fractal and fractional

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Some New Inequalities on Laplace–Stieltjes Transforms Involving Logarithmic Growth

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Abstract: This article is devoted to exploring the properties on the logarithmic growth of entire functions represented by Laplace–Stieltjes transforms of zero order. In order to describe the growth of Laplace–Stieltjes transforms more finely, we introduce some concepts of the logarithmic indexes of the maximum term and the center index of the maximum term of Laplace–Stieltjes transforms, and establish some new inequalities focusing on the above logarithmic indexes, the logarithmic order, the (lower) logarithmic type and the coefficients of Laplace–Stieltjes transforms. Moreover, we obtain two estimation forms on the (lower) logarithmic type of entire functions represented by Laplace–Stieltjes transform by applying these inequalities. One estimation is mainly by the center indexes of the maximum term, the other is by the logarithmic order, exponent and coefficients. Finally, we obtain the equivalence condition of entire functions with the perfectly logarithmic linear growth. This result shows that the two estimation forms can be equivalent to some extent.

Keywords: logarithmic order; (lower) logarithmic type; Laplace–Stieltjes transform; inequalities

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Citation: Xu, H.; Li, H.; Xuan, Z. Some New Inequalities on Laplace–Stieltjes Transforms Involving Logarithmic Growth. *Fractal Fract.* **2022**, *6*, 233. https://doi.org/10.3390/fractalfract6050233

Academic Editor: Hari Mohan Srivastava

Received: 15 March 2022
Accepted: 19 April 2022
Published: 22 April 2022

1. Introduction and Some Basic Notations

As we all know, the following transform

$$F(s) = \int_0^{+\infty} e^{sx} d\alpha(x), \qquad s = \sigma + it, \tag{1}$$

is usually called a Laplace–Stieltjes transform, if $\alpha(x)$ is a bounded variation on any finite interval $[0, Y](0 < Y < +\infty)$, and σ and t are real variables. Laplace–Stieltjes transform was first named after Pierre-Simon Laplace and Thomas Joannes Stieltjes, and is also an integral transform similar to the Laplace transform. Over the past 80 years or so, it has been used in many fields of mathematics, such as functional analysis, and certain areas of theoretical and applied probability.

Yu [1] in 1963 first studied the growth and convergence of Laplace–Stieltjes transforms (1) and gave the famous Valiron–Knopp–Bohr formula of the associated abscissas of bounded convergence, absolute convergence and uniform convergence of Laplace–Stieltjes transforms, and the Borel lines of entire functions represented by Laplace–Stieltjes transforms. After his wonderful results, many mathematicians had paid considerable attention focusing on the growth and the value distribution of analytic functions defined by Laplace–Stieltjes transforms convergent in the half-plane and whole complex plane, and obtained a series of classic and important results. For example, L. N. Shang, Z. S. Gao, Z. X. Xuan, etc. further investigated the value distributions of analytic functions of some kinds of growth defined by Laplace–Stieltjes transforms, and obtained some results about the singular direction and points of Laplace–Stieltjes transforms (see [2–5]); C. Singhal, G. S. Srivastava,

Y. Y. Kong, S. Y. Liu and H. Y. Xu studied the properties on the approximation of entire functions represented by Laplace-stieltjes transforms, and obtained some interesting theorems on the relationship between the error and growth (see [6–9]); O. Posiko and M. M. Sheremeta [10] in 2007 explored the relationships between the growth and the maximum term of Laplace–Stieltjes transform $\int_0^\infty f(x)e^{x\sigma}dF(x)$, where $f(x)\geq 0$, M. S. Dobushovskyi, M. M. Sheremeta [11,12] in 2017 and 2021, respectively, further analyzed the convergence and relative growth of such transform; Y. J. Bi and Y. Y. Huo [13] recently considered the growth of the double Laplace–Stieltjes transforms, and obtained some foundation growth theorems; Y. Y. Kong and his co-authors studied the growth of analytic functions defined by Laplace–Stieltjes transforms which converge in the half plane and the whole plane, and gave a great number of important theorems concerning the zero order, the generalized order, the finite and infinite order, and so on (see [14–21]).

In order to study the growth of Laplace–Stieltjes transform (1), we usually take a sequence $\{\lambda_n\}$ satisfying

$$0\leq\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots,\lambda_n\to\infty\ \ as\ \ n\to\infty, \tag{2}$$

and

$$\limsup_{n\to+\infty}(\lambda_{n+1}-\lambda_n)<+\infty. \tag{3}$$

And denote

$$A_n^*=\sup_{\lambda_n<x\leq\lambda_{n+1},-\infty<t<+\infty}\left|\int_{\lambda_n}^{x}e^{ity}d\alpha(y)\right|,$$

if Laplace–Stieltjes transform (1) satisfies

$$\limsup_{n\to+\infty}\frac{\log n}{\lambda_n}=D<\infty,\qquad \limsup_{n\to+\infty}\frac{\log A_n^*}{\lambda_n}=-\infty, \tag{4}$$

then in view of Refs. [1,22,23], we can conclude that $\sigma_u^F=+\infty$, i.e., $F(s)$ is analytic on the whole plane. For convenience, let L_∞ to denote the class of all the functions $F(s)$ of the form (1) which are analytic in the half plane $\Re s<+\infty$ and the sequence $\{\lambda_n\}$ satisfy (2)–(4).

Let

$$M_u(\sigma,F)=\sup_{0<x<+\infty,-\infty<t<+\infty}\left|\int_0^x e^{(\sigma+it)y}d\alpha(y)\right|,$$

and

$$\mu(\sigma,F)=\max_{n\in\mathbb{N}}\{A_n^*e^{\lambda_n\sigma}\}(\sigma<+\infty),\quad \nu(\sigma,F)=\max_{n}\{\lambda_n|\mu(\sigma,F)=A_n^*e^{\lambda_n\sigma}\}.$$

Usually, we utilize the order and the type to estimate the growth of $F(s)$, which are defined as follows.

Definition 1 (see [19]). *If $F(s)\in L_\infty$ and*

$$\limsup_{\sigma\to+\infty}\frac{\log^+\log^+M_u(\sigma,F)}{\sigma}=\rho,$$

we call $F(s)$ is of order ρ in the whole plane; if

$$\liminf_{\sigma\to+\infty}\frac{\log^+\log^+M_u(\sigma,F)}{\sigma}=\tau,$$

we call $F(s)$ is of lower order τ in the whole plane, where $\log^+x=\max\{\log x,0\}$.

