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Barrier Option Pricing in the Sub-Mixed Fractional Brownian Motion with Jump Environment

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Abstract: This paper investigates the pricing formula for barrier options where the underlying asset is driven by the sub-mixed fractional Brownian motion with jump. By applying the corresponding Itô’s formula, the B-S type PDE is derived by a self-financing strategy. Furthermore, the explicit pricing formula for barrier options is obtained through converting the PDE to the Cauchy problem. Numerical experiments are conducted to test the impact of the barrier price, the Hurst index, the jump intensity and the volatility on the value of barrier option respectively.

Keywords: barrier options; sub-mixed fractional Brownian motion; jump diffusion

1. Introduction

Barrier option is a path-dependent exotic option, whose value depends not only on the price of the underlying asset, but also on whether the price of the underlying asset touches the preset barrier price within the effective execution period of the option. For its cheaper premiums against the corresponding vanilla options, barrier options can be seen everywhere in global exchanges and over-the-counter markets. Many companies use various barrier options to hedge risks. In addition, the studies on barrier option pricing can also promote the research of many structured financial products, such as convertible bonds, bank-triggered financial products and so on. For the above reasons, the pricing of barrier options has always been a topical issue [1–4]. If the option right terminates (starts) when the underlying asset price touches the given barrier price, it is called a knock-out (in) kind; it is called a down (up) option, if the initial underlying asset price is above (below) the barrier price [5]. Therefore, single-barrier options which this paper discusses include eight types: down (up)-and-out (in) call (put) options.

In 1973, Merton [6] gave the closed solution of down-and-out European call options. Later, Reiner and Rubinstein [7] extended the pricing formulas of other European barrier options in 1991. However, these studies are under the Black–Scholes model [8] (the B-S model) which assumes the underlying asset price follows the logarithmic normal distribution. However, in recent years the self-similarity and long-range dependence has been found in the financial asset through numbers of the financial empirical studies [9,10], which is inconsistent with the B-S model. Then, Necula [11] studied the extended B-S model, where the assets price is driven by the fractional Brownian motion (fBm) instead of the Brownian motion. The fBm was first proposed by Kolmogorov [12], which exhibits self-similarity and long-range dependence. Since then, a volume of research on option-pricing models with fBm have been conducted, such as [13–15].

However, the fBm is neither a Markov process nor a semi-martinale, except degenerating into the Brownian motion. Although we can use Wick-self-financing strategies to analyze the fBm [16,17], Björk and Hult [18] found the application of the fBm has little economic sense, which limited its applicability in financial market. Therefore, other
processes are proposed to describe the fluctuation of financial assets, such as the sub-fractional Brownian motion (sub-fBm) [19] and the sub-mixed fractional Brownian motion (sub-mixed fBm) [20].

The sub-fBm preserves most properties of the fBm, but it has the characteristics of non-stationary second-order moment increment and faster convergence [21]. Moreover, the sub-mixed fBm is a combination of the Brownian motion and the sub-fBm. When the Hurst index $H \in [0.75, 1)$, the sub-mixed fBm becomes a semi martingale, which is equivalent to the Brownian motion [22]. At the same time, inspired by Merton [23] and other recent research [24–26], this paper introduces the jump diffusion process to describe the jump points of asset price caused by unsystematic risk factors, which is usually ignored in the pricing of barrier options. The purpose of this paper is to obtain the pricing formula of barrier options where the underlying asset is driven by the sub-mixed fBm and the compensated Poisson process.

The remainder of this paper is organized as follows: In Section 2, some necessary preliminary knowledge about the sub-fBm will be presented. In Section 3, we obtain the processes are proposed to describe the fluctuation of financial assets, such as the sub-mixed fBm [20].

1. There are two kinds of assets in the financial market: risk-free assets (bonds) and risky assets (stocks).

2. Preliminaries

Let $\{\Omega, \mathcal{F}, \mathbb{P}\}$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions.

Definition 1. The sub-mixed fBm $\xi_t^H = \{\xi_t^H(\alpha, \beta)\}_{t \geq 0}$ is a linear combination of the Brownian motion $\{B_t\}_{t \geq 0}$ and the sub-fBm $\{B_t^H\}_{t \geq 0}$ which can be expressed as:

$$\xi_t^H(\alpha, \beta) = \alpha B_t + \beta B_t^H, \forall t \geq 0,$$

where $H$ is the Hurst index, $\alpha$ and $\beta$ are positive constants, $\{B_t\}_{t \geq 0}$ and $\{B_t^H\}_{t \geq 0}$ are independent of each other.

Lemma 1. The sub-mixed fBm $\xi_t^H = \{\xi_t^H(\alpha, \beta)\}_{t \geq 0}$ has the following properties [20]:

1. $\{\xi_t^H(\alpha, \beta)\}_{t \geq 0}$ is a central Gaussian process.
2. When $t = 0$, $\xi_0^H(\alpha, \beta) = \alpha B_0 + \beta B_0^H = 0$.
3. $\forall t, s \geq 0$, the covariance of $\xi_t^H(\alpha, \beta)$ and $\xi_s^H(\alpha, \beta)$ is

$$\text{Cov}(\xi_t^H(\alpha, \beta), \xi_s^H(\alpha, \beta)) = a^2(t \wedge s) + \frac{\beta^2}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right),$$

where $t \wedge s = \frac{1}{2}(t + s - |t-s|)$.
4. $\forall t \geq 0$, $E \left( (\xi_t^H(\alpha, \beta))^2 \right) = a^2 t + \beta^2 (2 - 2^{2H-1}) t^{2H}$.

