *for each* $\varsigma \in [0,\ 1]$. *Hence,* $\widetilde{\mathfrak{U}}(\varkappa)$ *is* χ*-pre-invex F-IV-F.*

3. Main Results

For the purpose of giving improved versions of the $H.H$ inequalities on the fuzzy-interval space, let us first retrospect the generalized $H.H$-type inequality for χ-pre-invex

F-IV-Fs. To prove the upcoming results, we need the following assumption regarding the function $\Lambda : [t, v] \times [t, v] \to \mathbb{R},$ which plays an important role in upcoming results.
Condition C. [16]

$$\Lambda(\mu, \varkappa + s\Lambda(\mu, \varkappa)) = (1 - s)\Lambda(\mu, \varkappa),$$
$$\Lambda(\varkappa, \varkappa + s\Lambda(\mu, \varkappa)) = -s\Lambda(\mu, \varkappa).$$

Note that $\forall\, \varkappa,\ \mu \in [t, v]$ and $s \in [0,\ 1]$, then from Condition C we have

$$\Lambda(\varkappa + s_2\Lambda(\mu, \varkappa), \varkappa + s_1\Lambda(\mu, \varkappa)) = (s_2 - s_1)\Lambda(\mu, \varkappa).$$

Clearly for $s = 0$, we have $\Lambda(\mu, \varkappa) = 0$ if and only if $\mu = \varkappa$, for all $\varkappa,\ \mu \in [t, v]$. For the application of Condition C, see [23,42,44].

Theorem 3. *Let* $\widetilde{\mathfrak{U}} : [t,\ t + \Lambda(v, t)] \to \mathbb{F}_0$ *be a* χ*-pre-invex F-IV-F on* $[t,\ t + \Lambda(v, t)]$, *whose* ς*-cuts define the family of IV-Fs* $\mathfrak{U}_\varsigma : [t,\ t + \Lambda(v, t)] \subset \mathbb{R} \to \mathcal{K}_C^+$ *are given by* $\mathfrak{U}_\varsigma(\varkappa) = [\mathfrak{U}_*(\varkappa, \varsigma),\ \mathfrak{U}^*(\varkappa, \varsigma)]$ *for all* $\varkappa \in\ [t,\ t + \Lambda(v, t)]$ *and for all* $\varsigma \in [0,\ 1]$. *If* Λ *satisfies Condition C and* $\widetilde{\mathfrak{U}} \in L([t,\ t + \Lambda(v, t)], \mathbb{F}_0)$, *then*

$$\begin{aligned} \frac{1}{\alpha\chi\left(\frac{1}{2}\right)}\widetilde{\mathfrak{U}}\left(\frac{2t+\Lambda(v,t)}{2}\right) &\preccurlyeq \frac{\Gamma(\alpha)}{(\Lambda(v,t))^\alpha}\left[\mathcal{I}_{t^+}^\alpha\ \widetilde{\mathfrak{U}}(t + \Lambda(v, t))\widetilde{+}\mathcal{I}_{t+\Lambda(v,t)^-}^\alpha\ \widetilde{\mathfrak{U}}(t)\right] \\ &\preccurlyeq \widetilde{\mathfrak{U}}(t)\widetilde{+}\widetilde{\mathfrak{U}}(t + \Lambda(v, t))\int_0^1 s^{\alpha-1}[\chi(s) - \chi(1 - s)]ds \\ &\preccurlyeq \widetilde{\mathfrak{U}}(t)\widetilde{+}\widetilde{\mathfrak{U}}(v)\int_0^1 s^{\alpha-1}[\chi(s) - \chi(1 - s)]ds. \end{aligned} \tag{18}$$

If $\widetilde{\mathfrak{U}}(\varkappa)$ *is pre-incave* F-IV-F, *then*

$$\begin{aligned} \frac{1}{\alpha\chi\left(\frac{1}{2}\right)}\widetilde{\mathfrak{U}}\left(\frac{2t+\Lambda(v,t)}{2}\right) &\succcurlyeq \frac{\Gamma(\alpha)}{(\Lambda(v,t))^\alpha}\left[\mathcal{I}_{t^+}^\alpha\ \widetilde{\mathfrak{U}}(t + \Lambda(v, t))\widetilde{+}\mathcal{I}_{t+\Lambda(v,t)^-}^\alpha\ \widetilde{\mathfrak{U}}(t)\right] \\ &\succcurlyeq \widetilde{\mathfrak{U}}(t)\widetilde{+}\widetilde{\mathfrak{U}}(t + \Lambda(v, t))\int_0^1 s^{\alpha-1}[\chi(s) - \chi(1 - s)]ds. \\ &\succcurlyeq \widetilde{\mathfrak{U}}(t)\widetilde{+}\widetilde{\mathfrak{U}}(v)\int_0^1 s^{\alpha-1}[\chi(s) - \chi(1 - s)]ds. \end{aligned} \tag{19}$$

Proof. Let $\widetilde{\mathfrak{U}} : [t,\ t + \Lambda(v, t)] \to \mathbb{F}_0$ be a χ-pre-invex F-IV-F. If Condition C holds then, by hypothesis, we have

$$\frac{1}{\chi\left(\frac{1}{2}\right)}\widetilde{\mathfrak{U}}\left(\frac{2t + \Lambda(v, t)}{2}\right) \preccurlyeq \widetilde{\mathfrak{U}}(t + (1 - s)\Lambda(v, t))\widetilde{+}\widetilde{\mathfrak{U}}(t + s\Lambda(v, t)).$$

Therefore, for every $\varsigma \in [0,\ 1]$, we have

$$\begin{aligned} \frac{1}{\chi\left(\frac{1}{2}\right)}\mathfrak{U}_*\left(\frac{2t+\Lambda(v,t)}{2},\ \varsigma\right) &\le \mathfrak{U}_*(t + (1 - s)\Lambda(v, t),\ \varsigma) + \mathfrak{U}_*(t + s\Lambda(v, t),\ \varsigma), \\ \frac{1}{\chi\left(\frac{1}{2}\right)}\mathfrak{U}^*\left(\frac{2t+\Lambda(v,t)}{2},\ \varsigma\right) &\le \mathfrak{U}^*(t + (1 - s)\Lambda(v, t),\ \varsigma) + \mathfrak{U}^*(t + s\Lambda(v, t), \varsigma). \end{aligned}$$

Multiplying both sides by $s^{\alpha-1}$ and integrating the obtained result with respect to s over $(0, 1)$, we have

$$\begin{aligned} &\frac{1}{\chi\left(\frac{1}{2}\right)}\int_0^1 s^{\alpha-1}\mathfrak{U}_*\left(\frac{2t+\Lambda(v,t)}{2},\ \varsigma\right)ds \\ &\le \int_0^1 s^{\alpha-1}\mathfrak{U}_*(t + (1 - s)\Lambda(v, t), \varsigma)ds + \int_0^1 s^{\alpha-1}\mathfrak{U}_*(t + s\Lambda(v, t),\ \varsigma)ds, \\ &\frac{1}{\chi\left(\frac{1}{2}\right)}\int_0^1 s^{\alpha-1}\mathfrak{U}^*\left(\frac{2t+\Lambda(v,t)}{2}, \varsigma\right)ds \\ &\le \int_0^1 s^{\alpha-1}\mathfrak{U}^*(t + (1 - s)\Lambda(v, t),\ \varsigma)ds + \int_0^1 s^{\alpha-1}\mathfrak{U}^*(t + s\Lambda(v, t), \varsigma)ds. \end{aligned} \tag{20}$$

Let $\mu = t + (1-s)\Lambda(v,t)$ and $= t + s\Lambda(v,t)$. Then, we have

$$\begin{aligned}
\frac{1}{\alpha\chi\left(\frac{1}{2}\right)}\mathfrak{U}_*\left(\frac{2t+\Lambda(v,t)}{2},\,\varsigma\right) &\le \frac{1}{(\Lambda(v,t))^\alpha}\int_t^{t+\Lambda(v,t)}(t+\Lambda(v,t)-\mu)^{\alpha-1}\mathfrak{U}_*(\mu,\varsigma)d\mu \\
&\quad+\frac{1}{(\Lambda(v,t))^\alpha}\int_t^{t+\Lambda(v,t)}(\varkappa-t)^{\alpha-1}\mathfrak{U}_*(\varkappa,\varsigma)d\varkappa \\
\frac{1}{\alpha\chi\left(\frac{1}{2}\right)}\mathfrak{U}^*\left(\frac{2t+\Lambda(v,t)}{2},\,\varsigma\right) &\le \frac{1}{(\Lambda(v,t))^\alpha}\int_t^{t+\Lambda(v,t)}(t+\Lambda(v,t)-\mu)^{\alpha-1}\mathfrak{U}^*(\mu,\varsigma)d\mu \\
&\quad+\frac{1}{(\Lambda(v,t))^\alpha}\int_t^{t+\Lambda(v,t)}(\varkappa-t)^{\alpha-1}\mathfrak{U}^*(\varkappa,\varsigma)d\varkappa, \\
&\le \frac{\Gamma(\alpha)}{(\Lambda(v,t))^\alpha}\left[\mathcal{I}_{t^+}^\alpha\ \mathfrak{U}_*(t+\Lambda(v,t),\,\varsigma)+\mathcal{I}_{t+\Lambda(v,t)^-}^\alpha\ \mathfrak{U}_*(t,\,\varsigma)\right] \\
&\le \frac{\Gamma(\alpha)}{(\Lambda(v,t))^\alpha}\left[\mathcal{I}_{t^+}^\alpha\ \mathfrak{U}^*(t+\Lambda(v,t),\,\varsigma)+\mathcal{I}_{t+\Lambda(v,t)^-}^\alpha\ \mathfrak{U}^*(t,\,\varsigma)\right],
\end{aligned} \tag{21}$$

that is

$$\begin{aligned}
&\frac{1}{\alpha\chi\left(\frac{1}{2}\right)}\left[\mathfrak{U}_*\left(\frac{2t+\Lambda(v,t)}{2},\,\varsigma\right),\,\mathfrak{U}^*\left(\frac{2t+\Lambda(v,t)}{2},\,\varsigma\right)\right] \\
\le_I\ &\frac{\Gamma(\alpha)}{(\Lambda(v,t))^\alpha}\left[\left[\mathcal{I}_{t^+}^\alpha\ \mathfrak{U}_*(t+\Lambda(v,t),\,\varsigma)+\mathcal{I}_{t+\Lambda(v,t)^-}^\alpha\ \mathfrak{U}_*(t,\,\varsigma)\right],\,\left[\mathcal{I}_{t^+}^\alpha\ \mathfrak{U}^*(t+\Lambda(v,t),\,\varsigma)+\mathcal{I}_{t+\Lambda(v,t)^-}^\alpha\ \mathfrak{U}^*(t,\,\varsigma)\right]\right],
\end{aligned}$$

thus,

$$\frac{1}{\alpha\chi\left(\frac{1}{2}\right)}\ \mathfrak{U}_\varsigma\left(\frac{2t+\Lambda(v,t)}{2}\right) \le_I \frac{\Gamma(\alpha)}{(\Lambda(v,t))^\alpha}\left[\mathcal{I}_{t^+}^\alpha\ \mathfrak{U}_\varsigma(t+\Lambda(v,t))+\mathcal{I}_{t+\Lambda(v,t)^-}^\alpha\ \mathfrak{U}_\varsigma(t)\right]. \tag{22}$$

In a similar way as above, we have

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{(\Lambda(v,t))^\alpha}\left[\mathcal{I}_{t^+}^\alpha\ \mathfrak{U}_\varsigma(t+\Lambda(v,t))+\mathcal{I}_{t+\Lambda(v,t)^-}^\alpha\ \mathfrak{U}_\varsigma(t)\right] \\
&\le_I [\mathfrak{U}_\varsigma(t)+\mathfrak{U}_\varsigma(t+\Lambda(v,t))]\int_0^1 s^{\alpha-1}[\chi(s)-\chi(1-s)]ds.
\end{aligned} \tag{23}$$

Combining (20) and (21), we have

$$\begin{aligned}
\frac{1}{\alpha\chi\left(\frac{1}{2}\right)}\mathfrak{U}_\varsigma\left(\frac{2t+\Lambda(v,t)}{2}\right) &\le_I \frac{\Gamma(\alpha)}{(\Lambda(v,t))^\alpha}\left[\mathcal{I}_{t^+}^\alpha\ \mathfrak{U}_\varsigma(t+\Lambda(v,t))+\mathcal{I}_{t+\Lambda(v,t)^-}^\alpha\ \mathfrak{U}_\varsigma(t)\right] \\
&\le_I [\mathfrak{U}_\varsigma(t)+\mathfrak{U}_\varsigma(t+\Lambda(v,t))]\int_0^1 s^{\alpha-1}[\chi(s)-\chi(1-s)]ds,
\end{aligned}$$

that is

$$\begin{aligned}
\frac{1}{\alpha\chi\left(\frac{1}{2}\right)}\widetilde{\mathfrak{U}}\left(\frac{2t+\Lambda(v,t)}{2}\right) &\preccurlyeq \frac{\Gamma(\alpha)}{(\Lambda(v,t))}\left[\mathcal{I}_{t^+}^\alpha\ \widetilde{\mathfrak{U}}(t+\Lambda(v,t))\widetilde{+}\mathcal{I}_{t+\Lambda(v,t)^-}^\alpha\ \widetilde{\mathfrak{U}}(t)\right] \\
&\preccurlyeq \left[\widetilde{\mathfrak{U}}(t)\widetilde{+}\widetilde{\mathfrak{U}}(t+\Lambda(v,t))\right]\int_0^1 s^{\alpha-1}[\chi(s)-\chi(1-s)]ds \\
&\preccurlyeq \widetilde{\mathfrak{U}}(t)\widetilde{+}\widetilde{\mathfrak{U}}(v)\int_0^1 s^{\alpha-1}[\chi(s)-\chi(1-s)]ds.
\end{aligned}$$

the theorem has been proved. □

Remark 4. *If one attempts to take $\alpha = 1$, then from inequality (18) one achieves the result for χ-pre-invex F-IV-F, see [41]:*

$$\frac{1}{2\chi\left(\frac{1}{2}\right)}\ \widetilde{\mathfrak{U}}\left(\frac{2t+\Lambda(v,t)}{2}\right) \preccurlyeq \frac{1}{\Lambda(v,t)}\ (FR)\int_t^{t+\Lambda(v,t)}\widetilde{\mathfrak{U}}(\varkappa)d \preccurlyeq \left[\widetilde{\mathfrak{U}}(t)\ \widetilde{+}\ \widetilde{\mathfrak{U}}(t+\Lambda(v,t))\right]\int_0^1\chi(s)ds. \tag{24}$$

