Dynamic Analysis and Bifurcation Study on Fractional-Order Tri-Neuron Neural Networks Incorporating Delays

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Article

Abstract: In this manuscript, we principally probe into a class of fractional-order tri-neuron neural networks incorporating delays. Making use of fixed point theorem, we prove the existence and uniqueness of solution to the fractional-order tri-neuron neural networks incorporating delays. By virtue of a suitable function, we prove the uniformly boundedness of the solution to the fractional-order tri-neuron neural networks incorporating delays. With the aid of the stability theory and bifurcation knowledge of fractional-order differential equation, a new delay-independent condition to guarantee the stability and creation of Hopf bifurcation of the fractional-order tri-neuron neural networks incorporating delays is established. Taking advantage of the mixed controller that contains state feedback and parameter perturbation, the stability region and the time of onset of Hopf bifurcation of the fractional-order trineuron neural networks incorporating delays are successfully controlled. Software simulation plots are displayed to illustrate the established key results. The obtained conclusions in this article have important theoretical significance in designing and controlling neural networks.

Keywords: fractional-order tri-neuron neural networks; stability; Hopf bifurcation; Hopf bifurcation control

1. Introduction

After the classical work of Hopfield [1] on neural networks, many scholars pay much attention to the study of the dynamics of different types of neural networks since neural networks have displayed extraordinary application value in various fields such as associate memory, automatic control, image processing, biological engineering, pattern recognition and so on [2–5]. Generally speaking, delay often occurs in the signal transmission among different neurons due to the finite propagation velocity in neural networks [6]. A natural problem arises: What is the impact of time delay on the dynamical behavior of neural networks? During the past decades, in order to reveal the influence of time delay on various dynamics of neural networks, a great deal of researchers from mathematics and engineering have made great efforts to explore the dynamics of delayed neural networks and lots of valuable fruits have been achieved. For instance, Wang et al. [7] explored the fixed-time synchronization problem of delayed complex-valued BAM neural networks by using pinning control and adaptive pinning control. Gan et al. [8] discussed the anti-synchronization of a class of BAM neural networks involving the Markov scheduling protocol. Syed Ali et al. [9] set up a novel sufficient condition to ensure the global Mittag–Leffler stability of impulsive fractional-order delayed complex-valued BAM neural networks. Chen et al. [10] studied the finite-time stabilization of fractional-order fuzzy quaternion-valued BAM neural networks by applying direct quaternion approach. Making use of basic theories on fractional calculus, inequality skills of fuzzy logic and reduction to absurdity, Chen et al. [10] established the sufficient criteria that guarantee the finite-time stabilization of the fractional-order fuzzy...
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Here we notice that most of the literature (e.g., [1–9,11–16]) only involves integer-order delayed neural networks, not considering the fractional-order cases. Since the introduction of fractional-order derivative by Leibniz and L’Hospital [17] three hundred years ago, fractional calculus gradually arouse much attention from many scholars. The fractional derivative has more greater advantages than integer-order ones. It can adequately display the global correlation and better description of long-term development process and memory [18]. Recent research shows that fractional calculus has been widely used in many areas such as electrical engineering, wave theory in physics, electronic information, biological technique, control engineering, neural systems and so on [19–21]. Thus, the investigation of the dynamics of fractional-order dynamical systems has attracted much interest from numerous scholars and fruitful results have been achieved. For instance, Zhang et al. [22] explored the multistability of fractional-order competitive neural networks with delays. Li et al. [23] made a detailed analysis on the complete and finite-time synchronization of fractional-order fuzzy neural networks by virtue of nonlinear feedback control. Padmaja and Balasubramaniam [24] set up a new delay and order-dependent passivity criteria for impulsive fractional-order neural networks involving proportional delays. Ke [25] proded into the Mittag–Leffler stability and asymptotic α-periodicity of fractional-order delayed inertial neural networks. In details, one can see [26–37].

In fractional-order differential equations, time delay is a vital factor that affects the dynamical behavior of systems. In particular, delay-induced Hopf bifurcation is an important topic in fractional-order differential systems. In order to design, optimize and control neural networks to serve humanity, we need to explore the impact of time delay on the dynamics of delayed fractional-order neural networks, especially, we need to investigate the impact of time delay on Hopf bifurcation phenomenon of delayed fractional-order neural networks. By adjusting the value of time delay, we can enlarge or narrow the stability region of neural network systems, then postpone or advance the time of onset of Hopf bifurcation of neural networks. However, many works on delay-induced Hopf bifurcation mainly focus on the integer-order delayed neural networks and there are few works on the fractional-order delayed neural networks. Recently, there has been some literature on delay-induced Hopf bifurcation of fractional-order delayed dynamical systems. For example, Xu et al. [38] established a sufficient condition to ensure the stability and the onset of Hopf bifurcation of fractional-order six-neuron BAM neural networks with multi-delays. Huang et al. [39] explored the Hopf bifurcation of fractional-order quaternion-valued neural networks. Xiao et al. [40] investigated the fractional-order PD control of Hopf bifurcations in fractional-order small-world networks with delays. For more related studies, see Refs. [41–47]. Although some works on Hopf bifurcation of fractional-order delayed neural networks have been carried out, many challenging problems are expected to be solved. Stimulated by the idea, in this present work, we are to deal with the following three aspects: (i) Explore the existence and uniqueness, the boundedness of solution of the involved neural networks with single delay; (ii) Set up a sufficient criterion ensuring the stability and the creation of Hopf bifurcation of the involved neural networks with single delay; (iii) Make use of a suitable mixed controller that contains state feedback and parameter perturbation to control the time of onset of Hopf bifurcation for the involved neural networks with single delay.

In 2006, Yan [41] investigated the following tri-neuron network system:

\[
\begin{align*}
\dot{w}_1(t) &= -\nu w_1(t) + h(w_1(t)) + h_{12}(w_2(t - \rho)) + h_{13}(w_3(t - \rho)), \\
\dot{w}_2(t) &= -\nu w_2(t) + h(w_2(t)) + h_{21}(w_1(t - \rho)) + h_{23}(w_3(t - \rho)), \\
\dot{w}_3(t) &= -\nu w_3(t) + h(w_3(t)) + h_{31}(w_1(t - \rho)) + h_{32}(w_2(t - \rho)),
\end{align*}
\]  

(1)
where \( w_1(t), w_2(t), w_3(t) \) stand for the voltage on the input of the first neuron at time \( t(t > 0) \), the voltage on the input of the second neuron at time \( t(t > 0) \), the voltage on the input of the third neuron at time \( t(t > 0) \), respectively; \( \nu > 0 \) stands for the rate with which the three neurons will reset its potential to the resting state in isolation when disconnected from the network and external inputs. \( h, h_{12}, h_{13}, h_{21}, h_{23}, h_{31}, h_{32} \) are the sigmoidal activation functions of the different neurons. \( \rho > 0 \) represents the transmission delay of the signal along the axon of the neuron. In details, one can see [41–43]. By analyzing the characteristic equation of system (1) and selecting the time delay as bifurcation parameter, Yan [41] established the delay-independent stability and Hopf bifurcation condition ensuring the stability and the creation of Hopf bifurcation of system (1). In addition, making use of the normal form and center manifold theorem, Yan [41] explored the Hopf bifurcation peculiarities.

On the basis of the analysis above, we will deal with the delay-induced Hopf bifurcation of fractional order neural networks. Inspired by the above viewpoint, we modify model (1) as the following fractional-order form

\[
\begin{align*}
\frac{d^r w_1(t)}{dt^r} &= -\nu w_1(t) + h(w_1(t)) + h_{12}(w_2(t - \rho)) + h_{13}(w_3(t - \rho)), \\
\frac{d^r w_2(t)}{dt^r} &= -\nu w_2(t) + h(w_2(t)) + h_{21}(w_1(t - \rho)) + h_{23}(w_3(t - \rho)), \\
\frac{d^r w_3(t)}{dt^r} &= -\nu w_3(t) + h(w_3(t)) + h_{31}(w_1(t - \rho)) + h_{32}(w_2(t - \rho)),
\end{align*}
\]  

(2)

where \( r \in (0, 1] \) is a constant. We give the initial value of system (2) as follows:

\[
\begin{align*}
w_1(\phi) &= w_1(\phi), \phi \in [-\rho, t_0], \\
w_2(\phi) &= w_2(\phi), \phi \in [-\rho, t_0], \\
w_3(\phi) &= w_3(\phi), \phi \in [-\rho, t_0],
\end{align*}
\]  