3. Asset Pricing Model

In this paper, the following assumptions are hold:

1. There are two kinds of assets in the financial market: risk-free assets (bonds) and risky assets (stocks).
2. The stock price $S_t$ is driven by the sub-mixed fBm with jump:

$$dS_t = (\mu - q) S_t dt + S_t d\xi_t^H(\alpha, \beta) + \gamma S_t dJ_t$$

$$= (\mu - q) S_t dt + \alpha S_t dB_t + \beta S_t dB_t^H + \gamma S_t dJ_t,$$  \hspace{1cm} (1)
Assume that $Y_t = \zeta_t^H (\alpha, \beta) + \gamma_t$ with the initial value zero, and $f(t, Y_t)$ is second-order differentiable. Then, the Itô’s formula of the sub-mixed fBm with jump can be expressed as follows:

\[
\begin{align*}
  f(t, Y_t) &= f(0, 0) + \int_0^t \left[ \frac{\partial f}{\partial s} - \lambda \gamma \frac{\partial f}{\partial Y} \right] ds + \int_0^t \left[ \frac{\alpha^2}{2} + \left( 2 - 2^{2H-1} \right) H \beta^2 s^{2H-1} \right] \frac{\partial^2 f}{\partial Y^2} ds \\
  &\quad + \alpha \int_0^t \frac{\partial f}{\partial Y} dB_s + \beta \int_0^t \frac{\partial f}{\partial Y} dB^{Ht}_s + \sum_{s \leq t} \left[ f(s, Y_s) - f(s-, Y_{s-}) \right] \\
  &= f(0, 0) + \int_0^t \left[ \frac{\partial f}{\partial s} - \lambda \gamma \frac{\partial f}{\partial Y} \right] ds + \int_0^t \left[ \frac{\alpha^2}{2} + \left( 2 - 2^{2H-1} \right) H \beta^2 s^{2H-1} \right] \frac{\partial^2 f}{\partial Y^2} ds \\
  &\quad + \alpha \int_0^t \frac{\partial f}{\partial Y} dB_s + \beta \int_0^t \frac{\partial f}{\partial Y} dB^{Ht}_s + \gamma \int_0^t \frac{\partial f}{\partial Y} dJ_s.
\end{align*}
\]

**Proof.** According to the sub-mixed fBm Itô’s formula [20] and the jump process analysis method [27], we have

\[
\begin{align*}
  f(t, Y_t) &= f(0, 0) + \int_0^t \frac{\partial f}{\partial s} ds + \int_0^t \frac{\partial f}{\partial s} dY^c_s + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial s^2} (dY^c_s)^2 + \sum_{s \leq t} \left[ f(s, Y_s) - f(s-, Y_{s-}) \right] \\
  &= f(0, 0) + \int_0^t \left[ \frac{\partial f}{\partial s} - \lambda \gamma \frac{\partial f}{\partial Y} \right] ds + \int_0^t \left[ \frac{\alpha^2}{2} + \left( 2 - 2^{2H-1} \right) H \beta^2 s^{2H-1} \right] \frac{\partial^2 f}{\partial Y^2} ds \\
  &\quad + \alpha \int_0^t \frac{\partial f}{\partial Y} dB_s + \beta \int_0^t \frac{\partial f}{\partial Y} dB^{Ht}_s + \sum_{s \leq t} \left[ f(s, Y_s) - f(s-, Y_{s-}) \right].
\end{align*}
\]

The following identities are used:

\[
\begin{align*}
  dY^c_t &= \alpha dB_t + \beta dB^{Ht}_t - \lambda \gamma dt, \\
  (dY^c_t)^2 &= \left[ \alpha^2 + 2 \left( 2 - 2^{2H-1} \right) H \beta^2 t^{2H-1} \right] dt
\end{align*}
\]

and $Y_t^c = \alpha B_t + \beta B^{Ht}_t - \lambda \gamma t$ is the continuous part of $Y_t$.

If $g(x)$ is second order differentiable. Given that Poisson process $\{N_t\}_{t \geq 0}$ with intensity $\lambda$ has the second-order moment increments $< dN_t, dN_t > = \lambda dt$, by generalized Itô’s formula we obtain

\[
\sum_{s \leq t} \left[ g(N_s) - g(N_{s-}) \right] = \int_0^t \frac{\partial g}{\partial N} dN_s + \frac{\lambda}{2} \int_0^t \frac{\partial^2 g}{\partial N^2} ds.
\]

Combining $Y_t = \zeta_t^H (\alpha, \beta) + \gamma N_t = \alpha B_t + \beta B^{Ht} + \gamma N_t - \lambda \gamma t$, we arrive at

\[
\sum_{s \leq t} \left[ f(s, Y_s) - f(s, Y_{s-}) \right] = \gamma \int_0^t \frac{\partial f}{\partial Y} dN_s + \frac{\lambda \gamma^2}{2} \int_0^t \frac{\partial^2 f}{\partial Y^2} ds.
\]
Substitute (4) back to (3), which yields
\[
\begin{align*}
f(t, Y_t) &= f(0, 0) + \int_0^t \left\{ \frac{\partial f}{\partial t} + \left[ \frac{\alpha^2}{2} + \frac{\lambda \gamma^2}{2} + \left( 2 - 2^{2H-1} \right) H B^2 t^{2H-1} \right] \frac{\partial^2 f}{\partial Y^2} \right\} ds \\
& \quad + \alpha \int_0^t \frac{\partial f}{\partial Y} dB_s + \beta \int_0^t \frac{\partial f}{\partial Y} dB^H_s + \gamma \int_0^t \frac{\partial f}{\partial Y} d\gamma_t.
\end{align*}
\]

\[\square\]

**Theorem 2.** The stock price satisfying (1) has the following explicit solution:
\[
S_t = S_0 \exp \left\{ (\mu - q)t - \left[ \left( \frac{\alpha^2}{2} + \frac{\lambda \gamma^2}{2} \right) t + \left( 1 - 2^{2H-2} \right) \beta^2 t^{2H} \right] + \alpha B_t + \beta B^H_t + \gamma J_t \right\}.
\]

**Proof.** Let \( f(t, Y_t) = S_0 \exp \left\{ (\mu - q)t - \left[ \left( \frac{\alpha^2}{2} + \frac{\lambda \gamma^2}{2} \right) t + \left( 1 - 2^{2H-2} \right) \beta^2 t^{2H} \right] + Y_t \right\}. \)

