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Nonlocal Coupled System for (k,φ)-Hilfer Fractional Differential Equations

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Abstract: In this paper, we study a coupled system consisting of (k,φ)-Hilfer fractional differential equations of the order $(1,2]$, supplemented with nonlocal coupled multi-point boundary conditions. The existence and uniqueness of the results are established via Banach's contraction mapping principle, the Leray–Schauder alternative and Krasnosel'skiĭ's fixed-point theorem. Numerical examples are constructed to illustrate the obtained results.

Keywords: (k,φ)-Hilfer fractional derivative; Riemann–Liouville fractional derivative; Caputo fractional derivative; existence; uniqueness; fixed-point theorems

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1. Introduction

Fractional differential equations have recently been applied as a valuable tool in the modeling of many physical phenomena. There has been substantial theoretical development in fractional calculus and fractional differential equations in recent years; see the monographs [1–9]. Usually, fractional derivative operators are defined via fractional integral operators and depend on Euler's gamma function. Different definitions of fractional derivative operators, such as Riemann–Liouville, Caputo, Erdélyi–Kober, Hadamard and Hilfer fractional operators to name a few, have been proposed in the literature. In [10], the Riemann–Liouville fractional integral operator, with the help of the generalized Euler's k gamma function, was extended to the k-Riemann–Liouville fractional integral operator. Based on this integral operator, in [11], the k-Riemann–Liouville fractional derivative was defined. We refer to [12–17] and the references cited therein for some results on the k-Riemann–Liouville fractional derivative. The φ-Riemann–Liouville fractional integral and φ-Riemann–Liouville fractional derivative were introduced in [2]. In addition the φ-Hilfer fractional derivative was defined in [18]. In [19], the (k,φ)-Riemann–Liouville fractional integral was defined, and in [11], the (k,φ)-Riemann–Liouville fractional derivative operators were defined. Very recently, in [20], the (k,φ)-Hilfer fractional derivative operator was introduced, and several of its properties were studied.

Moreover, in [20], the authors studied the following (k,φ)-Hilfer fractional nonlinear initial value problem of the form

$$\begin{cases} {}^{k,H}D_{c+}^{\alpha,\beta;\varphi}w(\theta) = f(\theta,w(\theta)), \;\; \theta\in(c,d],\; 0<\alpha<k,\; 0\le\beta\le 1, \\ {}^{k}I^{k-\theta_k;\varphi}w(c) = w_c\in\mathbb{R}, \;\; \theta_k=\alpha+\beta(k-\alpha), \end{cases} \tag{1}$$

where ${}^{k,H}D^{\alpha,\beta;\varphi}$ is the (k,φ)-Hilfer fractional derivative operator of order α, $0<\alpha\leq 1$ and parameter β, $0\leq\beta\leq 1$, and $f:[c,d]\times\mathbb{R}\to\mathbb{R}$ is a continuous function. An existence and uniqueness result was proved via Banach's fixed-point theorem.

Recently, motivated by the paper [20], we initiated in [21] the study of boundary value problems for (k,φ)-Hilfer fractional derivative of order in $(1,2]$ of the form

$$\begin{cases} {}^{k,H}D^{\overline{\alpha},\overline{\beta};\varphi}w(\theta)=f(\theta,w(\theta)), \quad \theta\in(c,d], \\ w(c)=0, \quad w(d)=\sum_{i=1}^{m}\lambda_i\, w(\xi_i), \end{cases} \tag{2}$$

where ${}^{k,H}D^{\overline{\alpha},\overline{\beta};\varphi}$ is the (k,φ)-Hilfer fractional derivative of order $\overline{\alpha}$, $1<\overline{\alpha}<2$ and parameter $\overline{\beta}$, $0\leq\overline{\beta}\leq 1$, $f:[c,d]\times\mathbb{R}\to\mathbb{R}$ is a continuous function, $\lambda_i\in\mathbb{R}$, and $c<\xi_i<d, i=1,2,\ldots,m$. The existence and uniqueness of the results were proved by using Banach's and Krasnosel'skiǐ's fixed-point theorems, as well as the Laray–Schauder nonlinear alternative.

As far as we know, in the literature there is no other paper dealing with the (k,φ)-Hilfer fractional derivative operator of the order $\overline{\alpha}\in(1,2]$, and therefore, this new area of research needs further exploration. Thus, our main contribution in this paper is to enrich this new research area with new results in other directions. In the present work, we discuss the existence and uniqueness of solutions for a coupled system of nonlinear fractional differential equations involving the (k,φ)-Hilfer fractional derivative operator of order $\overline{\alpha}$, $1<\overline{\alpha}\leq 2$ and parameter $\overline{\beta}$, $0\leq\overline{\beta}\leq 1$ of the form

$$\begin{cases} {}^{k,H}D^{\overline{\alpha},\overline{\beta};\varphi}w(\theta)=f(\theta,w(\theta),z(\theta)), \quad \theta\in(c,d], \\ {}^{k,H}D^{\alpha_1,\beta_1;\varphi}z(\theta)=f_1(\theta,w(\theta),z(\theta)), \quad \theta\in(c,d], \\ w(c)=0, \quad w(d)=\sum_{i=1}^{m}\lambda_i\, z(\xi_i), \\ z(c)=0, \quad z(d)=\sum_{j=1}^{k}\mu_j\, w(\eta_j), \end{cases} \tag{3}$$

where ${}^{k,H}D^{\overline{\alpha},\overline{\beta};\varphi}$, ${}^{k,H}D^{\alpha_1,\beta_1;\varphi}$ denote the (k,φ)-Hilfer fractional derivative operator of orders $\overline{\alpha},\alpha_1$, $1<\overline{\alpha},\alpha_1<2$ and parameters $\overline{\beta},\beta_1$, $0\leq\overline{\beta},\beta_1\leq 1$, respectively, $f,f_1:[c,d]\times\mathbb{R}\to\mathbb{R}$ are continuous functions, $\lambda_i,\mu_j\in\mathbb{R}$, and $a<\xi_i,\eta_j<b, i=1,2,\ldots,m, j=1,2,\ldots,k$. The existence and uniqueness of the results are proved by using Banach's contraction mapping principle, the Leray–Schauder alternative and Krasnosel'skiǐ's fixed-point theorem.

Numerical examples are constructed, illustrating the applicability of our obtained theoretical results.

Nonlocal conditions are considered to be more plausible than the classical conditions, as they can correctly describe certain features of physical problems.

We organize the remaining part of this work as follows. In Section 2, an auxiliary result concerning a linear variant of the system (3) is presented. This lemma is the basic key in transforming the given system iinto an equivalent fixed-point problem. The main results are presented in Section 3, while Section 5 is devoted to illustrative examples. The study of coupled systems of fractional differential equations is a significant area of investigation, as such systems often occur in applications. The work in this paper is new and enriches the literature on coupled systems of (k,φ)-Hilfer fractional differential equations. The used methods are standard, but their configuration in the present problem is new.

2. Preliminaries

Definition 1 ([2])**.** *Suppose that $h\in L^1([c,d],\mathbb{R})$. Then, the Riemann–Liouville fractional integral is defined by*

$$I^{\alpha}_{c+}h(\theta)=\frac{1}{\Gamma(\alpha)}\int_c^{\theta}(\theta-s)^{\alpha-1}h(s)ds, \ \ \alpha>0,\ \theta>c. \tag{4}$$

Here, $\Gamma(\cdot)$ is the classical Euler gamma function.

Definition 2 ([2])**.** *Let $h \in C([c,d],\mathbb{R})$. Then the Riemann–Liouville fractional derivative operator of order $\alpha > 0$ is defined by*

$$ {}^{RL}D^{\alpha}_{c+}h(\theta) = D^n I^{n-\alpha}_{c+}h(\theta) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{d\theta^n}\int_c^{\theta}(\theta - s)^{n-\alpha-1}h(s)ds, \;\; \theta > c, \tag{5}$$

where $n-1 < \alpha \leq n$ and $n \in \mathbb{N}$.

Definition 3 ([2])**.** *Let $h \in C^n([c,d],\mathbb{R})$. Then the Caputo fractional derivative operator of order $\alpha > 0$ is defined by*

$$ {}^{C}D^{\alpha}_{c+}h(\theta) = I^{n-\alpha}_{c+}D^n f(\theta) = \frac{1}{\Gamma(n-\alpha)}\int_c^{\theta}(\theta - s)^{n-\alpha-1}h^{(n)}(s)ds, \;\; \theta > c, \tag{6}$$

where $n-1 < \alpha \leq n$ and $n \in \mathbb{N}$.

Definition 4 ([10])**.** *Let $h \in L^1([c,d],\mathbb{R})$ and $k,\alpha \in \mathbb{R}^+$. Then the k-Riemann–Liouville fractional derivative of order α of the function h is given by*

$$ {}^{k}I^{\alpha}_{c+}h(\theta) = \frac{1}{k\Gamma_k(\alpha)}\int_c^{\theta}(\theta - s)^{\frac{\alpha}{k}-1}h(s)ds, \tag{7}$$

where Γ_k is the k-gamma function for $z \in \mathbb{C}$ with $\Re(z) > 0$ and $k \in \mathbb{R}, k > 0$ which is defined in [22] by

$$\Gamma_k(z) = \int_0^{\infty} s^{z-1}e^{-\frac{s^k}{k}}ds.$$

It is well known that

$$\Gamma(x) = \lim_{k\to 1}\Gamma_k(x), \;\; \Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right) \text{ and } \Gamma_k(x+k) = x\Gamma_k(x).$$

Definition 5 ([11])**.** *Let $h \in L^1([c,d],\mathbb{R})$ and $k,\alpha \in \mathbb{R}^+$. Then the k-Riemann–Liouville fractional derivative of order $\overline{\alpha}$ of the function h is given by*

$$ {}^{k,RL}D^{\overline{\alpha}}_{c+}h(\theta) = \left(k\frac{d}{d\theta}\right)^n {}^{k}I^{nk-\overline{\alpha}}_{c+}h(\theta), \;\; n = \left\lceil \frac{\overline{\alpha}}{k} \right\rceil, \tag{8}$$

where $\left\lceil \frac{\overline{\alpha}}{k} \right\rceil$ is the ceiling function of $\frac{\overline{\alpha}}{k}$.