Definition 2 (see [19]). *If $F(s) \in L_\infty$, and is of order $\rho(0 < \rho < \infty)$, then we define the type and the lower type of Laplace–Stieltjes transform $F(s)$, respectively,*

$$\limsup_{\sigma\to+\infty} \frac{\log^+ M_u(\sigma, F)}{e^{\sigma\rho}} = T, \quad \liminf_{\sigma\to+\infty} \frac{\log^+ M_u(\sigma, F)}{e^{\sigma\rho}} = t.$$

For $0 < \rho(F) < \infty$, Luo and Kong [19] in 2012 discussed the properties on entire functions represented by a Laplace–Stieltjes transform of finite order, and obtained

Theorem 1 (see [19,20]). *If $F(s) \in L_\infty$, and is of order $\rho(0 < \rho < \infty)$ and of type T, then*

$$\rho = \limsup_{n\to+\infty} \frac{\lambda_n \log \lambda_n}{-\log A_n^*}, \quad T = \limsup_{n\to+\infty} \frac{\lambda_n}{\rho e}(A_n^*)^{\frac{\rho}{\lambda_n}};$$

Furthermore, if $\lambda_n \sim \lambda_{n-1}$ and

$$\psi(n) = \frac{\log A_n^* - \log A_{n+1}^*}{\lambda_{n+1} - \lambda_n}$$

form a non-decreasing function of n, then

$$t = \liminf_{n\to+\infty} \frac{\lambda_n}{\rho e}(A_n^*)^{\frac{\rho}{\lambda_n}}.$$

Remark 1. *For $\rho(F) = 0$, we can see that the (lower) order and the (lower) type cannot better characterize the growth of the maximum module $M_u(\sigma, F)$ of (1).*

In view of Remark 1, Xu and Liu [9] in 2019 investigated the growth of Laplace–Stieltjes transforms for the case $\rho(F) = 0$, by using the concepts of the logarithmic order and the logarithmic type below.

Definition 3 (see [9]). *If $F(s) \in L_\infty$, and is of order $\rho = 0$, and*

$$\limsup_{\sigma\to+\infty} \frac{\log^+ \log^+ M_u(\sigma, F)}{\log \sigma} = \rho_l, \quad 1 \le \rho_l \le +\infty,$$

then ρ_l is called the logarithmic order of $F(s)$ of zero order. Furthermore, if $1 \le \rho_l < +\infty$, we define the logarithmic type T_l and the lower logarithmic type t_l of $F(s)$, respectively,

$$\limsup_{\sigma\to+\infty} \frac{\log^+ M_u(\sigma, F)}{\sigma^{\rho_l}} = T_l, \quad \liminf_{\sigma\to+\infty} \frac{\log^+ M_u(\sigma, F)}{\sigma^{\rho_l}} = t_l.$$

Remark 2. *We say that $F(s)$ is of perfectly logarithmic linear growth if and only if $0 < t_l = T_l < \infty$ and $1 < \rho_l < \infty$. Obviously, $T_l = \infty$ as $\rho_l = 1$.*

Theorem 2 (see ([9], Theorem 1.5)). *If Laplace–Stieltjes transform $F(s) \in L_\infty$, and is of zero order and of logarithmic order ρ_l, then*

$$\rho_l = \limsup_{\sigma\to+\infty} \frac{\log^+ \log^+ M_u(\sigma, F)}{\log \sigma} = \limsup_{\sigma\to+\infty} \frac{\log^+ \log^+ \mu(\sigma, F)}{\log \sigma}. \tag{5}$$

Furthermore, we have

$$\rho_l = \frac{1}{1 - \limsup\limits_{n\to+\infty} \frac{\log \lambda_n}{\log\log(A_n^*)^{-1}}}. \tag{6}$$

Theorem 3 (see ([9], Theorem 1.6))**.** *If Laplace–Stieltjes transform $F(s) \in L_\infty$, and is of zero order and of logarithmic order $\rho_l (1 < \rho_l < +\infty)$ and logarithmic type T_l, then*

$$T_l = \frac{(\rho_l - 1)^{\rho_l - 1}}{\rho_l^{\rho_l}} \limsup_{n\to+\infty} \frac{\log \frac{1}{A_n^*}}{\left(\frac{1}{\lambda_n} \log \frac{1}{A_n^*}\right)^{\rho_l}}.$$

Remark 3. *In fact, in view of Theorem 3 and Lemma 1, we have*

$$\limsup_{\sigma\to+\infty} \frac{\log^+ \mu(\sigma, F)}{\sigma^{\rho_l}} = T_l, \quad \liminf_{\sigma\to+\infty} \frac{\log^+ \mu(\sigma, F)}{\sigma^{\rho_l}} = t_l,$$

and

$$\limsup_{n\to+\infty} \frac{\lambda_n}{\rho_l \left[\frac{1}{\rho_l - 1} \log (A_n^*)^{-\frac{\rho_l}{\lambda_n}}\right]^{\rho_l - 1}} = T_l.$$

Motivated by Theorems 2 and 3, one may ask the following questions.

Question 1. *What will happened to the parameters ρ_l, λ_n, A_n^*, if $F(s)$ is of the lower logarithmic type t_l, or $F(s)$ is of perfectly logarithmic linear growth?*

Question 2. *What can be said about the correlation between the logarithmic growth and the center index $\nu(\sigma, F)$ of the maximum term $\mu(\sigma, F)$ of Laplace–Stieltjes transform with zero order?*

In view of the above questions, we will study the properties of logarithmic growth of entire functions defined by Laplace–Stieltjes transforms convergent in the whole plane, including the lower logarithmic type t_l, and the relations about the logarithmic type T_l, the lower logarithmic type t_l, $\nu(\sigma, F)$, λ_n and A_n^*. *As far as we know, it appears that the study of the logarithmic growth of Laplace–Stieltjes transforms has seldom been involved in the literature before now.* The paper is organized as follows. In Section 2, we will discuss the lower logarithmic type t_l of entire functions defined by Laplace–Stieltjes transforms. In Section 3, we will study the relation among the logarithmic order ρ_l, logarithmic type T_l, lower logarithmic type t_l and the center index $\nu(\sigma, F)$ of the maximum term. In Section 4, we will establish the expression of the (lower) logarithmic type by the logarithmic order ρ_l, λ_n, A_n^*, and also obtain some equivalence conditions between the (lower) logarithmic type $T_l(t_l)$ and $\nu(\sigma, F)$. Finally, the conclusions of this paper will be presented in Section 5.