An application of Theorem 1 yields
\[
\frac{\partial f}{\partial t} + \left[ \frac{\alpha^2}{2} + \frac{\lambda \gamma^2}{2} + \left( 2 - 2^{2H-1} \right) H B^2 t^{2H-1} \right] \frac{\partial^2 f}{\partial Y^2} dt + \frac{\partial f}{\partial Y} dY_t,
\]
\[
= (\mu - q)f(t, Y_t) dt + f(t, Y_t) dY_t,
\]
\[
\quad = (\mu - q)f(t, Y_t) dt + f(t, Y_t) d\xi^H_t(\alpha, \beta) + \gamma f(t, Y_t) dJ_t,
\]
where
\[
\frac{\partial f}{\partial t} = \left( \mu - q \right) \left[ \left( \frac{\alpha^2}{2} + \frac{\lambda \gamma^2}{2} \right) t + \left( 1 - 2^{2H-2} \right) \beta^2 t^{2H} \right] f(t, Y_t),
\]
\[
\frac{\partial f}{\partial Y} = f(t, Y_t) \text{ and } \frac{\partial^2 f}{\partial Y^2} = f(t, Y_t).
\]

Comparing (1) and (5), we can deduce \( dS_t = df(t, Y_t) \), where the values are the same \( f(0, Y_0) = S_0 \). Therefore,
\[
S_t = f(t, Y_t)
\]
\[
= S_0 \exp \left\{ (\mu - q)t - \left[ \left( \frac{\alpha^2}{2} + \frac{\lambda \gamma^2}{2} \right) t + \left( 1 - 2^{2H-2} \right) \beta^2 t^{2H} \right] + \alpha B_t + \beta B^H_t + \gamma J_t \right\}.
\]

\[\square\]

4. Pricing Formula for Barrier Options

With the explicit solution of the stock price \( S_t \) in hand, in this section the pricing formula for barrier options can be derived.

**Theorem 3.** Assuming that the underlying asset price \( S_t \) follows (1), then the value of contingent claims \( V_t = V(t, S_t) \) satisfies the following PDE:
\[
\frac{\partial V}{\partial t} + (r - q)S_t \frac{\partial V}{\partial S} + \left[ \frac{\alpha^2}{2} + \frac{\lambda \gamma^2}{2} + \left( 2 - 2^{2H-1} \right) H B^2 t^{2H-1} \right] S_t \frac{\partial^2 V}{\partial S^2} - rV_t = 0.
\]

**Proof.** Using the self-financing strategy \( \theta_t = (\theta^1_t, \theta^2_t) \), we hold a number of \( \theta^1_t \) bonds and \( \theta^2_t \) stocks to build the wealth process, whose value at time \( t \) is
\[
V_t = \theta^1_t M_t + \theta^2_t S_t, \quad (6)
\]
Combining (1) and (2), we obtain
\[
dV_t = \theta_1^T dM_t + \theta_2^T dS_t + \theta_3^T \sigma S_t dt
\]
\[
= \left( r \theta_1^T M_t + \mu \theta_2^T S_t \right) dt + \theta_3^T \left( \sigma dB_t + \beta dB_t^H + \gamma dH_t \right).
\]

(7)

At the same time, by applying Theorems 1 and 2, we have
\[
dV_t = \frac{\partial V_t}{\partial t} dt + \frac{\partial V_t}{\partial S_t} dS_t + \frac{1}{2} \frac{\partial^2 V_t}{\partial S_t^2} (dS_t)^2
\]
\[
= \left\{ \frac{\partial V_t}{\partial t} + (\mu - q) S_t \frac{\partial V_t}{\partial S_t} + \left[ \frac{\alpha^2}{2} + \frac{\lambda \gamma^2}{2} + \left( 2 - 2H^{-1} \right) H \beta^2 \iota^2H^{-1} \right] S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} \right\} dt + S_t \frac{\partial V_t}{\partial S_t} \left( \sigma dB_t + \beta dB_t^H + \gamma dH_t \right),
\]
where \((dS_t)^2 = S_t^2 \left[ (\alpha^2 + \lambda \gamma^2) + 2 \left( 2 - 2H^{-1} \right) H \beta^2 \iota^2H^{-1} \right] dt.
\]
Combining (7) and (8), \(\theta_1^T\) and \(\theta_2^T\) are given
\[
\left\{ \begin{array}{l}
\theta_1^T = (r M_t)^{-1} \left\{ \frac{\partial V_t}{\partial t} - q S_t \frac{\partial V_t}{\partial S_t} + \left[ \frac{\alpha^2}{2} + \frac{\lambda \gamma^2}{2} + \left( 2 - 2H^{-1} \right) H \beta^2 \iota^2H^{-1} \right] S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} \right\} \\
\theta_2^T = \frac{\partial V_t}{\partial S_t}
\end{array} \right.
\]
\]
\]
(9)

From (6), we obtain
\[
\theta_1^T = \frac{V_t - \theta_2^T S_t}{M_t}.
\]
(10)
Combining (9) and (10), Theorem 3 is proved.