If one attempt to take $\chi(s) = s$, then from inequality (18) one achieves the result for pre-invex F-IV-F, see [24]:

$$\widetilde{\mathfrak{U}}\left(\frac{2t+\Lambda(v,t)}{2}\right) \preccurlyeq \frac{\Gamma(\alpha+1)}{2(\Lambda(v,t))^\alpha}\left[\mathcal{I}_{t^+}^\alpha\ \widetilde{\mathfrak{U}}(t+\Lambda(v,t))\widetilde{+}\mathcal{I}_{t+\Lambda(v,t)^-}^\alpha\ \widetilde{\mathfrak{U}}(t)\right] \preccurlyeq \frac{\widetilde{\mathfrak{U}}(t)\widetilde{+}\widetilde{\mathfrak{U}}(t+\Lambda(v,t))}{2}. \tag{25}$$

Let one attempt to take $\alpha = 1$ *and* $\chi(s) = s$*. Then, from inequality (18) one acquires the result for pre-invex- IV-F given in [41]:*

$$\widetilde{\mathfrak{U}}\left(\frac{2t+\Lambda(v,t)}{2}\right) \preccurlyeq \frac{1}{\Lambda(v,t)}\ (FR)\int_{t}^{t+\Lambda(v,t)} \widetilde{\mathfrak{U}}(\varkappa)d\varkappa \preccurlyeq \frac{\widetilde{\mathfrak{U}}(t)\widetilde{+}\widetilde{\mathfrak{U}}(v)}{2}. \tag{26}$$

If one attempt to take $\mathfrak{U}_*(\varkappa,\ \varsigma) = \mathfrak{U}^*(\varkappa,\ \varsigma)$ *and* $\varsigma = 1$*, then, from inequality (18) one acquires coming inequality given in [42]:*

$$\begin{aligned}\frac{1}{\alpha\chi\left(\frac{1}{2}\right)}\mathfrak{U}\left(\frac{2t+\Lambda(v,t)}{2}\right) &\leq \frac{\Gamma(\alpha)}{(\Lambda(v,t))^{\alpha}}\left[\mathcal{I}_{t^+}^{\alpha}\ \mathfrak{U}(t+\Lambda(v,t)) + \mathcal{I}_{t+\Lambda(v,t)^-}^{\alpha}\ \mathfrak{U}(t)\right] \\ &\leq [\mathfrak{U}(t)+\mathfrak{U}(t+\Lambda(v,t))]\int_0^1 s^{\alpha-1}[\chi(s)-\chi(1-s)]ds.\end{aligned} \tag{27}$$

Let one attempt to take $\alpha = 1 = \varsigma$ *and* $\mathfrak{U}_*(\varkappa,\ \varsigma) = \mathfrak{U}^*(\varkappa,\ \varsigma)$*. Then, from inequality (18) one acquires coming inequality given in [43]:*

$$\frac{1}{2\chi\left(\frac{1}{2}\right)}\ \mathfrak{U}\left(\frac{2t+\Lambda(v,t)}{2}\right) \leq \frac{1}{\Lambda(v,t)}\ (R)\int_{t}^{t+\Lambda(v,t)} \mathfrak{U}(\varkappa)d\varkappa \leq [\mathfrak{U}(t)+\mathfrak{U}(t+\Lambda(v,t))]\int_0^1 \chi(s)ds. \tag{28}$$

Example 2. *If we attempt to take* $= \frac{1}{2}$*,* $\chi(s) = s$*, for all* $s \in [0,\ 1]$ *and the F-IV-F* $\widetilde{\mathfrak{U}} : [t,\ t+\Lambda(v,t)] = [2,\ 2+\Lambda(3,2)] \to \mathbb{F}_0$*, defined by*

$$\widetilde{\mathfrak{U}}(\varkappa)(\theta) = \begin{cases} \frac{\theta}{2-\varkappa^{\frac{1}{2}}},\ \theta \in \left[0,\ 2-\varkappa^{\frac{1}{2}}\right] \\ \frac{2\left(2-\varkappa^{\frac{1}{2}}\right)-\theta}{2-\varkappa^{\frac{1}{2}}},\ \theta \in \left(2-\varkappa^{\frac{1}{2}},\ 2\left(2-\varkappa^{\frac{1}{2}}\right)\right] \\ 0,\ \text{otherwise}. \end{cases} \tag{29}$$

Then, for each $\varsigma \in [0,\ 1]$*, we have* $\mathfrak{U}_{\varsigma}(\varkappa) = \left[\varsigma\left(2-\varkappa^{\frac{1}{2}}\right), (2-\varsigma)\left(2-\varkappa^{\frac{1}{2}}\right)\right]$*. Since left and right end-point functions* $\mathfrak{U}_*(\varkappa,\varsigma) = \varsigma\left(2-\varkappa^{\frac{1}{2}}\right)$*,* $\mathfrak{U}^*(\varkappa,\ \varsigma) = (2-\varsigma)\left(2-\varkappa^{\frac{1}{2}}\right)$*, are* χ*-pre-invex functions for each* $\varsigma \in [0,\ 1]$*, then* $\widetilde{\mathfrak{U}}(\varkappa)$ *is* χ*-pre-invex F-IV-F. We clearly see that* $\widetilde{\mathfrak{U}} \in L([t,\ t+\Lambda(v,t)],\mathbb{F}_0)$ *and*

$$\begin{aligned} \frac{1}{\alpha\chi\left(\frac{1}{2}\right)}\mathfrak{U}_*\left(\frac{2t+\Lambda(v,t)}{2},\ \varsigma\right) &= \mathfrak{U}_*\left(\frac{5}{2},\ \varsigma\right) = \varsigma\frac{4-\sqrt{10}}{8} \\ \frac{1}{\alpha\chi\left(\frac{1}{2}\right)}\mathfrak{U}^*\left(\frac{2t+\Lambda(v,t)}{2},\ \varsigma\right) &= \mathfrak{U}^*\left(\frac{5}{2},\ \varsigma\right) = (2-\varsigma)\frac{4-\sqrt{10}}{8}, \\ \frac{\mathfrak{U}_*(t,\varsigma)+\mathfrak{U}_*(t+\Lambda(v,t),\ \varsigma)}{2}\int_0^1 s^{\alpha-1}[\chi(s)-\chi(1-s)]ds &= \varsigma\left(4-\sqrt{2}-\sqrt{3}\right) \\ \frac{\mathfrak{U}^*(t,\varsigma)+\mathfrak{U}^*(t+\Lambda(v,t),\ \varsigma)}{2}\int_0^1 s^{\alpha-1}[\chi(s)-\chi(1-s)]ds &= (2-\varsigma)\left(4-\sqrt{2}-\sqrt{3}\right). \end{aligned} \tag{30}$$

Note that

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{(\Lambda(v,t))^{\alpha}}\left[\mathcal{I}^{\alpha}_{t^{+}}\ \mathfrak{U}_*(t+\Lambda(v,t),\ \varsigma)+\mathcal{I}^{\alpha}_{t+\Lambda(v,t)^{-}}\ \mathfrak{U}_*(t,\varsigma)\right] \\
&=\frac{\Gamma\left(\frac{1}{2}\right)}{2}\frac{1}{\sqrt{\pi}}\int_2^3(3-\varkappa)^{\frac{-1}{2}}\ .\ \varsigma\left(2-\varkappa^{\frac{1}{2}}\right)d\varkappa \\
&+\frac{\Gamma\left(\frac{1}{2}\right)}{2}\frac{1}{\sqrt{\pi}}\int_2^3(\varkappa-2)^{\frac{-1}{2}}\ .\ \varsigma\left(2-\varkappa^{\frac{1}{2}}\right)d\varkappa \\
&=\frac{1}{2}\varsigma\left[\frac{7393}{10{,}000}+\frac{9501}{10{,}000}\right] \\
&=\varsigma\frac{8447}{20{,}000}. \\
&\frac{\Gamma(\alpha)}{(\Lambda(v,t))^{\alpha}}\left[\mathcal{I}^{\alpha}_{t^{+}}\ \mathfrak{U}^*(t+\Lambda(v,t),\ \varsigma)+\mathcal{I}^{\alpha}_{t+\Lambda(v,t)^{-}}\ \mathfrak{U}^*(t,\varsigma)\right] \\
&=\frac{\Gamma\left(\frac{1}{2}\right)}{2}\frac{1}{\sqrt{\pi}}\int_2^3(3-\varkappa)^{\frac{-1}{2}}\ .\ (2-\varsigma)\left(2-\varkappa^{\frac{1}{2}}\right)d\varkappa \\
&+\frac{\Gamma\left(\frac{1}{2}\right)}{2}\frac{1}{\sqrt{\pi}}\int_2^3(\varkappa-2)^{\frac{-1}{2}}\ .\ (2-\varsigma)\left(2-\varkappa^{\frac{1}{2}}\right)d\varkappa \\
&=\frac{1}{2}(2-\varsigma)\left[\frac{7393}{10{,}000}+\frac{9501}{10{,}000}\right] \\
&=(2-\varsigma)\frac{8447}{20{,}000}.
\end{aligned}$$

Therefore

$$\left[\varsigma\frac{4-\sqrt{10}}{8},(2-\varsigma)\frac{4-\sqrt{10}}{8}\right]\leq_I\left[\varsigma\frac{8447}{20{,}000},(2-\varsigma)\frac{8447}{20{,}000}\right]\leq_I\left[\varsigma\left(4-\sqrt{2}-\sqrt{3}\right),(2-\varsigma)\left(4-\sqrt{2}-\sqrt{3}\right)\right],$$

and Theorem 3 is verified.

We get various fuzzy-interval-fractional-integral inequalities connected to fuzzy-interval-fractional-$H{\cdot}H$ inequalities from Theorems 4 and 5 via products of two χ-pre-invex *F-IV-Fs*.

Theorem 4. *Let* $\widetilde{\mathfrak{U}},\widetilde{\mathcal{H}}:[t,\ t+\Lambda(v,t)]\to\mathbb{F}_0$ *be* χ_1*-pre-invex and* χ_2*-pre-invex F-IV-Fs on* $[t,\ t+\Lambda(v,t)]$*, respectively, whose* ς*-cuts* $\mathfrak{U}_\varsigma,\ \mathcal{H}_\varsigma:[t,\ t+\Lambda(v,t)]\subset\mathbb{R}\to\mathcal{K}_C^+$ *are defined by* $\mathfrak{U}_\varsigma(\varkappa)=[\mathfrak{U}_*(\varkappa,\varsigma),\ \mathfrak{U}^*(\varkappa,\varsigma)]$ *and* $\mathcal{H}_\varsigma(\varkappa)=[\mathcal{H}_*(\varkappa,\varsigma),\ \mathcal{H}^*(\varkappa,\varsigma)]$ *for all* $\varkappa\in[t,\ t+\Lambda(v,t)]$ *and for all* $\varsigma\in[0,\ 1]$*. If* Λ *satisfies Condition C and* $\widetilde{\mathfrak{U}}\widetilde{\times}\widetilde{\mathcal{H}}\in\ L([t,\ t+\Lambda(v,t)],\mathbb{F}_0)$*, then*

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{(\Lambda(v,t))^{\alpha}}\left[\mathcal{I}^{\alpha}_{t^{+}}\ \widetilde{\mathfrak{U}}(t+\Lambda(v,t))\widetilde{\times}\widetilde{\mathcal{H}}(t+\Lambda(v,t))+\mathcal{I}^{\alpha}_{t+\Lambda(v,t)^{-}}\widetilde{\mathfrak{U}}(t)\widetilde{\times}\widetilde{\mathcal{H}}(t)\right] \\
&\preccurlyeq\widetilde{\Delta}(t,t+\Lambda(v,t))\int_0^1 s^{\alpha-1}[\chi_1(s)\chi_2(s)+\chi_1(1-s)\chi_2(1-s)]ds \\
&+\widetilde{\nabla}(t,t+\Lambda(v,t))\int_0^1 s^{\alpha-1}[\chi_1(s)\chi_2(1-s)+\chi_1(1-s)\chi_2(s)]ds.
\end{aligned} \tag{31}$$

where $\widetilde{\Delta}(t,t+\Lambda(v,t))=\widetilde{\mathfrak{U}}(t)\widetilde{\times}\widetilde{\mathcal{H}}(t)\ \widetilde{+}\ \widetilde{\mathfrak{U}}(t+\Lambda(v,t))\widetilde{\times}\widetilde{\mathcal{H}}(t+\Lambda(v,t))$, $\widetilde{\nabla}(t,t+\Lambda(v,t))=\widetilde{\mathfrak{U}}(t)\widetilde{\times}\widetilde{\mathcal{H}}(t+\Lambda(v,t))\ \widetilde{+}\ \widetilde{\mathfrak{U}}(t+\Lambda(v,t))\widetilde{\times}\widetilde{\mathcal{H}}(t)$, *and* $\Delta_\varsigma(t,t+\Lambda(v,t))=[\Delta_*((t,t+\Lambda(v,t)),\ \varsigma),\ \Delta^*((t,t+\Lambda(v,t)),\ \varsigma)]$ *and* $\nabla_\varsigma(t,t+\Lambda(v,t))=[\nabla_*((t,t+\Lambda(v,t)),\ \varsigma),\ \nabla^*((t,t+\Lambda(v,t)),\ \varsigma)]$.

