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Article

Novel Approaches for Differentiable Convex Functions via the Proportional Caputo-Hybrid Operators

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Abstract: This study is built on the relationship between inequality theory and fractional analysis. Thanks to the new fractional operators and based on the proportional Caputo-hybrid operators, integral inequalities containing new approaches are obtained for differentiable convex functions. In the findings section, firstly, an integral identity is obtained and various integral inequalities are obtained based on this identity. The peculiarity of the results is that a hybrid operator has been used in inequality theory, which includes the derivative and integral operators together.

Keywords: Hadamard type inequalities; Caputo derivative; the proportional Caputo-hybrid operator

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1. Introduction

The definition of convex functions, which is one of the important concepts of inequality theory, has been the focus of attention of many researchers because of the inequality and the aesthetic structure it contains. The concept of inequalities and convexity have continued to exist in the literature as inseparable pairs. In [1], Pečarić et al. have presented a collection of classical and novel inequalities that can be widely applied in statistics, physics, engineering sciences, applied sciences, and numerical analysis, as well as mathematical analysis. Many famous inequalities such as Hermite-Hadamard, Ostrowski, Simpson, Minkowski, Jensen and Young inequality and their various versions have become motivational points in the work of researchers. In [2], the authors have proved some new Hadamard type inequalities for $m-$ and $(\alpha, m)-$functions. Further studies on $m-$convex functions that are general forms of the convexity and the star-shaped functions, were obtained by Dragomir and Toader in [3,4].

Another modification of convex functions is the concept of convexity in coordinates. We recommend the articles [5–10] to the readers for obtaining more details on the concept of coordinates convexity, the properties, and inclusions, different types of convexity in coordinates, and various Hadamard-type integral inequalities obtained for these function classes.

As in all branches of mathematics, fractional analysis, which has recently shown its effect in many fields such as physics, engineering sciences, mathematical biology, modeling, control theory, chaos theory and optimization and has brought new orientations to all of these fields, is based on the calculation of derivatives and integrals of arbitrary order. The main motivation point of fractional analysis, which has its origins as long as classical analysis, is to explain physical and mathematical phenomena with the help of operators of fractional order. While fractional analysis sometimes provides convenience in the solutions of real world problems, it sometimes causes serious optimization problems in the error amounts of the solutions. The feature that gives this movement to fractional calculus is the algebraic structures and properties of the new fractional derivative and associated integral operators. The kernel structures of fractional derivative and integral operators have differences in terms of features such as singularity, locality, and general form.

We present to your attention some studies [11–27], including the innovative and motivating findings of various researchers on inequalities, disease models, real world problems and so on, through new concepts of fractional calculus.

2. Preliminaries

Let's start with the definition of the concept of a convex function, which attracts attention in many fields such as statistics, convex programming, numerical analysis and approximation theory.

Definition 1 ([1])**.** *The function* $\Upsilon : [c_1, c_2] \to \Re$, *is said to be convex, if we have*

$$\Upsilon(\lambda r + (1-\lambda)s) \leq \lambda \Upsilon(r) + (1-\lambda)\Upsilon(s)$$

for all $r, s \in [c_1, c_2]$ *and* $\lambda \in [0,1]$.

Besides, the concept of convex function has many useful properties, it also forms the basis of the Hermite-Hadamard (HH) inequality, one of the well-known fundamental and famous inequalities in the literature. The HH inequality, which has the potential to produce lower and upper bounds to the mean value of a convex function in the Cauchy sense, has inspired many researchers in mathematical analysis with its applications. The statement of this inequality is as follows.

If a mapping $\Upsilon : J \subseteq \Re \to \Re$ is a convex function on J and $r, s \in J$, $r < s$, then

$$\Upsilon\left(\frac{r+s}{2}\right) \leq \frac{1}{s-r}\int_r^s \Upsilon(\lambda)d\lambda \leq \frac{\Upsilon(r)+\Upsilon(s)}{2}.$$

We recommend that readers refer to papers [2–10,28–31] for versions of the HH inequality for different kinds of convex functions, its modification to co-ordinates, and its expansions with the help of various new fractional integral operators.

The definition of the Riemann-Liouville fractional integral operator can be given as following (see [32]):

Definition 2. *Let* $\kappa \in L[\mu, \nu]$. *The Riemann-Liouville fractional integrals* $J^{\alpha}_{\mu^+}$ *and* $J^{\alpha}_{\nu^-}$ *of order* $\alpha > 0$ *with* $\mu \geq 0$ *are defined by*

$$J^{\alpha}_{\mu^+}\kappa(x) = \frac{1}{\Gamma(\alpha)}\int_{\mu}^{x}(x-\rho)^{\alpha-1}\kappa(\rho)d\rho, \ x > \mu$$

and

$$J^{\alpha}_{\nu^-}\kappa(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\nu}(\rho-x)^{\alpha-1}\kappa(\rho)d\rho, \ x < \nu$$

respectively, where $\Gamma(\alpha)$ *is the Gamma function defined by* $\Gamma(\alpha) = \int_0^{\infty} e^{-\rho}\rho^{\alpha-1}d\rho$ *and* $J^0_{\mu^+}\kappa(x) = J^0_{\nu^-}\kappa(x) = \kappa(x)$.

The following definition is very important for fractional calculus (see [33]).