Theorem 4. Suppose that the underlying asset price \(S_t\) satisfies (1), then at time \(t\) the value of the down-and-out call option \(C_{do}(t, S_t)\) with the fixed strike price \(K\), the fixed barrier \(B\) and the maturity time \(T\) is given
\[
C_{do}(t, S_t) = S_t e^{-q(T-t)} N(d_1) - Ke^{-r(T-t)} N(d_2)
\]
\[
- \left( \frac{S_t}{B} \right)^{\kappa(t)} \left[ \frac{B^2}{S_t} e^{-q(T-t)} N(d_3) - Ke^{-r(T-t)} N(d_4) \right].
\]
The following identities are used:
\[
N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt, \text{which denotes the cumulative probability of standard normal distribution;}
\]
\[
d_1 = \frac{\ln \frac{S_t}{K} + \left( r - q + \frac{\alpha^2}{2} + \frac{\lambda \gamma^2}{2} \right)(T-t) + (1 - 2H^{-2}) \beta^2 (2H^2 - H^2)}{\sqrt{(\alpha^2 + \lambda \gamma^2)(T-t) + (2 - 2H^{-1}) \beta^2 (2H^2 - H^2)}};
\]
\[
d_2 = d_1 - \sqrt{(\alpha^2 + \lambda \gamma^2)(T-t) + (2 - 2H^{-1}) \beta^2 (2H^2 - H^2)};
\]
\[
d_3 = \frac{\ln \frac{B^2}{S_t} + \left( r - q + \frac{\alpha^2}{2} + \frac{\lambda \gamma^2}{2} \right)(T-t) + (1 - 2H^{-2}) \beta^2 (2H^2 - H^2)}{\sqrt{(\alpha^2 + \lambda \gamma^2)(T-t) + (2 - 2H^{-1}) \beta^2 (2H^2 - H^2)}};
\]
\[
d_4 = d_3 - \sqrt{(\alpha^2 + \lambda \gamma^2)(T-t) + (2 - 2H^{-1}) \beta^2 (2H^2 - H^2)};
\]
\[
\kappa(t) = 1 - \frac{2(r-q)(T-t)}{(\alpha^2 + \lambda \gamma^2)(T-t) + (2 - 2H^{-1}) \beta^2 (2H^2 - H^2)}.\]
Proof. Let \( V(t, S_t) = C_{do}(t, S_t) = C_{do} \). Then according to Theorem 3, the value of the down-and-out call option \( C_{do}(t, S_t) \) is given by

\[
\frac{\partial C_{do}}{\partial t} + (r - q)S_t \frac{\partial C_{do}}{\partial S} + \left[ \frac{\alpha^2}{2} + \frac{\lambda \gamma^2}{2} + (2 - 2^{2H-1})H \beta^2 2^H \right] \frac{\partial^2 C_{do}}{\partial S^2} - r C_{do} = 0,
\]

with the initial condition \( C_{do}(T, S_T) = (S_T - K)^+ \), \( B < S_t < +\infty \), and the boundary condition \( C_{do}(t, B) = 0 \), \( 0 \leq t \leq T \).

Let

\[
x = \ln \frac{S_t}{B}, \quad C_{do}(t, S_t) = B \hat{C}(t, x).
\]

Then,

\[
\frac{\partial C_{do}}{\partial t} = B \frac{\partial \hat{C}}{\partial t}, \quad \frac{\partial C_{do}}{\partial S} = B \frac{\partial \hat{C}}{\partial x} \frac{\partial x}{\partial S} = B \frac{\partial \hat{C}}{\partial x} \text{ and } \frac{\partial^2 C_{do}}{\partial S^2} = \frac{B}{S_t^2} \left[ \frac{\partial^2 \hat{C}}{\partial x^2} - \frac{\partial \hat{C}}{\partial x} \right].
\]

Therefore, we can deduce that

\[
\frac{\partial \hat{C}}{\partial t} + (r - q) \frac{\partial \hat{C}}{\partial x} + \left[ \frac{\alpha^2}{2} + \frac{\lambda \gamma^2}{2} + (2 - 2^{2H-1})H \beta^2 2^H \right] \left( \frac{\partial^2 \hat{C}}{\partial x^2} - \frac{\partial \hat{C}}{\partial x} \right) - r \hat{C} = 0,
\]

with the initial condition \( \hat{C}(T, \ln \frac{S_T}{B}) = \left( e^x - \frac{K}{B} \right)^+ \), \( 0 < x < +\infty \), and the boundary condition \( \hat{C}(0, t) = 0 \), \( 0 \leq t \leq T \).

Next, let

\[
\omega(\tau, \eta) = \hat{C}(x, t)e^{b(t)}, \quad \tau = c(t), \quad \eta = x + a(t),
\]

where \( a(t), b(t) \) and \( c(t) \) are undetermined functions about \( t \), which are first order differentiable. Then, we derive

\[
\frac{\partial \hat{C}}{\partial t} = e^{-b(t)} \left[ a'(t) \frac{\partial \omega}{\partial \eta} + c'(t) \frac{\partial \omega}{\partial \tau} - b'(t) \omega \right], \quad \frac{\partial \hat{C}}{\partial x} = e^{-b(t)} \frac{\partial \omega}{\partial \eta} \frac{\partial \hat{C}}{\partial x} = e^{-b(t)} \frac{\partial^2 \omega}{\partial \eta^2}
\]

and

\[
c'(t) \frac{\partial \omega}{\partial \tau} + \sigma(t) \frac{\partial^2 \omega}{\partial \eta^2} + \left[ r - q + a'(t) - \sigma(t) \right] \frac{\partial \hat{C}}{\partial x} - \left[ r + b'(t) \right] \omega = 0,
\]

where \( \sigma(t) = \frac{\alpha^2}{2} + \frac{\lambda \gamma^2}{2} + (2 - 2^{2H-1})H \beta^2 2^H \). In order for the solution, let

\[
\begin{aligned}
c'(t) + \sigma(t) &= 0, \\
r - q + a'(t) - \sigma(t) &= 0, \\
r + b'(t) &= 0, \\
a(T) &= b(T) = c(T) = 0,
\end{aligned}
\]

(14)

to convert (13) into the heat equation.

From (14), \( a(t), b(t) \) and \( c(t) \) are given

\[
\begin{aligned}
a(t) &= \int_{t}^{T} r - q - c(s) ds = \left( r - q - \frac{\alpha^2}{2} - \frac{\lambda \gamma^2}{2} \right) (T - t) - \left( 1 - 2^{2H-2} \right) \beta^2 \left( T^{2H} - t^{2H} \right), \\
b(t) &= \int_{t}^{T} r ds = r (T - t), \\
c(t) &= \left( \frac{\alpha^2}{2} + \frac{\lambda \gamma^2}{2} \right) (T - t) + \left( 1 - 2^{2H-2} \right) \beta^2 \left( T^{2H} - t^{2H} \right).
\end{aligned}
\]

(15)
Substitute (15) into (13), then the value of the down-and-out call option $C_{d}(t, S_{t})$ is given by
\[
\frac{\partial \omega}{\partial \tau} = \frac{\partial^{2} \omega}{\partial \eta^{2}},
\]
with the initial condition $\omega(0, \eta) = (e^{\eta} - K)^{+}$, \(0 < \eta < +\infty\), and the boundary condition $\omega(\tau, a(t)) = 0$, \(0 \leq t \leq T\).