(3)

where \( t_0 > 0 \) is a constant. In order to achieve our goal in this paper, we prepare the following necessary assumptions:

(\( Q_1 \)) \( h, h_{12}, h_{13}, h_{21}, h_{23}, h_{31}, h_{32} \in C^1, h(0) = h_{12}(0) = h_{13}(0) = h_{21}(0) = h_{23}(0) = h_{31}(0) = h_{32}(0) = 0. \)

(\( Q_2 \)) There exist positive constants \( H, H_{12}, H_{13}, H_{21}, H_{23}, H_{31}, H_{32} \) such that

\[
\begin{align*}
|h(t_1) - h(t_2)| &\leq H|t_1 - t_2|, \\
h_{12}(t_1) - h_{12}(t_2) |\leq H_{12}|t_1 - t_2|, \\
h_{13}(t_1) - h_{13}(t_2) |\leq H_{13}|t_1 - t_2|, \\
h_{21}(t_1) - h_{21}(t_2) |\leq H_{21}|t_1 - t_2|, \\
h_{23}(t_1) - h_{23}(t_2) |\leq H_{23}|t_1 - t_2|, \\
h_{31}(t_1) - h_{31}(t_2) |\leq H_{31}|t_1 - t_2|, \\
h_{32}(t_1) - h_{32}(t_2) |\leq H_{32}|t_1 - t_2|.
\end{align*}
\]

(\( Q_3 \)) There exist positive constants \( \mathcal{H}, \mathcal{H}_{12}, \mathcal{H}_{13}, \mathcal{H}_{21}, \mathcal{H}_{23}, \mathcal{H}_{31}, \mathcal{H}_{32} \) such that

\[
\begin{align*}
|h(t)| &\leq \mathcal{H}, \\
h_{12}(t) |\leq \mathcal{H}_{12}, \\
h_{13}(t) |\leq \mathcal{H}_{13}, \\
h_{21}(t) |\leq \mathcal{H}_{21}, \\
h_{23}(t) |\leq \mathcal{H}_{23}, \\
h_{31}(t) |\leq \mathcal{H}_{31}, \\
h_{32}(t) |\leq \mathcal{H}_{32}
\end{align*}
\]

for all \( t \in \mathbb{R} \).

We arrange the basic structure of the article as follows. Section 2 lists some related basic theories on fractional-order dynamical system. Section 3 gives the proof existence and uniqueness of solution of system (2). Section 4 shows the proof of boundedness of solution.
of system (2). Section 5 establishes the delay-independent sufficient condition to ensure the stability and the creation of Hopf bifurcation of system (2). Section 6 seeks a suitable mixed controller to control the Hopf bifurcation of system (2). Section 7 carries out software simulations to support the derived key conclusions. Section 8 completes this work.

2. Prerequisite Theory

In this segment, we list some related theory on fractional-order systems. Let \( R_+ = \{ x | x \geq 0, x \in R \} \).

Definition 1 ([44]). Define the Caputo fractional order derivative as follows:

\[
\mathcal{D}^\alpha \kappa (\varphi) = \frac{1}{\Gamma(l - r)} \int_{0}^{\varphi} \frac{\kappa^{(l)}(u)}{(\varphi - u)^{r-l+1}} du,
\]

where \( \kappa(\varphi) \in ([0, \infty), R), \Gamma(\varphi) = \int_{0}^{\infty} e^{-u} u^{l-1} du, \varphi \geq 0 \) and \( l \in Z^+, r \in (r-1, r) \).

The Laplace transform of Caputo-type fractional order derivative is given by

\[
\mathcal{L}\{ \mathcal{D}^\alpha g(t); s \} = s^{r} G(s) - \sum_{j=0}^{n-1} s^{r-j-1} g^{(j)}(0), r \in (n-1, n), n \in Z^+,
\]

where \( G(s) = \mathcal{L}\{ g(t) \} \). If \( g^{(i)}(0) = 0, j = 1, 2, \ldots, n \), then \( \mathcal{L}\{ \mathcal{D}^\alpha g(t); s \} = s^{r} G(s) \).

Definition 2 ([45]). \((w_{1*}, w_{2*}, w_{3*})\) is an equilibrium point of model (2) provided that

\[
\begin{align*}
-w_{1*} + h(w_{1*}) + h_{12}(w_{2*}) + h_{13}(w_{3*}) &= 0, \\
-w_{2*} + h(w_{2*}) + h_{21}(w_{1*}) + h_{23}(w_{3*}) &= 0, \\
-w_{3*} + h(w_{3*}) + h_{31}(w_{1*}) + h_{32}(w_{2*}) &= 0.
\end{align*}
\]

Lemma 1 ([46]). Give the fractional-order system as follows:

\[
\frac{d^\alpha u(t)}{dt^\alpha} = g(t, u(t)), u(t_0) = u_0, t_0 > 0,
\]

where \( t_0 \in R, r \in (0, 1], g : [t_0, \infty) \times \varphi \to R^m, \varphi \subset R_+ \), then system (5) has a unique solution defined on \([t_0, \infty)\) if \( g(t, u) \) satisfies the local Lipschitz condition with respect to \( u \).

Lemma 2 ([47]). Assume that \( u(t) \in C_[t_0, \infty) \) and satisfies

\[
\begin{align*}
&D^\alpha u(t) = -c_1 u(t) + c_2, \\
&u(t_0) = u_0,
\end{align*}
\]

where \( 0 < r < 1, c_1, c_2 \in R, c_1 \neq 0, t_0 \geq 0 \). Then,

\[
u(t) \leq \left( u(t_0) - \frac{c_2}{c_1} \right) e^{r} - c_1 (t - t_0) + \frac{c_2}{c_1}.
\]

Lemma 3 ([48]). Consider the following fractional-order model:

\[
\frac{d^\alpha u(t)}{dt^\alpha} = g(t, u(t)), u(0) = u_0,
\]

where \( r \in (0, 1], g(t, u(t)) : R^* \times R^m \to R^m \). Denote \( u_* \) the equilibrium point of system (8). Then \( u_* \) is locally asymptotically stable provided that every eigenvalue \( \lambda \) of \( \frac{dg(t, u)}{du}|_{u=u_*} \) satisfies |arg(\lambda)| > \frac{\pi}{2}.\]
Lemma 4 ([49]). Give the following fractional-order system as follows:

\[
\begin{align*}
\frac{d^1 P_1(t)}{dt} &= \tau_1 P_1(t - \rho_{11}) + \tau_2 P_2(t - \rho_{12}) + \cdots + \tau_m P_m(t - \rho_{1m}), \\
\frac{d^2 P_1(t)}{dt^2} &= \tau_1 P_1(t - \rho_{21}) + \tau_2 P_2(t - \rho_{22}) + \cdots + \tau_m P_m(t - \rho_{2m}), \\
\vdots \\
\frac{d^n P_m(t)}{dt^n} &= \tau_1 P_1(t - \rho_{nm}) + \tau_2 P_2(t - \rho_{2m}) + \cdots + \tau_m P_m(t - \rho_{nm}), \\
\end{align*}
\]

where \( r_i \in (0,1) (i = 1, 2, \cdots, m) \). Set

\[
\Delta(\sigma) = \begin{bmatrix}
\sigma^{p_{11}} & -\tau_{12} e^{-\sigma \rho_{12}} & \cdots & -\tau_{1m} e^{-\sigma \rho_{1m}} \\
-\tau_{21} e^{-\sigma \rho_{21}} & \sigma^{p_{22}} & \cdots & -\tau_{2m} e^{-\sigma \rho_{2m}} \\
\vdots & \vdots & \ddots & \vdots \\
-\tau_{m1} e^{-\sigma \rho_{m1}} & -\tau_{m2} e^{-\sigma \rho_{m2}} & \cdots & \sigma^{\nu} - \tau_{mm} e^{-\sigma \rho_{mm}}
\end{bmatrix}.
\]

Then the equilibrium point of system (9) is asymptotically stable in Liapunov sense if every root of \( \text{det}(\Delta(\sigma)) = 0 \) has negative real parts.

3. Existence and Uniqueness

Theorem 1. Denote \( \Psi = \{(w_1,w_2,w_3) \in R^3 : \max\{|w_1|,|w_2|,|w_3| \} < \mathcal{W} \} \), where \( \mathcal{W} > 0 \) stands for a constant. Then for arbitrary \( \mathcal{W}_{t_0} = (w_{1t_0},w_{2t_0},w_{3t_0}) \in \Psi \) and for all \( t \geq t_0 \), then system (2) with the initial value \( \mathcal{W}_{t_0} \) has a unique solution \( \mathcal{W}(t) \in \Psi \).