2. The Lower Logarithmic Type of Laplace–Stieltjes Transform

We first give the following lemma, which is used to prove our two main theorems.

Lemma 1 (see ([19], Lemma 2.1))**.** *If Laplace–Stieltjes transform $F(s) \in L_\infty$, for any $\sigma(-\infty < \sigma < +\infty)$ and $\varepsilon(> 0)$, we have*

$$\frac{1}{2}\mu(\sigma, F) \le M_u(\sigma, F) \le C\mu((1 + 2\varepsilon)\sigma, F),$$

where C is a constant.

In fact, we obtain the main result about the lower logarithmic type of Laplace–Stieltjes transform $F(s)$ in the case $\rho(F) = 0$ as follows.

Theorem 4. *If Laplace–Stieltjes transform $F(s) \in L_\infty$, and is of logarithmic order $\rho_l (1 < \rho_l < +\infty)$, and of lower logarithmic type t_l, and if $\lambda_n \sim \lambda_{n+1}$ and the function*

$$\psi(n) = \frac{\log A_n^* - \log A_{n+1}^*}{\lambda_{n+1} - \lambda_n} \tag{7}$$

is a non-decreasing function of n, then

$$\liminf_{n\to+\infty}\frac{\lambda_n}{\rho_l\left[\frac{1}{\rho_l-1}\log(A_n^*)^{-\frac{\rho_l}{\lambda_n}}\right]^{\rho_l-1}}=t_l.$$

Remark 4. *Obviously, Theorem 4 is a good supplement of Theorems 2 and 3.*

In order to prove Theorem 4, we only give the proof of Theorems 5 and 6 below.

Theorem 5. *If Laplace–Stieltjes transform $F(s)\in L_\infty$, and is of logarithmic order $\rho_l(1<\rho_l<+\infty)$ and lower logarithmic type t_l, and if $\lambda_n\sim\lambda_{n+1}$, then*

$$\liminf_{n\to+\infty}\frac{\lambda_n}{\rho_l\left[\frac{1}{\rho_l-1}\log(A_n^*)^{-\frac{\rho_l}{\lambda_n}}\right]^{\rho_l-1}}\le t_l.$$

Proof. Set

$$\theta=\liminf_{n\to+\infty}\frac{\lambda_{n-1}}{\rho_l\left[\frac{1}{\rho_l-1}\log(A_n^*)^{-\frac{\rho_l}{\lambda_n}}\right]^{\rho_l-1}}. \tag{8}$$

Assume that $0<\theta<\infty$, for any given ε such that $0<\varepsilon<\theta$, we have from (8) that there exists a positive integer $n_0(\varepsilon)$ such that for all $n>n_0(\varepsilon)$,

$$\log A_n^*>-\frac{\rho_l-1}{\rho_l}\left[\frac{\lambda_{n-1}}{\rho_l(\theta-\varepsilon)}\right]^{\frac{1}{\rho_l-1}}\lambda_n. \tag{9}$$

Thus, it follows by Lemma 1 and (9) that

$$\log M_u(\sigma,F)>-\frac{\rho_l-1}{\rho_l}\left[\frac{\lambda_{n-1}}{\rho_l(\theta-\varepsilon)}\right]^{\frac{1}{\rho_l-1}}\lambda_n+\lambda_n\sigma-\log 2. \tag{10}$$

Taking $\sigma=\left[\frac{\lambda_{n-1}}{\rho_l(\theta-\varepsilon)}\right]^{\frac{1}{\rho_l-1}}$, we have from (10) that

$$\log M_u(\sigma,F)>(1+o(1))(\theta-\varepsilon)\sigma^{\rho_l},\quad for\quad n>n_0(\varepsilon). \tag{11}$$

In view of (11), and combining the definition of lower logarithmic type t_l, we have $t_l\ge\theta$. If $\theta=0$, the conclusion holds obviously. In the case $\theta=\infty$, similar to the above argument, we can also obtain the inequality when we replace $\theta-\varepsilon$ by an arbitrarily large number.

Therefore, this completes the proof of Theorem 5. □

Theorem 6. *If Laplace–Stieltjes transform $F(s)\in L_\infty$, and is of logarithmic order $\rho_l(1<\rho_l<+\infty)$ and lower logarithmic type t_l, and if the function (7) form a non-decreasing function of n, then*

$$\liminf_{n\to+\infty}\frac{\lambda_n}{\rho_l\left[\frac{1}{\rho_l-1}\log(A_n^*)^{-\frac{\rho_l}{\lambda_n}}\right]^{\rho_l-1}}\ge t_l.$$

Proof. Assume that $0<t_l<+\infty$. From the assumption of Theorem 6, and in view of Definition 3 and Lemma 1, for any given small number $\varepsilon(0<\varepsilon<t_l)$, there exists a fixed $\sigma_0>0$ such that for all $\sigma>\sigma_0$,

$$\log\mu(\sigma,F)>(t_l-\varepsilon)\sigma^{\rho_l},$$

that is,

$$\log A_n^*+\lambda_n\sigma>(t_l-\varepsilon)\sigma^{\rho_l}. \tag{12}$$

Let $\sigma > \sigma_0$ and let n_1 and $n_2(n_2 - 1)$ be two consecutive maximum terms, then

$$\log A^*_{n_2} + \lambda_{n_2}\sigma > (t_l - \varepsilon)\sigma^{\rho_l} \tag{13}$$

for all σ satisfying $\psi(n_2 - 1) \le \sigma < \psi(n_2)$. Let $n_1 \le n < n_2 - 1$, we have

$$\psi(n_1) = \psi(n_1 + 1) = \cdots = \psi(n) = \cdots = \psi(n_2 - 1), \tag{14}$$

and

$$A^*_n e^{\lambda_n \sigma} = A^*_{n_2} e^{\lambda_{n_2}\sigma}, \quad for \quad \sigma = \psi(n). \tag{15}$$