Notice that if we just consider PDE with its initial condition, the solution can be obtained by the Poisson formula
\[
\omega(\tau, \eta) = \frac{1}{2\sqrt{\pi \tau}} \int_{-\infty}^{+\infty} \varphi(z) e^{-\frac{(z-\eta)^2}{4\tau}} dz.
\]

To deal with the boundary condition, let $G(z) = \varphi(z) e^{-\frac{(z-\eta)^2}{4\tau}}$ when $z > 0$. Then we extend $G(z)$ to become an odd function in the whole real number field
\[
G(z) = \begin{cases} 
\varphi(z) e^{-\frac{(z-\eta)^2}{4\tau}}, & z > 0, \\
-\varphi(-z) e^{-\frac{(z+\eta)^2}{4\tau}}, & z \leq 0.
\end{cases}
\]

Comparing the above equation and the original initial condition in (16), we obtain extended initial condition which contains the boundary condition
\[
\varphi(z) = \begin{cases} 
(e^{z} - K)^{+}, & z > 0, \\
-(e^{-z} - \frac{e^{\eta}}{K})^{+} e^{-\frac{4\eta^2}{\tau}} z, & z \leq 0.
\end{cases}
\]

Therefore, (16) will be transformed into a Cauchy problem
\[
\frac{\partial \omega}{\partial \tau} = \frac{\partial^{2} \omega}{\partial \eta^{2}},
\]
with the initial condition $\omega(0, \eta) = \varphi(\eta)$, \(-\infty < \eta < +\infty\).

According to (17),
\[
\omega(\tau, \eta) = \frac{1}{2\sqrt{\pi \tau}} \int_{-\infty}^{+\infty} \varphi(z) e^{-\frac{(z-\eta)^2}{4\tau}} dz
\]
\[
= \frac{1}{2\sqrt{\pi \tau}} \int_{\ln \frac{K}{B}}^{+\infty} (e^{z} - \frac{K}{B}) e^{-\frac{(z-\eta)^2}{4\tau}} dz - \frac{1}{2\sqrt{\pi \tau}} \int_{-\infty}^{-\ln \frac{K}{B}} (e^{-z} - \frac{K}{B}) e^{-\frac{(z+\eta)^2}{4\tau}} dz
\]
\[
= \frac{1}{2\sqrt{\pi \tau}} \int_{\ln \frac{K}{B}}^{+\infty} e^{z} e^{-\frac{(z-\eta)^2}{4\tau}} dz - \frac{1}{2\sqrt{\pi \tau}} \int_{0}^{+\infty} e^{-\frac{(z+\eta)^2}{4\tau}} dz
\]
\[
= I_{2} + I_{3} + I_{4}.
\]

For convenience, define $N(x)$ as follows: $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$, which represents the cumulative probability of standard normal distribution.

For $I_{1}$,
\[
I_{1} = \frac{1}{2\sqrt{\pi \tau}} \int_{\ln \frac{K}{B}}^{+\infty} e^{z} e^{-\frac{(z-\eta)^2}{4\tau}} dz = e^{\tau+\eta} \frac{1}{2\sqrt{\pi \tau}} \int_{\ln \frac{K}{B}}^{+\infty} e^{-\frac{(z-\eta)^2}{4\tau}} dz.
\]
Let \( t = \frac{z - \eta - 2\tau}{\sqrt{2\tau}} \), then
\[
I_1 = e^{t + \eta} \frac{1}{\sqrt{2\pi}} \int_{\frac{\eta}{\sqrt{2\tau}}}^{+\infty} e^{-\frac{z^2}{2\tau}} dz = e^{t + \eta} N(d_1),
\]
where \( d_1 = \eta + 2\tau - \ln \frac{K}{B}. \)

Similarly, let \( t = \frac{z - \eta}{\sqrt{2\tau}} \) and obtain
\[
I_2 = \frac{1}{\sqrt{2\pi}} \int_{\frac{\eta}{\sqrt{2\tau}}}^{+\infty} e^{-\frac{(z-\eta)^2}{2\tau}} dz = -\frac{K}{B} \frac{1}{\sqrt{2\pi}} \int_{\frac{\eta}{\sqrt{2\tau}}}^{+\infty} e^{-\frac{z^2}{2\tau}} dz = -\frac{K}{B} N(d_2),
\]
where \( d_2 = \frac{\eta - \ln \frac{K}{B}}{\sqrt{2\tau}} = d_1 - \sqrt{2\tau}. \)

For \( I_3 \),
\[
I_3 = \frac{1}{\sqrt{2\pi}} \int_{\frac{z+\eta}{\sqrt{2\tau}}}^{+\infty} e^{-\frac{(z+\eta)^2}{2\tau}} dz = -e^{\eta} \frac{1}{\sqrt{2\pi}} \int_{\frac{\eta+\eta}{\sqrt{2\tau}}}^{+\infty} e^{-\frac{z^2}{2\tau}} dz = -e^\eta N(d_3),
\]
where \( d_3 = \frac{2a(t) + 2\tau - \eta - \ln \frac{K}{B}}{\sqrt{2\tau}}. \)