Proof. Let \( \mathcal{W} = (w_1,w_2,w_3) \) and \( \mathcal{W} = (\tilde{w}_1,\tilde{w}_2,\tilde{w}_3) \) and \( \Pi(\mathcal{W}) = (\Pi_1(\mathcal{W}),\Pi_2(\mathcal{W}),\Pi_3(\mathcal{W})) \), where

\[
\begin{align*}
\Pi_1(\mathcal{V}) &= -\nu w_1(t) + h(w_1(t)) + h_{12}(\tilde{w}_2(t - \rho)) + h_{13}(w_3(t - \rho)), \\
\Pi_2(\mathcal{V}) &= -\nu w_2(t) + h(w_2(t)) + h_{21}(\tilde{w}_1(t - \rho)) + h_{23}(w_3(t - \rho)), \\
\Pi_3(\mathcal{V}) &= -\nu w_3(t) + h(w_3(t)) + h_{31}(w_1(t - \rho)) + h_{32}(\tilde{w}_2(t - \rho)).
\end{align*}
\]

\( \forall \ \mathcal{W}, \mathcal{W} \in \Psi, \) by virtue of (Q2) and (Q3), we have

\[
|\Pi(\mathcal{W}) - \Pi(\mathcal{W})| \\
= \sum_{i=1}^{3} |\Pi_i(\mathcal{W}) - \Pi_i(\mathcal{W})| \\
= |[-\nu \tilde{w}_1(t) + h(\tilde{w}_1(t)) + h_{12}(w_2(t - \rho)) + h_{13}(w_3(t - \rho))] \\
-[-\nu \tilde{w}_1(t) + h(\tilde{w}_1(t)) + h_{12}(\tilde{w}_2(t - \rho)) + h_{13}(\tilde{w}_3(t - \rho))]| \\
+|[-\nu w_1(t) + h(w_1(t)) + h_{12}(\tilde{w}_2(t - \rho)) + h_{13}(w_3(t - \rho))] \\
-[-\nu \tilde{w}_1(t) + h(\tilde{w}_1(t)) + h_{12}(w_2(t - \rho)) + h_{13}(\tilde{w}_3(t - \rho))]| \\
+|[-\nu \tilde{w}_2(t) + h(\tilde{w}_2(t)) + h_{21}(w_1(t - \rho)) + h_{23}(w_3(t - \rho))] \\
-[-\nu w_2(t) + h(w_2(t)) + h_{21}(\tilde{w}_1(t - \rho)) + h_{23}(\tilde{w}_3(t - \rho))]| \\
+|[-\nu \tilde{w}_3(t) + h(\tilde{w}_3(t)) + h_{31}(w_1(t - \rho)) + h_{32}(w_2(t - \rho))] \\
-[-\nu w_3(t) + h(w_3(t)) + h_{31}(\tilde{w}_1(t - \rho)) + h_{32}(\tilde{w}_2(t - \rho))]| \\
\leq \nu |w_1(t) - \tilde{w}_1(t)| + H_1 |w_1(t) - \tilde{w}_1(t)| + H_{12} |w_2(t - \rho) - \tilde{w}_2(t - \rho)| \\
+H_{13} |w_3(t - \rho) - \tilde{w}_3(t - \rho)| + \nu |w_2(t) - \tilde{w}_2(t)| + H |w_2(t) - \tilde{w}_2(t)| \\
+H_{21} |w_1(t - \rho) - \tilde{w}_1(t - \rho)| + H_{23} |w_3(t - \rho) - \tilde{w}_3(t - \rho)| \\
+\nu |w_3(t) - \tilde{w}_3(t)| + H |w_3(t) - \tilde{w}_3(t)| \\
+H_{31} |w_1(t - \rho) - \tilde{w}_1(t - \rho)| + H_{32} |w_2(t - \rho) - \tilde{w}_2(t - \rho)| \\
\leq A_1 |w_1(t) - \tilde{w}_1(t)| + A_2 |w_2(t) - \tilde{w}_2(t)| + A_3 |w_3(t) - \tilde{w}_3(t)|,
\]

where

\[
\begin{align*}
A_1 &= \nu + H + H_{21} + H_{31}, \\
A_2 &= \nu + H + H_{12} + H_{32}, \\
A_3 &= \nu + H + H_{13} + H_{23}.
\end{align*}
\]
By (12), we obtain
\[ ||\Pi(W) - \Pi(\hat{W})|| \leq A||W - \hat{W}||, \tag{14} \]
where
\[ A = \max \{ A_1, A_2, A_3 \}. \tag{15} \]

By virtue of Lemma 1, we easily know that Theorem 1 is true. The proof completes. \( \square \)

**Remark 1.** According to Lemma 1, we know that \( \Pi(W) \) satisfies the local Lipschitz condition with respect to \( W \). By virtue of fixed point theorem, we can conclude that Theorem 1 is correct.

### 4. Boundedness

In this segment, we will prove the boundedness of the solution to system (2). Denote \( \Psi_+ = \{(w_1, w_2, w_3) \in \Psi : w_1, w_2, w_3 \in R_+ \} \).

**Theorem 2.** All solutions of system (2) that start with \( \Psi_+ \) are uniformly bounded.

**Proof.** Define
\[ V(t) = w_1(t) + w_2(t) + w_3(t). \tag{16} \]

It follows from (2) that:
\[
\frac{d^r V(t)}{dt^r} = \frac{d^r w_1(t)}{dt^r} + \frac{d^r w_2(t)}{dt^r} + \frac{d^r w_3(t)}{dt^r} \\
= -v w_1(t) + h(w_1(t)) + h_{12}(w_2(t - \rho)) + h_{13}(w_3(t - \rho)) \\
- v w_2(t) + h(w_2(t)) + h_{21}(w_1(t - \rho)) + h_{23}(w_3(t - \rho)) \\
- v w_3(t) + h(w_3(t)) + h_{31}(w_1(t - \rho)) + h_{32}(w_2(t - \rho)) \\
\leq -v V(t) + H_0, \tag{17} \]

where
\[ H_0 = 3H_* + H_{12*} + H_{13*} + H_{21*} + H_{23*} + H_{31*} + H_{32*}. \tag{18} \]

According to Lemma 2, we get
\[ V(t) \leq \left( V_0 - \frac{H_0}{v} \right) E_r[-v(t - t_0)^r] + \frac{H_0}{v} \rightarrow \frac{H_0}{v}, t \rightarrow \infty. \tag{19} \]

Therefore, all solutions to system (2) which begin with \( \Psi_+ \) are uniformly bounded.

The proof finishes. \( \square \)

### 5. Bifurcation Study

In this segment, we will analyze the stability and the creation of Hopf bifurcation of system (2). By \( Q_1 \), we can easily know that system (2) has a unique zero equilibrium point \( W_0(0,0,0) \). The linear system of model (2) at \( W_0(0,0,0) \) is:
\[
\begin{align*}
\frac{d^r w_1(t)}{dt^r} &= -a_1 w_1(t) + a_{12} w_2(t - \rho) + a_{13} w_3(t - \rho), \\
\frac{d^r w_2(t)}{dt^r} &= -a_2 w_2(t) + a_{21} w_1(t - \rho) + a_{23} w_3(t - \rho), \\
\frac{d^r w_3(t)}{dt^r} &= -a_3 w_3(t) + a_{31} w_1(t - \rho) + a_{32} w_2(t - \rho),
\end{align*} \tag{20} \]
where $\alpha = v - h'(0)$, $a_{12} = h^{12}(0), a_{13} = h^{13}(0), a_{21} = h^{21}(0), a_{23} = h^{23}(0), a_{31} = h^{31}(0), a_{32} = h^{32}(0)$. The corresponding characteristic equation for (20) is given by:

$$\det \begin{bmatrix} s^2 + \alpha & -a_{12}e^{-\nu} & -a_{13}e^{-\nu} \\ -a_{21}e^{-\nu} & s^2 + \alpha & -a_{23}e^{-\nu} \\ -a_{31}e^{-\nu} & -a_{32}e^{-\nu} & s^2 + \alpha \end{bmatrix} = 0. \quad (21)$$