Definition 3. *Let* $\alpha > 0$ *and* $\alpha \notin \{1, 2, 3, \ldots\}$, $n = [\alpha] + 1$, $f \in AC^n[a, b]$, *the space of functions having n-th derivatives absolutely continuous. The left-sided and right-sided Caputo fractional derivatives of order* α *are defined as follows:*

$$\left({}^{c}D^{\alpha}_{a^+}f\right)(x) = \frac{1}{\Gamma(n-\alpha)}\int\limits_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}dt \quad (x > a)$$

and

$$\left({}^{c}D^{\alpha}_{b^-}f\right)(x) = \frac{1}{\Gamma(n-\alpha)} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}}dt \qquad (x < b).$$

If $\alpha = n \in \{1, 2, 3, \dots\}$ and usual derivative $f^{(n)}(x)$ of order n exists, then Caputo fractional derivative $\left({}^{c}D^{\alpha}_{a^+}f\right)(x)$ coincides with $f^{(n)}(x)$ whereas $\left({}^{c}D^{\alpha}_{b^-}f\right)(x)$ coincides with $f^{(n)}(x)$ with exactness to a constant multiplier $(-1)^n$. In particular we have

$$\left({}^{c}D^{0}_{a^+}f\right)(x) = \left({}^{c}D^{0}_{b^-}f\right)(x) = f(x),$$

where $n = 1$ and $\alpha = 0$.

The proportional Caputo hybrid operator, which was put forward as a non-local and singular operator containing both derivative and integral operator parts in its definition, and which is a simple linear combination of the Riemann-Liouville integral and the Caputo derivative operators, is defined as follows (see [34]).

Definition 4. *Let $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I°. Also let f and f' are locally L^1 functions on I. Then, the proportional Caputo-hybrid operator may be defined as*

$${}^{CPC}_{0}D^{\alpha}_{t}f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \left(K_1(\alpha)f(\tau) + K_0(\alpha)f'(\tau)\right)(t-\tau)^{-\alpha}d\tau$$

where $\alpha \in [0, 1]$ and K_0 and K_1 are functions satisfing

$$\lim_{\alpha\to 0^+} K_0(\alpha) = 0; \quad \lim_{\alpha\to 1^-} K_0(\alpha) = 1; \quad K_0(\alpha) \neq 0, \quad \alpha \in (0, 1]; \tag{1}$$

$$\lim_{\alpha\to 0^+} K_1(\alpha) = 0; \quad \lim_{\alpha\to 1^-} K_1(\alpha) = 1; \quad K_1(\alpha) \neq 0, \quad \alpha \in [0, 1). \tag{2}$$

Remark 1 (See [34])**.** *We originally wrote this paper using the specific case*

$$\begin{aligned} K_0(\alpha, t) &= \alpha t^{1-\alpha} \\ K_1(\alpha, t) &= (1-\alpha)t^{\alpha} \end{aligned}$$

which is afforded special attention in [35].

The main purpose of this study is to obtain an integral identity that is likely to contribute to the field of inequalities with the help of the proportional-Caputo hybrid operators and to prove new integral inequalities for the class of differentiable convex functions based on this identity.

3. Findings

An important integral identity is emboided in the following lemma that will be useful to prove the main findings as follows:

Lemma 1. *Let $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be a twice differentiable function on I°. Also let f and f' are locally L^1 functions on I. Then, the following equality holds:*

$$\begin{aligned} &K_1(\alpha)\int_0^1 t^{1-\alpha}f'(ta+(1-t)x)dt + K_0(\alpha)\int_0^1 t^{1-\alpha}f''(ta+(1-t)x)dt \\ &+K_1(\alpha)\int_0^1 t^{1-\alpha}f'(tx+(1-t)b)dt + K_0(\alpha)\int_0^1 t^{1-\alpha}f''(tx+(1-t)b)dt \end{aligned}$$

$$
\begin{aligned}
= \quad & -\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a}-\frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x} \\
& +\Gamma(2-\alpha)\left(\frac{{}_{a}^{CPC}D_{x}^{\alpha}f(x)}{(x-a)^{2-\alpha}}+\frac{{}_{x}^{CPC}D_{b}^{\alpha}f(b)}{(b-x)^{2-\alpha}}\right)
\end{aligned}
$$

where $\alpha \in [0,1]$, $a < x < b$ and K_0 and K_1 are functions satisfing the conditions (1) and (2).

Proof. With the above assumptions it can be written that

$$
\begin{aligned}
& \int_0^1 t^{1-\alpha} f'(ta+(1-t)x)dt \\
= \quad & t^{1-\alpha}\frac{f(ta+(1-t)x)}{a-x}\bigg|_0^1-\int_0^1 \frac{f(ta+(1-t)x)}{a-x}(1-\alpha)t^{-\alpha}dt \\
= \quad & -\frac{f(a)}{x-a}+\frac{1-\alpha}{x-a}\int_0^1 t^{-\alpha} f(ta+(1-t)x)dt \\
= \quad & -\frac{f(a)}{x-a}+\frac{1-\alpha}{x-a}\int_x^a \left(\frac{x-u}{x-a}\right)^{-\alpha} f(u)\frac{du}{a-x} \\
= \quad & -\frac{f(a)}{x-a}+\frac{1-\alpha}{(x-a)^{2-\alpha}}\int_a^x (x-u)^{-\alpha} f(u)du. \qquad (3)
\end{aligned}
$$

Similarly, we have

$$
\int_0^1 t^{1-\alpha} f''(ta+(1-t)x)dt = -\frac{f'(a)}{x-a}+\frac{1-\alpha}{(x-a)^{2-\alpha}}\int_a^x (x-u)^{-\alpha} f'(u)du. \qquad (4)
$$