Thus, it follows from (12)–(15) that

$$\frac{\lambda_n}{\rho_l\left[\frac{1}{\rho_l-1}\log(A^*_n)^{-\frac{\rho_l}{\lambda_n}}\right]^{\rho_l-1}} > \frac{\lambda_n}{\frac{\rho_l^{\rho_l}}{(\rho_l-1)^{\rho_l-1}}\left[-\frac{1}{\lambda_n}(t_l-\varepsilon)\sigma^{\rho_l}+\sigma\right]^{\rho_l-1}}. \tag{16}$$

Let $\sigma = \left(\frac{\lambda_n}{\rho_l(t_l-\varepsilon)}\right)^{(\rho_l-1)^{-1}}$, and let $n \to +\infty$, it follows from (16) that

$$\liminf_{n\to+\infty} \frac{\lambda_n}{\rho_l\left[\frac{1}{\rho_l-1}\log(A^*_n)^{-\frac{\rho_l}{\lambda_n}}\right]^{\rho_l-1}} \ge t_l.$$

Besides, the conclusion holds obviously if $t_l = 0$. By using the same argument as in the above, we can also prove the inequality in the case $t_l = \infty$ when we replace $t_l - \varepsilon$ by an arbitrarily large number.

Therefore, this completes the proof of Theorem 6. □

3. Some Inequalities on the Maximum Term Index

In order to further explore the properties of logarithmic growth of Laplace–Stieltjes transform $F(s)$, we first introduce the following indicators. Let $F(s) \in L_\infty$ be of logarithmic order ρ_l. Here and below, unless otherwise specified, we always assume $1 < \rho_l < +\infty$. Thus, we define

$$V = \limsup_{\sigma\to+\infty} \frac{\nu(\sigma, F)}{\sigma^{\rho_l-1}}, \quad v = \liminf_{\sigma\to+\infty} \frac{\nu(\sigma, F)}{\sigma^{\rho_l-1}},$$

and

$$H = \limsup_{\sigma\to+\infty} \frac{\log\mu(\sigma, F)}{\sigma\nu(\sigma, F)}, \quad h = \liminf_{\sigma\to+\infty} \frac{\log\mu(\sigma, F)}{\sigma\nu(\sigma, F)}.$$

Obviously, we have $v \le V$ and $h \le H$. As for the further relationship between them, we have

Theorem 7. *If Laplace–Stieltjes transform $F(s) \in L_\infty$, and is of logarithmic order $\rho_l(1 < \rho_l < +\infty)$, logarithmic type T_l and lower logarithmic type t_l. Then we have*

$$v \le \rho_l t_l \le v\left[\rho_l - (\rho_l - 1)\left(\frac{v}{V}\right)^{\frac{1}{\rho_l-1}}\right] \le V, \tag{17}$$

and

$$v \le V\left[\frac{\rho_l V - V}{\rho_l V - v}\right]^{\rho_l-1} \le \rho_l T_l \le V. \tag{18}$$

Remark 5. *In view of $\frac{\rho_l V - V}{\rho_l V - v} \ge 1$, by combining with (17) and (18), we have*

$$v \le \rho_l t_l \le \rho_l T_l \le V.$$

To prove this result, we require the following lemma.

Lemma 2 (see ([20], Lemma 2.2))**.** *If Laplace–Stieltjes transform $F(s) \in L_\infty$, then we have*

$$\log \mu(\sigma, F) = \log \mu(\sigma_0, F) + \int_{\sigma_0}^{\sigma} \nu(t, F)dt,$$

for $\sigma_0 > 0$.

Proof of Theorem 7. In view of $v \le V$ and $\rho_l > 1$, it follows that $0 \le \frac{v}{V} \le 1$ and $\frac{\rho_l V - V}{\rho_l V - v} \ge 1$. Thus,

$$v \le V \left[\frac{\rho_l V - V}{\rho_l V - v} \right]^{\rho_l - 1} \tag{19}$$

holds obviously. Define $f(x) = \rho_l x - (\rho_l - 1)x^{\frac{\rho_l}{\rho_l - 1}} - 1, 0 \le x \le 1, \rho_l > 1$. Since $f'(x) = \rho_l \left(1 - x^{\frac{1}{\rho_l - 1}}\right) \ge 0$, then $f(x)$ is a increasing function in $0 \le \rho_l \le 1$. Thus, $f(x) \le f(1) = 0$. Replaced x by $\frac{v}{V}$, we can easily prove that

$$v \left[\rho_l - (\rho_l - 1) \left(\frac{v}{V} \right)^{\frac{1}{\rho_l - 1}} \right] \le V. \tag{20}$$

In view of the definitions of v and V, we have that for any $\varepsilon > 0$

$$(v - \varepsilon)\sigma^{\rho_l - 1} < \nu(\sigma, F) < (V + \varepsilon)\sigma^{\rho_l - 1}, \quad for \quad \sigma > \sigma_0(\varepsilon). \tag{21}$$

By Lemma 2, for any $\sigma \ge \sigma_0 > 0$ and $\eta \ge 1$, it follows that

$$\begin{aligned} \log \mu(\sigma \eta^{\frac{1}{\rho_l}}, F) &= \log \mu(\sigma_0, F) + \int_{\sigma_0}^{\sigma \eta^{\frac{1}{\rho_l}}} \nu(t, F)dt \\ &= O(1) + \int_{\sigma_0}^{\sigma} \nu(t, F)dt + \int_{\sigma}^{\sigma \eta^{\frac{1}{\rho_l}}} \nu(t, F)dt \end{aligned} \tag{22}$$

holds for any fixed positive number $\sigma_0 > 0$. Since $\nu(\sigma, F)$ is an increasing function of σ, we have from (21) and (22) that

$$\log \mu(\sigma \eta^{\frac{1}{\rho_l}}, F) < O(1) + \frac{V + \varepsilon}{\rho_l} \sigma^{\rho_l} + \nu(\sigma \eta^{\frac{1}{\rho_l}}, F)(\eta^{\frac{1}{\rho_l}} - 1)\sigma, \tag{23}$$

for all $\sigma > \sigma_0$. In view of Remark 1.2, and let $\sigma \to +\infty$, it follows from (23) that

$$T_l \le \frac{V}{\eta \rho_l} + \frac{V(\eta^{\frac{1}{\rho_l}} - 1)}{\eta^{\frac{1}{\rho_l}}}, \tag{24}$$

$$t_l \le \frac{V}{\eta \rho_l} + \frac{v(\eta^{\frac{1}{\rho_l}} - 1)}{\eta^{\frac{1}{\rho_l}}}. \tag{25}$$