Proof. Since $\widetilde{\mathfrak{U}},\ \widetilde{\mathcal{H}}$ both are χ_1-pre-invex and χ_2-pre-invex *F-IV-F* then, for each $\varsigma\in[0,\ 1]$ we have

$$\begin{aligned}
\mathfrak{U}_*(t+(1-s)\Lambda(v,t),\ \varsigma)&=\mathfrak{U}_*(t+\Lambda(v,t)+s\Lambda(t,t+\Lambda(v,t)),\ \varsigma) \\
&\leq\chi_1(s)\mathfrak{U}_*(t,\varsigma)+\chi_1(1-s)\mathfrak{U}_*(t+\Lambda(v,t),\ \varsigma) \\
\mathfrak{U}^*(t+(1-s)\Lambda(v,t),\ \varsigma)&=\mathfrak{U}^*(t+\Lambda(v,t)+s\Lambda(t,t+\Lambda(v,t)),\ \varsigma) \\
&\leq\chi_1(s)\mathfrak{U}^*(t,\varsigma)+\chi_1(1-s)\mathfrak{U}^*(t+\Lambda(v,t),\ \varsigma).
\end{aligned}$$

and

$$\begin{array}{c} \mathcal{H}_*(t+(1-s)\Lambda(v,t),\ \varsigma) = \mathcal{H}_*(t+(1-s)\Lambda(v,t),\ \varsigma) \\ \leq \chi_2(s)\mathcal{H}_*(t,\varsigma) + \chi_2(1-s)\mathcal{H}_*(t+\Lambda(v,t),\ \varsigma) \\ \mathcal{H}^*(t+(1-s)\Lambda(v,t),\ \varsigma) = \mathcal{H}^*(t+(1-s)\Lambda(v,t),\ \varsigma) \\ \leq \chi_2(s)\mathcal{H}^*(t,\varsigma) + \chi_2(1-s)\mathcal{H}^*(t+\Lambda(v,t),\ \varsigma). \end{array} \tag{32}$$

From the definition of χ-pre-invex *F-IV-Fs* it follows that $\widetilde{0} \preccurlyeq \widetilde{\mathfrak{U}}(\varkappa)$ and $\widetilde{0} \preccurlyeq \widetilde{\mathcal{H}}()$, so

$$\begin{array}{c} \mathfrak{U}_*(t+(1-s)\Lambda(v,t),\ \varsigma) \times \mathcal{H}_*(t+(1-s)\Lambda(v,t),\ \varsigma) \\ \leq \chi_1(s)\chi_2(s)\mathfrak{U}_*(t,\varsigma) \times \mathcal{H}_*(t,\varsigma) + \chi_1(1-s)\chi_2(1-s)\mathfrak{U}_*(t+\Lambda(v,t),\ \varsigma) \times \mathcal{H}_*(t+\Lambda(v,t),\ \varsigma) \\ +\chi_1(s)\chi_2(1-s)\mathfrak{U}_*(t,\varsigma) \times \mathcal{H}_*(t+\Lambda(v,t),\ \varsigma) + \chi_1(1-s)\chi_2(s)\mathfrak{U}_*(t+\Lambda(v,t),\ \varsigma) \times \mathcal{H}_*(t,\varsigma) \\ \mathfrak{U}^*(t+(1-s)\Lambda(v,t),\ \varsigma) \times \mathcal{H}^*(t+(1-s)\Lambda(v,t),\ \varsigma) \\ \leq \chi_1(s)\chi_2(s)\mathfrak{U}^*(t,\varsigma) \times \mathcal{H}^*(t,\varsigma) + \chi_1(1-s)\chi_2(1-s)\mathfrak{U}^*(t+\Lambda(v,t),\ \varsigma) \times \mathcal{H}^*(t+\Lambda(v,t),\ \varsigma) \\ +\chi_1(s)\chi_2(1-s)\mathfrak{U}^*(t,\varsigma) \times \mathcal{H}^*(t+\Lambda(v,t),\ \varsigma) + \chi_1(1-s)\chi_2(s)\mathfrak{U}^*(t+\Lambda(v,t),\ \varsigma) \times \mathcal{H}^*(t,\varsigma). \end{array} \tag{33}$$

Analogously, we have

$$\begin{array}{c} \mathfrak{U}_*(t+s\Lambda(v,t),\ \varsigma)\mathcal{H}_*(t+s\Lambda(v,t),\ \varsigma) \\ \leq \chi_1(1-s)\chi_2(1-s)\mathfrak{U}_*(t,\varsigma) \times \mathcal{H}_*(t,\varsigma) + \chi_1(s)\chi_2(s)\mathfrak{U}_*(t+\Lambda(v,t),\ \varsigma) \times \mathcal{H}_*(t+\Lambda(v,t),\ \varsigma) \\ +\chi_1(1-s)\chi_2(s)\mathfrak{U}_*(t,\varsigma) \times \mathcal{H}_*(t+\Lambda(v,t),\ \varsigma) + \chi_1(s)\chi_2(1-s)\mathfrak{U}_*(t+\Lambda(v,t),\ \varsigma) \times \mathcal{H}_*(t,\varsigma) \\ \mathfrak{U}^*(t+s\Lambda(v,t),\ \varsigma) \times \mathcal{H}^*(t+s\Lambda(v,t),\ \varsigma) \\ \leq \chi_1(1-s)\chi_2(1-s)\mathfrak{U}^*(t,\varsigma) \times \mathcal{H}^*(t,\varsigma) + \chi_1(s)\chi_2(s)\mathfrak{U}^*(t+\Lambda(v,t),\ \varsigma) \times \mathcal{H}^*(t+\Lambda(v,t),\ \varsigma) \\ +\chi_1(1-s)\chi_2(s)\mathfrak{U}^*(t,\varsigma) \times \mathcal{H}^*(t+\Lambda(v,t),\ \varsigma) + \chi_1(s)\chi_2(1-s)\mathfrak{U}^*(t+\Lambda(v,t),\ \varsigma) \times \mathcal{H}^*(t,\varsigma). \end{array} \tag{34}$$

Adding (27) and (28), we have

$$\begin{array}{c} \mathfrak{U}_*(t+(1-s)\Lambda(v,t),\ \varsigma) \times \mathcal{H}_*(t+(1-s)\Lambda(v,t),\ \varsigma) \\ +\mathfrak{U}_*(t+s\Lambda(v,t),\ \varsigma) \times \mathcal{H}_*(t+s\Lambda(v,t),\ \varsigma) \\ \leq [\chi_1(s)\chi_2(s) + \chi_1(1-s)\chi_2(1-s)][\mathfrak{U}_*(t,\varsigma) \times \mathcal{H}_*(t,\varsigma) + \mathfrak{U}_*(t+\Lambda(v,t),\ \varsigma) \times \mathcal{H}_*(t+\Lambda(v,t),\ \varsigma)] \\ +[\chi_1(s)\chi_2(1-s) + \chi_1(1-s)\chi_2(s)][\mathfrak{U}_*(t+\Lambda(v,t),\ \varsigma) \times \mathcal{H}_*(t,\varsigma) + \mathfrak{U}_*(t,\varsigma) \times \mathcal{H}_*(t+\Lambda(v,t),\ \varsigma)] \\ \mathfrak{U}^*(t+(1-s)\Lambda(v,t),\ \varsigma) \times \mathcal{H}^*(t+(1-s)\Lambda(v,t),\ \varsigma) \\ +\mathfrak{U}^*(t+s\Lambda(v,t),\ \varsigma) \times \mathcal{H}^*(t+s\Lambda(v,t),\ \varsigma) \\ \leq [\chi_1(s)\chi_2(s) + \chi_1(1-s)\chi_2(1-s)][\mathfrak{U}^*(t,\varsigma) \times \mathcal{H}^*(t,\varsigma) + \mathfrak{U}^*(t+\Lambda(v,t),\ \varsigma) \times \mathcal{H}^*(t+\Lambda(v,t),\ \varsigma)] \\ +[\chi_1(s)\chi_2(1-s) + \chi_1(1-s)\chi_2(s)][\mathfrak{U}^*(t+\Lambda(v,t),\ \varsigma) \times \mathcal{H}^*(t,\varsigma) + \mathfrak{U}^*(t,\varsigma) \times \mathcal{H}^*(t+\Lambda(v,t),\ \varsigma)]. \end{array} \tag{35}$$

Taking multiplication of (29) with $s^{\alpha-1}$ and integrating the obtained result with respect to s over (0, 1), we have

$$\begin{array}{c} \int_0^1 s^{\alpha-1}\mathfrak{U}_*(t+(1-s)\Lambda(v,t),\ \varsigma) \times \mathcal{H}_*(t+(1-s)\Lambda(v,t),\ \varsigma) \\ +s^{\alpha-1}\mathfrak{U}_*(t+s\Lambda(v,t),\ \varsigma) \times \mathcal{H}_*(t+s\Lambda(v,t),\ \varsigma)ds \\ \leq \Delta_*((t,t+\Lambda(v,t)),\ \varsigma)\int_0^1 s^{\alpha-1}[\chi_1(s)\chi_2(s) + \chi_1(1-s)\chi_2(1-s)]ds \\ +\nabla_*((t,t+\Lambda(v,t)),\ \varsigma)\int_0^1 s^{\alpha-1}[\chi_1(s)\chi_2(1-s) + \chi_1(1-s)\chi_2(s)]ds \\ \int_0^1 s^{\alpha-1}\mathfrak{U}^*(t+(1-s)\Lambda(v,t),\ \varsigma) \times \mathcal{H}^*(t+(1-s)\Lambda(v,t),\ \varsigma) \\ +s^{\alpha-1}\mathfrak{U}^*(t+s\Lambda(v,t),\ \varsigma) \times \mathcal{H}^*(t+s\Lambda(v,t),\ \varsigma)ds \\ \leq \Delta^*((t,t+\Lambda(v,t)),\ \varsigma)\int_0^1 s^{\alpha-1}[\chi_1(s)\chi_2(s) + \chi_1(1-s)\chi_2(1-s)]ds \\ +\nabla^*((t,t+\Lambda(v,t)),\ \varsigma)\int_0^1 s^{\alpha-1}[\chi_1(s)\chi_2(1-s) + \chi_1(1-s)\chi_2(s)]ds. \end{array}$$

It follows that,

$$\begin{array}{c} \frac{\Gamma(\alpha)}{(\Lambda(v,t))^\alpha}\left[\mathcal{I}_{t^+}^{\alpha}\ \mathfrak{U}_*(t+\Lambda(v,t),\ \varsigma) \times \mathcal{H}_*(t+\Lambda(v,t),\ \varsigma) + \mathcal{I}_{t+\Lambda(v,t)^-}^{\alpha}\ \mathfrak{U}_*(t,\varsigma) \times \mathcal{H}_*(t,\varsigma)\right] \\ \leq \Delta_*((t,t+\Lambda(v,t)),\ \varsigma)\int_0^1 s^{\alpha-1}[\chi_1(s)\chi_2(s) + \chi_1(1-s)\chi_2(1-s)]ds \\ +\nabla_*((t,t+\Lambda(v,t)),\ \varsigma)\int_0^1 s^{\alpha-1}[\chi_1(s)\chi_2(1-s) + \chi_1(1-s)\chi_2(s)]ds \\ \frac{\Gamma(\alpha)}{(\Lambda(v,t))^\alpha}\left[\mathcal{I}_{t^+}^{\alpha}\ \mathfrak{U}^*(t+\Lambda(v,t),\ \varsigma) \times \mathcal{H}^*(t+\Lambda(v,t),\ \varsigma) + \mathcal{I}_{t+\Lambda(v,t)^-}^{\alpha}\ \mathfrak{U}^*(t,\varsigma) \times \mathcal{H}^*(t,\varsigma)\right] \\ \leq\ \Delta^*((t,t+\Lambda(v,t)),\ \varsigma)\int_0^1 s^{\alpha-1}[\chi_1(s)\chi_2(s) + \chi_1(1-s)\chi_2(1-s)]ds \\ +\nabla^*((t,t+\Lambda(v,t)),\ \varsigma)\int_0^1 s^{\alpha-1}[\chi_1(s)\chi_2(1-s) + \chi_1(1-s)\chi_2(s)]ds. \end{array}$$

It follows that

$$\frac{\Gamma(\alpha)}{(\Lambda(v,t))^{\alpha}}[\mathcal{I}_{t^+}^{\alpha}\ \mathfrak{U}_*(t+\Lambda(v,t),\ \varsigma)\times\mathcal{H}_*(t+\Lambda(v,t),\ \varsigma)+\mathcal{I}_{t+\Lambda(v,t)^-}^{\alpha}\ \mathfrak{U}_*(t,\varsigma)\times\mathcal{H}_*(t,\varsigma),\ \mathcal{I}_{t^+}^{\alpha}\ \mathfrak{U}^*(t+\Lambda(v,t),\ \varsigma)\times$$
$$\mathcal{H}^*(t+\Lambda(v,t),\ \varsigma)+\mathcal{I}_{t+\Lambda(v,t)^-}^{\alpha}\ \mathfrak{U}^*(t,\varsigma)\times\mathcal{H}^*(t,\ \varsigma)]$$
$$\leq_I [\Delta_*((t,t+\Lambda(v,t)),\ \varsigma),\ \Delta^*((t,t+\Lambda(v,t)),\ \varsigma)]\int_0^1 s^{\alpha-1}[\chi_1(s)\chi_2(s)+\chi_1(1-s)\chi_2(1-s)]ds$$
$$+[\nabla_*((t,t+\Lambda(v,t)),\ \varsigma),\ \nabla^*((t,t+\Lambda(v,t)),\ \varsigma)]\int_0^1 s^{\alpha-1}[\chi_1(s)\chi_2(1-s)+\chi_1(1-s)\chi_2(s)]ds,$$

that is

$$\frac{\Gamma(\alpha)}{(\Lambda(v,t))^{\alpha}}\left[\mathcal{I}_{t^+}^{\alpha}\ \mathfrak{U}_\varsigma(t+\Lambda(v,t))\times\mathcal{H}_\varsigma(t+\Lambda(v,t))+\mathcal{I}_{t+\Lambda(v,t)^-}^{\alpha}\mathfrak{U}_\varsigma(t)\times\mathcal{H}_\varsigma(t)\right]$$
$$\leq_I \Delta_\varsigma(t,t+\Lambda(v,t))\int_0^1 s^{\alpha-1}[\chi_1(s)\chi_2(s)+\chi_1(1-s)\chi_2(1-s)]ds$$
$$+\nabla_\varsigma(t,t+\Lambda(v,t))\int_0^1 s^{\alpha-1}[\chi_1(s)\chi_2(1-s)+\chi_1(1-s)\chi_2(s)]ds.$$