For \( I_4 \), let \( t = \frac{z + \eta - 2a(t)}{\sqrt{2\tau}} \), we derive
\[
I_4 = \frac{1}{\sqrt{2\pi}} \int_{\frac{\eta}{\sqrt{2\tau}}}^{+\infty} e^{-\frac{(z+a(t))^2}{2\tau}} dz = -e^{\eta} \frac{1}{\sqrt{2\pi}} \int_{\frac{\eta+\eta}{\sqrt{2\tau}}}^{+\infty} e^{-\frac{z^2}{2\tau}} dz = -e^{\eta} N(d_4),
\]
where \( d_4 = \frac{2a(t) - \eta - \ln \frac{K}{B}}{\sqrt{2\tau}} = d_3 - \sqrt{2\tau}. \)

Substitute (11) and (12) in them, then
\[
I_1 = \frac{S_t}{B} e^{(r-q)(T-t)} N(d_1);
\]
\[
I_2 = -\frac{K}{B} N(d_2);
\]
\[
I_3 = -e^{\frac{(r-q)(T-t) - \ln \frac{S_t}{B}}{\sqrt{2\tau}}} N(d_3) = -\frac{S_t}{B} \frac{1}{S_t} \frac{1}{B} \frac{1}{N(d_3)} = -e^{(r-q)(T-t)} \left( \frac{S_t}{B} \right)^{1 - (r-q)(T-t)} N(d_3);
\]
\[
I_4 = \frac{K}{B} e^{\frac{(r-q)(T-t)(\ln \frac{S_t}{B})}{\sqrt{2\tau}}} N(d_4) = \frac{K}{B} \left( \frac{S_t}{B} \right)^{1 - (r-q)(T-t)} N(d_4).
\]
By combining $I_1$, $I_2$, $I_3$, and $I_4$, we have

$$C_{do}(t, S_t) = B\tilde{C}(t, x) = Be^{-r(T-t)}\omega(\tau, \eta) = Be^{-r(T-t)}(I_1 + I_2 + I_3 + I_4)$$

$$= S_t e^{-q(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$- \left( \frac{S_t}{B} \right)^{\kappa(t)} \left[ \frac{B^2}{S_t} e^{-q(T-t)}N(d_3) - Ke^{-r(T-t)}N(d_4) \right],$$

where $\kappa(t) = 1 - \frac{2(r-q)(T-t)}{(\alpha^2 + \lambda\gamma^2)(T-t) + (2 - 2^{2H-1})\beta^2(T^{2H} - t^{2H})}$. \quad \Box

**Theorem 5.** Suppose that the underlying asset price $S_t$ satisfies (1), then at time $t$ the following parity formula between the values of the down-and-out call option $C_{do}(t, S_t)$ and the down-and-out put option $P_{do}(t, S_t)$, if the options have the same fixed strike price $K$ and the same maturity time $T$:

$$C_{do}(t, S_t) + Ke^{-r(T-t)}N(d_6) \left[ N(d_6) - \left( \frac{S_t}{B} \right)^{\kappa(t)} N(d_8) \right] = P_{do}(t, S_t) + S_t e^{-q(T-t)}N(d_5) N(d_7)$$
where \( N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt \), which denotes the cumulative probability of standard normal distribution;

\[
ds_5 = \ln \frac{S_t}{B} + \left( r - q + \frac{\alpha^2}{2} + \frac{\lambda \gamma^2}{2} \right) (T - t) + \frac{1 - 2^{2H-2} \beta^2 (T^{2H} - t^{2H})}{\sqrt{\alpha^2 + \lambda \gamma^2}} (T - t) + (2 - 2^{2H-1}) \beta^2 (T^{2H} - t^{2H})
\]

\[
ds_6 = d_5 - \sqrt{(\alpha^2 + \lambda \gamma^2)} (T - t) + (2 - 2^{2H-1}) \beta^2 (T^{2H} - t^{2H})
\]

\[
ds_7 = \ln \frac{B}{S_t} + \left( r - q + \frac{\alpha^2}{2} + \frac{\lambda \gamma^2}{2} \right) (T - t) + \frac{1 - 2^{2H-2} \beta^2 (T^{2H} - t^{2H})}{\sqrt{\alpha^2 + \lambda \gamma^2}} (T - t) + (2 - 2^{2H-1}) \beta^2 (T^{2H} - t^{2H})
\]

\[
ds_8 = d_7 - \sqrt{(\alpha^2 + \lambda \gamma^2)} (T - t) + (2 - 2^{2H-1}) \beta^2 (T^{2H} - t^{2H})
\]

\[
\kappa(t) = 1 - \frac{2(r - q)(T - t)}{(\alpha^2 + \lambda \gamma^2)} (T - t) + (2 - 2^{2H-1}) \beta^2 (T^{2H} - t^{2H})
\]

Proof. Let \( W_{do}(t, S_t) = C_{do}(t, S_t) - P_{do}(t, S_t) \), which is the difference between the value of the down-and-out call option \( C_{do}(t, S_t) \) and the down-and-out put option \( P_{do}(t, S_t) \) at the moment of \( t \). Notice that \( W_{do}(t, S_t) \) satisfies the following PDE:

\[
\frac{\partial W_{do}}{\partial t} + (r - q)S_t \frac{\partial W_{do}}{\partial S} + \left[ \frac{\alpha^2}{2} + \frac{\lambda \gamma^2}{2} + (2 - 2^{2H-1}) 2 \beta^2 T^{2H-1} \right] S_t^2 \frac{\partial^2 W_{do}}{\partial S^2} - rW_{do} = 0,
\]

with the initial condition \( W_{do}(T, S_T) = S_T - K, \ B < S_t < +\infty, \) and the boundary condition \( W_{do}(t, B) = 0, \ 0 \leq t \leq T \).