Thus, let $\eta = 1$ in (24), and let $\eta = (\frac{V}{v})^{\frac{\rho_l}{\rho_l - 1}}$ in (25), we have

$$\rho_l T_l \le V, \quad \rho_l t_l \le v \left[\rho_l - (\rho_l - 1) \left(\frac{v}{V} \right)^{\frac{1}{\rho_l - 1}} \right]. \tag{26}$$

By combining with the first inequality and (22), we also obtain that

$$T_l \geq \frac{v}{\eta\rho_l} + \frac{V(\eta^{\frac{1}{\rho_l}} - 1)}{\eta}, \tag{27}$$

$$t_l \geq \frac{v}{\eta\rho_l} + \frac{v(\eta^{\frac{1}{\rho_l}} - 1)}{\eta}. \tag{28}$$

Thus, let $\eta = 1$ in (28), and let $\eta = \left(\frac{\rho_l V - V}{\rho_l V - v}\right)^{\rho_l}$ in (27), we have

$$\rho_l t_l \geq v, \quad \rho_l T_l \geq V\left(\frac{\rho_l V - V}{\rho_l V - v}\right)^{\rho_l - 1}. \tag{29}$$

By combining with (19), (20), (26) and (29), we can prove the conclusions of Theorem 7 easily.

Therefore, this completes the proof of Theorem 7. □

Next, the following results show the relations among the quotas v, V, h and H.

Theorem 8. *If Laplace–Stieltjes transform $F(s) \in L_\infty$, and is of logarithmic order $\rho_l (1 < \rho_l < +\infty)$. Then we have*

$$\frac{v}{\rho_l V} \leq h \leq H \leq \frac{V}{\rho_l v}. \tag{30}$$

Proof. By making use of Lemma 2 and (21), and combining with the definitions of h and H, we can prove the conclusions of Theorem 8 easily. □

Theorem 9. *If Laplace–Stieltjes transform $F(s) \in L_\infty$, and is of logarithmic order $\rho_l (1 < \rho_l < +\infty)$, logarithmic type T_l and lower logarithmic type t_l, $(0 < t_l \leq T_l < +\infty)$. Then we have*

(i)

$$\lim_{\sigma \to +\infty} \frac{\log M_u(\sigma, F)}{\sigma^{\rho_l}} = T_l \Longleftrightarrow \lim_{\sigma \to +\infty} \frac{\nu(\sigma, F)}{\sigma^{\rho_l - 1}} = \rho_l T_l;$$

(ii)

$$0 < t_l \leq T_l < +\infty \Longleftrightarrow 0 < v \leq V < +\infty.$$

Remark 6. *From Theorem 9 (i), we can see that $t_l = T_l \Longleftrightarrow v = V = \rho_l T_l$.*

Proof. (i) $\Longrightarrow$: If

$$\lim_{\sigma \to +\infty} \frac{\log M_u(\sigma, F)}{\sigma^{\rho_l}} = T_l,$$

in view of Lemma 1, it follows $t_l = T_l$ and

$$\lim_{\sigma \to +\infty} \frac{\log \mu(\sigma, F)}{\sigma^{\rho_l}} = T_l.$$

By Lemma 2, for $\eta = 1 + \vartheta$, $\vartheta > 0$, we have

$$\begin{aligned} T_l\left[(\sigma\eta^{\frac{1}{\rho_l}})^{\rho_l} - \sigma^{\rho_l} + o(\sigma^{\rho_l})\right] &= \log \mu\left(\sigma\eta^{\frac{1}{\rho_l}}, F\right) - \log \mu(\sigma, F) \\ &= \int_\sigma^{\sigma\eta^{\frac{1}{\rho_l}}} \nu(t, F)dt < \nu\left(\sigma\eta^{\frac{1}{\rho_l}}, F\right)\sigma(\eta^{\frac{1}{\rho_l}} - 1). \end{aligned} \tag{31}$$

Dividing $(\sigma\eta^{\frac{1}{\rho_l}})^{\rho_l - 1}$ into two side of (31), and let $\sigma \to +\infty$, we have

$$\liminf_{\sigma \to +\infty} \frac{\nu\left(\sigma\eta^{\frac{1}{\rho_l}}, F\right)}{(\sigma\eta^{\frac{1}{\rho_l}})^{\rho_l - 1}} \geq \frac{T_l(\eta - 1)}{(\eta^{\frac{1}{\rho_l}} - 1)\eta^{\frac{\rho_l - 1}{\rho_l}}} = \frac{\vartheta T_l}{\left[(1 + \vartheta) - (1 + \vartheta)^{\frac{\rho_l - 1}{\rho_l}}\right]}. \tag{32}$$

By applying L'Hospital's rule, and let $\vartheta \to 0^+$, it is easy to obtain

$$\lim_{\vartheta \to 0+} \frac{\vartheta T_l}{\left[(1+\vartheta) - (1+\vartheta)^{\frac{\rho_l - 1}{\rho_l}}\right]} = \rho_l T_l. \tag{33}$$

Thus, in view of (32) and (33), we have $v \geq \rho_l T_l$. Similarly, let $\eta = 1 - \vartheta, 0 < \vartheta < 1$, we have

$$\begin{aligned} T_l\left[\sigma^{\rho_l} - (\sigma\eta^{\frac{1}{\rho_l}})^{\rho_l} + o(\sigma^{\rho_l})\right] &= \log \mu(\sigma, F) - \log \mu\left(\sigma\eta^{\frac{1}{\rho_l}}, F\right) \\ &= \int_{\sigma\eta^{\frac{1}{\rho_l}}}^{\sigma} \nu(t, F)dt > \nu\left(\sigma\eta^{\frac{1}{\rho_l}}, F\right)\sigma(1 - \eta^{\frac{1}{\rho_l}}), \end{aligned}$$

and

$$V = \limsup_{\sigma \to +\infty} \frac{\nu\left(\sigma\eta^{\frac{1}{\rho_l}}, F\right)}{(\sigma\eta^{\frac{1}{\rho_l}})^{\rho_l - 1}} \leq \rho_l T_l.$$