It follows from (21) that:

$$C_1(s) + C_2(s)e^{-2\nu s} + C_3(s)e^{-3\nu s} = 0, \quad (22)$$

where

$$\begin{cases} C_1(s) = s^3 + \varphi_1 s^2 + \varphi_2 s + \varphi_3, \\ C_2(s) = \varphi_4 s^2 + \varphi_5, \\ C_3(s) = \varphi_6, \end{cases} \quad (23)$$

where

$$\begin{cases} \varphi_1 = 3\alpha, \\ \varphi_2 = 3\alpha^2, \\ \varphi_3 = \alpha^3, \\ \varphi_4 = -\alpha(a_{12}a_{21} + a_{13}a_{31} + a_{23}a_{32}), \\ \varphi_5 = -\alpha(a_{12}a_{21} + a_{13}a_{31} + a_{23}a_{32}), \\ \varphi_6 = -\alpha(a_{12}a_{21} + a_{13}a_{31} + a_{23}a_{32}). \end{cases} \quad (24)$$

We rewrite (22) as:

$$C_1(s)e^{\varphi s} + C_2(s)e^{-\nu s} + \varphi_6 e^{-2\nu s} = 0. \quad (25)$$

Let $s = i\theta = \theta(\cos \frac{\varphi s}{2} + i \sin \frac{\varphi s}{2})$ be the root of Equation (25) and let $C_{IR}(s)$ and $C_{II}(s)$ denote the real part and the imaginary part of $C_j(s) (j = 1, 2)$. Then one has

$$\begin{cases} C_{IR}(s) = \theta^3 \sin \frac{\varphi s}{2} + \varphi_1 \theta^2 \cos \varphi r \pi + \varphi_2 \theta \cos \frac{r \pi}{2} + \varphi_3, \\ C_{II}(s) = \theta^3 \cos \frac{r \pi}{2} + \varphi_1 \theta^2 \sin \varphi r \pi + \varphi_2 \theta \sin \frac{r \pi}{2}, \\ C_{2R}(s) = \varphi_4 \theta \cos \frac{r \pi}{2} + \varphi_5, \\ C_{2I}(s) = \varphi_4 \theta \sin \frac{r \pi}{2}. \end{cases} \quad (26)$$

Let

$$\begin{cases} d_1 = \cos \frac{3\varphi s}{2}, \\ d_2 = \varphi_1 \cos \varphi r \pi, \\ d_3 = \varphi_2 \cos \frac{r \pi}{2}, \\ d_4 = \varphi_3, \\ d_5 = \sin \frac{3\varphi s}{2}, \\ d_6 = \varphi_1 \sin \varphi r \pi, \\ d_7 = \varphi_2 \sin \frac{r \pi}{2}, \\ d_8 = \varphi_4 \cos \frac{r \pi}{2}, \\ d_9 = \varphi_5, \\ d_{10} = \varphi_4 \sin \frac{r \pi}{2}. \end{cases} \quad (27)$$

then (26) becomes

$$\begin{cases} C_{IR}(s) = d_1 \theta^3 + d_2 \theta^2 r + d_3 \theta r^2 + d_4, \\ C_{II}(s) = d_5 \theta^3 + d_6 \theta^2 r + d_7 \theta r^2, \\ C_{2R}(s) = d_8 \theta r^2 + d_9, \\ C_{2I}(s) = d_{10} \theta r^2. \end{cases} \quad (28)$$
By virtue of (25) and (28), we have:

\[
[(d_1 \theta^{3r} + d_2 \theta^{2r} + d_3 \theta^r + d_4) + i(d_5 \theta^{3r} + d_6 \theta^{2r} + d_7 \theta^r)](\cos \theta \rho + i \sin \theta \rho) + [(d_8 \theta^r + d_9) + i d_{10} \theta^r](\cos \theta \rho - i \sin \theta \rho) + \phi_6(\cos 2\theta \rho - i \sin 2\theta \rho) = 0. \tag{29}
\]

It follows from (29) that:

\[
\begin{align*}
D_1 \cos \theta \rho - D_2 \sin \theta \rho &= -\phi_6 \cos 2\theta \rho, \\
D_3 \cos \theta \rho + D_4 \cos \theta \rho &= \phi_6 \sin 2\theta \rho,
\end{align*}
\]

where

\[
\begin{align*}
D_1 &= d_1 \theta^{3r} + d_2 \theta^{2r} + (d_3 + d_8) \theta^r + d_4 + d_9, \\
D_2 &= d_5 \theta^{3r} + d_6 \theta^{2r} + (d_7 + d_{10}) \theta^r, \\
D_3 &= d_5 \theta^{3r} + d_6 \theta^{2r} + (d_7 - d_{10}) \theta^r, \\
D_4 &= d_1 \theta^{3r} + d_2 \theta^{2r} + (d_3 - d_8) \theta^r + d_4 - d_9.
\end{align*}
\]

In view of (30), we get:

\[
[D_1 \cos \theta \rho - D_2 \sin \theta \rho]^2 + [D_3 \cos \theta \rho + D_4 \cos \theta \rho]^2 = \phi_6^2, \tag{32}
\]

which leads to:

\[
(D_1^2 + D_2^2) \cos^2 \theta \rho + (D_3^2 + D_4^2) \sin^2 \theta \rho + 2(D_3 D_4 - D_1 D_2) \cos \theta \rho \sin \theta \rho = \phi_6^2. \tag{33}
\]

By (33), we have:

\[
(D_1^2 + D_2^2) \cos^2 \theta \rho + (D_3^2 + D_4^2) \sin^2 \theta \rho - \phi_6^2 = -2(D_3 D_4 - D_1 D_2) \cos \theta \rho \sin \theta \rho, \tag{34}
\]

which leads to

\[
[(D_1^2 + D_2^2) \cos^2 \theta \rho + (D_3^2 + D_4^2) \sin^2 \theta \rho - \phi_6^2]^2 = 4(D_3 D_4 - D_1 D_2)^2 \cos^2 \theta \rho \sin^2 \theta \rho. \tag{35}
\]

It follows from (35) that:

\[
[(D_1^2 + D_2^2) \cos^2 \theta \rho + (D_3^2 + D_4^2) \sin^2 \theta \rho - \phi_6^2]^2 = 4(D_3 D_4 - D_1 D_2)^2 \cos^2 \theta \rho \sin^2 \theta \rho. \tag{36}
\]

Namely,

\[
[(D_1^2 + D_2^2) \cos^2 \theta \rho + (D_3^2 + D_4^2) \sin^2 \theta \rho - \phi_6^2]^2 = 4(D_3 D_4 - D_1 D_2)^2 \cos^2 \theta \rho \sin^2 \theta \rho. \tag{37}
\]

Then

\[
B_1 \cos^4 \theta \rho + B_2 \cos^2 \theta \rho + B_3 = 0, \tag{38}
\]

where

\[
\begin{align*}
B_1 &= (D_1^2 + D_2^2 - D_3^2 - D_4^2)^2 + 4(D_3 D_4 - D_1 D_2)^2, \\
B_2 &= 2(D_1^2 + D_2^2) \cos^2 \theta \rho + (D_3^2 + D_4^2)(D_2^2 + D_4^2 - \phi_6^2), \\
B_3 &= (D_2^2 + D_4^2 - \phi_6^2)^2.
\end{align*}
\]

Assume that:

\[
(Q_4) \ B_2 < 0, B_2^2 \geq 4B_1 B_3
\]

holds, then by (38), we get

\[
\cos^2 \theta \rho = \frac{-B_2 + \sqrt{B_2^2 - 4B_1 B_3}}{2B_1}, \tag{40}
\]

or

\[
\cos^2 \theta \rho = \frac{-B_2 - \sqrt{B_2^2 - 4B_1 B_3}}{2B_1}. \tag{41}
\]
According to (40) and (41), we get:

\[
\cos \theta \rho = \pm \left( \frac{-B_2 + \sqrt{B_2^2 - 4B_1B_3}}{2B_1} \right)^{\frac{1}{2}},
\]

or

\[
\cos \theta \rho = \pm \left( \frac{-B_2 - \sqrt{B_2^2 - 4B_1B_3}}{2B_1} \right)^{\frac{1}{2}}.
\]