Thus,

$$\frac{\Gamma(\alpha)}{(\Lambda(v,t))^{\alpha}}\left[\mathcal{I}_{t^+}^{\alpha}\ \widetilde{\mathfrak{U}}(t+\Lambda(v,t))\widetilde{\times}\widetilde{\mathcal{H}}(t+\Lambda(v,t))+\mathcal{I}_{t+\Lambda(v,t)^-}^{\alpha}\widetilde{\mathfrak{U}}(t)\widetilde{\times}\widetilde{\mathcal{H}}(t)\right]$$
$$\preccurlyeq \widetilde{\Delta}(t,t+\Lambda(v,t))\int_0^1 s^{\alpha-1}[\chi_1(s)\chi_2(s)+\chi_1(1-s)\chi_2(1-s)]ds$$
$$+\widetilde{\nabla}(t,t+\Lambda(v,t))\int_0^1 s^{\alpha-1}[\chi_1(s)\chi_2(1-s)+\chi_1(1-s)\chi_2(s)]ds.$$

and the theorem has been established. □

Theorem 5. *Let $\widetilde{\mathfrak{U}},\widetilde{\mathcal{H}}\ :[t,\ t+\Lambda(v,t)]\to\mathbb{F}_0$ be two χ_1-pre-invex and χ_2-pre-invex F-IV-Fs, respectively, whose ς-cuts define the family of IV-Fs $\mathfrak{U}_\varsigma,\ \mathcal{H}_\varsigma:[t,\ t+\Lambda(v,t)]\subset\mathbb{R}\to\mathcal{K}_C^+$ are given by $\mathfrak{U}_\varsigma(\varkappa)=[\mathfrak{U}_*(\varkappa,\varsigma),\ \mathfrak{U}^*(\varkappa,\varsigma)]$ and $\mathcal{H}_\varsigma(\varkappa)=[\mathcal{H}_*(\varkappa,\varsigma),\ \mathcal{H}^*(\varkappa,\varsigma)]$ for all $\varkappa\in[t,\ t+\Lambda(v,t)]$ and for all $\varsigma\in[0,\ 1]$. If Λ satisfies Condition C and $\widetilde{\mathfrak{U}}\widetilde{\times}\widetilde{\mathcal{H}}\in L([t,\ t+\Lambda(v,t)],\mathbb{F}_0)$, then*

$$\frac{1}{\alpha\chi_1\left(\frac{1}{2}\right)\chi_2\left(\frac{1}{2}\right)}\widetilde{\mathfrak{U}}\left(\frac{2t+\Lambda(v,t)}{2}\right)\widetilde{\times}\widetilde{\mathcal{H}}\left(\frac{2t+\Lambda(v,t)}{2}\right)\preccurlyeq\frac{\Gamma(\alpha)}{(\Lambda(v,t))^{\alpha}}\left[\mathcal{I}_{t^+}^{\alpha}\ \widetilde{\mathfrak{U}}(t+\Lambda(v,t))\widetilde{\times}\widetilde{\mathcal{H}}(v)\widetilde{+}\mathcal{I}_{t+\Lambda(v,t)^-}^{\alpha}\ \widetilde{\mathfrak{U}}(t)\widetilde{\times}\widetilde{\mathcal{H}}(t)\right]$$
$$\widetilde{+}\widetilde{\nabla}(t,t+\Lambda(v,t))\int_0^1\left[s^{\alpha-1}+(1-s)^{\alpha-1}\right]\chi_1(s)\chi_2(1-s)ds \tag{36}$$
$$\widetilde{+}\widetilde{\Delta}(t,t+\Lambda(v,t))\int_0^1\left[s^{\alpha-1}+(1-s)^{\alpha-1}\right]\chi_1(1-s)\chi_2(1-s)ds.$$

where $\widetilde{\Delta}(t,t+\Lambda(v,t))=\widetilde{\mathfrak{U}}(t)\widetilde{\times}\widetilde{\mathcal{H}}(t)\ \widetilde{+}\ \widetilde{\mathfrak{U}}(v)\widetilde{\times}\widetilde{\mathcal{H}}(v)$, $\widetilde{\nabla}(t,t+\Lambda(v,t))=\widetilde{\mathfrak{U}}(t)\widetilde{\times}\widetilde{\mathcal{H}}(v)\ \widetilde{+}\ \widetilde{\mathfrak{U}}(v)\widetilde{\times}\widetilde{\mathcal{H}}(t)$, and $\Delta_\varsigma(t,t+\Lambda(v,t))=[\Delta_*((t,t+\Lambda(v,t)),\ \varsigma),\ \Delta^*((t,t+\Lambda(v,t)),\ \varsigma)]$ and $\nabla_\varsigma(t,t+\Lambda(v,t))=[\nabla_*((t,t+\Lambda(v,t)),\ \varsigma),\ \nabla^*((t,t+\Lambda(v,t)),\ \varsigma)]$.

Proof. Consider $\widetilde{\mathfrak{U}}, \widetilde{\mathcal{H}} : [t,\ t+\Lambda(v,t)] \to \mathbb{F}_0$ are χ_1-pre-invex and χ_2-pre-invex *F-IV-Fs*. Then, by hypothesis, for each $\varsigma \in [0,\ 1]$, we have

$$\begin{aligned}
&\mathfrak{U}_*\left(\tfrac{2t+\Lambda(v,t)}{2},\varsigma\right) \times \mathcal{H}_*\left(\tfrac{2t+\Lambda(v,t)}{2},\varsigma\right)\\
&\mathfrak{U}^*\left(\tfrac{2t+\Lambda(v,t)}{2},\varsigma\right) \times \mathcal{H}^*\left(\tfrac{2t+\Lambda(v,t)}{2},\varsigma\right)\\
&\le \chi_1\left(\tfrac{1}{2}\right)\chi_2\left(\tfrac{1}{2}\right)\left[\begin{array}{c} \mathfrak{U}_*(t+(1-s)\Lambda(v,t),\ \varsigma) \times \mathcal{H}_*(t+(1-s)\Lambda(v,t),\ \varsigma) \\ +\mathfrak{U}_*(t+(1-s)\Lambda(v,t),\ \varsigma) \times \mathcal{H}_*(t+s\Lambda(v,t),\ \varsigma) \end{array}\right]\\
&\quad + \chi_1\left(\tfrac{1}{2}\right)\chi_2\left(\tfrac{1}{2}\right)\left[\begin{array}{c} \mathfrak{U}_*(t+s\Lambda(v,t),\ \varsigma) \times \mathcal{H}_*(t+(1-s)\Lambda(v,t),\ \varsigma) \\ +\mathfrak{U}_*(t+s\Lambda(v,t),\ \varsigma) \times \mathcal{H}_*(t+s\Lambda(v,t),\ \varsigma) \end{array}\right]\\
&\le \chi_1\left(\tfrac{1}{2}\right)\chi_2\left(\tfrac{1}{2}\right)\left[\begin{array}{c} \mathfrak{U}^*(t+(1-s)\Lambda(v,t),\ \varsigma) \times \mathcal{H}^*(t+(1-s)\Lambda(v,t),\ \varsigma) \\ +\mathfrak{U}^*(t+(1-s)\Lambda(v,t),\ \varsigma) \times \mathcal{H}^*(t+s\Lambda(v,t),\ \varsigma) \end{array}\right]\\
&\quad + \chi_1\left(\tfrac{1}{2}\right)\chi_2\left(\tfrac{1}{2}\right)\left[\begin{array}{c} \mathfrak{U}^*(t+s\Lambda(v,t),\ \varsigma) \times \mathcal{H}^*(t+(1-s)\Lambda(v,t),\ \varsigma) \\ +\mathfrak{U}^*(t+s\Lambda(v,t),\ \varsigma) \times \mathcal{H}^*(t+s\Lambda(v,t),\ \varsigma) \end{array}\right],\\
&\le \chi_1\left(\tfrac{1}{2}\right)\chi_2\left(\tfrac{1}{2}\right)\left[\begin{array}{c} \mathfrak{U}_*(t+(1-s)\Lambda(v,t),\ \varsigma) \times \mathcal{H}_*(t+(1-s)\Lambda(v,t),\ \varsigma) \\ +\mathfrak{U}_*(t+s\Lambda(v,t),\ \varsigma) \times \mathcal{H}_*(t+s\Lambda(v,t),\ \varsigma) \end{array}\right]\\
&\quad + \chi_1\left(\tfrac{1}{2}\right)\chi_2\left(\tfrac{1}{2}\right)\left[\begin{array}{c} (\chi_1(s)\mathfrak{U}_*(t,\ \varsigma) + \chi_1(1-s)\mathfrak{U}_*(t+\Lambda(v,t),,\ \varsigma)) \\ \times(\chi_2(1-s)\mathcal{H}_*(t,\ \varsigma) + \chi_2(s)\mathcal{H}_*(t+\Lambda(v,t),,\ \varsigma)) \\ +(\chi_1(1-s)\mathfrak{U}_*(t,\ \varsigma) + \chi_1(s)\mathfrak{U}_*(t+\Lambda(v,t),,\ \varsigma)) \\ \times(\chi_2(s)\mathcal{H}_*(t,\ \varsigma) + \chi_2(1-s)\mathcal{H}_*(t+\Lambda(v,t),,\ \varsigma)) \end{array}\right]\\
&\le \chi_1\left(\tfrac{1}{2}\right)\chi_2\left(\tfrac{1}{2}\right)\left[\begin{array}{c} \mathfrak{U}^*(t+(1-s)\Lambda(v,t),\ \varsigma) \times \mathcal{H}^*(t+(1-s)\Lambda(v,t),\ \varsigma) \\ +\mathfrak{U}^*(t+s\Lambda(v,t),\ \varsigma) \times \mathcal{H}^*(t+s\Lambda(v,t),\ \varsigma) \end{array}\right]\\
&\quad + \chi_1\left(\tfrac{1}{2}\right)\chi_2\left(\tfrac{1}{2}\right)\left[\begin{array}{c} (\chi_1(s)\mathfrak{U}^*(t,\ \varsigma) + \chi_1(1-s)\mathfrak{U}^*(t+\Lambda(v,t),,\ \varsigma)) \\ \times(\chi_2(1-s)\mathcal{H}^*(t,\ \varsigma) + \chi_2(s)\mathcal{H}^*(t+\Lambda(v,t),,\ \varsigma)) \\ +(\chi_1(1-s)\mathfrak{U}^*(t,\ \varsigma) + \chi_1(s)\mathfrak{U}^*(t+\Lambda(v,t),,\ \varsigma)) \\ \times(\chi_2(s)\mathcal{H}^*(t,\ \varsigma) + \chi_2(1-s)\mathcal{H}^*(t+\Lambda(v,t),,\ \varsigma)) \end{array}\right],\\
&= \chi_1\left(\tfrac{1}{2}\right)\chi_2\left(\tfrac{1}{2}\right)\left[\begin{array}{c} \mathfrak{U}_*(t+(1-s)\Lambda(v,t),\ \varsigma) \times \mathcal{H}_*(t+(1-s)\Lambda(v,t),\ \varsigma) \\ +\mathfrak{U}_*(t+s\Lambda(v,t),\ \varsigma) \times \mathcal{H}_*(t+s\Lambda(v,t),\ \varsigma) \end{array}\right]\\
&+ \chi_1\left(\tfrac{1}{2}\right)\chi_2\left(\tfrac{1}{2}\right)\left[\begin{array}{c} \{\chi_1(s)\chi_2(1-s) + \chi_1(1-s)\chi_2(s)\}\nabla_*((t,t+\Lambda(v,t)),\ \varsigma) \\ +\{\chi_1(s)\chi_2(s) + \chi_1(1-s)\chi_2(1-s)\}\Delta_*((t,t+\Lambda(v,t)),\ \varsigma) \end{array}\right]\\
&= \chi_1\left(\tfrac{1}{2}\right)\chi_2\left(\tfrac{1}{2}\right)\left[\begin{array}{c} \mathfrak{U}^*(t+(1-s)\Lambda(v,t),\ \varsigma) \times \mathcal{H}^*(t+(1-s)\Lambda(v,t),\ \varsigma) \\ +\mathfrak{U}^*(t+s\Lambda(v,t),\ \varsigma) \times \mathcal{H}^*(t+s\Lambda(v,t),\ \varsigma) \end{array}\right]\\
&+ \chi_1\left(\tfrac{1}{2}\right)\chi_2\left(\tfrac{1}{2}\right)\left[\begin{array}{c} \{\chi_1(s)\chi_2(1-s) + \chi_1(1-s)\chi_2(s)\}\nabla^*((t,t+\Lambda(v,t)),\ \varsigma) \\ +\{\chi_1(s)\chi_2(s) + \chi_1(1-s)\chi_2(1-s)\}\Delta^*((t,t+\Lambda(v,t)),\ \varsigma) \end{array}\right].
\end{aligned} \tag{37}$$

Taking multiplication of (30) with $s^{\alpha-1}$ and integrating over $(0,\ 1)$, we get

$$\begin{aligned}
&\frac{1}{\alpha\chi_1\left(\frac{1}{2}\right)\chi_2\left(\frac{1}{2}\right)}\mathfrak{U}_*\left(\tfrac{2t+\Lambda(v,t)}{2},\varsigma\right) \times \mathcal{H}_*\left(\tfrac{2t+\Lambda(v,t)}{2},\varsigma\right)\\
&\le \frac{\Gamma(\alpha)}{(\Lambda(v,t))^\alpha}\left[\mathcal{I}^\alpha_{t^+}\ \mathfrak{U}_*(t+\Lambda(v,t)) \times \mathcal{H}_*(t+\Lambda(v,t)) + \mathcal{I}^\alpha_{t+\Lambda(v,t)^-}\ \mathfrak{U}_*(t) \times \mathcal{H}_*(t)\right]\\
&\quad +\nabla_*((t,t+\Lambda(v,t)),\ \varsigma)\int_0^1\left[s^{\alpha-1} + (1-s)^{\alpha-1}\right]\chi_1(s)\chi_2(1-s)ds\\
&\quad +\Delta_*((t,t+\Lambda(v,t)),\ \varsigma)\int_0^1\left[s^{\alpha-1} + (1-s)^{\alpha-1}\right]\chi_1(1-s)\chi_2(1-s)ds\\
&\frac{1}{\alpha\chi_1\left(\frac{1}{2}\right)\chi_2\left(\frac{1}{2}\right)}\mathfrak{U}^*\left(\tfrac{2t+\Lambda(v,t)}{2},\varsigma\right) \times \mathcal{H}^*\left(\tfrac{2t+\Lambda(v,t)}{2},\varsigma\right)\\
&\le \frac{\Gamma(\alpha)}{(\Lambda(v,t))^\alpha}\left[\mathcal{I}^\alpha_{t^+}\ \mathfrak{U}^*(t+\Lambda(v,t)) \times \mathcal{H}^*(t+\Lambda(v,t)) + \mathcal{I}^\alpha_{t+\Lambda(v,t)^-}\ \mathfrak{U}^*(t) \times \mathcal{H}^*(t)\right]\\
&\quad +\nabla^*((t,t+\Lambda(v,t)),\ \varsigma)\int_0^1\left[s^{\alpha-1} + (1-s)^{\alpha-1}\right]\chi_1(s)\chi_2(1-s)ds\\
&\quad +\Delta^*((t,t+\Lambda(v,t)),\ \varsigma)\int_0^1\left[s^{\alpha-1} + (1-s)^{\alpha-1}\right]\chi_1(1-s)\chi_2(1-s)ds,
\end{aligned} \tag{38}$$