By combining with $v \leq V$, we have

$$v = V = \lim_{\sigma \to +\infty} \frac{\nu(\sigma, F)}{\sigma^{\rho_l - 1}} = \rho_l T_l.$$

Now, we will prove the sufficiency of Theorem 9 (i). Let $\lim\limits_{\sigma \to +\infty} \frac{\nu(\sigma,F)}{\sigma^{\rho_l - 1}} = \rho_l T_l$, in view of the definitions of v and V, we have $v = V = \rho_l T_l$. By combining with Remark 5, we obtain that $t_l = T_l$, that is,

$$\lim_{\sigma \to +\infty} \frac{\log M_u(\sigma, F)}{\sigma^{\rho_l}} = T_l.$$

Therefore, this completes the conclusion (i) of Theorem 9.

(ii) We first prove the sufficiency of Theorem 9 (ii). Let $0 < v \leq V < +\infty$. In view of Remark 5 and $0 < \rho_l < +\infty$, it follows that $T_l < +\infty$ and $t_l > 0$. Furthermore, in view of Theorem 7 (i), we can obtain that $t_l = T_l$ if $v = V$. Thus, the sufficiency of Theorem 9 (ii) is proved.

Next, we will prove the necessity of Theorem 9 (ii). Let $0 < t_l \leq T_l < +\infty$. Then it follows that $v > 0$ and $V < +\infty$. Otherwise, if $v = 0$, then we have from (25) that $T_l \geq \eta\rho_l t_l$. This is a contradiction since $T_l < +\infty$ and η is arbitrary. Similarly, if $V = +\infty$, then we have from (27) that $T_l \geq \frac{V(\eta^{\frac{1}{\rho_l}} - 1)}{\eta}$. This is a contradiction since $T_l < +\infty$ and $\eta > 1$. Besides, in view of Theorem 7 (i), we can obtain that $v = V$ if $t_l = T_l$. Thus, the necessity of Theorem 9 (ii) is proved.

Therefore, we complete the proof of Theorem 9. □

4. Applications

In this section, we will establish some results to reveal the relationship between the logarithm order ρ_l, the logarithm type T_l, the lower logarithm type t_l, the form exponent λ_n and the form coefficients A_n^* of Laplace–Stieltjes transformation of small growth, by applying the inequalities given in Sections 1 and 2. Denote

$$w = \liminf_{n \to +\infty} \frac{\lambda_n}{\left[\frac{1}{\lambda_{n+1} - \lambda_n} \log \frac{A_n^*}{A_{n+1}^*}\right]^{\rho_l - 1}}, \quad W = \limsup_{n \to +\infty} \frac{\lambda_n}{\left[\frac{1}{\lambda_{n+1} - \lambda_n} \log \frac{A_n^*}{A_{n+1}^*}\right]^{\rho_l - 1}}.$$

Theorem 10. *If Laplace–Stieltjes transform $F(s) \in L_\infty$, and is of logarithmic order $\rho_l(1 < \rho_l < +\infty)$, logarithmic type T_l and lower logarithmic type t_l. If $\lambda_n \sim \lambda_{n+1}$ and*

$$\sum_{m=n_0}^{n-1} \lambda_m^k(\lambda_{m+1} - \lambda_m) \sim \frac{\lambda_n^{k+1}}{k+1}, \quad (k \geq 0, n_0 \geq 1), \tag{34}$$

then we have

$$w \leq \rho_l t_l \leq \rho_l T_l \leq W. \tag{35}$$

The following example shows that the inequalities in (35) are best possible to some extent.

Example 1. *Let $\lambda_n = n$ and $\alpha(x)$ satisfy*

$$\alpha(x) = 1 + e^{-1} + e^{-2^2} + \cdots + e^{-(n-1)^2}, \quad (n-1 < x < n, n = 1, 2, \ldots,).$$

Then (1) can be expressed as the form

$$F(s) = \sum_{n=1}^{\infty} e^{-n^2} e^{sn}.$$

In view of Theorems 2-4, by simple calculation, we have $\rho_l(F) = 2$, $T_l(F) = t_l(F) = \frac{1}{4}$ and $w = W = \frac{1}{2}$. Thus, this shows that the equal sign situation in (35) can be attained.

Proof. Assume that $0 < w \leq W < +\infty$. From the definitions of w and W, for a fixed positive integer n_0, then we obtain that for any $\varepsilon > 0$, the following inequalities

$$\frac{1}{W} - \varepsilon < \frac{\left[\frac{1}{\lambda_{n+1}-\lambda_n} \log \frac{A_n^*}{A_{n+1}^*}\right]^{\rho_l - 1}}{\lambda_n} < \frac{1}{w} + \varepsilon \tag{36}$$

hold for all $n \geq n_0$. Thus, for any positive integer $m \geq n_0$, we have

$$\left[(\frac{1}{W} - \varepsilon)\lambda_m\right]^{\frac{1}{\rho_l - 1}} (\lambda_{m+1} - \lambda_m) < \log A_m^* - \log A_{m+1}^* < \left[(\frac{1}{w} + \varepsilon)\lambda_m\right]^{\frac{1}{\rho_l - 1}} (\lambda_{m+1} - \lambda_m). \tag{37}$$

Let $m = n_0, n_0 + 1, \ldots, n-1$ in (37), adding them, then it follows that

$$\begin{aligned} (\frac{1}{W} - \varepsilon)^{\frac{1}{\rho_l - 1}} \sum_{m=n_0}^{n-1} \lambda_m^{\frac{1}{\rho_l - 1}} (\lambda_{m+1} - \lambda_m) &< \log A_{n_0}^* - \log A_n^* \\ &< (\frac{1}{w} + \varepsilon)^{\frac{1}{\rho_l - 1}} \sum_{m=n_0}^{n-1} \lambda_m^{\frac{1}{\rho_l - 1}} (\lambda_{m+1} - \lambda_m). \end{aligned} \tag{38}$$