Multiplying (3) by $\frac{K_1(\alpha)}{\Gamma(1-\alpha)}$ and (4) by $\frac{K_0(\alpha)}{\Gamma(1-\alpha)}$ respectively and adding them side by side, we get

$$
\begin{aligned}
& \frac{K_1(\alpha)}{\Gamma(1-\alpha)}\int_0^1 t^{1-\alpha} f'(ta+(1-t)x)dt+\frac{K_0(\alpha)}{\Gamma(1-\alpha)}\int_0^1 t^{1-\alpha} f''(ta+(1-t)x)dt \\
= \quad & \frac{K_1(\alpha)}{\Gamma(1-\alpha)}\left(-\frac{f(a)}{x-a}+\frac{1-\alpha}{(x-a)^{2-\alpha}}\int_a^x (x-u)^{-\alpha} f(u)du\right) \\
& +\frac{K_0(\alpha)}{\Gamma(1-\alpha)}\left(-\frac{f'(a)}{x-a}+\frac{1-\alpha}{(x-a)^{2-\alpha}}\int_a^x (x-u)^{-\alpha} f'(u)du\right) \\
= \quad & -\frac{1}{(x-a)\Gamma(1-\alpha)}\left(K_1(\alpha)f(a)+K_0(\alpha)f'(a)\right) \qquad (5) \\
& +\frac{1-\alpha}{(x-a)^{2-\alpha}\Gamma(1-\alpha)}\left[\int_a^x \left(K_1(\alpha)f(u)+K_0(\alpha)f'(u)\right)(x-u)^{-\alpha}du\right] \\
= \quad & -\frac{1}{(x-a)\Gamma(1-\alpha)}\left(K_1(\alpha)f(a)+K_0(\alpha)f'(a)\right)+\frac{1-\alpha}{(x-a)^{2-\alpha}}\,{}_{a}^{CPC}D_{x}^{\alpha}f(x).
\end{aligned}
$$

Also, it is easy to see that

$$\begin{aligned}
&\int_0^1 t^{1-\alpha} f'(tx+(1-t)b)dt \\
=\;& t^{1-\alpha}\frac{f(tx+(1-t)b)}{x-b}\bigg|_0^1 - \int_0^1 \frac{f(tx+(1-t)b)}{x-b}(1-\alpha)t^{-\alpha}dt \\
=\;& -\frac{f(x)}{b-x}+\frac{1-\alpha}{b-x}\int_0^1 t^{-\alpha} f(tx+(1-t)b)dt \qquad (6)\\
=\;& -\frac{f(x)}{b-x}+\frac{1-\alpha}{b-x}\int_b^x \left(\frac{b-u}{b-x}\right)^{-\alpha} f(u)\frac{du}{x-b} \\
=\;& -\frac{f(x)}{b-x}+\frac{1-\alpha}{(b-x)^{2-\alpha}}\int_x^b (b-u)^{-\alpha} f(u)du
\end{aligned}$$

and similarly

$$\int_0^1 t^{1-\alpha} f''(tx+(1-t)b)dt = -\frac{f'(x)}{b-x}+\frac{1-\alpha}{(b-x)^{2-\alpha}}\int_x^b (b-u)^{-\alpha} f'(u)du. \qquad (7)$$

Multiplying (7) by $\frac{K_1(\alpha)}{\Gamma(1-\alpha)}$ and (7) by $\frac{K_0(\alpha)}{\Gamma(1-\alpha)}$ respectively and adding them side by side, we get

$$\begin{aligned}
&\frac{K_1(\alpha)}{\Gamma(1-\alpha)}\int_0^1 t^{1-\alpha} f'(tx+(1-t)b)dt+\frac{K_0(\alpha)}{\Gamma(1-\alpha)}\int_0^1 t^{1-\alpha} f''(tx+(1-t)b)dt \\
=\;& \frac{K_1(\alpha)}{\Gamma(1-\alpha)}\left(-\frac{f(x)}{b-x}+\frac{1-\alpha}{(b-x)^{2-\alpha}}\int_x^b (b-u)^{-\alpha} f(u)du\right) \\
&+\frac{K_0(\alpha)}{\Gamma(1-\alpha)}\left(-\frac{f'(x)}{b-x}+\frac{1-\alpha}{(b-x)^{2-\alpha}}\int_x^b (b-u)^{-\alpha} f'(u)du\right) \\
=\;& -\frac{1}{(b-x)\Gamma(1-\alpha)}\big(K_1(\alpha)f(x)+K_0(\alpha)f'(x)\big) \qquad (8)\\
&+\frac{1-\alpha}{(b-x)^{2-\alpha}\Gamma(1-\alpha)}\left[\int_x^b \big(K_1(\alpha)f(u)+K_0(\alpha)f'(u)\big)(b-u)^{-\alpha}du\right] \\
=\;& -\frac{1}{(b-x)\Gamma(1-\alpha)}\big(K_1(\alpha)f(x)+K_0(\alpha)f'(x)\big)+\frac{1-\alpha}{(b-x)^{2-\alpha}}\,{}_x^{CPC}D_b^{\alpha}f(b).
\end{aligned}$$

By adding (5) and (8) and multiplying with $\Gamma(1-\alpha)$, we get the desired result.