Suppose that (38) has twelve real roots \( \theta_j (j = 1, 2, \cdots, 6) \). By (42) and (43), we have:

\[
\rho_j^1 = \frac{1}{\rho_j} \left\{ \arccos \left[ \left. \left( \frac{-B_2 + \sqrt{B_2^2 - 4B_1B_3}}{2B_1} \right)^{\frac{1}{2}} \right| + 2l\pi \right] \right\},
\]

\[
\rho_j^2 = \frac{1}{\rho_j} \left\{ \arccos \left[ \left. \left( \frac{-B_2 + \sqrt{B_2^2 - 4B_1B_3}}{2B_1} \right)^{\frac{1}{2}} \right| + 2l\pi \right] \right\},
\]

\[
\rho_j^3 = \frac{1}{\rho_j} \left\{ \arccos \left[ \left. \left( \frac{-B_2 - \sqrt{B_2^2 - 4B_1B_3}}{2B_1} \right)^{\frac{1}{2}} \right| + 2l\pi \right] \right\},
\]

\[
\rho_j^4 = \frac{1}{\rho_j} \left\{ \arccos \left[ \left. \left( \frac{-B_2 - \sqrt{B_2^2 - 4B_1B_3}}{2B_1} \right)^{\frac{1}{2}} \right| + 2l\pi \right] \right\},
\]

where \( l = 0, 1, 2, \cdots, j = 1, 2, \cdots, 12 \). Let

\[
\rho_0 = \min_{k=1,2,3,4} \{ \rho_j^0 \}, \theta_0 = \theta |_{\rho = \rho_0}.
\]

In the sequel, the hypothesis is given:

(Q5) \( S_{11}S_{21} + S_{12}S_{22} > 0 \), where

\[
\begin{align*}
S_{11} &= 3\rho_0^{3r - 1} \cos \left( \frac{3r - 1}{2} \pi \right) + 2\rho_1 \rho_0^{2r-1} \cos \left( \frac{2r - 1}{2} \pi \right) + r\rho_4 \rho_0^{r-1} \cos \left( \frac{r - 1}{2} \pi \right), \\
S_{12} &= 3\rho_0^{3r - 1} \sin \left( \frac{3r - 1}{2} \pi \right) + 2\rho_1 \rho_0^{2r-1} \sin \left( \frac{2r - 1}{2} \pi \right) + r\rho_4 \rho_0^{r-1} \sin \left( \frac{r - 1}{2} \pi \right), \\
S_{21} &= -\rho_0 \rho_0^{r+1} \cos \left( \frac{2r - 1}{2} \pi \right) \sin \theta_0 \rho_0 + \rho_4 \rho_0^{r-1} \sin \left( \frac{2r - 1}{2} \pi \right) \cos \theta_0 \rho_0, \\
S_{22} &= \rho_0 \rho_0^{r+1} \sin \left( \frac{2r - 1}{2} \pi \right) \sin \theta_0 \rho_0 + \rho_4 \rho_0^{r-1} \cos \left( \frac{2r - 1}{2} \pi \right) \cos \theta_0 \rho_0.
\end{align*}
\]

Lemma 5. Let \( s(\rho) = i_1(\rho) + i_2(\rho) \) be the root of Equation (22) at \( \rho = \rho_0 \) such that \( i_1(\rho_0) = 0, i_2(\rho_0) = \omega_0, \) then \( \text{Re} \left[ \frac{d}{dp} \right]_{p=\rho_0, \theta=\theta_0} > 0. \)

Proof. It follows from Equation (22) that:
(3r^3s^{-1} + 2r\varepsilon_1 s^{2r-1} + r\varepsilon_2 s^{r-1}) \frac{ds}{d\rho} + r\varepsilon_4 s^{r-1} e^{-2sp} \frac{ds}{d\rho} - 2e^{-2sp} (\frac{ds}{d\rho} + s) (q_4 s^r + \varepsilon_5) - 3\varepsilon_6 e^{-3sp} (\frac{ds}{d\rho} + s) = 0. \quad (50)

Then
\[
\frac{ds}{d\rho} = \frac{S_1(s)}{S_2(s)} - \frac{\rho}{s},
\]
where
\[
\begin{cases}
S_1(s) = 3r^3s^{-1} + 2r\varepsilon_1 s^{2r-1} + r\varepsilon_2 s^{r-1} e^{-2sp}, \\
S_2(s) = 2se^{-2sp} (q_4 s^r + \varepsilon_5) + 3\varepsilon_6 e^{-3sp}.
\end{cases}
\]

Then
\[
\text{Re}\left\{ \left( \frac{ds}{d\rho} \right)^{-1} \right\} = \text{Re}\left\{ \frac{S_1(s)}{S_2(s)} \right\} = \frac{S_{11}S_{22} + S_{12}S_{21}}{S_{21}^2 + S_{22}^2}. \quad (53)
\]

By (Q_3), one gets:
\[
\text{Re}\left\{ \left( \frac{ds}{d\rho} \right)^{-1} \right\} \bigg|_{\rho=\rho_0, \theta=\theta_0} > 0, \quad (54)
\]
which completes the proof. \( \square \)

Now we give the following hypothesis:
(Q_6) The following inequalities hold:
\[
\begin{cases}
G_1 = e_1 > 0, \\
G_2 = \text{det} \begin{bmatrix} e_1 & 1 \\ e_3 + e_5 + e_6 & e_2 + e_4 \end{bmatrix} > 0, \\
G_3 = (e_3 + e_5 + e_6) G_2 > 0.
\end{cases} \quad (55)
\]

**Lemma 6.** If (Q_6) is satisfied, then the zero equilibrium point of system (2) with \( \rho = 0 \) is locally asymptotically stable.

**Proof.** When \( \rho = 0 \), then Equation (22) takes the form:
\[
\lambda^3 + e_1 \lambda^2 + (e_2 + e_4) \lambda + e_3 + e_5 + e_6 = 0. \quad (56)
\]

In view of (Q_6), one knows that all roots \( \lambda_j \) of (54) satisfy \( |\arg(\lambda_j)| > \frac{\pi}{2} (j = 1, 2, 3) \). Thus Lemma 6 is true. The proof ends. \( \square \)

Based on the analysis above, we can easily derive the following conclusion.

**Theorem 3.** Assume that (Q_1) - (Q_6) are fulfilled, then the zero equilibrium point \( W_0(0,0,0) \) of system (2) is locally asymptotically stable if \( 0 \leq \rho < \rho_0 \) and a Hopf bifurcation arises near \( W_0(0,0,0) \) if \( \rho = \rho_0 \).

6. Bifurcation Control

In this segment, we will explore the control issue of the Hopf bifurcation of system (2) via mixed controller including state feedback and parameter perturbation. Following the idea of Yuan et al. [50], we establish the following controlled system:
\[
\begin{align*}
\frac{d'w_1(t)}{dt'} &= \varepsilon_1 [-\nu w_1(t) + h(w_1(t)) + h_{12}(w_2(t - \rho)) + h_{13}(w_3(t - \rho))] + \varepsilon_2 w_1(t), \\
\frac{d'w_2(t)}{dt'} &= \varepsilon_1 [-\nu w_2(t) + h(w_2(t)) + h_{21}(w_1(t - \rho)) + h_{23}(w_3(t - \rho))] + \varepsilon_2 w_2(t), \\
\frac{d'w_3(t)}{dt'} &= \varepsilon_1 [-\nu w_3(t) + h(w_3(t)) + h_{31}(w_1(t - \rho)) + h_{32}(w_2(t - \rho))] + \varepsilon_2 w_3(t),
\end{align*}
\quad (57)
\]
where \( \varepsilon_1 \) and \( \varepsilon_2 \) are feedback gain parameters and real constants. It is easy to see that system (57) has a unique equilibrium point \( W_0(0,0,0) \). The linear system of model (57) at \( W_0(0,0,0) \) is:

\[
\begin{align*}
\begin{bmatrix}
\frac{d^r w_1(t)}{dt^r} \\
\frac{d^r w_2(t)}{dt^r} \\
\frac{d^r w_3(t)}{dt^r}
\end{bmatrix}
&= -\beta w_1(t) + b_{12}w_2(t - \rho) + b_{13}w_3(t - \rho), \\
&= -\beta w_2(t) + b_{21}w_1(t - \rho) + b_{23}w_3(t - \rho), \\
&= -\beta w_3(t) + b_{31}w_1(t - \rho) + b_{32}w_2(t - \rho),
\end{align*}
\]

(58)

where \( \beta = \varepsilon_1 (\nu - h^r(0)), b_{12} = \varepsilon_1 h^{12}(0), b_{13} = \varepsilon_1 h^{13}(0), b_{21} = \varepsilon_1 h^{21}(0), b_{23} = \varepsilon_1 h^{23}(0), b_{31} = \varepsilon_1 h^{31}(0), b_{32} = \varepsilon_1 h^{32}(0) \). The corresponding characteristic equation for (58) is:

\[
\begin{bmatrix}
\delta' + \beta & -b_{12}e^{-\delta p} & -b_{13}e^{-\delta p} \\
-b_{21}e^{-\delta p} & \delta' + \beta & -b_{23}e^{-\delta p} \\
-b_{31}e^{-\delta p} & -b_{32}e^{-\delta p} & \delta' + \beta
\end{bmatrix}
= 0.
\]