Definition 6 ([2])**.** *Let $h \in L^1([c,d],\mathbb{R})$ and an increasing function $\varphi : [c,d] \to \mathbb{R}$ with $\varphi'(\theta) \neq 0$ for all $\theta \in [c,d]$. Then the φ-Riemann–Liouville fractional integral of the function h is given by*

$$ I^{\overline{\alpha};\varphi}h(\theta) = \frac{1}{\Gamma_k(\overline{\alpha})}\int_c^{\theta}\varphi'(s)(\varphi(\theta) - \varphi(s))^{\overline{\alpha}-1}h(s)ds. \tag{9}$$

Definition 7 ([2])**.** *Let $n-1 < \overline{\alpha} \leq n$, $\varphi \in C^n([c,d],\mathbb{R})$, $\varphi'(\theta) \neq 0, \theta \in [c,d]$, and $h \in C([c,d],\mathbb{R})$. Then the φ-Riemann–Liouville fractional derivative of the function h of order $\overline{\alpha}$ is given by*

$$ {}^{RL}D^{\overline{\alpha};\varphi}h(\theta) = \left(\frac{1}{\varphi'(\theta)}\frac{d}{dt}\right)^n I^{n-\overline{\alpha};\varphi}_{c+}h(\theta). \tag{10}$$

Definition 8 ([23])**.** *Let $n-1 < \overline{\alpha} \leq n, \varphi \in C^n([c,d],\mathbb{R})$, $\varphi'(\theta) \neq 0, \theta \in [c,d]$, and $h \in C([c,d],\mathbb{R})$. Then the φ-Caputo fractional derivative of the function h of order α is given by*

$$^{C}D^{\overline{\alpha};\varphi}h(\theta) = I_{c+}^{n-\overline{\alpha};\varphi}\left(\frac{1}{\varphi'(\theta)}\frac{d}{d\theta}\right)^{n} h(\theta), \tag{11}$$

respectively.

Definition 9 ([18])**.** *Let $n-1 < \overline{\alpha} \leq n$, $\varphi \in C^n([c,d],\mathbb{R})$, $\varphi'(\theta) \neq 0, \theta \in [c,d]$, and $h \in C([c,d],\mathbb{R})$. Then the φ-Hilfer fractional derivative of the function $h \in C([c,d],\mathbb{R})$ of order $\overline{\alpha} \in (n-1,n]$ and type $\overline{\beta} \in [0,1]$ and $\varphi \in C^n([c,d],\mathbb{R})$, $\varphi'(\theta) \neq 0, \theta \in [c,d]$, is defined by*

$$^{H}D^{\overline{\alpha},\overline{\beta};\varphi}h(\theta) = I_{c+}^{\beta(n-\overline{\alpha});\varphi}\left(\frac{1}{\varphi'(\theta)}\frac{d}{d\theta}\right)^{n} I_{c+}^{(1-\overline{\beta})(n-\overline{\alpha});\varphi}h(\theta). \tag{12}$$

Definition 10 ([19])**.** *Let $h \in L^1([c,d],\mathbb{R})$ and $k > 0$. Then the (k,φ)-Riemann–Liouville fractional integral of order $\overline{\alpha} > 0$ ($\overline{\alpha} \in \mathbb{R}$) of the function h is given by*

$$^{k}I_{c+}^{\overline{\alpha};\varphi}h(\theta) = \frac{1}{k\Gamma_k(\overline{\alpha})}\int_c^\theta \varphi'(s)(\varphi(\theta)-\varphi(s))^{\frac{\overline{\alpha}}{k}-1}h(s)ds. \tag{13}$$

Definition 11 ([20])**.** *Let $\overline{\alpha}, k \in \mathbb{R}^+ = (0,\infty)$, $\beta \in [0,1]$, $\varphi \in C^n([c,d],\mathbb{R})$, $\varphi'(\theta) \neq 0$, $\theta \in [c,d]$ and $h \in C^n([c,d],\mathbb{R})$. Then the (k,φ)-Hilfer fractional derivative of the function h of order $\overline{\alpha}$ and type $\overline{\beta}$, is defined by*

$$^{k,H}D^{\overline{\alpha},\overline{\beta};\varphi}h(\overline{\theta}) = I_{c+}^{\overline{\beta}(nk-\overline{\alpha});\varphi}\left(\frac{k}{\varphi'(\theta)}\frac{d}{d\theta}\right)^{n} {}^{k}I_{c+}^{(1-\overline{\beta})(nk-\overline{\alpha});\varphi}h(\theta), \;\; n = \left\lceil \frac{\overline{\alpha}}{k} \right\rceil. \tag{14}$$

Note the following:

1. For $\overline{\beta} = 0$ and $\overline{\beta} = 1$, (14) reduces to

$$^{k,RL}D^{\overline{\alpha};\varphi}h(\theta) = \left(\frac{k}{\varphi'(\theta)}\frac{d}{d\theta}\right)^{n} {}^{k}I_{c+}^{(1-\overline{\beta})(nk-\overline{\alpha})}h(\theta), \tag{15}$$

and

$$^{k,C}D^{\overline{\alpha};\varphi}h(\theta) = I_{c+}^{nk-\overline{\alpha};\varphi}\left(\frac{k}{\varphi'(\theta)}\frac{d}{d\theta}\right)^{n} h(\theta), \tag{16}$$

which are, respectively, the (k,φ)-Riemann–Liouville and (k,φ)-Caputo fractional derivatives. If, in addition, we take in (15), $\varphi(\theta) = \theta$, then we obtain the k-Riemann–Liouville fractional derivative defined in [11], while if we take $\varphi(\theta) = \theta$ in (16), then we obtain the k-Caputo fractional derivative

$$^{k,C}D^{\overline{\alpha};\varphi}h(\theta) = I_{c+}^{nk-\overline{\alpha};\varphi}\left(k\frac{d}{d\theta}\right)^{n} h(\theta). \tag{17}$$

2. If $\varphi(\theta) = \theta^\rho$ then (14) reduces to the k-Hilfer–Katugampola fractional derivative operator. If, in addition, $\overline{\beta} = 0$, then (14) reduces to the k-Katugampola fractional derivative, while if $\overline{\beta} = 1$ reduces to the k-Caputo–Katugampola fractional derivative operator, respectively.
3. If $\varphi(\theta) = \log\theta$, then (14) reduces to the k-Hilfer–Hadamard fractional derivative operator. If, in addition, $\overline{\beta} = 0$, then (14) reduces to the k-Hadamard fractional derivative, while if $\overline{\beta} = 1$ then (14) reduces to the k-Caputo–Hadamard fractional derivative operator.

Remark 1. *If we put $\theta_k = \overline{\alpha} + \overline{\beta}(nk-\overline{\alpha})$, then we obtain $(1-\overline{\beta})(nk-\overline{\alpha}) = nk - \theta_k$ and $\overline{\beta}(nk-\overline{\alpha}) = \theta_k - \overline{\alpha}$. Hence*

$$^{k,H}D^{\overline{\alpha},\overline{\beta};\varphi}h(\theta) \;\; = \;\; {}^{k}I_{c+}^{\theta_k-\overline{\alpha};\varphi}\left(\frac{k}{\varphi'(\theta)}\frac{d}{d\theta}\right)^{n} {}^{k}I_{c+}^{nk-\theta_k;\varphi}h(\theta)$$

$$= {}^kI_{c+}^{\theta_k-\overline{\alpha};\varphi}\left({}^{k,RL}D^{\theta_k,\varphi}h\right)(\theta).$$

which means that the (k,φ)-Hilfer fractional derivative can be defined in the form of the (k,φ)-Riemann–Liouville fractional derivative.

Note that for $\beta \in [c,d]$ and $n-1 < \frac{\overline{\alpha}}{k} \le n$, we have $n-1 < \frac{\theta_k}{k} \le n$.