It follows that

$$\begin{array}{c}\frac{1}{\alpha\chi_1\left(\frac{1}{2}\right)\chi_2\left(\frac{1}{2}\right)}\,\mathfrak{U}_\varsigma\left(\frac{2t+\Lambda(v,t)}{2}\right)\times\mathcal{H}_\varsigma\left(\frac{2t+\Lambda(v,t)}{2}\right)\\ \leq_I \frac{\Gamma(\alpha)}{(\Lambda(v,t))^\alpha}\left[\mathcal{I}^\alpha_{t^+}\,\mathfrak{U}_\varsigma(t+\Lambda(v,t))\times\mathcal{H}_\varsigma(t+\Lambda(v,t))+\mathcal{I}^\alpha_{t+\Lambda(v,t)^-}\,\mathfrak{U}_\varsigma(t)\times\mathcal{H}_\varsigma(t)\right]\\ +\nabla_\varsigma(t,t+\Lambda(v,t))\int_0^1\left[s^{\alpha-1}+(1-s)^{\alpha-1}\right]\chi_1(1-s)\chi_2(1-s)ds\\ +\Delta_\varsigma(t,t+\Lambda(v,t))\int_0^1\left[s^{\alpha-1}+(1-s)^{\alpha-1}\right]\chi_1(1-s)\chi_2(1-s)ds,\end{array}$$

that is

$$\begin{array}{c}\frac{1}{\alpha\chi_1\left(\frac{1}{2}\right)\chi_2\left(\frac{1}{2}\right)}\,\widetilde{\mathfrak{U}}\left(\frac{2t+\Lambda(v,t)}{2}\right)\widetilde{\times}\widetilde{\mathcal{H}}\left(\frac{2t+\Lambda(v,t)}{2}\right)\\ \preccurlyeq \frac{\Gamma(\alpha)}{(\Lambda(v,t))^\alpha}\left[\mathcal{I}^\alpha_{t^+}\,\widetilde{\mathfrak{U}}(t+\Lambda(v,t))\widetilde{\times}\widetilde{\mathcal{H}}(t+\Lambda(v,t))\widetilde{+}\mathcal{I}^\alpha_{t+\Lambda(v,t)^-}\,\widetilde{\mathfrak{U}}(t)\widetilde{\times}\widetilde{\mathcal{H}}(t)\right]\\ \widetilde{+}\widetilde{\nabla}(t,t+\Lambda(v,t))\int_0^1\left[s^{\alpha-1}+(1-s)^{\alpha-1}\right]\chi_1(s)\chi_2(1-s)ds\\ \widetilde{+}\widetilde{\Delta}(t,t+\Lambda(v,t))\int_0^1\left[s^{\alpha-1}+(1-s)^{\alpha-1}\right]\chi_1(1-s)\chi_2(1-s)ds.\end{array}$$

Hence, the required result. □

Theorems 6 and 7 are, respectively, linked to the right and left halves of the standard $H{\cdot}H$-Fejér inequality. The left and right component of the conventional $H{\cdot}H$-Fejér inequality are obtained using a fuzzy-Riemann–Liouville-fractional integral inequalities, and these are known as the first and second fuzzy-fractional-$H{\cdot}H$-Fejér inequalities, respectively. Firstly, we obtain second $H{\cdot}H$-Fejér inequality.

Theorem 6. *Let $\widetilde{\mathfrak{U}}:[t,\ t+\Lambda(v,t)]\to\mathbb{F}_0$ be a χ-pre-invex F-IV-F with $t<v$, whose ς-cuts define the family of IV-Fs $\mathfrak{U}_\varsigma:[t,\ t+\Lambda(v,t)]\subset\mathbb{R}\to\mathcal{K}_C^+$ are given by $\mathfrak{U}_\varsigma(\varkappa)=[\mathfrak{U}_*(\varkappa,\varsigma),\ \mathfrak{U}^*(\varkappa,\varsigma)]$ for all $\varkappa\in\ [t,\ t+\Lambda(v,t)]$ and for all $\varsigma\in[0,\ 1]$. If $\widetilde{\mathfrak{U}}\in L([t,\ t+\Lambda(v,t)],\mathbb{F}_0)$ and $\Omega:[t,\ t+\Lambda(v,t)]\to\mathbb{R},\ \Omega(\varkappa)\geq 0,$ and symmetric with respect to $\frac{2t+\Lambda(v,t)}{2}$, then*

$$\begin{array}{c}\frac{\Gamma(\alpha)}{(\Lambda(v,t))^\alpha}\left[\mathcal{I}^\alpha_{t^+}\,\widetilde{\mathfrak{U}}\Omega(t+\Lambda(v,t))\widetilde{+}\mathcal{I}^\alpha_{t+\Lambda(v,t)^-}\,\widetilde{\mathfrak{U}}\Omega(t)\right]\\ \preccurlyeq\left(\widetilde{\mathfrak{U}}(t)\widetilde{+}\widetilde{\mathfrak{U}}(t+\Lambda(v,t))\right)\int_0^1 s^{\alpha-1}[\chi(s)+\chi(1-s)]\Omega(t+s\Lambda(v,t))ds\\ \preccurlyeq\left(\widetilde{\mathfrak{U}}(t)\widetilde{+}\widetilde{\mathfrak{U}}(v)\right)\int_0^1 s^{\alpha-1}[\chi(s)+\chi(1-s)]\Omega(t+s\Lambda(v,t))ds.\end{array}\tag{39}$$

If $\widetilde{\mathfrak{U}}$ is pre-incave F-IV-F, then inequality (31) is reversed.

Proof. Let $\widetilde{\mathfrak{U}}$ be a χ-pre-invex *F-IV-F* and $s^{\alpha-1}\Omega(t+(1-s)\Lambda(v,t))\geq 0$. Then, for each $\varsigma\in[0,\ 1]$, we have

$$\begin{array}{c}s^{\alpha-1}\mathfrak{U}_*(t+(1-s)\Lambda(v,t),\ \varsigma)\Omega(t+(1-s)\Lambda(v,t))\\ \leq s^{\alpha-1}(\chi(s)\mathfrak{U}_*(t,\ \varsigma)+\chi(1-s)\mathfrak{U}_*(t+\Lambda(v,t),\ \varsigma))\Omega(t+(1-s)\Lambda(v,t))\\ s^{\alpha-1}\mathfrak{U}^*(t+(1-s)\Lambda(v,t),\ \varsigma)\Omega(t+(1-s)\Lambda(v,t))\\ \leq s^{\alpha-1}(\chi(s)\mathfrak{U}^*(t,\ \varsigma)+\chi(1-s)\mathfrak{U}^*(t+\Lambda(v,t),\ \varsigma))\Omega(t+(1-s)\Lambda(v,t)),\end{array}\tag{40}$$

and

$$\begin{array}{c}s^{\alpha-1}\mathfrak{U}_*(t+s\Lambda(v,t),\ \varsigma)\Omega(t+s\Lambda(v,t))\\ \leq s^{\alpha-1}(\chi(1-s)\mathfrak{U}_*(t,\ \varsigma)+\chi(s)\mathfrak{U}_*(t+\Lambda(v,t),\ \varsigma))\Omega(t+s\Lambda(v,t))\\ s^{\alpha-1}\mathfrak{U}^*(t+s\Lambda(v,t),\ \varsigma)\Omega(t+s\Lambda(v,t))\\ \leq s^{\alpha-1}(\chi(1-s)\mathfrak{U}^*(t,\ \varsigma)+\chi(s)\mathfrak{U}^*(t+\Lambda(v,t),\ \varsigma))\Omega(t+s\Lambda(v,t)).\end{array}\tag{41}$$

After adding (32) and (33), and integrating over $[0,\ 1]$, we get

$$\begin{aligned}
&\int_0^1 s^{\alpha-1}\mathfrak{U}_*(t+(1-s)\Lambda(v,t),\ \varsigma)\Omega(t+(1-s)\Lambda(v,t))ds \\
&+\int_0^1 s^{\alpha-1}\mathfrak{U}_*(t+s\Lambda(v,t),\ \varsigma)\Omega(t+s\Lambda(v,t))ds \\
&\leq \int_0^1 \begin{bmatrix} s^{\alpha-1}\mathfrak{U}_*(t,\ \varsigma)\{\chi(s)\Omega(t+(1-s)\Lambda(v,t))+\chi(1-s)\Omega(t+s\Lambda(v,t))\} \\ +s^{\alpha-1}\mathfrak{U}_*(t+\Lambda(v,t),\varsigma)\{\chi(1-s)\Omega(t+(1-s)\Lambda(v,t))+\chi(s)\Omega(t+s\Lambda(v,t))\} \end{bmatrix} ds, \\
&= \mathfrak{U}_*(t,\ \varsigma)\int_0^1 s^{\alpha-1}[\chi(s)+\chi(1-s)]\Omega(t+(1-s)\Lambda(v,t))\ ds \\
&+\mathfrak{U}_*(t+\Lambda(v,t),\ \varsigma)\int_0^1 s^{\alpha-1}[\chi(s)+\chi(1-s)]\Omega(t+s\Lambda(v,t))\ ds, \\
&\int_0^1 s^{\alpha-1}\mathfrak{U}^*(t+s\Lambda(v,t),\ \varsigma)\Omega(t+s\Lambda(v,t))ds \\
&+\int_0^1 s^{\alpha-1}\mathfrak{U}^*(t+(1-s)\Lambda(v,t),\ \varsigma)\Omega(t+(1-s)\Lambda(v,t))ds \\
&\leq \int_0^1 \begin{bmatrix} s^{\alpha-1}\mathfrak{U}^*(t,\ \varsigma)\{\chi(s)\Omega(t+(1-s)\Lambda(v,t))+\chi(1-s)\Omega(t+s\Lambda(v,t))\} \\ +s^{\alpha-1}\mathfrak{U}^*(t+\Lambda(v,t),\ \varsigma)\{\chi(1-s)\Omega(t+(1-s)\Lambda(v,t))+\chi(s)\Omega(t+s\Lambda(v,t))\} \end{bmatrix} ds, \\
&= \mathfrak{U}^*(t,\ \varsigma)\int_0^1 s^{\alpha-1}[\chi(s)+\chi(1-s)]\Omega(t+(1-s)\Lambda(v,t))\ ds \\
&+\mathfrak{U}^*(t+\Lambda(v,t),\ \varsigma)\int_0^1 s^{\alpha-1}[\chi(s)+\chi(1-s)]\Omega(t+s\Lambda(v,t))\ ds.
\end{aligned} \tag{42}$$

Taking the right-hand side of inequality (34), we have

$$\begin{aligned}
&\int_0^1 s^{\alpha-1}\mathfrak{U}_*(t+(1-s)\Lambda(v,t),\ \varsigma)\Omega(t+s\Lambda(v,t))ds \\
&+\int_0^1 s^{\alpha-1}\mathfrak{U}_*(t+s\Lambda(v,t),\ \varsigma)\Omega(t+s\Lambda(v,t))ds \\
&= \frac{1}{(\Lambda(v,t))^\alpha}\int_t^{t+\Lambda(v,t)}(-t)^{\alpha-1}\mathfrak{U}_*(2t+\Lambda(v,t)-,\varsigma)\Omega(\varkappa)d \\
&+\frac{1}{(\Lambda(v,t))^\alpha}\int_t^{t+\Lambda(v,t)}(-t)^{\alpha-1}\mathfrak{U}_*(\varkappa,\varsigma)\Omega(\varkappa)d \\
&= \frac{1}{(\Lambda(v,t))^\alpha}\int_t^{t+\Lambda(v,t)}(t+\Lambda(v,t)-)^{\alpha-1}\mathfrak{U}_*(\varkappa,\varsigma)\Omega(2t+\Lambda(v,t)-)d \\
&+\frac{1}{(\Lambda(v,t))^\alpha}\int_t^{t+\Lambda(v,t)}(-t)^{\alpha-1}\mathfrak{U}_*(\varkappa,\varsigma)\Omega(\varkappa)d \\
&= \frac{\Gamma(\alpha)}{(\Lambda(v,t))^\alpha}\Big[\mathcal{I}_{t^+}^\alpha\ \mathfrak{U}_*\Omega(t+\Lambda(v,t))+\mathcal{I}_{t+\Lambda(v,t)^-}^\alpha\ \mathfrak{U}_*\Omega(t)\Big], \\
&\int_0^1 s^{\alpha-1}\mathfrak{U}^*(t+(1-s)\Lambda(v,t),\ \varsigma)\Omega(t+s\Lambda(v,t))ds \\
&+\int_0^1 s^{\alpha-1}\mathfrak{U}^*(t+s\Lambda(v,t),\ \varsigma)\Omega(t+s\Lambda(v,t))ds \\
&= \frac{\Gamma(\alpha)}{(\Lambda(v,t))^\alpha}\Big[\mathcal{I}_{t^+}^\alpha\ \mathfrak{U}^*\Omega(t+\Lambda(v,t))+\mathcal{I}_{t+\Lambda(v,t)^-}^\alpha\ \mathfrak{U}^*\Omega(t)\Big].
\end{aligned} \tag{43}$$

From (35), we have

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{(\Lambda(v,t))^\alpha}\Big[\mathcal{I}_{t^+}^\alpha\ \mathfrak{U}_*\Omega(t+\Lambda(v,t))+\mathcal{I}_{t+\Lambda(v,t)^-}^\alpha\ \mathfrak{U}_*\Omega(t)\Big] \\
&\leq \frac{\mathfrak{U}_*(t,\ \varsigma)+\mathfrak{U}_*(t+\Lambda(v,t),\ \varsigma)}{2}\int_0^1 s^{\alpha-1}\begin{bmatrix}\chi(s)\\+\chi(1-s)\end{bmatrix}\Omega(t+s\Lambda(v,t)) \\
&\frac{\Gamma(\alpha)}{(\Lambda(v,t))^\alpha}\Big[\mathcal{I}_{t^+}^\alpha\ \mathfrak{U}^*\Omega(t+\Lambda(v,t))+\mathcal{I}_{t+\Lambda(v,t)^-}^\alpha\ \mathfrak{U}^*\Omega(t)\Big] \\
&\leq \frac{\mathfrak{U}^*(t,\ \varsigma)+\mathfrak{U}^*(t+\Lambda(v,t),\ \varsigma)}{2}\int_0^1 s^{\alpha-1}\begin{bmatrix}\chi(s)\\+\chi(1-s)\end{bmatrix}\Omega(t+s\Lambda(v,t))\ ,
\end{aligned} \tag{44}$$

that is

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{(\Lambda(v,t))^\alpha}\Big[\mathcal{I}_{t^+}^\alpha\ \mathfrak{U}_*\Omega(t+\Lambda(v,t))+\mathcal{I}_{t+\Lambda(v,t)^-}^\alpha\ \mathfrak{U}_*\Omega(t),\ \mathcal{I}_{t^+}^\alpha\ \mathfrak{U}^*\Omega(t+\Lambda(v,t))+\mathcal{I}_{t+\Lambda(v,t)^-}^\alpha\ \mathfrak{U}^*\Omega(t)\Big] \\
&\leq_I \left[\frac{\mathfrak{U}_*(t,\ \varsigma)+\mathfrak{U}_*(t+\Lambda(v,t),\ \varsigma)}{2},\ \frac{\mathfrak{U}^*(t,\ \varsigma)+\mathfrak{U}^*(t+\Lambda(v,t),\ \varsigma)}{2}\right]\int_0^1 s^{\alpha-1}[\chi(s)+\chi(1-s)]\Omega(t+s\Lambda(v,t))ds,
\end{aligned}$$