(59)

It follows from (58) that

\[
\mathcal{H}_1(s) + \mathcal{H}_2(s)e^{-2\delta p} + \mathcal{H}_3(s)e^{-3\delta p} = 0,
\]

(60)

where

\[
\begin{align*}
\mathcal{H}_1(s) &= s^r + \delta_1 s^{2r} + \delta_2 s^r + \delta_3, \\
\mathcal{H}_2(s) &= \delta_4 s^r + \delta_5, \\
\mathcal{H}_3(s) &= \delta_6,
\end{align*}
\]

(61)

where

\[
\begin{align*}
\delta_1 &= 3\beta, \\
\delta_2 &= 3\beta^2, \\
\delta_3 &= \beta^3, \\
\delta_4 &= -(b_{12}b_{21} + b_{13}b_{31} + b_{23}b_{32}), \\
\delta_5 &= -a(b_{12}b_{21} + b_{13}b_{31} + b_{23}b_{32}), \\
\delta_6 &= -(b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32}).
\end{align*}
\]

(62)

We rewrite (60) as:

\[
\mathcal{H}_1(s)e^{\delta p} + \mathcal{H}_2(s)e^{-\delta p} + \delta_6 e^{-2\delta p} = 0.
\]

(63)

Let \( s = i\theta = \theta (\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}) \) be the root of Equation (63) and let \( \mathcal{H}_{1R}(s) \) and \( \mathcal{H}_{1I}(s) \) denote the real part and the imaginary part of \( \mathcal{H}_i(s) (i = 1, 2) \). Then one has

\[
\begin{align*}
\mathcal{H}_{1R}(s) &= \theta^{3r} \cos \frac{3r\pi}{2} + \delta_1 \theta^{2r} \cos \frac{r\pi}{2} + \delta_2 \theta^r \cos \frac{r\pi}{2} + \delta_3, \\
\mathcal{H}_{1I}(s) &= \theta^{3r} \sin \frac{3r\pi}{2} + \delta_1 \theta^{2r} \sin \frac{r\pi}{2} + \delta_2 \theta^r \sin \frac{r\pi}{2}, \\
\mathcal{H}_{2R}(s) &= \delta_4 \theta^r \cos \frac{r\pi}{2} + \delta_5, \\
\mathcal{H}_{2I}(s) &= \delta_4 \theta^r \sin \frac{r\pi}{2}.
\end{align*}
\]

(64)
Let
\[
\begin{cases}
  e_1 = \cos \frac{3r\pi}{2}, \\
  e_2 = \delta_1 \cos r\pi, \\
  e_3 = \delta_2 \cos \frac{r\pi}{2}, \\
  e_4 = \delta_3, \\
  e_5 = \sin \frac{3r\pi}{2}, \\
  e_6 = \delta_1 \sin r\pi, \\
  e_7 = \delta_2 \sin \frac{r\pi}{2}, \\
  e_8 = \delta_4 \cos \frac{r\pi}{2}, \\
  e_9 = \delta_5, \\
  e_{10} = \delta_4 \sin \frac{r\pi}{2},
\end{cases}
\]

then (64) becomes
\[
\begin{align*}
  \mathcal{H}_{1R}(s) &= e_1 \theta^3 + e_2 \theta^2 r + e_3 \theta r + e_4, \\
  \mathcal{H}_{1I}(s) &= e_5 \theta^3 + e_6 \theta^2 r + e_7 \theta r, \\
  \mathcal{H}_{2R}(s) &= e_8 \theta^3 + e_9, \\
  \mathcal{H}_{2I}(s) &= e_{10} \theta r.
\end{align*}
\]

By virtue of (63) and (66), we have:
\[
\begin{align*}
  &\left[(e_1 \theta^3 + e_2 \theta^2 r + e_3 \theta r + e_4) + i(e_5 \theta^3 + e_6 \theta^2 r + e_7 \theta r)\right](\cos \theta p + i \sin \theta p) \\
  &+ \left[(e_8 \theta^3 + e_9) + i e_{10} \theta r\right](\cos \theta p - i \sin \theta p) + \delta_6 (\cos 2\theta p - i \sin 2\theta p) = 0.
\end{align*}
\]

It follows from (67) that
\[
\begin{align*}
  \begin{cases}
    I_1 \cos \theta p - I_2 \sin \theta p &= -\delta_6 \cos 2\theta p, \\
    I_3 \cos \theta p + I_4 \cos \theta p &= \delta_6 \sin 2\theta p,
  \end{cases}
\end{align*}
\]

where
\[
\begin{align*}
  I_1 &= e_1 \theta^3 + e_2 \theta^2 r + (e_3 + e_8) \theta r + e_4 + e_9, \\
  I_2 &= e_5 \theta^3 + e_6 \theta^2 r + (e_7 + e_{10}) \theta r, \\
  I_3 &= e_8 \theta^3 + e_9, \\
  I_4 &= e_1 \theta^3 + e_2 \theta^2 r + (e_3 - e_8) \theta r + e_4 - e_9.
\end{align*}
\]

In view of (69), we get:
\[
[I_1 \cos \theta p - I_2 \sin \theta p]^2 + [I_3 \cos \theta p + I_4 \cos \theta p]^2 = \delta_6^2, 
\]

which leads to:
\[
(I_1^2 + I_2^2) \cos^2 \theta p + (I_3^2 + I_4^2) \sin^2 \theta p + 2(I_3 I_4 - I_1 I_2) \cos \theta p \sin \theta p = \delta_6^2. 
\]

By (71), we have:
\[
(I_1^2 + I_2^2) \cos^2 \theta p + (I_3^2 + I_4^2) \sin^2 \theta p - \delta_6^2 = -2(I_3 I_4 - I_1 I_2) \cos \theta p \sin \theta p, 
\]

which leads to:
\[
|\left(I_1^2 + I_2^2\right) \cos^2 \theta p + \left(I_3^2 + I_4^2\right) \sin^2 \theta p - \delta_6^2| = 4\left(I_3 I_4 - I_1 I_2\right)^2 \cos^2 \theta p \sin^2 \theta p. 
\]

It follows from (73) that:
\[
|\left(I_1^2 + I_2^2\right) \cos^2 \theta p + \left(I_3^2 + I_4^2\right) \sin^2 \theta p - \delta_6^2| = 4\left(I_3 I_4 - I_1 I_2\right)^2 \cos^2 \theta p \sin^2 \theta p. 
\]
Namely,
\[(I_2^2 + I_3^2) \cos^2 \varphi + (I_2^2 + I_4^2) \sin^2 \varphi - \delta_0^2] = 4(I_3 I_4 - I_1 I_2)^2 \cos^2 \varphi \sin^2 \varphi. \tag{75}\]

Then,
\[\mathcal{J}_1 \cos^4 \varphi + \mathcal{J}_2 \cos^2 \varphi + \mathcal{J}_3 = 0, \tag{76}\]

where
\[
\begin{cases}
\mathcal{J}_1 = (I_2^2 + I_3^2 - I_2^2 - I_4^2)^2 + 4(I_3 I_4 - I_1 I_2)^2, \\
\mathcal{J}_2 = 2(I_2^2 + I_3^2) \cos^2 \varphi + (I_2^2 + I_4^2)(I_2^2 + I_4^2 - \delta_0^2), \\
\mathcal{J}_3 = (I_2^2 + I_4^2 - \delta_0^2)^2.
\end{cases} \tag{77}\]

Assume that
\[(Q_7) \mathcal{J}_2 < 0, \mathcal{J}_2^2 \geq 4 \mathcal{J}_1 \mathcal{J}_3 \]
holds, then by (76), we get:
\[\cos^2 \varphi = \frac{-\mathcal{J}_2 + \sqrt{\mathcal{J}_2^2 - 4 \mathcal{J}_1 \mathcal{J}_3}}{2 \mathcal{J}_1} \tag{78}\]
or
\[\cos^2 \varphi = \frac{-\mathcal{J}_2 - \sqrt{\mathcal{J}_2^2 - 4 \mathcal{J}_1 \mathcal{J}_3}}{2 \mathcal{J}_1} \tag{79}\]

According to (78) and (79), we can get:
\[\cos \varphi = \pm \left(\frac{-\mathcal{J}_2 + \sqrt{\mathcal{J}_2^2 - 4 \mathcal{J}_1 \mathcal{J}_3}}{2 \mathcal{J}_1}\right)^{\frac{1}{2}} \tag{80}\]
or
\[\cos \varphi = \pm \left(\frac{-\mathcal{J}_2 - \sqrt{\mathcal{J}_2^2 - 4 \mathcal{J}_1 \mathcal{J}_3}}{2 \mathcal{J}_1}\right)^{\frac{1}{2}} \tag{81}\]