Lemma 1 ([20])**.** *Let $\mu, k \in \mathbb{R}^+ = (0,\infty)$ and $n = \left\lceil \frac{\mu}{k} \right\rceil$. Assume that $h \in C^n([c,d],\mathbb{R})$ and ${}^kI_{c+}^{nk-\mu;\varphi}h \in C^n([c,d],\mathbb{R})$. Then*

$${}^kI^{\mu;\varphi}\left({}^{k,RL}D^{\mu;\varphi}h(\theta)\right) = h(\theta) - \sum_{j=1}^{n}\frac{(\varphi(\theta)-\varphi(c))^{\frac{\mu}{k}-j}}{\Gamma_k(\mu-jk+k)}\left[\left(\frac{k}{\varphi'(\theta)}\frac{d}{d\theta}\right)^{n-j}{}^kI_{c+}^{nk-\mu;\varphi}h(\theta)\right]_{\theta=c}.$$

Lemma 2 ([20])**.** *Let $\overline{\alpha}, k \in \mathbb{R}^+ = (0,\infty)$ with $\overline{\alpha} < k$, $\overline{\beta} \in [c,d]$ and $\theta_k = \overline{\alpha} + \overline{\beta}(k-\overline{\alpha})$. Then*

$${}^kI^{\theta_k,\varphi}\left({}^{k,RL}D^{\theta_k,\varphi}h\right)(\theta) = {}^kI^{\overline{\alpha};\varphi}\left({}^{k,H}D^{\overline{\alpha},\overline{\beta};\varphi}h\right)(\theta), \;\; h \in C^n([c,d],\mathbb{R}).$$

Now we prove an auxiliary result concerning a linear variant of the system (3).

Lemma 3. *Let $c < d$, $k > 0$, $1 < \overline{\alpha}, \alpha_1 \le 2$, $\overline{\beta}, \beta_1 \in [0,1]$, $\theta_k = \overline{\alpha} + \overline{\beta}(2k-\overline{\alpha})$, $q_k = \alpha_1 + \beta_1(2k-\alpha_1)$, $h \in C^2([c,d],\mathbb{R})$ and*

$$A := A_1A_4 - A_2A_3 \neq 0. \tag{18}$$

Then the function unique solution of the nonlocal (k,φ)-Hilfer fractional system

$$\begin{cases} {}^{k,H}D^{\overline{\alpha},\overline{\beta};\varphi}w(\theta) = h(\theta), \quad \theta \in (c,d], \\ {}^{k,H}D^{\alpha_1,\beta_1;\varphi}z(\theta) = h_1(\theta), \quad \theta \in (c,d], \\ w(c) = 0, \quad w(d) = \sum_{i=1}^{m}\lambda_i\, z(\xi_i), \\ z(c) = 0, \quad z(d) = \sum_{j=1}^{k}\mu_j\, w(\eta_j), \end{cases} \tag{19}$$

is given by

$$\begin{aligned} w(\theta) &= {}^kI^{\overline{\alpha};\varphi}h(\theta) + \frac{(\varphi(\theta)-\varphi(c))^{\frac{\theta_k}{k}-1}}{A\Gamma_k(\theta_k)}\left[A_4\left(\sum_{i=1}^{m}\lambda_i\,{}^kI^{\alpha_1;\varphi}h_1(\xi_i) - {}^kI^{\overline{\alpha};\varphi}h(d)\right)\right. \\ &\left.+A_2\left(\sum_{j=1}^{k}\mu_j\,{}^kI^{\overline{\alpha};\varphi}h(\eta_j) - {}^kI^{\alpha_1;\varphi}h_1(d)\right)\right], \end{aligned} \tag{20}$$

and

$$\begin{aligned} z(\theta) &= {}^kI^{\alpha_1;\varphi}h_1(\theta) + \frac{(\varphi(\theta)-\varphi(c))^{\frac{q_k}{k}-1}}{A\Gamma_k(q_k)}\left[A_1\left(\sum_{j=1}^{k}\mu_j\,{}^kI^{\overline{\alpha};\varphi}h(\eta_j) - {}^kI^{\alpha_1;\varphi}h_1(d)\right)\right. \\ &\left.+A_3\left(\sum_{i=1}^{m}\lambda_i\,{}^kI^{\alpha_1;\varphi}h_1(\xi_i) - {}^kI^{\overline{\alpha};\varphi}h(d)\right)\right], \end{aligned} \tag{21}$$

where

$$A_1 = \frac{(\varphi(d)-\varphi(c))^{\frac{\theta_k}{k}-1}}{\Gamma_k(\theta_k)}, \;\; A_2 = \sum_{i=1}^{m}\lambda_i\,\frac{(\varphi(\xi_i)-\varphi(c))^{\frac{q_k}{k}-1}}{\Gamma_k(q_k)}$$

$$A_3 = \sum_{j=1}^{k} \mu_j \frac{(\varphi(\eta_j) - \varphi(c))^{\frac{\theta_k}{k}-1}}{\Gamma_k(\theta_k)}, \quad A_4 = \frac{(\varphi(d) - \varphi(c))^{\frac{q_k}{k}-1}}{\Gamma_k(q_k)}. \tag{22}$$

Proof. Let w be a solution of the system (19). Operating fractional integral ${}^kI^{\bar{\alpha};\varphi}$ on both sides of the first equation in (19) and using Lemma 1, we obtain

$$w(\theta) = {}^kI^{\bar{\alpha};\varphi}h(\theta) + c_0\frac{(\varphi(\theta) - \varphi(c))^{\frac{\theta_k}{k}-1}}{\Gamma_k(\theta_k)} + c_1\frac{(\varphi(\theta) - \varphi(c))^{\frac{\theta_k}{k}-2}}{\Gamma_k(\theta_k - k)}, \tag{23}$$

where

$$c_0 = \left[\left(\frac{k}{\varphi'(\theta)}\frac{d}{d\theta}\right){}^kI^{2k-\theta_k;\varphi}w(\theta)\right]_{\theta=c}, \quad c_1 = \left[{}^kI^{2k-\theta_k;\varphi}w(\theta)\right]_{\theta=c}.$$

In the same process, let z be a solution of the system (19). Taking fractional integral ${}^kI^{\alpha_1;\varphi}$ on both sides of the second equation in (19) and using Lemma 1, we obtain

$$z(\theta) = {}^kI^{\alpha_1;\varphi}h_1(\theta) + d_0\frac{(\varphi(\theta) - \varphi(c))^{\frac{q_k}{k}-1}}{\Gamma_k(q_k)} + d_1\frac{(\varphi(\theta) - \varphi(c))^{\frac{q_k}{k}-2}}{\Gamma_k(q_k - k)}, \tag{24}$$

where

$$d_0 = \left[\left(\frac{k}{\varphi'(\theta)}\frac{d}{d\theta}\right){}^kI^{2k-q_k;\varphi}z(\theta)\right]_{\theta=c}, \quad d_1 = \left[{}^kI^{2k-q_k;\varphi}z(\theta)\right]_{\theta=c}.$$

Due to the boundary conditions $w(c) = 0$ and $z(c) = 0$, we obtain $c_1 = 0$ and $d_1 = 0$, since $\frac{\theta_k}{k} - 2 < 0$, $\frac{q_k}{k} - 2 < 0$ by Remark 1. From the second boundary conditions $w(d) = \sum_{i=1}^{m} \lambda_i\, z(\xi_i)$ and $z(d) = \sum_{j=1}^{k} \mu_j\, w(\eta_j)$ we obtain the system

$$\begin{aligned}
{}^kI^{\bar{\alpha};\varphi}h(d) + c_0\frac{(\varphi(d) - \varphi(c))^{\frac{\theta_k}{k}-1}}{\Gamma_k(\theta_k)} &= \sum_{i=1}^{m} \lambda_i\, {}^kI^{\alpha_1;\varphi}h_1(\xi_i) + d_0\sum_{i=1}^{m} \lambda_i \frac{(\varphi(\xi_i) - \varphi(c))^{\frac{q_k}{k}-1}}{\Gamma_k(q_k)}, \\
{}^kI^{\alpha_1;\varphi}h_1(d) + d_0\frac{(\varphi(d) - \varphi(c))^{\frac{q_k}{k}-1}}{\Gamma_k(q_k)} &= \sum_{j=1}^{k} \mu_j\, {}^kI^{\bar{\alpha};\varphi}h(\eta_j) + c_0\sum_{j=1}^{k} \mu_j \frac{(\varphi(\eta_j) - \varphi(c))^{\frac{\theta_k}{k}-1}}{\Gamma_k(\theta_k)},
\end{aligned}$$

or, using the notations (22)

$$\begin{aligned}
A_1c_0 - A_2d_0 &= \sum_{i=1}^{m} \lambda_i\, {}^kI^{\alpha_1;\varphi}h_1(\xi_i) - {}^kI^{\bar{\alpha};\varphi}h(d), \\
-A_3c_0 + A_4d_0 &= \sum_{j=1}^{k} \mu_j\, {}^kI^{\alpha;\varphi}h(\eta_j) - {}^kI^{\alpha_1;\varphi}h_1(d).
\end{aligned} \tag{25}$$

Solving the system (25) for c_0 and d_0, we have

$$\begin{aligned}
c_0 &= \frac{1}{A}\left[A_4\left(\sum_{i=1}^{m} \lambda_i\, {}^kI^{\alpha_1;\varphi}h_1(\xi_i) - {}^kI^{\bar{\alpha};\varphi}h(d)\right) + A_2\left(\sum_{j=1}^{k} \mu_j\, {}^kI^{\bar{\alpha};\varphi}h(\eta_j) - {}^kI^{\alpha_1;\varphi}h_1(d)\right)\right], \\
d_0 &= \frac{1}{A}\left[A_1\left(\sum_{j=1}^{k} \mu_j\, {}^kI^{\bar{\alpha};\varphi}h(\eta_j) - {}^kI^{\alpha_1;\varphi}h_1(d)\right) + A_3\left(\sum_{i=1}^{m} \lambda_i\, {}^kI^{\alpha_1;\varphi}h_1(\xi_i) - {}^kI^{\bar{\alpha};\varphi}h(d)\right)\right].
\end{aligned}$$

Substituting the values of c_0, c_1 and d_0, d_1 in (23) and (24), respectively, we obtain the solutions (20) and (21). We can prove easily the converse by direct computation. The proof is finished. □

3. Existence and Uniqueness Results

Let $X = C([c,d],\mathbb{R})$ be the Banach space of all continuous functions w from $[c,d]$ to $\mathbb{R}$ endowed with the norm $\|w\| = \max\{|w(\theta)|, \theta \in [c,d]\}$. The product space $(X \times X, \|(w,z)\|)$ is a Banach space with norm $\|(w,z)\| = \|w\| + \|z\|$.