□

Theorem 1. *Let $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be a twice differentiable function on I°. Also let f and f' are locally L^1 functions on I. If f' and f'' are convex on I, then the following inequality holds:*

$$\begin{aligned}
&\Bigg| -\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a} - \frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x} \\
&+\Gamma(2-\alpha)\left(\frac{{}^{CPC}_{a}D^{\alpha}_{x}f(x)}{(x-a)^{2-\alpha}} + \frac{{}^{CPC}_{x}D^{\alpha}_{b}f(b)}{(b-x)^{2-\alpha}}\right)\Bigg| \\
\leq\ &\frac{K_1(\alpha)|f'(a)|+K_0(\alpha)|f''(a)|}{(3-\alpha)} + \frac{K_1(\alpha)|f'(b)|+K_0(\alpha)|f''(b)|}{(3-\alpha)(2-\alpha)} \\
&+\frac{K_1(\alpha)|f'(x)|+K_0(\alpha)|f''(x)|}{(2-\alpha)}
\end{aligned}$$

where $\alpha \in [0,1]$, $a < x < b$ and K_0 and K_1 are functions satisfing the conditions (1) and (2).

Proof. From Lemma 1 and using properties of absolute value, we have

$$\begin{aligned}
&\Bigg| -\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a} - \frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x} \\
&+\Gamma(2-\alpha)\left(\frac{{}^{CPC}_{a}D^{\alpha}_{x}f(x)}{(x-a)^{2-\alpha}} + \frac{{}^{CPC}_{x}D^{\alpha}_{b}f(b)}{(b-x)^{2-\alpha}}\right)\Bigg| \\
\leq\ &K_1(\alpha)\int_0^1 t^{1-\alpha}\big|f'(ta+(1-t)x)\big|dt \\
&+K_0(\alpha)\int_0^1 t^{1-\alpha}\big|f''(ta+(1-t)x)\big|dt \\
&+K_1(\alpha)\int_0^1 t^{1-\alpha}\big|f'(tx+(1-t)b)\big|dt \\
&+K_0(\alpha)\int_0^1 t^{1-\alpha}\big|f''(tx+(1-t)b)\big|dt.
\end{aligned} \tag{9}$$

As f' and f'' are convex functions on I, one can write

$$\begin{aligned}
&\Bigg| -\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a} - \frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x} \\
&+\Gamma(2-\alpha)\left(\frac{{}^{CPC}_{a}D^{\alpha}_{x}f(x)}{(x-a)^{2-\alpha}} + \frac{{}^{CPC}_{x}D^{\alpha}_{b}f(b)}{(b-x)^{2-\alpha}}\right)\Bigg| \\
\leq\ &K_1(\alpha)\int_0^1 t^{1-\alpha}\big(t\big|f'(a)\big|+(1-t)\big|f'(x)\big|\big)dt \\
&+K_0(\alpha)\int_0^1 t^{1-\alpha}\big(t\big|f''(a)\big|+(1-t)\big|f''(x)\big|\big)dt \\
&+K_1(\alpha)\int_0^1 t^{1-\alpha}\big(t\big|f'(x)\big|+(1-t)\big|f'(b)\big|\big)dt \\
&+K_0(\alpha)\int_0^1 t^{1-\alpha}\big(t\big|f''(x)\big|+(1-t)\big|f''(b)\big|\big)dt.
\end{aligned}$$

Namely,

$$
\begin{aligned}
&\Bigg|-\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a}-\frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x} \\
&+\Gamma(2-\alpha)\left(\frac{{}_{a}^{CPC}D_x^{\alpha}f(x)}{(x-a)^{2-\alpha}}+\frac{{}_{x}^{CPC}D_b^{\alpha}f(b)}{(b-x)^{2-\alpha}}\right)\Bigg| \\
= \quad &\big(K_1(\alpha)\big|f'(x)\big|+K_0(\alpha)\big|f''(x)\big|\big)\int_0^1\Big(t^{1-\alpha}(1-t)+t^{2-\alpha}\Big)dt \\
&+\big(K_1(\alpha)\big|f'(a)\big|+K_0(\alpha)\big|f''(a)\big|\big)\int_0^1 t^{2-\alpha}dt \\
&+\big(K_1(\alpha)\big|f'(b)\big|+K_0(\alpha)\big|f''(b)\big|\big)\int_0^1 t^{1-\alpha}(1-t)dt.
\end{aligned}
$$

The proof is completed by making the necessary calculations. □

Corollary 1. *Under the assumptions of Theorem 1, if we choose $x=\frac{a+b}{2}$, then the following inequality holds:*

$$
\begin{aligned}
&\Bigg|-\frac{(K_1(\alpha)f(a)+K_0(\alpha)f'(a))}{b-a}-\frac{\left(K_1(\alpha)f\left(\frac{a+b}{2}\right)+K_0(\alpha)f'\left(\frac{a+b}{2}\right)\right)}{b-a} \\
&+2^{1-\alpha}\Gamma(2-\alpha)\left(\frac{{}_{a}^{CPC}D_{\frac{a+b}{2}}^{\alpha}f\left(\frac{a+b}{2}\right)}{(b-a)^{2-\alpha}}+\frac{{}_{\frac{a+b}{2}}^{CPC}D_b^{\alpha}f(b)}{(b-a)^{2-\alpha}}\right)\Bigg| \\
\leq \quad &\frac{K_1(\alpha)|f'(a)|+K_0(\alpha)|f''(a)|}{2(3-\alpha)} \\
&+\frac{K_1(\alpha)|f'(b)|+K_0(\alpha)|f''(b)|}{2(3-\alpha)(2-\alpha)} \\
&+\frac{K_1(\alpha)\left|f'\left(\frac{a+b}{2}\right)\right|+K_0(\alpha)\left|f''\left(\frac{a+b}{2}\right)\right|}{2(2-\alpha)}.
\end{aligned}
$$