By analogy with the solution procedure of (16), it can be obtained

\[
W_{do}(t, S_t) = S_t e^{-q(T-t)} N(d_5) - Ke^{-r(T-t)} N(d_6) - \left( \frac{S_t}{B} \right)^{\kappa(t)} \left[ \frac{B^2}{S_t^2} e^{-q(T-t)} N(d_7) - Ke^{-r(T-t)} N(d_8) \right].
\]

Combining the above equation and (19), Theorem 5 is proved.

\[\square\]

**Theorem 6.** Suppose that the underlying asset price \( S_t \) satisfies (1), then at time \( t \) the value of the down-and-out put option \( P_{do}(t, S_t) \) with the fixed strike price \( K \), the fixed barrier \( B \) and the maturity time \( T \) is

\[
P_{do}(t, S_t) = S_t e^{-q(T-t)} [N(d_1) - N(d_5)] - Ke^{-r(T-t)} [N(d_2) - N(d_6)]
\]

\[
- \left( \frac{S_t}{B} \right)^{\kappa(t)} \left\{ \frac{B^2}{S_t^2} e^{-q(T-t)} [N(d_3) - N(d_7)] - Ke^{-r(T-t)} [N(d_4) - N(d_8)] \right\},
\]

where \( N(x), d_1 \sim d_8 \) and \( \kappa(t) \) are given in Theorems 4 and 5.

Proof. By combining Theorems 4 and 5, Theorem 6 is easily proved.

\[\square\]

**Theorem 7.** Suppose that the underlying asset price \( S_t \) follows Equation (1), the maturity date is \( T \), the fixed strike price is \( K \) and the fixed barrier is \( B \), and then the value of the down-and-in call option \( C_{di}(t, S_t) \) and the value of the down-and-in put option \( P_{di}(t, S_t) \) at time \( t \) are given respectively:
\[ C_{dl}(t, S_t) = \left( \frac{S_t}{B} \right)^{\kappa(t)} \left[ \frac{B^2}{\alpha} e^{-q(T-t)} N(d_3) - Ke^{-r(T-t)} N(d_4) \right], \]
\[ P_{di}(t, S_t) = Ke^{-r(T-t)} N(-d_5) - S_t e^{-q(T-t)} N(-d_3) + \left( \frac{S_t}{B} \right)^{\kappa(t)} \left\{ \frac{B^2}{\alpha} e^{-q(T-t)} [N(d_3) - N(d_7)] - Ke^{-r(T-t)} [N(d_4) - N(d_8)] \right\}, \]

where \( N(x) \), \( d_3 \sim d_8 \) and \( \kappa(t) \) are detailed in Theorems 4 and 5.

**Proof.** When other conditions are the same, a portfolio with a put option and the corresponding option in will always be able to exercise one of their option right, which is equivalent to a vanilla option

\[ V_{\text{vanilla}}(t, S_t) = V_{\text{do}}(t, S_t) + V_{\text{di}}(t, S_t) = V_{\text{uo}}(t, S_t) + V_{\text{ui}}(t, S_t), \]

where \( V_{\text{vanilla}}(t, S_t) \) is the European option, \( V_{\text{do}}(t, S_t) \), \( V_{\text{di}}(t, S_t) \), \( V_{\text{uo}}(t, S_t) \) and \( V_{\text{ui}}(t, S_t) \) respectively denote the corresponding value of the down-and-out option, the down-and-in option, the up-and-out option and the up-and-in option.

Therefore, \( C_{dl}(t, S_t) = C_{\text{vanilla}}(t, S_t) - C_{do}(t, S_t); P_{di}(t, S_t) = P_{\text{vanilla}}(t, S_t) - P_{do}(t, S_t) \).

Using Corollary 1, Corollary 2, Theorems 4 and 6, Theorem 7 is proved.

Above all, the pricing formulas of all four types of downward barrier options have been given. Similarly, the pricing formulas corresponding to four types of upward barrier options can be deduced.

5. Numerical Experiment

In this section, numerical experiments are conducted to discuss the effects of the barrier price \( B \), the Hurst index \( H \), the jump intensity \( \lambda \) and volatility \( \alpha, \beta, \gamma \) on barrier options by MATLAB and R language software. In this section, we just take the down-and-out call option as an example for space constraints.

Firstly, parameters are assumed as follows:

\[ t = 0, \ T = 0.5, \ K = 100, \ H = 0.75, \ \alpha = \beta = \gamma = 0.4, \ \lambda = 1. \]

According to Theorem 4, the value of down-and-out call option \( C_{do}(t, S_t) \) under different barrier prices and stock prices can be obtained, which are given in Table 1.

<table>
<thead>
<tr>
<th>( B )</th>
<th>( S_0 = 120 )</th>
<th>( S_0 = 110 )</th>
<th>( S_0 = 100 )</th>
<th>( S_0 = 90 )</th>
<th>( S_0 = 80 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>115</td>
<td>6.350</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>110</td>
<td>12.112</td>
<td>0.000</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>100</td>
<td>22.445</td>
<td>11.591</td>
<td>0.000</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>95</td>
<td>26.969</td>
<td>16.541</td>
<td>5.534</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>90</td>
<td>31.042</td>
<td>20.926</td>
<td>10.551</td>
<td>0.000</td>
<td>–</td>
</tr>
<tr>
<td>85</td>
<td>34.654</td>
<td>24.731</td>
<td>15.095</td>
<td>5.086</td>
<td>–</td>
</tr>
<tr>
<td>80</td>
<td>37.803</td>
<td>27.956</td>
<td>19.149</td>
<td>9.510</td>
<td>0.000</td>
</tr>
<tr>
<td>75</td>
<td>40.495</td>
<td>30.613</td>
<td>22.702</td>
<td>13.264</td>
<td>4.507</td>
</tr>
<tr>
<td>70</td>
<td>42.748</td>
<td>32.730</td>
<td>25.757</td>
<td>16.351</td>
<td>8.275</td>
</tr>
<tr>
<td>65</td>
<td>44.587</td>
<td>34.352</td>
<td>28.324</td>
<td>18.798</td>
<td>11.321</td>
</tr>
<tr>
<td>60</td>
<td>46.049</td>
<td>35.533</td>
<td>30.430</td>
<td>20.653</td>
<td>13.680</td>
</tr>
</tbody>
</table>
Observing Table 1 and Figure 1, it can be seen that when the stock price is fixed, the value of down-and-out call option decreases with the growth of barrier price. When other conditions remain unchanged, with the rising barrier price, the possibility of down-and-out call option termination is increasing, so the option value will continue to decline. In particular, when the barrier price increases to the initial stock price, the option will be knocked out at once, which means it has no value any more.