In view of Lemma 3, we define an operator $T : X \times X \to X \times X$ by

$$T(w,z)(\theta) = \begin{pmatrix} T_1(w,z)(\theta) \\ T_2(w,z)(\theta) \end{pmatrix},$$

where

$$\begin{aligned} & T_1(w,z)(\theta) \\ = \; & {}^{k}I^{\overline{\alpha};\varphi} f(\theta, w(\theta), z(\theta)) \\ & + \frac{(\varphi(\theta)-\varphi(c))^{\frac{\theta_k}{k}-1}}{A\Gamma_k(\theta_k)} \Big[A_4 \Big(\sum_{i=1}^{m} \lambda_i \, {}^{k}I^{\alpha_1;\varphi} f_1(\xi_i, w(\xi_i), z(\xi_i)) - {}^{k}I^{\overline{\alpha};\varphi} f(d, w(d), z(d)) \Big) \\ & + A_2 \Big(\sum_{j=1}^{k} \mu_j \, {}^{k}I^{\overline{\alpha};\varphi} f(\eta_j, w(\eta_j), z(\eta_j)) - {}^{k}I^{\alpha_1;\varphi} f_1(d, w(d), z(d)) \Big) \Big], \end{aligned} \tag{26}$$

and

$$\begin{aligned} & T_2(w,z)(\theta) \\ = \; & {}^{k}I^{\alpha_1;\varphi} f_1(\theta, w(\theta), z(\theta)) \\ & + \frac{(\varphi(\theta)-\varphi(c))^{\frac{q_k}{k}-1}}{A\Gamma_k(q_k)} \Big[A_1 \Big(\sum_{j=1}^{k} \mu_j \, {}^{k}I^{\overline{\alpha};\varphi} f(\eta_j, w(\eta_j), z(\eta_j)) - {}^{k}I^{\alpha_1;\varphi} f_1(d, w(d), z(d)) \Big) \\ & + A_3 \Big(\sum_{i=1}^{m} \lambda_i \, {}^{k}I^{\alpha_1;\varphi} f_1(\xi_i, w(\xi_i), z(\xi_i)) - {}^{k}I^{\overline{\alpha};\varphi} f(d, w(d), z(d)) \Big) \Big]. \end{aligned} \tag{27}$$

For convenience, we put

$$\begin{aligned} Q_1 &= \frac{(\varphi(d)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} + \frac{(\varphi(d)-\varphi(c))^{\frac{\theta_k}{k}-1}}{|A|\Gamma_k(\theta_k)} \Big[A_4 \frac{(\varphi(d)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} \\ & \quad + A_2 \sum_{j=1}^{k} |\mu_j| \, \frac{(\varphi(\eta_j)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} \Big], \end{aligned} \tag{28}$$

$$\begin{aligned} Q_2 &= \frac{(\varphi(d)-\varphi(c))^{\frac{\theta_k}{k}-1}}{|A|\Gamma_k(\theta_k)} \Big[A_4 \sum_{i=1}^{m} |\lambda_i| \, \frac{(\varphi(\xi_i)-\varphi(c))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)} \\ & \quad + A_2 \frac{(\varphi(d)-\varphi(c))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)} \Big], \end{aligned} \tag{29}$$

$$\begin{aligned} Q_3 &= \frac{(\varphi(d)-\varphi(c))^{\frac{q_k}{k}-1}}{|A|\Gamma_k(q_k)} \Big[A_1 \sum_{j=1}^{k} |\mu_j| \, \frac{(\varphi(\eta_j)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} \\ & \quad + A_3 \frac{(\varphi(d)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} \Big], \end{aligned} \tag{30}$$

$$\begin{aligned} Q_4 &= \frac{(\varphi(d)-\varphi(c))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)} + \frac{(\varphi(d)-\varphi(c))^{\frac{q_k}{k}-1}}{|A|\Gamma_k(q_k)} \Big[A_1 \frac{(\varphi(d)-\varphi(c))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)} \\ & \quad + A_3 \sum_{i=1}^{m} |\lambda_i| \, \frac{(\varphi(\xi_i)-\varphi(c))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)} \Big], \end{aligned} \tag{31}$$

and

$$Q_1^* = Q_1 - \frac{(\varphi(d)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)}, \qquad Q_4^* = Q_4 - \frac{(\varphi(d)-\varphi(c))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)}. \tag{32}$$

3.1. Existence of a Unique Solution

Now, by applying Banach's contraction mapping principle [24], our first result is obtained.

Theorem 1. *Assume that $A \neq 0$ and $f, f_1 : [c,d] \times \mathbb{R}^2 \to \mathbb{R}$ are two functions for which there exist constants $m_i, n_i, i = 1,2$ such that for all $\theta \in [c,d]$ and $u_i, v_i \in \mathbb{R}, i = 1,2,$*

$$|f(\theta,u_1,u_2) - f(\theta,v_1,v_2)| \leq m_1|u_1 - v_1| + m_2|u_2 - v_2|$$

and

$$|f_1(\theta,u_1,u_2) - f_1(,v_1,v_2)| \leq n_1|u_1 - v_1| + n_2|u_2 - v_2|.$$

In addition, we suppose that

$$(Q_1 + Q_3)(m_1 + m_2) + (Q_2 + Q_4)(n_1 + n_2) < 1,$$

where $Q_i, i = 1,2,3,4$ are given by (28)–(31). Then, the nonlocal (k,φ)-Hilfer fractional system (3) has a unique solution.

Proof. Define $\sup_{\theta\in[c,d]} f(\theta,0,0) = N < \infty$ and $\sup_{\theta\in[c,d]} f_1(\theta,0,0) = N_1 < \infty$ such that

$$r \geq \frac{(Q_1+Q_3)N + (Q_2+Q_4)N_1}{1 - [(Q_1+Q_3)(m_1+m_2) + (Q_2+Q_4)(n_1+n_2)]}.$$

We show that $TB_r \subset B_r$, where $B_r = \{(w,z) \in X \times X : \|(w,z)\| \leq r\}$.
For $(w,z) \in B_r$, we have