Theorem 2. *Let $f: I\subseteq\mathbb{R}^+\to\mathbb{R}$ be a twice differentiable function on I°. Also let f and f' are locally L^1 functions on I. If $|f'|^q$ and $|f''|^q$ are convex on I, then the following inequality holds:*

$$
\begin{aligned}
&\Bigg|-\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a}-\frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x} \\
&+\Gamma(2-\alpha)\left(\frac{{}_{a}^{CPC}D_x^{\alpha}f(x)}{(x-a)^{2-\alpha}}+\frac{{}_{x}^{CPC}D_b^{\alpha}f(b)}{(b-x)^{2-\alpha}}\right)\Bigg| \\
\leq \quad &\frac{1}{((1-\alpha)p+1)^{\frac{1}{p}}2^{\frac{1}{q}}}\bigg[K_1(\alpha)\Big(|f'(a)|^q+|f'(x)|^q\Big)^{\frac{1}{q}} \\
&+K_0(\alpha)\Big(|f''(a)|^q+|f''(x)|^q\Big)^{\frac{1}{q}} \\
&+K_1(\alpha)\Big(|f'(x)|^q+|f'(b)|^q\Big)^{\frac{1}{q}} \\
&+K_0(\alpha)\Big(|f''(x)|^q+|f''(b)|^q\Big)^{\frac{1}{q}}\bigg]
\end{aligned}
$$

where $\alpha \in [0,1]$, $a < x < b$, $p,q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ and K_0 and K_1 are functions satisfing the conditions (1) and (2).

Proof. By applying Hölder's inequality to (9), we get

$$\begin{aligned}
&\bigg|-\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a}-\frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x} \\
&+\Gamma(2-\alpha)\left(\frac{{}^{CPC}_{a}D^{\alpha}_{x}f(x)}{(x-a)^{2-\alpha}}+\frac{{}^{CPC}_{x}D^{\alpha}_{b}f(b)}{(b-x)^{2-\alpha}}\right)\bigg| \\
\leq\ & K_1(\alpha)\left[\left(\int_0^1 t^{(1-\alpha)p}dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(ta+(1-t)x)|^q dt\right)^{\frac{1}{q}}\right] \\
&+K_0(\alpha)\left[\left(\int_0^1 t^{(1-\alpha)p}dt\right)^{\frac{1}{p}}\left(\int_0^1 |f''(ta+(1-t)x)|^q dt\right)^{\frac{1}{q}}\right] \\
&+K_1(\alpha)\left[\left(\int_0^1 t^{(1-\alpha)p}dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(tx+(1-t)b)|^q dt\right)^{\frac{1}{q}}\right] \\
&+K_0(\alpha)\left[\left(\int_0^1 t^{(1-\alpha)p}dt\right)^{\frac{1}{p}}\left(\int_0^1 |f''(tx+(1-t)b)|^q dt\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Using convexity of $|f'|^q$ and $|f''|^q$, we have

$$\begin{aligned}
&\bigg|-\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a}-\frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x} \\
&+\Gamma(2-\alpha)\left(\frac{{}^{CPC}_{a}D^{\alpha}_{x}f(x)}{(x-a)^{2-\alpha}}+\frac{{}^{CPC}_{x}D^{\alpha}_{b}f(b)}{(b-x)^{2-\alpha}}\right)\bigg| \\
\leq\ & \frac{1}{((1-\alpha)p+1)^{\frac{1}{p}}}\left\{K_1(\alpha)\left(\int_0^1\Big(t|f'(a)|^q+(1-t)|f'(x)|^q\Big)dt\right)^{\frac{1}{q}}\right. \\
&+K_0(\alpha)\left(\int_0^1\Big(t|f''(a)|^q+(1-t)|f''(x)|^q\Big)dt\right)^{\frac{1}{q}} \\
&+K_1(\alpha)\left(\int_0^1\Big(t|f'(x)|^q+(1-t)|f'(b)|^q\Big)dt\right)^{\frac{1}{q}} \\
&\left.+K_0(\alpha)\left(\int_0^1\Big(t|f''(x)|^q+(1-t)|f''(b)|^q\Big)dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

With simple calculations, we get

$$
\begin{aligned}
& \Bigg| -\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a} - \frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x} \\
& +\Gamma(2-\alpha)\left(\frac{{}_{a}^{CPC}D_x^{\alpha}f(x)}{(x-a)^{2-\alpha}} + \frac{{}_{x}^{CPC}D_b^{\alpha}f(b)}{(b-x)^{2-\alpha}}\right)\Bigg| \\
\leq \ & \frac{1}{((1-\alpha)p+1)^{\frac{1}{p}}}\left\{K_1(\alpha)\left(\frac{|f'(a)|^q+|f'(x)|^q}{2}\right)^{\frac{1}{q}}\right. \\
& +K_0(\alpha)\left(\frac{|f''(a)|^q+|f''(x)|^q}{2}\right)^{\frac{1}{q}} + K_1(\alpha)\left(\frac{|f'(x)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}} \\
& \left.+K_0(\alpha)\left(\frac{|f''(x)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}\right\}
\end{aligned}
$$