Then, in order to discuss the impact of the Hurst index \( H \) and the jump intensity \( \lambda \) on the option price, a new hypothesis is proposed as follows:

\[
\begin{align*}
& t = 0, \quad T = 0.5, \quad K = 100, \quad B = 70, \quad \alpha = \beta = \gamma = 0.4. \\
& \lambda = 0, \quad \lambda = 2, \quad \lambda = 4, \quad \lambda = 0, \quad \lambda = 2, \quad \lambda = 4, \quad \lambda = 0, \quad \lambda = 2, \quad \lambda = 4.
\end{align*}
\]

Take the different \( H, \lambda \), and other assumptions remain unchanged to obtain the option value under various conditions, as shown in Table 2.

Table 2. The value of down-and-out option for different Hurst index and jump intensity.

<table>
<thead>
<tr>
<th>( S_0 )</th>
<th>( H = 0.75 )</th>
<th>( H = 0.85 )</th>
<th>( H = 0.95 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda = 0 )</td>
<td>31.119</td>
<td>35.812</td>
<td>39.018</td>
</tr>
<tr>
<td>( \lambda = 2 )</td>
<td>36.621</td>
<td>39.501</td>
<td>39.045</td>
</tr>
<tr>
<td>( \lambda = 4 )</td>
<td>40.017</td>
<td>28.472</td>
<td>39.018</td>
</tr>
<tr>
<td>( \lambda = 0 )</td>
<td>29.761</td>
<td>31.879</td>
<td>31.141</td>
</tr>
<tr>
<td>( \lambda = 2 )</td>
<td>35.812</td>
<td>35.360</td>
<td>34.910</td>
</tr>
<tr>
<td>( \lambda = 4 )</td>
<td>39.017</td>
<td>24.500</td>
<td>30.857</td>
</tr>
</tbody>
</table>

Figure 2 is the variation diagram of the value of down-and-out call option with the different Hurst index and jump intensity, when \( S_0 \) is fixed at 100. The relationships between the value of down-and-out option and the Hurst index is positive. The larger the Hurst index is, the more stable the underlying asset price is. This means the price fluctuation will be smaller, which denotes the corresponding option value will be smaller.

At the same time, the value of down-and-out option and the jump intensity change in the same direction. The jump intensity represents the unsystematic risk. When it increases, the underlying asset will has more intense fluctuations, which means higher upper limit and invariant lower bound. Therefore, the option value will rise.
Finally, for rigorousness, we verify the positive correlation between the volatility and the value of down-and-out call option, where $\alpha$, $\beta$, and $\gamma$ are different. Assume that the parameter selection is as follows:

$$t = 0, \ T = 0.5, \ K = 100, \ B = 70, \ H = 0.75, \ \lambda = 1.$$ 

Let $\hat{\sigma} = (\alpha, \beta, \gamma)$, and make

$$\hat{\sigma}_1 = (0.1, 0.15, 0.2); \ \hat{\sigma}_2 = (0.2, 0.25, 0.3); \ \hat{\sigma}_3 = (0.3, 0.35, 0.4); \ \hat{\sigma}_4 = (0.4, 0.45, 0.5).$$

According to Theorem 4, the value of down-and-out call option under different volatility can be obtained, which are shown in Table 3. The value of down-and-out call option increases with the rise of the volatility, which is consistent with the fact.

Table 3. The value of down-and-out option against the volatility of the underlying asset.

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>$\hat{\sigma}_1 = (0.1, 0.15, 0.2)$</th>
<th>$\hat{\sigma}_2 = (0.2, 0.25, 0.3)$</th>
<th>$\hat{\sigma}_3 = (0.3, 0.35, 0.4)$</th>
<th>$\hat{\sigma}_4 = (0.4, 0.45, 0.5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>120</td>
<td>25.323</td>
<td>28.441</td>
<td>32.360</td>
<td>36.149</td>
</tr>
<tr>
<td>115</td>
<td>20.918</td>
<td>24.467</td>
<td>28.506</td>
<td>32.092</td>
</tr>
<tr>
<td>105</td>
<td>12.963</td>
<td>17.176</td>
<td>21.222</td>
<td>24.559</td>
</tr>
<tr>
<td>100</td>
<td>9.581</td>
<td>13.917</td>
<td>17.807</td>
<td>20.867</td>
</tr>
<tr>
<td>95</td>
<td>6.704</td>
<td>10.948</td>
<td>14.549</td>
<td>17.259</td>
</tr>
<tr>
<td>90</td>
<td>4.385</td>
<td>8.281</td>
<td>11.443</td>
<td>13.729</td>
</tr>
<tr>
<td>85</td>
<td>2.637</td>
<td>5.910</td>
<td>8.477</td>
<td>10.267</td>
</tr>
<tr>
<td>80</td>
<td>1.416</td>
<td>3.803</td>
<td>5.623</td>
<td>6.856</td>
</tr>
<tr>
<td>75</td>
<td>0.608</td>
<td>1.886</td>
<td>2.838</td>
<td>3.467</td>
</tr>
</tbody>
</table>

6. Conclusions

This paper investigated the barrier option pricing model in the environment of the sub-mixed fractional Brownian motion with jump intensity. Through the self-financing strategy, we derive the B-S type PDE of the derivatives. Then, the value of the down-and-out option is obtained by applying transformation techniques. Meanwhile, the parity formula between barrier call option and barrier put option can be given by a similar method. Next, using the linear relationship between the knock-out option and the knock-in option, the value of the knock-in option can be deduced. In Section 5, the numerical experiment is carried out where we take the down-and-out call option as an example. According to the results shown in Figures 1 and 2, and Table 3, the following relationships can be found: The barrier
price and Hurst index are inversely related to the value of the down-and-out call option, while the jump intensity and volatility are positively correlated with it. In the following research, the Asian barrier options can be considered to extend the model in this paper.

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