In view of (34) and (38), for all $n \geq n_0$, then we obtain that

$$(\frac{1}{W} - \varepsilon)^{\frac{1}{\rho_l - 1}} \frac{\lambda_n^{\frac{1}{\rho_l - 1} + 1}}{\frac{\rho_l}{\rho_l - 1}} < \log A_{n_0}^* - \log A_n^* < (\frac{1}{w} + \varepsilon)^{\frac{1}{\rho_l - 1}} \frac{\lambda_n^{\frac{1}{\rho_l - 1} + 1}}{\frac{\rho_l}{\rho_l - 1}}. \tag{39}$$

Thus, it follows from (39) that

$$w \leq \liminf_{n \to +\infty} \frac{\lambda_n}{\left[\frac{1}{\rho_l - 1} \log(A_n^*)^{-\frac{\rho_l}{\lambda_n}}\right]^{\rho_l - 1}} \leq \limsup_{n \to +\infty} \frac{\lambda_n}{\left[\frac{1}{\rho_l - 1} \log(A_n^*)^{-\frac{\rho_l}{\lambda_n}}\right]^{\rho_l - 1}} \leq W. \tag{40}$$

By combining with Remark 3 and Theorem 4, we have from (40) that

$$w \le \rho_l t_l \le \rho_l T_l \le W. \tag{41}$$

If $w = 0$ or $W = +\infty$, the conclusions (41) are obvious. If $w = +\infty$, then $W = +\infty$. We can obtain (40) by replacing w, W by an arbitrarily large number. If $W = 0$, then $w = 0$. We also obtain (38) by replacing $\frac{1}{W} - \varepsilon$ by an arbitrarily large number. Thus, we can obtain (41) in either case.

Therefore, this completes the proof of Theorem 10. □

Theorem 11. *If Laplace–Stieltjes transform $F(s) \in L_\infty$, and is of logarithmic order $\rho_l (1 < \rho_l < +\infty)$ and logarithmic type $T_l (0 < T_l < +\infty)$. If the sequence $\{\lambda_n\}$ satisfy (34) and $\psi(n)$ form a non-decreasing function of $n (\ge n_0)$, then we have*

$$\rho_l T_l \le W \le \left(\frac{\rho_l}{\rho_l - 1}\right)^{\rho_l - 1} \rho_l T_l < e\rho_l T_l. \tag{42}$$

Proof. From the assumptions of Theorem 11, and the definitions of logarithmic type T_l, for any given $\varepsilon > 0$, there exists a positive integer $n_0(\varepsilon)$ such that for all $n > n_0(\varepsilon)$, we have

$$\lambda_n^{\frac{\rho_l}{\rho_l - 1}} < -\frac{\rho_l}{\rho_l - 1}(\rho_l T_l + \varepsilon)^{\frac{1}{\rho_l - 1}} \log A_n^*. \tag{43}$$

Thus, it follows that

$$A_n^* < \exp\left[-\frac{\rho_l - 1}{\rho_l}(\rho_l T_l + \varepsilon)^{-\frac{1}{\rho_l - 1}} \lambda_n^{\frac{\rho_l}{\rho_l - 1}}\right], \quad n > n_0(\varepsilon), \tag{44}$$

and

$$\log A_m^* + \log \frac{A_{m+1}^*}{A_m^*} + \cdots + \log \frac{A_n^*}{A_{n-1}^*} < -\frac{\rho_l - 1}{\rho_l}(\rho_l T_l + \varepsilon)^{-\frac{1}{\rho_l - 1}} \lambda_n^{\frac{\rho_l}{\rho_l - 1}}, \quad m > n_0(\varepsilon). \tag{45}$$

By combining with the non-decreasing function $\psi(m)$, we obtain

$$\log A_m^* - (\lambda_n - \lambda_m)\psi(n-1) < -\frac{\rho_l - 1}{\rho_l}(\rho_l T_l + \varepsilon)^{-\frac{1}{\rho_l - 1}} \lambda_n^{\frac{\rho_l}{\rho_l - 1}},$$

that is,

$$\frac{\lambda_n}{\psi(n-1)^{\rho_l - 1}} < \left(\frac{\rho_l - 1}{\rho_l}\right)^{\rho_l - 1}(\rho_l T_l + \varepsilon)\left(\frac{\lambda_n - \lambda_m}{\lambda_n}\right)^{\rho_l - 1}(1 + o(1)). \tag{46}$$

In view of $\psi(n-1) \le \psi(n)$, and let $n \to +\infty$, then we obtain from (46) that

$$W = \limsup_{n \to +\infty} \frac{\lambda_n}{\left[\frac{1}{\lambda_{n+1} - \lambda_n} \log \frac{A_n^*}{A_{n+1}^*}\right]^{\rho_l - 1}} \le \left(\frac{\rho_l - 1}{\rho_l}\right)^{\rho_l - 1} \rho_l T_l. \tag{47}$$

By combining with the fact that $e^x > 1 + x$ for $0 < x < +\infty$, it follows from (47) that

$$W \le \left(\frac{\rho_l - 1}{\rho_l}\right)^{\rho_l - 1} \rho_l T_l < e\rho_l T_l. \tag{48}$$

Thus, we can obtain (42) from (35) and (48) immediately.

Therefore, we complete the proof of Theorem 11. □

Theorem 12. *If Laplace–Stieltjes transform $F(s) \in L_\infty$, and is of logarithmic order $\rho_l (1 < \rho_l < +\infty)$. If λ_n satisfy $\lambda_n \sim \lambda_{n+1}$ and $\psi(n)$ form a non-decreasing function of $n(\geq n_0)$, then we have*

$$w = v \quad \text{and} \quad W = V. \tag{49}$$

Proof. From the assumptions of Theorem 12, and the definitions of v and V, for any positive number ε, there exists $\sigma_0(\varepsilon)$ such that for all $\sigma > \sigma_0(\varepsilon)$,

$$v - \varepsilon < \frac{\nu(\sigma, F)}{\sigma^{\rho_l - 1}} < V + \varepsilon. \tag{50}$$