Suppose that (76) has twelve real roots \(\varphi_j \) \((j = 1, 2, \ldots, 6)\). By (80) and (81), we have:
\[\rho_{11} = \frac{1}{\rho_i} \left\{ \arccos \left[ \left(\frac{-\mathcal{J}_2 + \sqrt{\mathcal{J}_2^2 - 4 \mathcal{J}_1 \mathcal{J}_3}}{2 \mathcal{J}_1}\right)^{\frac{1}{2}} + 2l \pi \right] \right\}, \tag{82}\]
\[\rho_{12} = \frac{1}{\rho_i} \left\{ \arccos \left[ -\left(\frac{-\mathcal{J}_2 + \sqrt{\mathcal{J}_2^2 - 4 \mathcal{J}_1 \mathcal{J}_3}}{2 \mathcal{J}_1}\right)^{\frac{1}{2}} + 2l \pi \right] \right\}, \tag{83}\]
\[\rho_{13} = \frac{1}{\rho_i} \left\{ \arccos \left[ \left(\frac{-\mathcal{J}_2 - \sqrt{\mathcal{J}_2^2 - 4 \mathcal{J}_1 \mathcal{J}_3}}{2 \mathcal{J}_1}\right)^{\frac{1}{2}} + 2l \pi \right] \right\}, \tag{84}\]
\[\rho_{14} = \frac{1}{\rho_i} \left\{ \arccos \left[ -\left(\frac{-\mathcal{J}_2 - \sqrt{\mathcal{J}_2^2 - 4 \mathcal{J}_1 \mathcal{J}_3}}{2 \mathcal{J}_1}\right)^{\frac{1}{2}} + 2l \pi \right] \right\}. \tag{85}\]
where \( l = 0, 1, 2, \ldots, i = 1, 2, \ldots, 6 \). Let
\[
\rho_* = \min_{l = 1, 2, 3, 4} \{ r^0_{il} \}, \theta_0 = \theta|_{\rho = \rho_*}.
\] (86)

In the sequel, the hypothesis is given:
\[
(Q_8) \quad N_{11}N_{21} + N_{12}N_{22} > 0, \text{ where}
\]
\[
\begin{align*}
N_{11} &= 3r_0^3 \rho^{-1} \cos \left( \frac{(r - 1)\pi}{2} \right) + 2r_1 \rho^{-1} \cos \left( \frac{(r - 1)\pi}{2} \right) + 3r_2 \rho^{-1} \cos \left( \frac{(r - 1)\pi}{2} \right) \\
N_{12} &= 3r_0^3 \rho^{-1} \sin \left( \frac{(r - 1)\pi}{2} \right) + 2r_1 \rho^{-1} \sin \left( \frac{(r - 1)\pi}{2} \right) + 3r_2 \rho^{-1} \sin \left( \frac{(r - 1)\pi}{2} \right) \\
N_{21} &= -\rho_* \delta_0 \rho + \theta_0 \left( \delta_4 \rho_0 \cos \left( \frac{\pi}{2} \right) + \delta_5 \right) \cos 2\theta_0 \rho_* \\
N_{22} &= \rho_* \delta_0 \rho + \theta_0 \left( \delta_4 \rho_0 \cos \left( \frac{\pi}{2} \right) + \delta_5 \right) \cos 2\theta_0 \rho_* \\
&+ 3\rho_* \rho_0 \cos 3\theta_0 \rho_*.
\end{align*}
\] (87)

Lemma 7. Let \( s(\rho) = \varphi_1(\rho) + i \varphi_2(\rho) \) be the root of Equation (60) at \( \rho = \rho_* \) such that \( \varphi_1(\rho_*) = 0, \varphi_2(\rho_*) = 0 \), then \( \text{Re} \left[ \frac{ds}{d\varphi} \right]_{\rho = \rho_*} > 0 \).

Proof. It follows from Equation (60) that
\[
(3r_0^3 + 2r_1 \rho + r_2 \rho^2) \frac{ds}{d\varphi} + r_4 \rho^2 \frac{ds}{d\varphi} + 4e^{-2\rho} \left( \frac{ds}{d\varphi} + s \right) (\delta_4 \rho_0 + \delta_5) - 3\delta_0 e^{-3\rho} \left( \frac{ds}{d\varphi} + s \right) = 0.
\] (88)

Then
\[
\left[ \frac{ds}{d\varphi} \right]^{-1} = \frac{N_1(s)}{N_2(s)} - \frac{\rho}{s}.
\] (89)

where
\[
\begin{align*}
N_1(s) &= 3r_0^3 + 2r_1 \rho + r_2 \rho^2 + r_4 \rho^2 e^{-2\rho}, \\
N_2(s) &= 2s e^{-2\rho} (\delta_4 \rho_0 + \delta_5) + 3\delta_0 e^{-3\rho}.
\end{align*}
\] (90)

Then,
\[
\text{Re} \left[ \frac{ds}{d\varphi} \right]_{\rho = \rho_*} = \text{Re} \left[ \frac{N_1(s)}{N_2(s)} \right]_{\rho = \rho_*} = \frac{N_{11}N_{21} + N_{12}N_{22}}{N_{21}^2 + N_{22}^2}.
\] (91)

By \( Q_8 \), one gets
\[
\text{Re} \left[ \left[ \frac{ds}{d\varphi} \right]^{-1} \right]_{\rho = \rho_*} > 0,
\] (92)

which completes the proof. \( \square \)

Now we give the following hypothesis:
\[
(Q_9) \quad \text{The following inequalities hold:}
\]
\[
\begin{align*}
F_1 &= \delta_1 > 0, \\
F_2 &= \det \left[ \begin{array}{ccc}
\delta_1 & \delta_1 & 1 \\
\delta_3 + \delta_5 + \delta_6 & \delta_2 + \delta_4 & \\
\delta_3 + \delta_5 + \delta_6 & \delta_3 + \delta_5 + \delta_6
\end{array} \right] > 0, \\
F_3 &= (\delta_3 + \delta_5 + \delta_6)F_2 > 0.
\end{align*}
\] (93)
Lemma 8. If \((Q_9)\) is satisfied, then the zero equilibrium point of system (57) with \(\rho = 0\) is locally asymptotically stable.

Proof. When \(\rho = 0\), then Equation (60) takes the form:

\[
\lambda^3 + \delta_1 \lambda^2 + (\delta_2 + \delta_4) \lambda + \delta_3 + \delta_5 + \delta_6 = 0.
\]

In view of \((Q_9)\), one knows that all roots \(\lambda_j\) of (94) satisfy \(|\text{arg}(\lambda_j)| > \frac{\pi}{2}\) \((j = 1, 2, 3)\). Thus Lemma 8 is true. The proof ends. \(\square\)

Based on the discussion above, we can easily derive the following conclusion.

Theorem 4. Assume that \((Q_1), (Q_2), (Q_3), (Q_7), (Q_8), (Q_9)\) are fulfilled, then the zero equilibrium point \(W_0(0, 0, 0)\) of system (57) is locally asymptotically stable if \(0 \leq \rho < \rho_*\) and a Hopf bifurcation arises near \(W_0(0, 0, 0)\) if \(\rho = \rho_*\).

Remark 2. In 2006, Yan [41] investigated the Hopf bifurcation of integer-order delayed tri-neuron neural networks. In this present research, on the basis of Yan [41], we build a class of new fractional-order delayed tri-neuron neural networks. The existence and uniqueness, boundedness of the solution, stability and the onset of Hopf bifurcation, Hopf bifurcation control of fractional-order delayed tri-neuron neural networks is discussed in detail. The research approach of Yan [41] can be be applied to the fractional-order delayed tri-neuron neural networks to establish the key results of this article. Based on this standpoint, we think that our work replenishes the work of Yan [41].