$$\begin{aligned}
&|T_1(w,z)(\theta)| \\
\leq\ & {}^kI^{\overline{\alpha};\varphi}[|f(\theta,w(\theta),z(\theta)) - f(\theta,0,0)| + |f(\theta,0,0)|] \\
&+ \frac{(\varphi(\theta)-\varphi(c))^{\frac{\theta_k}{k}-1}}{|A|\Gamma_k(\theta_k)}\Big[A_4\Big(\sum_{i=1}^m |\lambda_i|\, {}^kI^{\alpha_1;\varphi}[h_1(\xi_i,w(\xi_i),z(\xi_i)) - f_1(\xi_i,0,0)| \\
&+ |f(\xi_i,0,0)|] + {}^kI^{\overline{\alpha};\varphi}[|f(d,w(d),z(d)) - f(d,0,0)| + |f(d,0,0)|]\Big) \\
&+ A_2\Big(\sum_{j=1}^k |\mu_j|\, {}^kI^{\overline{\alpha};\varphi}[f(\eta_j,w(\eta_j),z(\eta_j)) - f(\eta_j,0,0)| + |f(\eta_j,0,0)|] \\
&+ {}^kI^{\alpha_1;\varphi}[f_1(d,w(d),z(d)) - f_1(d,0,0)| + |f_1(d,0,0)|]\Big)\Big] \\
\leq\ & {}^kI^{\overline{\alpha};\varphi}[m_1\|w\| + m_2\|z\| + N](d) \\
&+ \frac{(\varphi(d)-\varphi(c))^{\frac{\theta_k}{k}-1}}{|A|\Gamma_k(\theta_k)}\Big[A_4\Big(\sum_{i=1}^m |\lambda_i|\, {}^kI^{\alpha_1;\varphi}[n_1\|w\| + n_2\|z\| + N_1](\xi_i) \\
&+ {}^kI^{\overline{\alpha};\varphi}[m_1\|w\| + m_2\|z\| + N](d)\Big) + A_2\Big(\sum_{j=1}^k |\mu_j|\, {}^kI^{\overline{\alpha};\varphi}[m_1\|w\| + m_2\|z\| + N](\eta_j) \\
&+ {}^kI^{\alpha_1;\varphi}[n_1\|w\| + n_2\|z\| + N_1](d)\Big)\Big] \\
\leq\ & \frac{(\varphi(d)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)}[m_1\|w\| + m_2\|z\| + N]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(\varphi(\theta)-\varphi(c))^{\frac{\theta_k}{k}-1}}{|A|\Gamma_k(\theta_k)}\Big[A_4\Big(\sum_{i=1}^{m}|\lambda_i|\,\frac{(\varphi(\xi_i)-\varphi(c))^{\frac{\alpha_1}{k}}}{\Gamma_k(\overline{\alpha}+k)}[n_1\|w\|+n_2\|z\|+N_1]\\
&+\frac{(\varphi(d)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)}[m_1\|w\|+m_2\|z\|+N]\Big)\\
&+A_2\Big(\sum_{j=1}^{k}|\mu_j|\,\frac{\varphi(\eta_j)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)}[m_1\|w\|+m_2\|z\|+N]\\
&+\frac{(\varphi(d)-\varphi(c))^{\frac{\alpha_1}{k}}}{\Gamma_k(\overline{\alpha}+k)}[n_1\|w\|+n_2\|z\|+N_1]\Big)\Big]\\
= & \Big\{\frac{(\varphi(d)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)}+\frac{(\varphi(\theta)-\varphi(c))^{\frac{\theta_k}{k}-1}}{|A|\Gamma_k(\theta_k)}\Big[A_4\frac{\varphi(d)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)}\\
&+A_2\sum_{j=1}^{k}|\mu_j|\,\frac{(\varphi(\eta_j)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)}\Big]\Big\}[m_1\|w\|+m_2\|z\|+N]\\
&+\Big\{\frac{(\varphi(\theta)-\varphi(c))^{\frac{\theta_k}{k}-1}}{|A|\Gamma_k(\theta_k)}\Big[A_4\sum_{i=1}^{m}|\lambda_i|\,\frac{(\varphi(\xi_i)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)}\\
&+A_2\frac{(\varphi(d)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)}\Big]\Big\}[n_1\|w\|+n_2\|z\|+N_1]\\
= & Q_1[m_1\|w\|+m_2\|z\|+N]+Q_2[n_1\|w\|+n_2\|z\|+N_1]\\
= & (Q_1m_1+Q_2n_1)\|w\|+(Q_1m_2+Q_2n_2)\|z\|+Q_1N+Q_2N_1\\
\leq & (Q_1m_1+Q_2n_1+Q_1m_2+Q_2n_2)r+Q_1N+Q_2N_1.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
|T_2(w,z)(\theta)| \leq & \Big\{\frac{(\varphi(\theta)-\varphi(c))^{\frac{q_k}{k}-1}}{|A|\Gamma_k(q_k)}\Big[A_1\sum_{j=1}^{k}|\mu_j|\,\frac{(\varphi(\eta_j)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)}\\
&+A_3\frac{(\varphi(d)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)}\Big\}[m_1\|w\|+m_2\|z\|+N]\\
&+\Big\{\frac{(\varphi(d)-\varphi(c))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)}+\frac{(\varphi(\theta)-\varphi(c))^{\frac{q_k}{k}-1}}{|A|\Gamma_k(q_k)}\Big[A_1\frac{(\varphi(d)-\varphi(c))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)}\\
&+A_3\sum_{i=1}^{m}|\lambda_i|\,\frac{(\varphi(\xi_i)-\varphi(c))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)}\Big\}[n_1\|w\|+n_2\|z\|+N_1]\\
= & Q_3(m_1\|w\|+m_2\|z\|+N)+Q_4(n_1\|w\|+n_2\|z\|+N_1)+\\
= & (Q_3m_1+Q_4n_1)\|w\|+(Q_3m_2+Q_4n_2)\|z\|+Q_3N+Q_4N_1\\
\leq & (Q_3m_1+Q_4n_1+Q_3m_2+Q_4n_2)r+Q_3N+Q_4N_1.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\|T(w,z)\| = & \|T_1(w,z)\|+\|T_2(w,z)\|\\
\leq & [(Q_1+Q_3)(m_1+M_2)+(Q_2+Q_4)(n_1+n_2)]r\\
&+(Q_1+Q_3)N+(Q_2+Q_4)N_1\leq r,
\end{aligned}
$$

which implies that $TB_r\subset B_r$.

Now for $(w_2,z_2),(w_1,z_1)\in X\times X$, and for any $\theta\in[c,d]$, we obtain

$$|T_1(w_2,z_2)(\theta)-T_1(w_1,z_1)(\theta)|$$

$$
\begin{aligned}
&\le {}^{k}I^{\overline{\alpha};\varphi}|f(\theta,w_2(\theta),z_2(\theta))-f(\theta,w_1(\theta),z_1(\theta)| \\
&\quad+\frac{(\varphi(\theta)-\varphi(c))^{\frac{\theta_k}{k}-1}}{|A|\Gamma_k(\theta_k)}\Big[A_4\Big(\sum_{i=1}^{m}|\lambda_i|\,{}^{k}I^{\alpha_1;\varphi}|f_1(\xi_i,w_2(\xi_i),z_2(\xi_i))-f_1(\xi_i,w_1(\xi_i),z_1(\xi_i)| \\
&\quad+{}^{k}I^{\overline{\alpha};\varphi}|f(d,w_2(d),z_2(d))-f(d,w_1(d),z_1(d)|\Big) \\
&\quad+A_2\Big(\sum_{j=1}^{k}|\mu_j|\,{}^{k}I^{\overline{\alpha};\varphi}|f(\eta_j,w_2(\eta_j),z_2(\eta_j))-f(\eta_j,w_1(\eta_j),z_1(\eta_j))| \\
&\quad+{}^{k}I^{\alpha_1;\varphi}|f_1(d,w_2(d),z_2(d))-f_1(d,w_1(d),z_1(d)|\Big)\Big] \\
&\le \Bigg\{\frac{(\varphi(d)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)}+\frac{(\varphi(\theta)-\varphi(c))^{\frac{\theta_k}{k}-1}}{|A|\Gamma_k(\theta_k)}\Big[A_4\frac{(\varphi(d)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} \\
&\quad+A_2\sum_{j=1}^{k}|\mu_j|\,\frac{(\varphi(\eta_j)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)}\Big]\Bigg\}(m_1\|w_2-w_1\|+m_2\|z_2-z_1\|) \\
&\quad+\Bigg\{\frac{(\varphi(\theta)-\varphi(c))^{\frac{\theta_k}{k}-1}}{|A|\Gamma_k(\theta_k)}\Big[A_4\sum_{i=1}^{m}|\lambda_i|\,\frac{(\varphi(\xi_i)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} \\
&\quad+A_2\frac{(\varphi(d)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)}\Big]\Bigg\}(n_1\|w_2-w_1\|+n_2\|z_2-z_1\|) \\
&= Q_1(m_1\|w_2-w_1\|+m_2\|z_2-z_1\|)+Q_2(n_1\|w_2-w_1\|+n_2\|z_2-z_1\|) \\
&= (Q_1m_1+Q_2n_1)\|w_2-w_1\|+(Q_1m_2+Q_2n_2)\|z_2-z_1\|,
\end{aligned}
$$

and consequently, we obtain

$$\|T_1(w_2,z_2)(\theta)-T_1(w_1,z_1)\|\le(Q_1m_1+Q_2n_1+Q_1m_2+Q_2n_2)[\|w_2-w_1\|+\|z_2-z_1\|]. \tag{33}$$

Similarly,

$$\|T_2(w_2,z_2)(\theta)-T_2(w_1,z_1)\|\le(Q_3m_1+Q_4n_1+Q_3m_2+Q_4n_2)[\|w_2-w_1\|+\|z_2-z_1\|]. \tag{34}$$

It follows from (33) and (34) that

$$\|T(w_2,z_2)(\theta)-T(w_1,z_1)(\theta)\|\le[(Q_1+Q_3)(m_1+m_2)+(Q_2+Q_4)(n_1+n_2)](\|w_2-w_1\|+\|z_2-z_1\|).$$

Since $(Q_1+Q_3)(m_1+m_2)+(Q_2+Q_4)(n_1+n_2)<1$, the operator T is a contraction. By Banach's contraction mapping principle, a unique solution of the operator T is gained, and this completes the proof. □

3.2. Existence Results

Our first existence results for the nonlocal (k,φ)-Hilfer fractional system (3) is based on the Leray–Schauder alternative [25].

Theorem 2. *Assume that $A\neq 0$ and $f, f_1:[c,d]\times\mathbb{R}^2\to\mathbb{R}$ are continuous functions for which there exist real constants $k_i,\ \nu_i\ \ge 0\ (i=1,2)$ and $k_0>0,\nu_0>0$ such that $\forall w_i\in\mathbb{R},\ (i=1,2)$. We have*

$$|f(\theta,w_1,w_2)|\le k_0+k_1|w_1|+k_2|w_2|,$$
$$|f_1(\theta,w_1,w_2)|\le \nu_0+\nu_1|w_1|+\nu_2|w_2|.$$

In addition, it is assumed that

$$(Q_1+Q_3)k_1+(Q_2+Q_4)\nu_1<1 \ \text{ and } \ (Q_1+Q_3)k_2+(Q_2+Q_4)\nu_2<1,$$

where $Q_i, i=1,2,3,4$ are given by (28)–(31). Then, there exists at least one solution for the nonlocal (k,φ)-Hilfer fractional system (3).

Proof. Note that the operator T is continuous since f and f_1 are continuous. Next, we will show that the operator T is completely continuous.