Since $\psi(n)$ is an increasing function of n, taking

$$\frac{\log A^*_{n-1} - \log A^*_n}{\lambda_n - \lambda_{n-1}} \leq \sigma < \frac{\log A^*_n - \log A^*_{n+1}}{\lambda_{n+1} - \lambda_n}, \tag{51}$$

then we have that $A^*_n e^{\lambda_n \sigma}$ is the maximum term for $\Re s = \sigma$, that is, $\lambda_n = \nu(\sigma, F)$. In view of (50) and (51), we have

$$(v - \varepsilon)\left[\frac{\log A^*_{n-1} - \log A^*_n}{\lambda_n - \lambda_{n-1}}\right]^{\rho_l - 1} < \lambda_n < (V + \varepsilon)\left[\frac{\log A^*_n - \log A^*_{n+1}}{\lambda_{n+1} - \lambda_n}\right]^{\rho_l - 1}, \tag{52}$$

for all $n > n_0$. Thus, let $n \to +\infty$, it follows from (52) that

$$V \geq \limsup_{n \to +\infty} \frac{\lambda_n}{\left[\frac{1}{\lambda_{n+1} - \lambda_n} \log \frac{A^*_n}{A^*_{n+1}}\right]^{\rho_l - 1}}, \quad v \leq \liminf_{n \to +\infty} \frac{\lambda_{n+1}}{\left[\frac{1}{\lambda_{n+1} - \lambda_n} \log \frac{A^*_n}{A^*_{n+1}}\right]^{\rho_l - 1}}. \tag{53}$$

and in view of $\lambda_n \sim \lambda_{n+1}$, the second inequality in (53) becomes

$$v \leq \liminf_{n \to +\infty} \frac{\lambda_n}{\left[\frac{1}{\lambda_{n+1} - \lambda_n} \log \frac{A^*_n}{A^*_{n+1}}\right]^{\rho_l - 1}}. \tag{54}$$

Obviously, (53) and (40) hold for $v = 0$ and $V = +\infty$. Besides, if $v = +\infty$, we can obtain (54) by replacing $v - \varepsilon$ by an arbitrary large number in (50). Similarly, we can obtain (53) for $V = 0$.

On the other hand, from the definition of V, we have $\nu(\sigma, F) > (V - \varepsilon)\sigma^{\rho_l - 1}$ for a sequence of values of $\sigma = \sigma_1, \sigma_2, \ldots$, tending to ∞. Thus, in view of (51), corresponding to the sequence $\{\sigma_n\}$, we obtain

$$\lambda_n > (V - \varepsilon)\left[\frac{1}{\lambda_n - \lambda_{n-1}} \log \frac{A^*_{n-1}}{A^*_n}\right]^{\rho_l - 1}.$$

In view of $\lambda_n \sim \lambda_{n+1}$, for a sequence of values of $n \to +\infty$, we have

$$\frac{\lambda_n}{\left[\frac{1}{\lambda_{n+1} - \lambda_n} \log \frac{A^*_n}{A^*_{n+1}}\right]^{\rho_l - 1}} > V - \varepsilon. \tag{55}$$

Hence, we obtain

$$\limsup_{n \to +\infty} \frac{\lambda_n}{\left[\frac{1}{\lambda_{n+1} - \lambda_n} \log \frac{A^*_n}{A^*_{n+1}}\right]^{\rho_l - 1}} \geq V. \tag{56}$$

Similar to the above argument, we have

$$\liminf_{n \to +\infty} \frac{\lambda_n}{\left[\frac{1}{\lambda_{n+1} - \lambda_n} \log \frac{A^*_n}{A^*_{n+1}}\right]^{\rho_l - 1}} \leq v. \tag{57}$$

Thus, in view of (54)–(56), we can obtain $w = v$ and $W = V$.

Therefore, this completes the proof of Theorem 12. □

Theorem 13. *If Laplace–Stieltjes transform* $F(s) \in L_\infty$*, and is of logarithmic order* $\rho_l (1 < \rho_l < +\infty)$*. If* λ_n *satisfy* $\lambda_n \sim \lambda_{n+1}$ *and* $\psi(n)$ *form a non-decreasing function of* $n(\geq n_0)$*. We have*

(i) $F(s)$ *is of perfectly logarithmic linear growth if, and only if,*

$$\lim_{n\to+\infty} \frac{\lambda_n}{\left[\frac{1}{\lambda_{n+1}-\lambda_n} \log \frac{A_n^*}{A_{n+1}^*}\right]^{\rho_l-1}} = \rho_l T_l;$$

(ii) if $0 < t_l \leq T_l < \infty$*, then* $0 < w \leq W < \infty$*.*

Proof. (i) From Theorem 9 (i) and Theorem 12, we can obtain Theorem 13 (i) easily.

(ii) Similar to the argument as in the proof of Theorem 9 (ii), and combining with the conclusions of Theorem 12, we can prove Theorem 13 (ii).

Therefore, this completes the proof of Theorem 13. □

5. Conclusions

In view of Theorems 7–13, we can see that these results reveal the relationships between the logarithmic growth and some indexes of entire functions represented by Laplace–Stieltjes transforms of finite logarithmic order ρ_l. In fact, Theorems 7–11 and Remark 5 exhibit the relationships concerning some indexes including $\rho_l, t_l, T_l, v, V, h, H$. These theorems show that the (lower) logarithmic type $T_l(t_l)$ of Laplace–Stieltjes transform can be bounded not only by the center indexes $\nu(\sigma, F)$ of the maximum terms (see Theorems 7 and 8), but also by the logarithmic order ρ_l, A_n^* and λ_n (see Theorems 10 and 11). Finally, Theorems 12 and 13 depict the equivalence conditions between the (lower) logarithmic type $T_l(t_l)$ and $\nu(\sigma, F)$ of Laplace–Stieltjes transforms with certain restricts. These are very obvious differences since the growth indexes are usual estimated by A_n^*, λ_n (can be founded in Theorems 1–3).

Author Contributions: Conceptualization, H.X.; writing—original draft preparation, H.L., H.X. and Z.X.; writing—review and editing, H.L. and H.X.; funding acquisition, H.L., H.X. and Z.X. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the National Natural Science Foundation of China (12161074), and the Opening Foundation of Key Laboratory of Jiangxi Province for Numerical Simulation and Emulation Techniques of Gannan Normal University.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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