7. Numerical Simulations

Example 1. Give the fractional order system as follows:

\[
\begin{align*}
\frac{d^{0.95}w_1(t)}{dt^{0.95}} &= -2w_1(t) + \tanh(w_1(t)) + \tanh(w_2(t - \rho)) + \tanh(w_3(t - \rho)), \\
\frac{d^{0.95}w_2(t)}{dt^{0.95}} &= -2w_2(t) + \tanh(w_2(t)) + \tanh(w_1(t - \rho)) + \tanh(w_3(t - \rho)), \\
\frac{d^{0.95}w_3(t)}{dt^{0.95}} &= -2w_3(t) + \tanh(w_3(t)) + \tanh(w_1(t - \rho)) + \tanh(w_2(t - \rho)).
\end{align*}
\]

Obviously, system (95) owns a unique equilibrium point \(W_0(0, 0, 0)\). By virtue of Matlab software, we derive \(\theta_0 = 0.9223\) and \(\rho_0 = 0.6\). Furthermore, we can verify that all the hypotheses \((Q_1)-(Q_5)\) of Theorem 3 are fulfilled. So the zero equilibrium point \(W_0(0, 0, 0)\) of system (95) is locally asymptotically stable if \(\rho \in [0,0.6)\) and system (95) will generates Hopf bifurcation phenomenon near \(W_0(0, 0, 0)\) when \(\rho\) crosses \(\rho_0 \approx 0.6\). Now we select \(\rho = 0.5\) is less than \(\rho_0 \approx 0.6\), then we get the simulation results that are displayed in Figure 1, which shows that the three variables \(w_1(t), w_2(t), w_3(t)\) gradually tend to zero when time \(t\) increases. Then we select \(\rho = 0.7\) is bigger than \(\rho_0 \approx 0.6\), then we get the simulation results that are displayed in Figure 2, which shows that the three variables \(w_1(t), w_2(t), w_3(t)\) keep a periodic oscillation (namely, Hopf bifurcation) near \(W_0(0, 0, 0)\) when time \(t\) increases. The bifurcation figures are given in Figures 3–5. Figure 3 shows the relation between \(\rho \) and \(w_1\), Figure 4 shows the relation between \(\rho \) and \(w_2\) and Figure 5 shows the relation between \(\rho \) and \(w_3\). From Figures 3–5, we can see that the bifurcation value \(\rho_0 \approx 0.6\). In addition, the relation of time delay \(\rho_0\) and the amplitude \(\theta_0\) is presented in Table 1.

Example 2. Give the fractional order system as follows:

\[
\begin{align*}
\frac{d^{0.95}w_1(t)}{dt^{0.95}} &= 0.6[-2w_1(t) + \tanh(w_1(t)) + h_{12}(w_2(t - \rho)) + \tanh(w_3(t - \rho))] + 0.4w_1(t), \\
\frac{d^{0.95}w_2(t)}{dt^{0.95}} &= 0.6[-2w_2(t) + \tanh(w_2(t)) + h_{21}(w_1(t - \rho)) + \tanh(w_3(t - \rho))] + 0.4w_2(t), \\
\frac{d^{0.95}w_3(t)}{dt^{0.95}} &= 0.6[-2w_3(t) + \tanh(w_3(t)) + h_{31}(w_1(t - \rho)) + \tanh(w_2(t - \rho))] + 0.4w_3(t).
\end{align*}
\]
Obviously, system (96) owns a unique equilibrium point $W_0(0,0,0)$. By virtue of Matlab software, we derive $\theta_0 = 1.9070$ and $\rho_* \approx 0.3$. Furthermore, we can verify that all the hypotheses $(Q_1), (Q_2), (Q_3), (Q_7), (Q_8)$ of Theorem 4 are fulfilled. So the zero equilibrium point $W_0(0,0,0)$ of system (96) is locally asymptotically stable if $\rho \in [0,0.3)$ and system (96) will generate the Hopf bifurcation phenomenon near $W_0(0,0,0)$ when $\rho$ crosses $\rho_* \approx 0.3$. Now, we select $\rho = 0.24$ as less than $\rho_* \approx 0.6$, then we get the simulation results that are displayed in Figure 6, which show that the three variables $w_1(t), w_2(t), w_3(t)$ gradually tend to zero when time $t$ increases. Then we select $\rho = 0.35$ as bigger than $\rho_* \approx 0.3$, then we get the simulation results that are displayed in Figure 7, which show that the three variables $w_1(t), w_2(t), w_3(t)$ keep a periodic oscillation (namely, Hopf bifurcation) near $W_0(0,0,0)$ when time $t$ increases. The bifurcation figures are given in Figures 8–10. Figure 8 shows the relation between $\rho$ and $w_1$, Figure 9 shows the relation between $\rho$ and $w_2$ and Figure 10 shows the relation between $\rho$ and $w_3$. From Figures 8–10, we can see that the bifurcation value $\rho_* \approx 0.3$. In addition, the relation of time delay $\rho_0$ and the amplitude $\rho_*$ are presented in Table 2.

![Figure 1. Cont.](image-url)
Figure 1. Computer simulation figures of system (95) with $\rho = 0.5 < \rho_0 \approx 0.6$. The three variables $w_1(t) \to 0, w_2(t) \to 0, w_3(t) \to 0$ when time $t \to \infty$.

Figure 2. Cont.
Figure 2. Computer simulation figures of system (95) with $\rho = 0.7 > \rho_0 \approx 0.6$. The Hopf bifurcation arises in the vicinity of $W_0(0, 0, 0)$ when time $t \to \infty$.

Figure 3. Bifurcation figure of system (95): $\rho$ versus $w_1$. The bifurcation value $\rho_0 \approx 0.6$.

Figure 4. Bifurcation figure of system (95): $\rho$ versus $w_2$. The bifurcation value $\rho_0 \approx 0.6$. 
**Figure 5.** Bifurcation figure of system (95): $\rho$ versus $w_3$. The bifurcation value $\rho_0 \approx 0.6$.

**Table 1.** The relation of $\theta_0$ and $\rho_0$ of system (95).

<table>
<thead>
<tr>
<th>$\theta_0$</th>
<th>$\rho_0$</th>
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<tbody>
<tr>
<td>3.6893</td>
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<td>2.3058</td>
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<td>0.6435</td>
<td>0.86</td>
</tr>
<tr>
<td>0.5705</td>
<td>0.97</td>
</tr>
</tbody>
</table>

**Figure 6.** Cont.
Figure 6. Computer simulation figures of system (96) with $\rho = 0.24 < \rho_c \approx 0.3$. The three variables $w_1(t) \to 0, w_2(t) \to 0, w_3(t) \to 0$ when time $t \to \infty$.

Figure 7. Cont.
Figure 7. Computer simulation figures of system (96) with $\rho = 0.35 > \rho_* \approx 0.3$. The Hopf bifurcation arises in the vicinity of $W_0(0,0,0)$ when time $t \to \infty$. 
Figure 8. Bifurcation figure of system (96): $\rho$ versus $w_1$. The bifurcation value $\rho^* \approx 0.3$.

Figure 9. Bifurcation figure of system (96): $\rho$ versus $w_2$. The bifurcation value $\rho^* \approx 0.3$.

Figure 10. Bifurcation figure of system (96): $\rho$ versus $w_3$. The bifurcation value $\rho^* \approx 0.3$. 
Table 2. The relation of $\vartheta_0$ and $\rho_*$ of system (96).

<table>
<thead>
<tr>
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<th>$\rho_*$</th>
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<tbody>
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<tr>
<td>2.4874</td>
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<tr>
<td>0.6086</td>
<td>0.94</td>
</tr>
</tbody>
</table>

**Remark 3.** In system (95), the bifurcation value $\rho_0 \approx 0.6$, in system (96) (namely, the controlled system of system (95)), the bifurcation value $\rho_0 \approx 0.3$. We can clearly know that the stability region of system (95) is narrowed and the time of generation of Hopf bifurcation in advance.

8. Conclusions

The investigation of neural networks has become a very important topic in mathematical and engineering areas. Revealing the intrinsic dynamic characteristics of neural networks has aroused the interest of various scholars. In this article, based on the earlier studies, we built a new fractional-order tri-neuron neural network incorporating delays. The existence and uniqueness, boundedness of the solution of the established fractional-order tri-neuron neural networks are investigated. The delay-independent condition to ensure the stability and the onset of the Hopf bifurcation of the involved fractional-order tri-neuron neural networks is derived. The bifurcation value is determined. By virtue of a mixed controller, which includes state feedback and parameter perturbation, the stability region and the time of onset of the Hopf bifurcation of the considered fractional-order tri-neuron neural networks are adjusted. The research shows that time delay is a crucial parameter which affects the stability region and the time of onset of the Hopf bifurcation of the studied fractional-order tri-neuron neural networks and its controlled neural networks. The derived results have some theoretical value in designing and controlling neural networks. Meanwhile, the exploration idea of this article can also be utilized to study a lot of fractional-order delayed neural networks and some related delayed dynamical systems. In addition, in many neural networks, there exist leakage delays. In this paper, we do not involve the Hopf bifurcation control issue of neural networks with leakage delays. We will focus on this topic in near future.

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