which is the desired result. □

Corollary 2. *Under the assumptions of Theorem 2, if we set $x = \frac{a+b}{2}$, then the following inequality holds:*

$$
\begin{aligned}
& \Bigg| -\frac{(K_1(\alpha)f(a)+K_0(\alpha)f'(a))}{b-a} - \frac{\left(K_1(\alpha)f\left(\frac{a+b}{2}\right)+K_0(\alpha)f'\left(\frac{a+b}{2}\right)\right)}{b-a} \\
& +2^{1-\alpha}\Gamma(2-\alpha)\left(\frac{{}_{a}^{CPC}D_{\frac{a+b}{2}}^{\alpha}f\left(\frac{a+b}{2}\right)}{(b-a)^{2-\alpha}} + \frac{{}_{\frac{a+b}{2}}^{CPC}D_b^{\alpha}f(b)}{(b-a)^{2-\alpha}}\right)\Bigg| \times 2^{1+1/q}((1-\alpha)p+1)^{\frac{1}{p}} \\
\leq \ & K_1(\alpha)\left(|f'(a)|^q + \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} + K_0(\alpha)\left(|f''(a)|^q + \left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} \\
& +K_1(\alpha)\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(b)|^q\right)^{\frac{1}{q}} + K_0(\alpha)\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q + |f''(b)|^q\right)^{\frac{1}{q}}.
\end{aligned}
$$

Theorem 3. *Let $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be a twice differentiable function on I°. Also let f and f' are locally L^1 functions on I. If $|f'|^q$ and $|f''|^q$ are convex on I, then the following inequality holds:*

$$
\begin{aligned}
& \Bigg| -\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a} - \frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x} \\
& +\Gamma(2-\alpha)\left(\frac{{}_{a}^{CPC}D_x^{\alpha}f(x)}{(x-a)^{2-\alpha}} + \frac{{}_{x}^{CPC}D_b^{\alpha}f(b)}{(b-x)^{2-\alpha}}\right)\Bigg| \times (2-\alpha)^{1-\frac{1}{q}} \\
\leq \ & \left\{K_1(\alpha)\left(\frac{|f'(a)|^q}{(3-\alpha)} + \frac{|f'(x)|^q}{(2-\alpha)(3-\alpha)}\right)^{\frac{1}{q}} + K_0(\alpha)\left(\frac{|f''(a)|^q}{(3-\alpha)} + \frac{|f''(x)|^q}{(2-\alpha)(3-\alpha)}\right)^{\frac{1}{q}}\right. \\
& \left.+K_1(\alpha)\left(\frac{|f'(x)|^q}{(3-\alpha)} + \frac{|f'(b)|^q}{(2-\alpha)(3-\alpha)}\right)^{\frac{1}{q}} + K_0(\alpha)\left(\frac{|f''(x)|^q}{(3-\alpha)} + \frac{|f''(b)|^q}{(2-\alpha)(3-\alpha)}\right)^{\frac{1}{q}}\right\}
\end{aligned}
$$

where $\alpha \in [0,1]$, $a < x < b$, $q \geq 1$ and K_0 and K_1 are functions satisfing the conditions (1) and (2).

Proof. By applying power mean inequality to (9), we get

$$\begin{aligned}
&\Bigg| -\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a} - \frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x} \\
&+\Gamma(2-\alpha)\left(\frac{{}_{a}^{CPC}D_x^{\alpha}f(x)}{(x-a)^{2-\alpha}} + \frac{{}_{x}^{CPC}D_b^{\alpha}f(b)}{(b-x)^{2-\alpha}}\right)\Bigg| \\
\leq\ & K_1(\alpha)\left(\int_0^1 t^{1-\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{1-\alpha}\left|f'(ta+(1-t)x)\right|^q dt\right)^{\frac{1}{q}} \\
&+K_0(\alpha)\left(\int_0^1 t^{1-\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{1-\alpha}\left|f''(ta+(1-t)x)\right|^q dt\right)^{\frac{1}{q}} \\
&+K_1(\alpha)\left(\int_0^1 t^{1-\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{1-\alpha}\left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \\
&+K_0(\alpha)\left(\int_0^1 t^{1-\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{1-\alpha}\left|f''(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Using convexity of $|f'|^q$ and $|f''|^q$, we have

$$\begin{aligned}
&\Bigg| -\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a} - \frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x} \\
&+\Gamma(2-\alpha)\left(\frac{{}_{a}^{CPC}D_x^{\alpha}f(x)}{(x-a)^{2-\alpha}} + \frac{{}_{x}^{CPC}D_b^{\alpha}f(b)}{(b-x)^{2-\alpha}}\right)\Bigg| \\
\leq\ & \left(\frac{1}{2-\alpha}\right)^{1-\frac{1}{q}}\Bigg\{K_1(\alpha)\left(\int_0^1 t^{1-\alpha}\left(t\left|f'(a)\right|^q+(1-t)\left|f'(x)\right|^q\right)dt\right)^{\frac{1}{q}} \\
&+K_0(\alpha)\left(\int_0^1 t^{1-\alpha}\left(t\left|f''(a)\right|^q+(1-t)\left|f''(x)\right|^q\right)dt\right)^{\frac{1}{q}} \\
&+K_1(\alpha)\left(\int_0^1 t^{1-\alpha}\left(t\left|f'(x)\right|^q+(1-t)\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}} \\
&+K_0(\alpha)\left(\int_0^1 t^{1-\alpha}\left(t\left|f''(x)\right|^q+(1-t)\left|f''(b)\right|^q\right)dt\right)^{\frac{1}{q}}\Bigg\}.
\end{aligned}$$