Hence

$$\begin{aligned}&\frac{\Gamma(\alpha)}{(\Lambda(v,t))^{\alpha}}\left[\mathcal{I}_{t^+}^{\alpha}\ \widetilde{\mathfrak{U}}\Omega(t+\Lambda(v,t))\widetilde{+}\mathcal{I}_{t+\Lambda(v,t)^-}^{\alpha}\ \widetilde{\mathfrak{U}}\Omega(t)\right]\\ \preccurlyeq\ &\frac{\widetilde{\mathfrak{U}}(t)\widetilde{+}\widetilde{\mathfrak{U}}(t+\Lambda(v,t))}{2}\int_0^1 s^{\alpha-1}[\chi(s)+\chi(1-s)]\Omega(t+s\Lambda(v,t))ds.\\ \preccurlyeq\ &\frac{\widetilde{\mathfrak{U}}(t)\widetilde{+}\widetilde{\mathfrak{U}}(v)}{2}\int_0^1 s^{\alpha-1}[\chi(s)+\chi(1-s)]\Omega(t+s\Lambda(v,t))ds.\end{aligned} \tag{45}$$

□

Theorem 7. *Let $\widetilde{\mathfrak{U}}:[t,\ t+\Lambda(v,t)]\to\mathbb{F}_0$ be a χ-pre-invex F-IV-F with $t<t+\Lambda(v,t)$, whose ς-cuts define the family of IV-Fs $\mathfrak{U}_\varsigma:[t,\ t+\Lambda(v,t)]\subset\mathbb{R}\to\mathcal{K}_C^+$ are given by $\mathfrak{U}_\varsigma(\varkappa)=[\mathfrak{U}_*(\varkappa,\varsigma),\ \mathfrak{U}^*(\varkappa,\varsigma)]$ for all $\varkappa\in[t,\ t+\Lambda(v,t)]$ and for all $\varsigma\in[0,\ 1]$. Let $\widetilde{\mathfrak{U}}\in L([t,\ t+\Lambda(v,t)],\mathbb{F}_0)$ and $\Omega:[t,\ t+\Lambda(v,t)]\to\mathbb{R},\ \Omega(\varkappa)\geq 0,$ symmetric with respect to $\frac{2t+\Lambda(v,t)}{2}$. If Λ satisfies Condition C, then*

$$\frac{1}{2\chi\left(\frac{1}{2}\right)}\widetilde{\mathfrak{U}}\left(\frac{2t+\Lambda(v,t)}{2}\right)\left[\mathcal{I}_{t^+}^{\alpha}\ \Omega(t+\Lambda(v,t))+\mathcal{I}_{t+\Lambda(v,t)^-}^{\alpha}\ \Omega(t)\right]\preccurlyeq\left[\mathcal{I}_{t^+}^{\alpha}\ \widetilde{\mathfrak{U}}\Omega(t+\Lambda(v,t))+\mathcal{I}_{t+\Lambda(v,t)^-}^{\alpha}\ \widetilde{\mathfrak{U}}\Omega(t)\right]. \tag{46}$$

If $\widetilde{\mathfrak{U}}$ is pre-incave F-IV-F, then inequality (36) is reversed.

Proof. Since $\widetilde{\mathfrak{U}}$ is a χ-pre-invex *F-IV-F*, then for $\varsigma\in[0,\ 1]$, we have

$$\begin{aligned}\mathfrak{U}_*\left(\tfrac{2t+\Lambda(v,t)}{2},\ \varsigma\right)&\leq\chi\left(\tfrac{1}{2}\right)(\mathfrak{U}_*(t+(1-s)\Lambda(v,t),\ \varsigma)+\mathfrak{U}_*(t+s\Lambda(v,t),\ \varsigma))\\ \mathfrak{U}^*\left(\tfrac{2t+\Lambda(v,t)}{2},\ \varsigma\right)&\leq\chi\left(\tfrac{1}{2}\right)(\mathfrak{U}^*(t+(1-s)\Lambda(v,t),\ \varsigma)+\mathfrak{U}^*(t+s\Lambda(v,t),\ \varsigma)).\end{aligned} \tag{47}$$

Since $\Omega(t+(1-s)\Lambda(v,t))=\Omega(t+s\Lambda(v,t))$, then by multiplying (37) by $s^{\alpha-1}\Omega(t+s\Lambda(v,t))$ and integrate it with respect to s over $[0,\ 1]$, we obtain

$$\begin{aligned}&\mathfrak{U}_*\left(\tfrac{2t+\Lambda(v,t)}{2},\ \varsigma\right)\int_0^1 s^{\alpha-1}\Omega(t+s\Lambda(v,t))ds\\ \leq\ &\chi\left(\tfrac{1}{2}\right)\begin{pmatrix}\int_0^1 s^{\alpha-1}\mathfrak{U}_*(t+(1-s)\Lambda(v,t),\ \varsigma)\Omega(t+s\Lambda(v,t))ds\\ +\int_0^1 s^{\alpha-1}\mathfrak{U}_*(t+s\Lambda(v,t),\ \varsigma)\Omega(t+s\Lambda(v,t))ds\end{pmatrix},\\ &\mathfrak{U}^*\left(\tfrac{2t+\Lambda(v,t)}{2},\ \varsigma\right)\int_0^1 s^{\alpha-1}\Omega(t+s\Lambda(v,t))ds\\ \leq\ &\chi\left(\tfrac{1}{2}\right)\begin{pmatrix}\int_0^1 s^{\alpha-1}\mathfrak{U}^*(t+(1-s)\Lambda(v,t),\ \varsigma)\Omega(t+s\Lambda(v,t))ds\\ +\int_0^1 s^{\alpha-1}\mathfrak{U}^*(t+s\Lambda(v,t),\ \varsigma)\Omega(t+s\Lambda(v,t))ds\end{pmatrix}.\end{aligned} \tag{48}$$

Let $=t+s\Lambda(v,t)$. Then, right-hand side of inequality (38), we have

$$\begin{aligned}&\int_0^1 s^{\alpha-1}\mathfrak{U}_*(t+(1-s)\Lambda(v,t),\ \varsigma)\Omega(t+s\Lambda(v,t))ds\\ &\quad+\int_0^1 s^{\alpha-1}\mathfrak{U}_*(t+s\Lambda(v,t),\ \varsigma)\Omega(t+s\Lambda(v,t))ds\\ =\ &\frac{1}{(\Lambda(v,t))^{\alpha}}\int_t^{t+\Lambda(v,t)}(-t)^{\alpha-1}\mathfrak{U}_*(2t+\Lambda(v,t)-,\varsigma)\Omega(\varkappa)d\\ &\quad+\frac{1}{(\Lambda(v,t))^{\alpha}}\int_t^{t+\Lambda(v,t)}(-t)^{\alpha-1}\mathfrak{U}_*(\varkappa,\varsigma)\Omega(\varkappa)d\\ =\ &\frac{1}{(\Lambda(v,t))^{\alpha}}\int_t^{t+\Lambda(v,t)}(-t)^{\alpha-1}\mathfrak{U}_*(\varkappa,\varsigma)\Omega(2t+\Lambda(v,t)-)d\\ &\quad+\frac{1}{(\Lambda(v,t))^{\alpha}}\int_t^{t+\Lambda(v,t)}(-t)^{\alpha-1}\mathfrak{U}_*(\varkappa,\varsigma)\Omega(\varkappa)d\\ =\ &\frac{\Gamma(\alpha)}{(\Lambda(v,t))^{\alpha}}\left[\mathcal{I}_{t^+}^{\alpha}\ \mathfrak{U}_*\Omega(t+\Lambda(v,t))+\mathcal{I}_{t+\Lambda(v,t)^-}^{\alpha}\ \mathfrak{U}_*\Omega(t)\right],\\ &\int_0^1 s^{\alpha-1}\mathfrak{U}^*(t+(1-s)\Lambda(v,t),\ \varsigma)\Omega(t+s\Lambda(v,t))ds\\ &\quad+\int_0^1 s^{\alpha-1}\mathfrak{U}^*(t+s\Lambda(v,t),\ \varsigma)\Omega(t+s\Lambda(v,t))ds\\ =\ &\frac{\Gamma(\alpha)}{(\Lambda(v,t))^{\alpha}}\left[\mathcal{I}_{t^+}^{\alpha}\ \mathfrak{U}^*\Omega(t+\Lambda(v,t))+\mathcal{I}_{t+\Lambda(v,t)^-}^{\alpha}\ \mathfrak{U}^*\Omega(t)\right].\end{aligned} \tag{49}$$

Then from (39), we have

$$\frac{1}{2\chi\left(\frac{1}{2}\right)}\mathfrak{U}_*\left(\frac{2t+\Lambda(v,t)}{2},\ \varsigma\right)\left[\mathcal{I}^{\alpha}_{t^+}\ \Omega(t+\Lambda(v,t))+\mathcal{I}^{\alpha}_{t+\Lambda(v,t)^-}\ \Omega(t)\right]\leq\left[\mathcal{I}^{\alpha}_{t^+}\ \mathfrak{U}_*\Omega(t+\Lambda(v,t))+\mathcal{I}^{\alpha}_{t+\Lambda(v,t)^-}\ \mathfrak{U}_*\Omega(t)\right]$$
$$\frac{1}{2\chi\left(\frac{1}{2}\right)}\mathfrak{U}^*\left(\frac{2t+\Lambda(v,t)}{2},\ \varsigma\right)\left[\mathcal{I}^{\alpha}_{t^+}\ \Omega(t+\Lambda(v,t))+\mathcal{I}^{\alpha}_{t+\Lambda(v,t)^-}\ \Omega(t)\right]\leq\left[\mathcal{I}^{\alpha}_{t^+}\ \mathfrak{U}^*\Omega(t+\Lambda(v,t))+\mathcal{I}^{\alpha}_{t+\Lambda(v,t)^-}\ \mathfrak{U}^*\Omega(t)\right],$$

from which, we have

$$\frac{1}{2\chi\left(\frac{1}{2}\right)}\left[\mathfrak{U}_*\left(\frac{2t+\Lambda(v,t)}{2},\ \varsigma\right),\ \mathfrak{U}^*\left(\frac{2t+\Lambda(v,t)}{2},\ \varsigma\right)\right]\left[\mathcal{I}^{\alpha}_{t^+}\ \Omega(t+\Lambda(v,t))+\mathcal{I}^{\alpha}_{t+\Lambda(v,t)^-}\ \Omega(t)\right]$$
$$\leq_I\left[\mathcal{I}^{\alpha}_{t^+}\ \mathfrak{U}_*\Omega(t+\Lambda(v,t))+\mathcal{I}^{\alpha}_{t+\Lambda(v,t)^-}\ \mathfrak{U}_*\Omega(t),\ \mathcal{I}^{\alpha}_{t^+}\ \mathfrak{U}^*\Omega(t+\Lambda(v,t))+\mathcal{I}^{\alpha}_{t+\Lambda(v,t)^-}\ \mathfrak{U}^*\Omega(t)\right],$$

it follows that

$$\frac{1}{2\chi\left(\frac{1}{2}\right)}\mathfrak{U}_\varsigma\left(\frac{2t+\Lambda(v,t)}{2}\right)\left[\mathcal{I}^{\alpha}_{t^+}\ \Omega(t+\Lambda(v,t))+\mathcal{I}^{\alpha}_{t+\Lambda(v,t)^-}\ \Omega(t)\right]\leq_I\left[\mathcal{I}^{\alpha}_{t^+}\ \mathfrak{U}_\varsigma\Omega(t+\Lambda(v,t))+\mathcal{I}^{\alpha}_{t+\Lambda(v,t)^-}\ \mathfrak{U}_\varsigma\Omega(t)\right],$$

that is

$$\frac{1}{2\chi\left(\frac{1}{2}\right)}\widetilde{\mathfrak{U}}\left(\frac{2t+\Lambda(v,t)}{2}\right)\left[\mathcal{I}^{\alpha}_{t^+}\ \Omega(t+\Lambda(v,t))\widetilde{+}\mathcal{I}^{\alpha}_{t+\Lambda(v,t)^-}\ \Omega(t)\right]\preccurlyeq\left[\mathcal{I}^{\alpha}_{t^+}\ \widetilde{\mathfrak{U}}\Omega(t+\Lambda(v,t))\widetilde{+}\mathcal{I}^{\alpha}_{t+\Lambda(v,t)^-}\ \widetilde{\mathfrak{U}}\Omega(t)\right]. \quad (50)$$

This completes the proof. □

Remark 5. *If one attempt to take $\Omega(\varkappa)=1$, then from (31) and (36) one achieves Theorem 3.*

If one attempt to take $\chi(s)=s$, then from (31) and (36) one achieves coming inequality, see [24]:

$$\begin{aligned}\widetilde{\mathfrak{U}}\left(\frac{2t+\Lambda(v,t)}{2}\right)\left[\mathcal{I}^{\alpha}_{t^+}\ \Omega(t+\Lambda(v,t))+\mathcal{I}^{\alpha}_{t+\Lambda(v,t)^-}\ \Omega(t)\right]&\preccurlyeq\left[\mathcal{I}^{\alpha}_{t^+}\ \widetilde{\mathfrak{U}}\Omega(t+\Lambda(v,t))\widetilde{+}\mathcal{I}^{\alpha}_{t+\Lambda(v,t)^-}\ \widetilde{\mathfrak{U}}\Omega(t)\right]\\&\preccurlyeq\frac{\widetilde{\mathfrak{U}}(t)\widetilde{+}\widetilde{\mathfrak{U}}(t+\Lambda(v,t))}{2}\left[\mathcal{I}^{\alpha}_{t^+}\ \Omega(t+\Lambda(v,t))+\mathcal{I}^{\alpha}_{t+\Lambda(v,t)^-}\ \Omega(t)\right].\\&\preccurlyeq\frac{\widetilde{\mathfrak{U}}(t)\widetilde{+}\widetilde{\mathfrak{U}}(v)}{2}\left[\mathcal{I}^{\alpha}_{t^+}\ \Omega(t+\Lambda(v,t))+\mathcal{I}^{\alpha}_{t+\Lambda(v,t)^-}\ \Omega(t)\right].\end{aligned} \quad (51)$$