For any bounded set $\Omega \subset X \times X$, there exist positive constants L_1 and L_2 such that

$$|f(\theta, w(\theta), z(\theta)| \leq L_1, \quad |f_1(\theta, w(\theta), z(\theta)| \leq L_2, \ \forall (w,z) \in \Omega.$$

Then, for any $(w,z) \in \Omega$, we have

$$\begin{aligned}
|T_1(w,z)(\theta)| \leq & \Bigg\{ \frac{(\varphi(d)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} + \frac{(\varphi(d)-\varphi(c))^{\frac{\theta_k}{k}-1}}{|A|\Gamma_k(\theta_k)}\Big[A_4 \frac{(\varphi(d)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} \\
& + A_2 \sum_{j=1}^{k} |\mu_j| \frac{(\varphi(\eta_j)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} \Bigg\} L_1 \\
& + \Bigg\{ \frac{(\varphi(d)-\varphi(c))^{\frac{\theta_k}{k}-1}}{|A|\Gamma_k(\theta_k)} \Big[A_4 \sum_{i=1}^{m} |\lambda_i| \frac{(\varphi(\xi_i)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} \\
& + A_2 \frac{(\varphi(d)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} \Big] \Bigg\} L_2,
\end{aligned}$$

which implies that

$$\|T_1(w,z)\| \leq Q_1 L_1 + Q_2 L_2.$$

Similarly, we obtain

$$\|T_2(w,z)\| \leq Q_3 L_1 + Q_4 L_2.$$

Hence,

$$\|T(w,z)\| = \|T_1(w,z)\| + \|T_2(w,z)\| \leq (Q_1+Q_3)L_1 + (Q_2+Q_4)L_2,$$

which means the uniformly bounded property of the operator T.

The equicontinuity of T is proved now. Let $\theta_1, \theta_2 \in [c,d]$ with $\theta_1 < \theta_2$. Then, we have

$$\begin{aligned}
& |T_1(w(\theta_2), z(\theta_2)) - T_1(w(\theta_1), z(\theta_1))| \\
\leq & \frac{1}{\Gamma_k(\overline{\alpha})} \Bigg| \int_c^{\theta_1} \varphi'(s)[(\varphi(\theta_2)-\varphi(s))^{\frac{\overline{\alpha}}{k}-1} - (\varphi(\theta_1)-\varphi(s))^{\frac{\overline{\alpha}}{k}-1}] f(s,w(s),z(s)) ds \\
& + \int_{\theta_1}^{\theta_2} \varphi'(s)(\varphi(\theta_2)-\varphi(s))^{\frac{\overline{\alpha}3}{k}-1} f(s,w(s),z(s)) ds \Bigg| \\
& + \frac{(\varphi(\theta_2)-\varphi(c))^{\frac{\theta_k}{k}-1} - (\varphi(\theta_1)-\varphi(c))^{\frac{\theta_k}{k}-1}}{|\Lambda|\Gamma_k(\theta_k)} \Big[A_4 \Big(\sum_{i=1}^{m} |\lambda_i| \, {}^k I^{\alpha_1;\varphi} |f_1(\xi_i, w(\xi_i), z(\xi_i))| \\
& + {}^k I^{\overline{\alpha};\varphi} |f(d, w(d), z(d))| \Big) + A_2 \Big(\sum_{j=1}^{k} |\mu_j| \, {}^k I^{\overline{\alpha};\varphi} |f(\eta_j, w(\eta_j), z(\eta_j))| \\
& + {}^k I^{\alpha_1;\varphi} |f_1(d, w(d), z(d))| \Big) \Big] \\
\leq & \frac{L_1}{\Gamma_k(\overline{\alpha}+k)} [2(\varphi(\theta_2)-\varphi(\theta_1))^{\frac{\overline{\alpha}}{k}} + |(\varphi(\theta_2)-\varphi(c))^{\frac{\overline{\alpha}}{k}} - (\varphi(\theta_1)-\varphi(c))^{\frac{\overline{\alpha}}{k}}|] \\
& + \frac{(\varphi(\theta_2)-\varphi(c))^{\frac{\theta_k}{k}-1} - (\varphi(\theta_1)-\varphi(c))^{\frac{\theta_k}{k}-1}}{|\Lambda|\Gamma_k(\theta_k)} \Big[A_4 \Big(\sum_{i=1}^{m} |\lambda_i| \frac{(\varphi(\xi_i)-\varphi(c))^{\frac{\alpha_1}{k}}}{\Gamma_k(\overline{\alpha}+k)} L_2
\end{aligned}$$

$$+\frac{(\varphi(d)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)}L_1\Big)+A_2\Big(\sum_{j=1}^{k}|\mu_j|\,\frac{(\varphi(\eta_j)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)}L_1+\frac{(\varphi(d)-\varphi(c))^{\frac{\alpha_1}{k}}}{\Gamma_k(\overline{\alpha}+k)}L_2\Big)\Big],$$

which is independent of (w,z) and tend to zero as $\theta_2-\theta_1\to 0$. Thus, $T_1(w,z)$ is equicontinuous.

Analogously, we can obtain that $T_2(w,z)$ is equicontinuous. Consequently the operator $T(w,z)$ is completely continuous.

Finally, we will show the boundedness of the set $\mathcal{E}=\{(w,z)\in X\times X:(w,z)=\lambda T(w,z),0\le\lambda\le 1\}$. Let $(w,z)\in\mathcal{E}$, then $(w,z)=\lambda T(w,z)$. For any $\theta\in[c,d]$, we have

$$w(\theta)=\lambda T_1(w,z)(\theta),\quad z(\theta)=\lambda T_2(w,z)(\theta).$$

Then

$$\begin{aligned}|w(\theta)| \le{}& \Bigg\{\frac{(\varphi(d)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)}+\frac{(\varphi(d)-\varphi(c))^{\frac{\theta_k}{k}-1}}{|A|\Gamma_k(\theta_k)}\Big[A_4\frac{\varphi(d)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)}\\&+A_2\sum_{j=1}^{k}|\mu_j|\,\frac{(\varphi(\eta_j)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)}\Bigg\}(k_0+k_1\|w\|+k_2\|z\|)\\&+\Bigg\{\frac{(\varphi(d)-\varphi(c))^{\frac{\theta_k}{k}-1}}{|A|\Gamma_k(\theta_k)}\Big[A_4\sum_{i=1}^{m}|\lambda_i|\,\frac{(\varphi(\xi_i)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)}\\&+A_2\frac{(\varphi(d)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)}\Big]\Bigg\}(\nu_0+\nu_1\|w\|+\nu_2\|z\|),\end{aligned}$$

and

$$\begin{aligned}|z(\theta)| \le{}& \Bigg\{\frac{(\varphi(d)-\varphi(c))^{\frac{q_k}{k}-1}}{|A|\Gamma_k(q_k)}\Big[A_1\sum_{j=1}^{k}|\mu_j|\,\frac{(\varphi(\eta_j)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)}\\&+A_3\frac{(\varphi(d)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)}\Bigg\}(k_0+k_1\|w\|+k_2\|z\|)\\&+\Bigg\{\frac{(\varphi(d)-\varphi(c))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)}+\frac{(\varphi(d)-\varphi(c))^{\frac{q_k}{k}-1}}{|A|\Gamma_k(q_k)}\Big[A_1\frac{(\varphi(d)-\varphi(c))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)}\\&+A_3\sum_{i=1}^{m}|\lambda_i|\,\frac{(\varphi(\xi_i)-\varphi(c))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)}\Bigg\}(\nu_0+\nu_1\|w\|+\nu_2\|z\|).\end{aligned}$$

Hence, we have

$$\|w\|\le Q_1(k_0+k_1\|w\|+k_2\|z\|)+Q_2(\nu_0+\nu_1\|w\|+\nu_2\|z\|)$$

and

$$\|z\|\le Q_3(k_0+k_1\|w\|+k_2\|z\|)+Q_4(\nu_0+\nu_1\|w\|+\nu_2\|z\|),$$

which imply that

$$\begin{aligned}\|w\|+\|z\| \le{}& (Q_1+Q_3)k_0+(Q_2+Q_4)\nu_0+[(Q_1+Q_3)k_1+(Q_2+Q_4)\nu_1]\|w\|\\&+[(Q_1+Q_3)k_2+(Q_2+Q_4)\nu_2]\|z\|.\end{aligned}$$

Consequently,

$$\|(w,z)\|\le\frac{(Q_1+Q_3)k_0+(Q_2+Q_4)\nu_0}{M_0},$$

for any $\theta \in [c, d]$, where M_0 is defined by

$$M_0 = \min\{1 - [(Q_1 + Q_3)k_1 + (Q_2 + Q_4)\nu_1], 1 - [(Q_1 + Q_3)k_2 + (Q_2 + Q_4)\nu_2]\},$$

which proves that $\mathcal{E}$ is bounded. Thus, by the Leray–Schauder alternative, the operator T has at least one fixed point. Hence, we gain at least one solution of the nonlocal (k, φ)-Hilfer fractional system (3) on $[a, b]$. The proof is complete. □

The second existence result in this subsection is based on Krasnosel'skiĭ's fixed-point theorem [26].