Making necessary calculations, we get

$$\begin{aligned}
&\Bigg| -\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a} - \frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x} \\
&+\Gamma(2-\alpha)\left(\frac{{}_a^{CPC}D_x^\alpha f(x)}{(x-a)^{2-\alpha}} + \frac{{}_x^{CPC}D_b^\alpha f(b)}{(b-x)^{2-\alpha}}\right)\Bigg| \\
\leq\ & \left(\frac{1}{2-\alpha}\right)^{1-\frac{1}{q}}\Bigg\{K_1(\alpha)\left(\frac{|f'(a)|^q}{3-\alpha}+\frac{|f'(x)|^q}{(2-\alpha)(3-\alpha)}\right)^{\frac{1}{q}} \\
&+K_0(\alpha)\left(\frac{|f''(a)|^q}{3-\alpha}+\frac{|f''(x)|^q}{(2-\alpha)(3-\alpha)}\right)^{\frac{1}{q}}+K_1(\alpha)\left(\frac{|f'(x)|^q}{3-\alpha}+\frac{|f'(b)|^q}{(2-\alpha)(3-\alpha)}\right)^{\frac{1}{q}} \\
&+K_0(\alpha)\left(\frac{|f''(x)|^q}{3-\alpha}+\frac{|f''(b)|^q}{(2-\alpha)(3-\alpha)}\right)^{\frac{1}{q}}\Bigg\}.
\end{aligned}$$

So, the proof is completed. □

Corollary 3. *Under the assumptions of Theorem 3, if we take $x = \frac{a+b}{2}$, then the following inequality is valid:*

$$\begin{aligned}
&\Bigg| -\frac{(K_1(\alpha)f(a)+K_0(\alpha)f'(a))}{b-a} - \frac{\left(K_1(\alpha)f\left(\frac{a+b}{2}\right)+K_0(\alpha)f'\left(\frac{a+b}{2}\right)\right)}{b-a} \\
&+2^{2-\alpha}\Gamma(2-\alpha)\left(\frac{{}_a^{CPC}D_{\frac{a+b}{2}}^\alpha f\left(\frac{a+b}{2}\right)}{(b-a)^{2-\alpha}} + \frac{{}_{\frac{a+b}{2}}^{CPC}D_b^\alpha f(b)}{(b-a)^{2-\alpha}}\right)\Bigg| \times (2-\alpha)^{1-\frac{1}{q}} \\
\leq\ & \Bigg\{K_1(\alpha)\left(\frac{|f'(a)|^q}{(3-\alpha)}+\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q}{(2-\alpha)(3-\alpha)}\right)^{\frac{1}{q}}+K_0(\alpha)\left(\frac{|f''(a)|^q}{(3-\alpha)}+\frac{\left|f''\left(\frac{a+b}{2}\right)\right|^q}{(2-\alpha)(3-\alpha)}\right)^{\frac{1}{q}} \\
&+K_1(\alpha)\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q}{(3-\alpha)}+\frac{|f'(b)|^q}{(2-\alpha)(3-\alpha)}\right)^{\frac{1}{q}}+K_0(\alpha)\left(\frac{\left|f''\left(\frac{a+b}{2}\right)\right|^q}{(3-\alpha)}+\frac{|f''(b)|^q}{(2-\alpha)(3-\alpha)}\right)^{\frac{1}{q}}\Bigg\}.
\end{aligned}$$

Theorem 4. *Let $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be a twice differentiable function on I°. Also let f and f' are locally L^1 functions on I. If $|f'|^q$ and $|f''|^q$ are convex on I, then the following inequality holds:*

$$\begin{aligned}
&\Bigg| -\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a} - \frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x} \\
&+\Gamma(2-\alpha)\left(\frac{{}_a^{CPC}D_x^\alpha f(x)}{(x-a)^{2-\alpha}} + \frac{{}_x^{CPC}D_b^\alpha f(b)}{(b-x)^{2-\alpha}}\right)\Bigg| \\
\leq\ & \frac{2(K_1(\alpha)+K_0(\alpha))}{p^2(1-\alpha)+p} + \frac{K_1(\alpha)}{2q}\left(|f'(a)|^q+2|f'(x)|^q+|f'(b)|^q\right) \\
&+\frac{K_0(\alpha)}{2q}\left(|f''(a)|^q+2|f''(x)|^q+|f''(b)|^q\right)
\end{aligned}$$

where $\alpha \in [0,1]$, $a < x < b$, $\frac{1}{p}+\frac{1}{q}=1$, $q > 1$ and K_0 and K_1 are functions satisfing (1) and (2).