Let one attempt to take $\chi(s)=s$ and $\alpha=1$. Then, from (31) and (36) one achieves coming inequality for pre-invex F-IV-F, see [41].

$$\widetilde{\mathfrak{U}}\left(\frac{2t+\Lambda(v,t)}{2}\right)\preccurlyeq\frac{1}{\int_t^{t+\Lambda(v,t)}\Omega(\varkappa)d}\ (FR)\int_t^{t+\Lambda(v,t)}\widetilde{\mathfrak{U}}(\varkappa)\Omega(\varkappa)d\preccurlyeq\frac{\widetilde{\mathfrak{U}}(t)\widetilde{+}\widetilde{\mathfrak{U}}(v)}{2}. \quad (52)$$

Let one attempt to take $\chi(s)=s$ and $\alpha=1=\Omega(\varkappa)$. Then, from (31) and (36) one achieves coming inequality for pre-invex F-IV-F given in [28]:

$$\widetilde{\mathfrak{U}}\left(\frac{2t+\Lambda(v,t)}{2}\right)\preccurlyeq(FR)\int_t^{t+\Lambda(v,t)}\widetilde{\mathfrak{U}}(\varkappa)d\preccurlyeq\frac{\widetilde{\mathfrak{U}}(t)\widetilde{+}\widetilde{\mathfrak{U}}(v)}{2}. \quad (53)$$

If one attempts to take $\mathfrak{U}_*(\varkappa,\ \varsigma)=\mathfrak{U}^*(\varkappa,\ \varsigma)$ and $1=\varsigma$ and $\chi(s)=s$, then from (31) and (36) one achieves coming inequality given in [27]:

$$\begin{aligned}\mathfrak{U}\left(\frac{2t+\Lambda(v,t)}{2}\right)\left[\mathcal{I}^{\alpha}_{t^+}\ \Omega(v)+\mathcal{I}^{\alpha}_{v^-}\ \Omega(t)\right]&\leq\left[\mathcal{I}^{\alpha}_{t^+}\ \mathfrak{U}\Omega(t+\Lambda(v,t))+\mathcal{I}^{\alpha}_{t+\Lambda(v,t)^-}\ \mathfrak{U}\Omega(t)\right]\\&\leq\frac{\mathfrak{U}(t)+\mathfrak{U}(t+\Lambda(v,t))}{2}\left[\mathcal{I}^{\alpha}_{t^+}\ \Omega(t+\Lambda(v,t))+\mathcal{I}^{\alpha}_{t+\Lambda(v,t)^-}\ \Omega(t)\right]\\&\leq\frac{\mathfrak{U}(t)+\mathfrak{U}(v)}{2}\left[\mathcal{I}^{\alpha}_{t^+}\ \Omega(t+\Lambda(v,t))+\mathcal{I}^{\alpha}_{t+\Lambda(v,t)^-}\ \Omega(t)\right].\end{aligned} \quad (54)$$

If one attempt to take $\mathfrak{U}_*(\varkappa,\ \varsigma)=\mathfrak{U}^*(\varkappa,\ \varsigma)$ and $\alpha=1=\varsigma$ and $\chi(s)=s$, then from (31) and (36) one achieves the classical H-Fejér inequality, see [38].

If one attempt to take $\mathfrak{U}_*(\varkappa,\ \varsigma) = \mathfrak{U}^*(\varkappa,\ \varsigma)$ and $\Omega(\varkappa) = \alpha = 1 = \varsigma$ and $\chi(s) = s$, then from (31) and (36) one achieves the classical HH-inequality.

Example 3. *If we attempt to take F-IV-F $\widetilde{\mathfrak{U}} : [0,\ 2] \to \mathbb{F}_0$ defined by,*

$$\widetilde{\mathfrak{U}}(\varkappa)(\theta) = \begin{cases} \frac{\theta}{2-\sqrt{\varkappa}},\ \theta \in \left[0,\ 2-\sqrt{\varkappa}\right], \\ \frac{2\left(2-\sqrt{\varkappa}\right)-\theta}{2-\sqrt{\varkappa}},\ \theta \in \left(2-\sqrt{\varkappa},\ 2\left(2-\sqrt{\varkappa}\right)\right], \\ 0,\ \text{otherwise}. \end{cases}$$

Then, for each $\varsigma \in [0,\ 1]$, we have $\mathfrak{U}_\varsigma(\varkappa) = \left[\varsigma\left(2-\sqrt{\varkappa}\right), (2-\varsigma)\left(2-\sqrt{\varkappa}\right)\right]$. Since end point functions $\mathfrak{U}_*(\varkappa,\varsigma), \mathfrak{U}^*(\varkappa,\varsigma)$ are χ-pre-invex functions for each $\varsigma \in [0,\ 1]$, then $\widetilde{\mathfrak{U}}(\varkappa)$ is χ-pre-invex F-IV-F. If

$$\Omega(\varkappa) = \begin{cases} \sqrt{\varkappa},\ \theta \in [0,1], \\ \sqrt{2-},\ \theta \in (1,\ 2], \end{cases}$$

then $\Omega(2-\varkappa) = \Omega(\varkappa) \geq 0$, for all $\varkappa \in [0,\ 2]$. Since $\mathfrak{U}_*(\varkappa,\varsigma) = \varsigma\left(2-\sqrt{\varkappa}\right)$ and $\mathfrak{U}^*(\varkappa,\varsigma) = (2-\varsigma)\left(2-\sqrt{\varkappa}\right)$. If $\chi(s) = s$ and $\alpha = \frac{1}{2}$, then we compute the following:

$$\begin{aligned} &\frac{\mathfrak{U}_*(t,\ \varsigma)+\mathfrak{U}_*(t+\Lambda(v,t),\ \varsigma)}{2}\int_0^1 s^{\alpha-1}[\chi(s)+\chi(1-s)]\Omega(t+s\Lambda(v,t)) = \frac{\pi}{\sqrt{2}}\varsigma\left(\frac{4-\sqrt{2}}{2}\right), \\ &\frac{\mathfrak{U}^*(t,\ \varsigma)+\mathfrak{U}^*(t+\Lambda(v,t),\ \varsigma)}{2}\int_0^1 s^{\alpha-1}[\chi(s)+\chi(1-s)]\Omega(t+s\Lambda(v,t)) = \frac{\pi}{\sqrt{2}}(2-\varsigma)\left(\frac{4-\sqrt{2}}{2}\right), \end{aligned} \tag{55}$$

$$\begin{aligned} &\frac{\mathfrak{U}_*(t,\ \varsigma)+\mathfrak{U}_*(t+\Lambda(t+\Lambda(v,t),t),\ \varsigma)}{2}\int_0^1 s^{\alpha-1}[\chi(s)+\chi(1-s)]\Omega(t+s\Lambda(v,t)) = \frac{\pi}{\sqrt{2}}\varsigma\left(\frac{4-\sqrt{2}}{2}\right), \\ &\frac{\mathfrak{U}^*(t,\ \varsigma)+\mathfrak{U}^*(t+\Lambda(v,t),\ \varsigma)}{2}\int_0^1 s^{\alpha-1}[\chi(s)+\chi(1-s)]\Omega(t+s\Lambda(v,t)) = \frac{\pi}{\sqrt{2}}(2-\varsigma)\left(\frac{4-\sqrt{2}}{2}\right), \end{aligned}$$

$$\begin{aligned} &\frac{\Gamma(\alpha)}{(\Lambda(v,t))^\alpha}\left[\mathcal{I}_{t^+}^\alpha\ \mathfrak{U}_*\Omega(t+\Lambda(v,t))\widetilde{+}\mathcal{I}_{t+\Lambda(v,t)^-}^\alpha\ \mathfrak{U}_*\Omega(t)\right] = \frac{1}{\sqrt{\pi}}\varsigma\left(2\pi+\frac{4-8\sqrt{2}}{3}\right), \\ &\frac{\Gamma(\alpha)}{(\Lambda(v,t))^\alpha}\left[\mathcal{I}_{t^+}^\alpha\ \mathfrak{U}^*\Omega(t+\Lambda(v,t))\widetilde{+}\mathcal{I}_{t+\Lambda(v,t)^-}^\alpha\ \mathfrak{U}^*\Omega(t)\right] = \frac{1}{\sqrt{\pi}}(2-\varsigma)\left(2\pi+\frac{4-8\sqrt{2}}{3}\right). \end{aligned} \tag{56}$$

From (44) and (45), we have $\frac{1}{\sqrt{\pi}}\left[\varsigma\left(2\pi+\frac{4-8\sqrt{2}}{3}\right),\ (2-\varsigma)\left(2\pi+\frac{4-8\sqrt{2}}{3}\right)\right]$ $\leq_I \frac{\pi}{\sqrt{2}}\left[\varsigma\left(\frac{4-\sqrt{2}}{2}\right),\ (2-\varsigma)\left(\frac{4-\sqrt{2}}{2}\right)\right]$, for each $\varsigma \in [0,\ 1]$.

Hence, (31) is verified.

For (36), we have

$$\begin{aligned} &\mathcal{I}_{t^+}^\alpha\ \mathfrak{U}_*\Omega(t+\Lambda(v,t)) + \mathcal{I}_{t+\Lambda(v,t)^-}^\alpha\ \mathfrak{U}_*\Omega(t) \\ &= \frac{1}{\sqrt{\pi}}\int_0^2 (2-\varkappa)^{\frac{-1}{2}}\ \Omega(\varkappa)\left(\varsigma\left(2-\sqrt{\varkappa}\right)\right)d\varkappa + \frac{1}{\sqrt{\pi}}\int_0^2 (\varkappa)^{\frac{-1}{2}}\ \Omega(\varkappa)\left(\varsigma\left(2-\sqrt{\varkappa}\right)\right)d\varkappa \\ &= \frac{1}{\sqrt{\pi}}\varsigma\left(\pi+\frac{8-8\sqrt{2}}{3}\right) + \frac{1}{\sqrt{\pi}}\varsigma\left(\pi-\frac{4}{3}\right) = \frac{1}{\sqrt{\pi}}\varsigma\left(2\pi+\frac{4-8\sqrt{2}}{3}\right) \\ &\mathcal{I}_{t^+}^\alpha\ \mathfrak{U}^*\Omega(t+\Lambda(v,t)) + \mathcal{I}_{t+\Lambda(v,t)^-}^\alpha\ \mathfrak{U}^*\Omega(t) \\ &= \frac{1}{\sqrt{\pi}}\int_0^2 (2-)^{\frac{-1}{2}}\ \Omega(\varkappa)\left((2-\varsigma)\left(2-\sqrt{\varkappa}\right)\right)d\varkappa + \frac{1}{\sqrt{\pi}}\int_0^2 (\varkappa)^{\frac{-1}{2}}\ \Omega(\varkappa)\left((2-\varsigma)\left(2-\sqrt{\varkappa}\right)\right)d\varkappa \\ &= \frac{1}{\sqrt{\pi}}(2-\varsigma)\left(\pi+\frac{8-8\sqrt{2}}{3}\right) + \frac{1}{\sqrt{\pi}}(2-\varsigma)\left(\pi-\frac{4}{3}\right) = \frac{1}{\sqrt{\pi}}(2-\varsigma)\left(2\pi+\frac{4-8\sqrt{2}}{3}\right). \end{aligned} \tag{57}$$

$$\begin{aligned} &\frac{1}{2\chi\left(\frac{1}{2}\right)}\mathfrak{U}_*\left(\frac{2t+\Lambda(v,t)}{2},\ \varsigma\right)\left[\mathcal{I}_{t^+}^\alpha\ \Omega(t+\Lambda(v,t)) + \mathcal{I}_{t+\Lambda(v,t)^-}^\alpha\ \Omega(t)\right] = \varsigma\sqrt{\pi}, \\ &\frac{1}{2\chi\left(\frac{1}{2}\right)}\mathfrak{U}^*\left(\frac{2t+\Lambda(v,t)}{2},\ \varsigma\right)\left[\mathcal{I}_{t^+}^\alpha\ \Omega(t+\Lambda(v,t)) + \mathcal{I}_{t+\Lambda(v,t)^-}^\alpha\ \Omega(t)\right] = (2-\varsigma)\sqrt{\pi}. \end{aligned} \tag{58}$$

From (46) and (47), we have $\sqrt{\pi}[\varsigma,\ (2-\varsigma)] \leq_I \frac{1}{\sqrt{\pi}}\left[\varsigma\left(2\pi+\frac{4-8\sqrt{2}}{3}\right),\ (2-\varsigma)\left(2\pi+\frac{4-8\sqrt{2}}{3}\right)\right]$, for each $\varsigma \in [0,\ 1]$.

4. Conclusions and Future Plan

We use fuzzy-interval-valued-fractional-integral operators with χ-pre-invex *F-IV-F* to infer various inclusions in the Hermite–Hadamard and Hermite–Hadamard–Fejér type inequalities in this paper. We show the relationships between the examined results and previously published ones to show their generic properties. In addition, three instances are given to demonstrate the accuracy of the results derived in the study. The point we wish to make here is that interval-valued analyses are commonly used in practical mathematics, particularly in the field of optimality analysis (see [10,26,29]). This important field in interval-valued analysis using fractional integral operators merits more investigation. In future, we will try to explore in this concept for p-pre-invex *F-IV-Fs* and generalized p-pre-invex *F-IV-Fs* by using fuzzy-interval-Katugampola-fractional operators.

Author Contributions: Funding acquisition, T.B.; Methodology, T.B.; Project administration, N.A.S.; Supervision, M.A.N.; Conceptualization, writing—original draft, writing—review & editing, M.B.K.; Funding acquisition, writing—review & editing, K.M.A. All authors contributed equally to the writing of this paper. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No data were used to support this study.