Theorem 3. *Let $f, f_1 : [c, d] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous functions satisfying (H_1). In addition, we assume the following:*
(H_3) There exist continuous nonnegative functions P and $Q \in C([c, d], \mathbb{R}^+)$ such that

$$|f(\theta, w, z)| \le P(\theta), \quad |f_1(\theta, w, z)| \le Q(\theta), \ \text{for each} \ (\theta, w, z) \in [c, d] \times \mathbb{R} \times \mathbb{R}.$$

Then, the nonlocal (k, φ)-Hilfer fractional system (3) has at least one solution on $[c, d]$, provided that

$$\Big[Q_1^* + Q_3\Big](m_1 + m_2) + \Big[Q_2 + Q_4^*\Big](n_1 + n_2) < 1. \tag{35}$$

Proof. Let the operator T be decomposed into four operators $T_{1,1}, T_{1,2}, T_{2,1}$ and $T_{2,2}$ as

$$\begin{aligned}
T_{1,1}(w, z)(\theta) &= {}^kI^{\bar{\alpha};\varphi} f(\theta, w(\theta), z(\theta)), \\
T_{1,2}(w, z)(\theta) &= \frac{(\varphi(\theta) - \varphi(c))^{\frac{\theta_k}{k} - 1}}{A\Gamma_k(\theta_k)} \Big[A_4\Big(\sum_{i=1}^{m} \lambda_i \, {}^kI^{\alpha_1;\varphi} f_1(\xi_i, w(\xi_i), z(\xi_i)) \\
&\quad - {}^kI^{\bar{\alpha};\varphi} f(d, w(d), z(d))\Big) \\
&\quad + A_2\Big(\sum_{j=1}^{k} \mu_j \, {}^kI^{\bar{\alpha};\varphi} f(\eta_j, w(\eta_j), z(\eta_j)) - {}^kI^{\alpha_1;\varphi} f_1(d, w(d), z(d))\Big)\Big], \\
T_{2,1}(w, z)(\theta) &= {}^kI^{\alpha_1;\varphi} f_1(\theta, w(\theta), z(\theta)), \\
T_{2,2}(w, z)(\theta) &= \frac{(\varphi(\theta) - \varphi(c))^{\frac{q_k}{k} - 1}}{A\Gamma_k(q_k)} \Big[A_1\Big(\sum_{j=1}^{k} \mu_j \, {}^kI^{\bar{\alpha};\varphi} f(\eta_j, w(\eta_j), z(\eta_j)) \\
&\quad - {}^kI^{\alpha_1;\varphi} f_1(d, w(d), z(d))\Big) \\
&\quad + A_3\Big(\sum_{i=1}^{m} \lambda_i \, {}^kI^{\alpha_1;\varphi} f_1(\xi_i, w(\xi_i), z(\xi_i)) - {}^kI^{\bar{\alpha};\varphi} f(d, w(d), z(d))\Big)\Big].
\end{aligned}$$

Note that $T_1 = T_{1,1} + T_{1,2}, \ \ T_2 = T_{2,1} + T_{2,2}$. Let $B_\rho = \{(w, z) \in X \times X : \|(w, z)\| \le \rho\}$ be a ball, where $\rho \ge (Q_1 + Q_3)\|P\| + (Q_2 + Q_4)\|Q\|$. For any $w = (w_1, w_2), z = (z_1, z_2) \in B_\rho$ we have, as in Theorem 2, that

$$|T_{1,1}(w_1, w_2)(\theta) + T_{1,2}(z_1, z_2)(\theta)| \le Q_1\|P\| + Q_2\|Q\|.$$

Similarly, we can find that

$$|T_{2,1}(w_1, w_2)(\theta) + T_{2,2}(z_1, z_2)(\theta)| \le Q_3\|P\| + Q_4\|Q\|.$$

Consequently we have

$$\|T_1 w + T_2 z\| \le (Q_1 + Q_3)\|P\| + (Q_2 + Q_4)\|\|Q\| < \rho.$$

This yields $T_1 w + T_2 z \in B_\rho$.

To show that the operator $(T_{1,2}, T_{2,2})$ is a contraction mapping, for $(w_1, w_2), (z_1, z_2) \in B_\rho$, we have, as in Theorem 1, that

$$\begin{aligned}
& |T_{1,2}(x_2, y_2)(\theta) - T_{1,2}(x_1, y_1)(\theta)| \\
\leq & \left\{ \frac{(\varphi(d)-\varphi(c))^{\frac{\theta_k}{k}-1}}{|A|\Gamma_k(\theta_k)} \Big[A_4 \frac{(\varphi(d)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} \right. \\
& \left. + A_2 \sum_{j=1}^{k} |\mu_j| \frac{(\varphi(\eta_j)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} \Big] \right\} (m_1 \|w_2 - w_1\| + m_2 \|z_2 - z_1\|) \\
& + \left\{ \frac{(\varphi(d)-\varphi(c))^{\frac{\theta_k}{k}-1}}{|A|\Gamma_k(\theta_k)} \Big[A_4 \sum_{i=1}^{m} |\lambda_i| \frac{(\varphi(\xi_i)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} \right. \\
& \left. + A_2 \frac{(\varphi(d)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} \Big] \right\} (n_1 \|w_2 - w_1\| + n_2 \|z_2 - z_1\|) \\
= & Q_1^*(m_1 \|w_2 - w_1\| + m_2 \|z_2 - z_1\|) + Q_2(n_1 \|w_2 - w_1\| + n_2 \|z_2 - z_1\|) \\
= & \Big[Q_1^* m_1 + Q_2 n_1 \Big] \|w_2 - w_1\| + \Big[Q_1^* m_2 + Q_2 n_2 \Big] \|z_2 - z_1\|, \qquad (36)
\end{aligned}$$

and

$$\begin{aligned}
& |T_{2,2}(w_1, z_1)(\theta) - T_{2,2}(w_2, z_2)(\theta)| \\
\leq & \Big[\Big(Q_3 m_1 + Q_4^* n_1 \Big] \|w_2 - w_1\| + \Big[Q_3 m_2 + Q_4^* n_2 \Big] \|z_2 - z_1\| \Big]. \qquad (37)
\end{aligned}$$

It follows form (36) and (37) that

$$\begin{aligned}
& \|(T_{1,2}, T_{2,2})(w_1, z_1) - (T_{1,2}, T_{2,2})(w_2, z_2)\| \\
\leq & \Big\{ \Big[Q_1^* + Q_3 \Big] (m_1 + m_2) + \Big[Q_2 + Q_4^* \Big] (n_1 + n_2) \Big\} (\|w_1 - w_2\| + \|z_1 - z_2\|),
\end{aligned}$$

which is a contraction by inequality (35).

The operator $(T_{1,1}, T_{2,1})$ is continuous by the continuity of f, f_1. Additionally, $(T_{1,1}, T_{2,1})$ is uniformly bounded on B_ρ since

$$\|T_{1,1}(w,z)\| \leq \frac{(\varphi(d)-\varphi(c))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} \|P\| \text{ and } \|T_{2,1}(w,z)\| \leq \frac{(\varphi(d)-\varphi(c))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)} \|Q\|.$$

Then we obtain the following fact:

$$\|(T_{1,1}, T_{2,1})(w,z)\| \leq \frac{(\varphi(d)-\varphi(c))^{\frac{\varphi}{k}}}{\Gamma_k(\varphi+k)} \|P\| + \frac{(\varphi(d)-\varphi(c))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)} \|Q\|,$$

which implies that the set $(T_{1,1}, T_{2,1})B_\rho$ is uniformly bounded. In the next step, we show that the set $(T_{1,1}, T_{2,1})B_\rho$ is equicontinuous. For $\theta_1, \theta_2 \in [c,d], \theta_1 < \theta_2$, and for any $(w,z) \in B_\rho$, we can prove that

$$\begin{aligned}
& |T_{1,1}(w,z)(\theta_2) - T_{1,1}(w,z)(\theta_1)| \\
\leq & \frac{1}{\Gamma_k(\overline{\alpha})} \left| \int_c^{\theta_1} \varphi'(s)[(\varphi(\theta_2)-\varphi(s))^{\frac{\overline{\alpha}}{k}-1} - (\varphi(\theta_1)-\varphi(s))^{\frac{\overline{\alpha}}{k}-1}] f(s, w(s), z(s)) ds \right. \\
& \left. + \int_{\theta_1}^{\theta_2} \varphi'(s)(\varphi(\theta_2)-\varphi(s))^{\frac{\alpha}{k}-1} f(s, w(s), z(s)) ds \right| \\
\leq & \frac{\|P\|}{\Gamma_k(\overline{\alpha}+k)} [2(\varphi(\theta_2)-\varphi(\theta_1))^{\frac{\overline{\alpha}}{k}} + |(\varphi(\theta_2)-\varphi(c))^{\frac{\overline{\alpha}}{k}} - (\varphi(\theta_1)-\varphi(c))^{\frac{\overline{\alpha}}{k}}|],
\end{aligned}$$

which tends to zero, as $\theta_1 \to \theta_2$ independently of $(w, z) \in B_\rho$.

Similarly, we can show that $|T_{2,1}(w,z)(\theta_2) - T_{2,1}(w,z)(\theta_1)| \to 0$ as $\theta_1 \to \theta_2$ independently of $(w, z) \in B_\rho$.

Thus, $|(T_{1,1}, T_{2,1})(w,z)(\theta_2) - (T_{1,1}, T_{2,1})(w,z)(\theta_1)|$ tends to zero, as $\theta_1 \to \theta_2$.

Therefore, $(T_{1,1}, T_{2,1})$ is equicontinuous. From the Arzelá–Ascoli theorem, we conclude that the operator $(Ta_{1,1}, T_{2,1})$ is compact on B_ρ. Thus, the hypotheses of Krasnosel'skiı̆'s fixed-point theorem are satisfied, and therefore, there exists at least one solution on $[c, d]$. The proof is finished. □

4. Illustrative Examples

Now, we give some examples to show the benefits of our results.