Proof. Taking into account the Young inequality as $mn \leq \frac{m^p}{p} + \frac{n^q}{q}$ in (9), we get

$$\begin{aligned}
&\Bigg| -\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a} - \frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x} \\
&+\Gamma(2-\alpha)\left(\frac{{}^{CPC}_{a}D^{\alpha}_{x}f(x)}{(x-a)^{2-\alpha}} + \frac{{}^{CPC}_{x}D^{\alpha}_{b}f(b)}{(b-x)^{2-\alpha}}\right)\Bigg| \\
\leq\ & K_1(\alpha)\left[\frac{1}{p}\int_0^1 t^{p(1-\alpha)}dt + \frac{1}{q}\int_0^1 \left|f'(ta+(1-t)x)\right|^q dt\right] \\
&+K_0(\alpha)\left[\frac{1}{p}\int_0^1 t^{p(1-\alpha)}dt + \frac{1}{q}\int_0^1 \left|f''(ta+(1-t)x)\right|^q dt\right] \\
&+K_1(\alpha)\left[\frac{1}{p}\int_0^1 t^{p(1-\alpha)}dt + \frac{1}{q}\int_0^1 \left|f'(tx+(1-t)b)\right|^q dt\right] \\
&+K_0(\alpha)\left[\frac{1}{p}\int_0^1 t^{p(1-\alpha)}dt + \frac{1}{q}\int_0^1 \left|f''(tx+(1-t)b)\right|^q dt\right].
\end{aligned}$$

Using convexity of $|f'|^q$ and $|f''|^q$, we have

$$\begin{aligned}
&\Bigg| -\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a} - \frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x} \\
&+\Gamma(2-\alpha)\left(\frac{{}^{CPC}_{a}D^{\alpha}_{x}f(x)}{(x-a)^{2-\alpha}} + \frac{{}^{CPC}_{x}D^{\alpha}_{b}f(b)}{(b-x)^{2-\alpha}}\right)\Bigg| \\
\leq\ & K_1(\alpha)\left[\frac{1}{p^2(1-\alpha)+p} + \frac{1}{q}\int_0^1 \Big(t\left|f'(a)\right|^q + (1-t)\left|f'(x)\right|^q\Big)dt\right] \\
&+K_0(\alpha)\left[\frac{1}{p^2(1-\alpha)+p} + \frac{1}{q}\int_0^1 \Big(t\left|f''(a)\right|^q + (1-t)\left|f''(x)\right|^q\Big)dt\right] \\
&+K_1(\alpha)\left[\frac{1}{p^2(1-\alpha)+p} + \frac{1}{q}\int_0^1 \Big(t\left|f'(x)\right|^q + (1-t)\left|f'(b)\right|^q\Big)dt\right] \\
&+K_0(\alpha)\left[\frac{1}{p^2(1-\alpha)+p} + \frac{1}{q}\int_0^1 \Big(t\left|f''(x)\right|^q + (1-t)\left|f''(b)\right|^q\Big)dt\right].
\end{aligned}$$

By making necessary computations, we get

$$\begin{aligned}
&\Bigg| -\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a} - \frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x} \\
&+\Gamma(2-\alpha)\left(\frac{{}^{CPC}_{a}D^{\alpha}_{x}f(x)}{(x-a)^{2-\alpha}} + \frac{{}^{CPC}_{x}D^{\alpha}_{b}f(b)}{(b-x)^{2-\alpha}}\right)\Bigg| \\
\leq\ & K_1(\alpha)\left[\frac{1}{p^2(1-\alpha)+p} + \frac{|f'(a)|^q+|f'(x)|^q}{2q}\right] \\
&+K_0(\alpha)\left[\frac{1}{p^2(1-\alpha)+p} + \frac{|f''(a)|^q+|f''(x)|^q}{2q}\right] \\
&+K_1(\alpha)\left[\frac{1}{p^2(1-\alpha)+p} + \frac{|f'(x)|^q+|f'(b)|^q}{2q}\right]
\end{aligned}$$

$$+K_0(\alpha)\left[\frac{1}{p^2(1-\alpha)+p}+\frac{|f''(x)|^q+|f''(b)|^q}{2q}\right]$$

which completes the proof. □

Corollary 4. *Under the assumptions of Theorem 4, if we select* $x=\frac{a+b}{2}$*, then we have*

$$\begin{aligned}&\Bigg|-\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{b-a}-\frac{K_1(\alpha)f\left(\frac{a+b}{2}\right)+K_0(\alpha)f'\left(\frac{a+b}{2}\right)}{b-a}\\&+2^{1-\alpha}\Gamma(2-\alpha)\left(\frac{{}_{a}^{CPC}D^{\alpha}_{\frac{a+b}{2}}f\left(\frac{a+b}{2}\right)}{(b-a)^{2-\alpha}}+\frac{{}_{\frac{a+b}{2}}^{CPC}D^{\alpha}_{b}f(b)}{(b-a)^{2-\alpha}}\right)\Bigg|\\ \le\ &\frac{(K_1(\alpha)+K_0(\alpha))}{p^2(1-\alpha)+p}+\frac{K_1(\alpha)}{4q}\left(|f'(a)|^q+2\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right)\\&+\frac{K_0(\alpha)}{4q}\left(|f''(a)|^q+2\left|f''\left(\frac{a+b}{2}\right)\right|^q+|f''(b)|^q\right).\end{aligned}$$

Remark 2. *Several special cases can be considered by choosing the functions* $K_0(\alpha)$ *and* $K_1(\alpha)$ *as in Remark 1.*

4. Conclusions

The main motivation point of studies in the field of inequality theory is to obtain new and general inequalities. Different kinds of convex functions, some classical inequalities such as Hölder's inequality, Power-mean inequality, Young's inequality, and basic mathematical analysis methods are used to create some known inequalities in the literature and various new versions of these inequalities. Recent developments in the field of fractional analysis have also affected the field of inequalities, and several new studies have been performed to optimize the bounds with the help of different fractional integral operators. Within the scope of the study, it is aimed to prove various new inequalities by using proportional Caputo-hybrid fractional operators for differentiable convex functions. With the help of this new operator, which we used for the first time in the field of inequalities, new results can be produced for different kinds of convex functions and different types of inequalities.

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