Acknowledgments: The authors would like to thank the Rector, COMSATS University Islamabad, Islamabad, Pakistan, for providing excellent research and This Research was supported by Taif University Researchers Supporting Project Number (TURSP-2020/217), Taif University, Taif, Saudi Arabia.

Conflicts of Interest: The authors declare that they have no competing interests.

References

1. Fu, H.; Saleem, M.S.; Nazeer, W.; Ghafoor, M.; Li, P. On Hermite-Hadamard type inequalities for n-polynomial convex stochastic processes. *AIMS Math.* **2021**, *6*, 6322–6339. [CrossRef]
2. Lv, Y.P.; Farid, G.; Yasmeen, H.; Nazeer, W.; Jung, C.Y. Generalization of some fractional versions of Hadamard inequalities via exponentially (α, h−m)-convex functions. *AIMS Math.* **2021**, *6*, 8978–8999. [CrossRef]
3. Chalco-Cano, Y.; Flores-Franulic, A.; Roman-Flores, H. Ostrowski type inequalities for interval-valued functions using generalized Hukuhara derivative. *Comput. Appl. Math.* **2012**, *31*, 457–472.
4. Chalco-Cano, Y.; Lodwick, W.A.; Condori-Equice, W. Ostrowski type inequalities and applications in numerical integration for interval-valued functions. *Soft Comput.* **2015**, *19*, 3293–3300. [CrossRef]
5. Roman-Flores, H.; Chalco-Cano, Y.; Lodwick, W.A. Some integral inequalities for interval-valued functions. *Comput. Appl. Math.* **2018**, *37*, 1306–1318. [CrossRef]
6. Barani, A.; Ghazanfari, A.G.; Dragomir, S.S. Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex. *J. Inequal. Appl.* **2012**, *2012*, 247. [CrossRef]
7. Moore, R.E. *Methods and Applications of Interval Analysis*; Society for Industrial & Applied Math: Philadelphia, PA, USA, 1979.
8. Khan, M.B.; Treanţă, S.; Budak, H. Generalized p-Convex Fuzzy-Interval-Valued Functions and Inequalities Based upon the Fuzzy-Order Relation. *Fractal Fract.* **2022**, *6*, 63. [CrossRef]
9. Moore, R.E.; Kearfott, R.B.; Cloud, M.J. *Introduction to Interval Analysis*; Society for Industrial & Applied Math: Philadelphia, PA, USA, 2009.
10. Khan, M.B.; Srivastava, H.M.; Mohammed, P.O.; Macías-Díaz, J.E.; Hamed, Y.S. Some new versions of integral inequalities for log-preinvex fuzzy-interval-valued functions through fuzzy order relation. *Alex. Eng. J.* **2022**, *61*, 7089–7101. [CrossRef]
11. Lupulescu, V. Fractional calculus for interval-valued functions. *Fuzzy Sets Syst.* **2015**, *265*, 63–85. [CrossRef]
12. Budak, H.; Tuna, T.; Sarikaya, M.Z. Fractional Hermite-Hadamard-type inequalities for interval-valued functions. *Proc. Amer. Math. Soc.* **2020**, *148*, 705–718. [CrossRef]
13. Khan, M.B.; Srivastava, H.M.; Mohammed, P.O.; Guirao, J.L.; Jawa, T.M. Fuzzy-interval inequalities for generalized preinvex fuzzy interval valued functions. *Math. Biosci. Eng.* **2022**, *19*, 812–835. [CrossRef] [PubMed]
14. Hanson, M.A. On sufficiency of the Kuhn-Tucker conditions. *J. Math. Anal. Appl.* **1981**, *80*, 545–550. [CrossRef]
15. Ben-Isreal, A.; Mond, B. What is invexity? *Anziam. J.* **1986**, *28*, 1–9. [CrossRef]
16. Mohan, S.R.; Neogy, S.K. On invex sets and preinvex functions. *J. Math. Anal. Appl.* **1995**, *189*, 901–908. [CrossRef]
17. Antczak, T. Mean value in invexity analysis. *Nonlinear Anal. Theory Methods Appl.* **2005**, *60*, 1473–1484. [CrossRef]
18. Nanda, S.; Kar, K. Convex fuzzy mappings. *Fuzzy Sets Syst.* **1992**, *48*, 129–132. [CrossRef]
19. Syau, Y.-R. On convex and concave fuzzy mappings. *Fuzzy Sets Syst.* **1999**, *103*, 163–168. [CrossRef]
20. Furukawa, N. Convexity and local Lipschitz continuity of fuzzy-valued mappings. *Fuzzy Sets Syst.* **1998**, *93*, 113–119. [CrossRef]
21. Goetschel, R., Jr.; Voxman, W. Elementary fuzzy calculus. *Fuzzy Sets Syst.* **1986**, *18*, 31–43. [CrossRef]

22. Yan, H.; Xu, J. A class of convex fuzzy mappings. *Fuzzy Sets Syst.* **2002**, *129*, 47–56. [CrossRef]
23. Noor, M.A. Fuzzy preinvex functions. *Fuzzy Sets Syst.* **1994**, *64*, 95–104. [CrossRef]
24. Khan, M.B.; Mohammed, P.O.; Noor, M.A.; Hamed, Y.S. New Hermite-Hadamard inequalities in fuzzy-interval fractional calculus and related inequalities. *Symmetry* **2021**, *13*, 673. [CrossRef]
25. Khan, M.B.; Mohammed, P.O.; Noor, M.A.; Alsharif, A.M.; Noor, K.I. New fuzzy-interval inequalities in fuzzy-interval fractional calculus by means of fuzzy order relation. *AIMS Math.* **2021**, *6*, 10964–10988. [CrossRef]
26. Khan, M.B.; Noor, M.A.; Mohammed, P.O.; Guirao, J.L.; Noor, K.I. Some Integral Inequalities for Generalized Convex Fuzzy-Interval-Valued Functions via Fuzzy Riemann Integrals. *Int. J. Comput. Intell. Syst.* **2021**, *14*, 158. [CrossRef]
27. Khan, M.B.; Noor, M.A.; Noor, K.I.; Chu, Y.M. New Hermite-Hadamard type inequalities for (h_1,h_2)-convex fuzzy-interval-valued functions. *Adv. Differ. Equ.* **2021**, *2021*, 149. [CrossRef]
28. Bede, B.; Gal, S.G. Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations. *Fuzzy Sets Syst.* **2005**, *151*, 581–599. [CrossRef]
29. Kaleva, O. Fuzzy differential equations. *Fuzzy Sets Syst.* **1987**, *24*, 301–317. [CrossRef]
30. Kulish, U.; Miranker, W. *Computer Arithmetic in Theory and Practice*; Academic Press: New York, NY, USA, 2014.
31. Khan, M.B.; Noor, M.A.; Al-Shomrani, M.M.; Abdullah, L. Some novel inequalities for LR-*h*-convex interval-valued functions by means of pseudo-order relation. *Math. Methods Appl. Sci.* **2022**, *45*, 1310–1340. [CrossRef]
32. Allahviranloo, T.; Salahshour, S.; Abbasbandy, S. Explicit solutions of fractional differential equations with uncertainty. *Soft Comput.* **2012**, *16*, 297–302. [CrossRef]
33. Hadamard, J. Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann. *J. De Mathématiques Pures Et Appliquées* **1893**, *7*, 171–215.
34. Hermite, C. Sur deux limites d'une intégrale définie. *Mathesis* **1883**, *3*, 82–97.
35. Moore, R.E. *Interval Analysis*; Prentice Hall: Englewood Cliffs, NJ, USA, 1966.
36. Stefanini, L.; Bede, B. Generalized Hukuhara differentiability of interval-valued functions and interval differential equations. *Nonlinear Anal. Theory Methods Appl.* **2010**, *71*, 1311–1328. [CrossRef]
37. Sarikaya, M.Z.; Set, E.; Yaldiz, H.; Basak, N. Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities. *Math. Comput. Model.* **2013**, *57*, 2403–2407. [CrossRef]
38. Fejer, L. Uber die Fourierreihen II. Math. Naturwise. *Anz Ung. Akad Wiss.* **1906**, *24*, 369–390.
39. Costa, T.; Román-Flores, H. Some integral inequalities for fuzzy-interval-valued functions. *Inf. Sci.* **2017**, *420*, 110–125. [CrossRef]
40. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006; Volume 204.
41. Khan, M.B.; Noor, M.A.; Abdullah, L.; Chu, Y.M. Some New Classes of Preinvex Fuzzy-Interval-Valued Functions and Inequalities. *Int. J. Comput. Intell. Syst.* **2021**, *14*, 1403–1418. [CrossRef]
42. Iscan, I. Hermite-Hadamard's inequalities for preinvex functions via fractional integrals and related fractional inequalities. *arXiv* **2012**, arXiv:1204.0272.
43. Noor, M.A.; Noor, K.I.; Awan, M.U.; Li, J. On Hermite-Hadamard Inequalities for h-Preinvex Functions. *Filomat* **2014**, *28*, 1463–1474. [CrossRef]
44. Matłoka, M. Inequalities for h-preinvex functions. *Appl. Math. Comput.* **2014**, *234*, 52–57. [CrossRef]
45. Khan, M.B.; Noor, M.A.; Abdeljawad, T.; Mousa, A.A.A.; Abdalla, B.; Alghamdi, S.M. LR-Preinvex Interval-Valued Functions and Riemann–Liouville Fractional Integral Inequalities. *Fractal Fract.* **2021**, *5*, 243. [CrossRef]
46. Liu, P.; Khan, M.B.; Noor, M.A.; Noor, K.I. New Hermite-Hadamard and Jensen inequalities for log-*s*-convex fuzzy-interval-valued functions in the second sense. *Complex Intell. Syst.* **2021**, *2021*, 1–15. [CrossRef]
47. Khan, M.B.; Noor, M.A.; Al-Bayatti, H.M.; Noor, K.I. Some new inequalities for LR-log-h-convex interval-valued functions by means of pseudo order relation. *Appl. Math. Inf. Sci.* **2021**, *15*, 459–470.
48. Sana, G.; Khan, M.B.; Noor, M.A.; Mohammed, P.O.; Chu, Y.-M. Harmonically convex fuzzy-interval-valued functions and fuzzy-interval Riemann–Liouville fractional integral inequalities. *Int. J. Comput. Intell. Syst.* **2021**, *14*, 1809–1822. [CrossRef]
49. Khan, M.B.; Mohammed, P.O.; Noor, M.A.; Abualnaja, K.M. Fuzzy integral inequalities on coordinates of convex fuzzy interval-valued functions. *Math. Biosci. Eng.* **2021**, *18*, 6552–6580. [CrossRef] [PubMed]
50. Khan, M.B.; Noor, M.A.; Noor, K.I. Some Inequalities for LR-(h_1,h_2)-Convex Interval-Valued Functions by Means of Pseudo Order Relation. *Int. J. Comput. Intell. Syst.* **2021**, *14*, 180. [CrossRef]
51. Khan, M.B.; Mohammed, P.O.; Noor, M.A.; Baleanu, D.; Guirao, J. Some New Fractional Estimates of Inequalities for LR-p-Convex Interval-Valued Functions by Means of Pseudo Order Relation. *Axioms* **2021**, *10*, 175. [CrossRef]
52. Khan, M.B.; Treanță, S.; Soliman, M.S.; Nonlaopon, K.; Zaini, H.G. Some Hadamard–Fejér Type Inequalities for LR-Convex Interval-Valued Functions. *Fractal Fract.* **2022**, *6*, 6. [CrossRef]
53. Khan, M.B.; Srivastava, H.M.; Mohammed, P.O.; Nonlaopon, K.; Hamed, Y.S. Some new Jensen, Schur and Hermite-Hadamard inequalities for log convex fuzzy interval-valued functions. *AIMS Math.* **2022**, *7*, 4338–4358. [CrossRef]
54. Khan, M.B.; Mohammed, P.O.; Machado, J.A.T.; Guirao, J.L. Integral Inequalities for Generalized Harmonically Convex Functions in Fuzzy-Interval-Valued Settings. *Symmetry* **2021**, *13*, 2352. [CrossRef]
55. Khan, M.B.; Srivastava, H.M.; Mohammed, P.O.; Baleanu, D.; Jawa, T.M. Fuzzy-interval inequalities for generalized convex fuzzy-interval-valued functions via fuzzy Riemann integrals. *AIMS Math.* **2022**, *7*, 1507–1535. [CrossRef]

56. Macías-Díaz, J.E.; Khan, M.B.; Noor, M.A.; Abd Allah, A.M.; Alghamdi, S.M. Hermite-Hadamard inequalities for generalized convex functions in interval-valued calculus. *AIMS Math.* **2022**, *7*, 4266–4292. [CrossRef]
57. Khan, M.B.; Zaini, H.G.; Treanță, S.; Soliman, M.S.; Nonlaopon, K. Riemann–Liouville Fractional Integral Inequalities for Generalized Pre-Invex Functions of Interval-Valued Settings Based upon Pseudo Order Relation. *Mathematics* **2022**, *10*, 204. [CrossRef]
58. Chauhan, S.; Khan, M.A.; Kumar, S. Unified fixed point theorems in fuzzy metric spaces via common limit range property. *J. Inequal. Appl.* **2013**, *2013*, 182. [CrossRef]
59. Roldán-López-de-Hierro, A.F.; Karapınar, E.; Manro, S. Some new fixed point theorems in fuzzy metric spaces. *J. Intell. Fuzzy Syst.* **2014**, *27*, 2257–2264. [CrossRef]
60. Jleli, M.; Karapınar, E.; Samet, B. On cyclic (ψ,)-contractions in Kaleva-Seikkala's type fuzzy metric spaces. *J. Intell. Fuzzy Syst.* **2014**, *27*, 2045–2053. [CrossRef]
61. Sevinik Adigüzel, R.; Aksoy, Ü.; Karapinar, E.; Erhan, İ.M. On the solution of a boundary value problem associated with a fractional differential equation. *Math. Methods Appl. Sci.* **2020**. [CrossRef]
62. Sevinik-Adıgüzel, R.; Aksoy, Ü.; Karapınar, E.; Erhan, İ.M. Uniqueness of solution for higher-order nonlinear fractional differential equations with multi-point and integral boundary conditions. *Rev. Real Acad. Cienc. Exactas Físicas Naturales. Ser. A Matemáticas* **2021**, *115*, 155. [CrossRef]
63. Adiguzel, R.S.; Aksoy, U.; Karapinar, E.; Erhan, I.M. On the solutions of fractional differential equations via Geraghty type hybrid contractions. *Appl. Comput. Math.* **2021**, *20*, 313–333.