Example 1. *Consider the following nonlocal coupled system for* (k, φ)*-Hilfer fractional differential equations of the form*

$$\begin{cases} {}^{\frac{7}{4},H}D^{\frac{3}{2},\frac{1}{4};\theta^5 e^{-\theta}} w(\theta) = f(\theta, w(\theta), z(\theta)), & \theta \in \left(\frac{1}{7}, \frac{10}{7}\right], \\ {}^{\frac{7}{4},H}D^{\frac{5}{3},\frac{3}{4};\theta^5 e^{-\theta}} z(\theta) = f_1(\theta, w(\theta), z(\theta)), & \theta \in \left(\frac{1}{7}, \frac{10}{7}\right], \\ w\left(\frac{1}{7}\right) = 0, \quad w\left(\frac{10}{7}\right) = \frac{1}{22} z\left(\frac{3}{7}\right) + \frac{3}{44} z\left(\frac{5}{7}\right) + \frac{5}{66} z\left(\frac{9}{7}\right), \\ z\left(\frac{1}{7}\right) = 0, \ z\left(\frac{10}{7}\right) = \frac{2}{33} w\left(\frac{2}{7}\right) + \frac{4}{55} w\left(\frac{4}{7}\right) + \frac{6}{77} w\left(\frac{6}{7}\right) + \frac{8}{99} w\left(\frac{8}{7}\right). \end{cases} \tag{38}$$

Here, we set $k = 7/4$, $\overline{\alpha} = 3/2$, $\overline{\beta} = 1/4$, $\alpha_1 = 5/3$, $\beta_1 = 3/4$, $\varphi(\theta) = \theta^5 e^{-\theta}$, $c = 1/7$, $d = 10/7$, $m = 3$, $\lambda_1 = 1/22$, $\lambda_2 = 3/44$, $\lambda_3 = 5/66$, $\xi_1 = 3/7$, $\xi_2 = 5/7$, $\xi_3 = 9/7$, $k = 4$, $\mu_1 = 2/33$, $\mu_2 = 4/55$, $\mu_3 = 6/77$, $\mu_4 = 8/99$, $\eta_1 = 2/7$, $\eta_2 = 4/7$, $\eta_3 = 6/7$, $\eta_4 = 8/7$. Then we can compute that $\theta_{\frac{7}{4}} = 2$, $q_{\frac{7}{4}} = 73/24$, $\Gamma_{\frac{7}{4}}(\theta_{\frac{7}{4}}) \approx 1.013291796$, $\Gamma_{\frac{7}{4}}(q_{\frac{7}{4}}) \approx 1.385078519$, $A_1 \approx 1.038187892$, $A_2 \approx 0.06296417348$, $A_3 \approx 0.2031397681$, $A_4 \approx 0.9380941984$, $A \approx 0.9611275107$, $\Gamma_{\frac{7}{4}}(\overline{\alpha} + 7/4) \approx 1.531211531$, $\Gamma_{\frac{7}{4}}(\alpha_1 + 7/4) \approx 1.671252637$, $Q_1 \approx 1.785558660$, $Q_2 \approx 0.1062650045$, $Q_3 \approx 0.2266775746$, $Q_4 \approx 1.698531866$, $Q_1^* \approx 0.9003865435$, $Q_4^* \approx 0.8596622443$.

(i) Consider the nonlinear unbounded functions $f, f_1 : [(1/7), (10/7)] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ presented as

$$f(\theta, w, z) = \frac{e^{-(7\theta-1)^2}}{12}\left(\frac{w^2 + 2|w|}{1 + |w|}\right) + \frac{1}{7(\theta+1)} \sin|z| + \frac{1}{2}\theta + \frac{2}{3}, \tag{39}$$

$$f_1(\theta, w, z) = \frac{\cos^2 \pi\theta}{10} \tan^{-1}|w| + \frac{1}{8(7\theta+1)^2}\left(\frac{3z^2 + 4|z|}{1 + |z|}\right) + \frac{1}{4}\theta + \frac{1}{5}. \tag{40}$$

Then we can find that

$$|f(\theta, w_1, z_1) - f(\theta, w_2, z_2)| \le \frac{1}{6}|w_1 - w_2| + \frac{1}{8}|z_1 - z_2|$$

and

$$|f_1(\theta, w_1, z_1) - f_1(\theta, w_2, z_2)| \le \frac{1}{10}|w_1 - w_2| + \frac{1}{8}|z_1 - z_2|,$$

for all $w_1, w_2, z_1, z_2 \in \mathbb{R}$. By choosing $m_1 = 1/6$, $m_2 = 1/8$, $n_1 = 1/10$ and $n_2 = 1/8$, we obtain $(Q_1 + Q_3)(m_1 + m_2) + (Q_2 + Q_4)(n_1 + n_2) \approx 0.9929815310 < 1$. By Theorem 1, we conclude that the nonlocal coupled system for (k, φ)-Hilfer fractional differential Equation (38) with f, f_1 defined by (39) and (40) respectively, has a unique solution on $[(1/7), (10/7)]$.

(ii) Let the nonlinear bounded functions $f, f_1 : [(1/7),(10/7)] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be given by

$$f(\theta,w,z) = \frac{1}{k}\sin^4 \pi\theta\left(\frac{|w|}{1+|w|}\right) + \frac{1}{7\theta+2}\tan^{-1}|z| + \frac{1}{4}, \tag{41}$$

$$f_1(\theta,w,z) = \frac{1}{6}\sin|w|e^{-(7\theta-1)^4} + \frac{1}{(7\theta+1)^2}\left(\frac{|z|}{1+|z|}\right) + \frac{1}{5}. \tag{42}$$

Note that f, f_1 are bounded by

$$|f(\theta,w,z)| \le \frac{1}{k}\sin^4 \pi\theta + \frac{\pi}{2(7\theta+2)} + \frac{1}{4}, \quad |f_1(\theta,w,z)| \le \frac{1}{6}e^{-(7\theta-1)^4} + \frac{1}{(7\theta+1)^2} + \frac{1}{5}.$$

In addition, functions f, f_1 satisfy

$$|f(\theta,w_1,z_1) - f(\theta,w_2,z_2)| \le \frac{1}{k}|w_1 - w_2| + \frac{1}{3}|z_1 - z_2|$$

and

$$|f_1(\theta,w_1,z_1) - f_1(\theta,w_2,z_2)| \le \frac{1}{6}|w_1 - w_2| + \frac{1}{4}|z_1 - z_2|.$$

By setting $m_1 = 1/k$, $m_2 = 1/3$, $n_1 = 1/6$ and $n_2 = 1/4$ and from $(Q_1+Q_3)(1/3) + (Q_2+Q_4)((1/6)+(1/4)) \approx 1.422744108$, we get $(Q_1+Q_3)((1/k)+(1/3)) + (Q_2+Q_4)((1/6)+(1/4)) > 1$, for all $k \in \mathbb{R}^+$, which means that we cannot obtain the uniqueness result to this problem by applying Theorem 1. However, if $k > 5.080474967$, then we have $[Q_1^* + Q_3](m_1+m_2) + [Q_2 + Q_4^*](n_1+n_2) < 1$. Therefore, using the conclusion of Theorem 3, the nonlocal coupled system for (k,φ)-Hilfer fractional differential Equation (38) with f, f_1 defined by (41) and (42) respectively, has at least one solution on $[(1/7),(10/7)]$.

(iii) Assume that the nonlinear functions $f, f_1 : [(1/7),(10/7)] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are expressed by

$$f(\theta,w,z) = \frac{1}{7t+1} + \frac{1}{4}e^{-z^2}\left(\frac{|w|^{181}}{1+w^{180}}\right) + \frac{z}{6}\sin^4 w, \tag{43}$$

$$f_1(\theta,w,z) = \frac{1}{14t+1} + \frac{2}{5\pi}w\tan^{-1}|z| + \frac{z^{150}}{3(1+|z|^{149})}\cos^6 w. \tag{44}$$

Next, we can compute the linear bounded of two above functions as

$$|f(\theta,w,z)| \le \frac{1}{2} + \frac{1}{4}|w| + \frac{1}{6}|z| \quad \text{and} \quad |f_1(\theta,w,z)| \le \frac{1}{3} + \frac{1}{5}|w| + \frac{1}{3}|z|.$$

Then, choosing $k_0 = 1/2$, $k_1 = 1/4$, $k_2 = 1/6$, $\nu_0 = 1/3$, $\nu_1 = 1/5$, $\nu_2 = 1/3$, we have $(Q_1+Q_3)k_1 + (Q_2+Q_4)\nu_1 \approx 0.8640184328 < 1$ and $(Q_1+Q_3)k_2 + (Q_2+Q_4)\nu_2 \approx 0.9369716625 < 1$. Therefore, by Theorem 2, the nonlocal coupled system for (k,φ)-Hilfer fractional differential Equation (38) with f, f_1 defined by (43) and (44), respectively, has at least one solution on $[(1/7),(10/7)]$.

5. Conclusions

In this paper, we presented the existence and uniqueness criteria for the solutions of a system of (k,φ)-Hilfer fractional differential equation complemented with nonlocal multi-point boundary conditions. The given nonlinear problem was converted into a fixed-point problem via an auxiliary lemma concerning a linear variant of the problem. Then we first proved the existence of a unique solution via Banach contraction mapping principle, and next we established two existence results by applying the Leray–Schauder alternative and Krasnosel'skiĭ's fixed-point theorem. Numerical examples were constructed to illustrate the obtained results. Our results are new in the given configuration and enrich the literature on coupled systems for (k,φ)-Hilfer fractional differential equations of the order $(1,2]$.

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