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Existence of Positive Solutions for a Singular Second-Order Changing-Sign Differential Equation on Time Scales

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Abstract: In this paper, we focus on the existence of positive solutions for a boundary value problem of the changing-sign differential equation on time scales. By constructing a translation transformation and combining with the properties of the solution of the nonhomogeneous boundary value problem, we transfer the changing-sign problem to a positone problem, then by means of the known fixed-point theorem, several sufficient conditions for the existence of positive solutions are established for the case in which the nonlinear term of the equation may change sign.

Keywords: positive solutions; changing sign; fixed-point theorem; time scales; cone

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Citation: Tian, H.; Zhang, X.; Wu, Y.; Wiwatanapataphee, B. Existence of Positive Solutions for a Singular Second-Order Changing-Sign Differential Equation on Time Scales. *Fractal Fract.* **2022**, *6*, 315. https://doi.org/10.3390/fractalfract6060315

Academic Editor: Natália Martins

Received: 14 May 2022
Accepted: 2 June 2022
Published: 3 June 2022

1. Introduction

In this paper, we consider the existence of positive solutions for the following Sturm–Liouville boundary value problem with changing-sign term on time scales

$$\begin{cases} (z^{\triangle})^{\nabla}(t) + a(t)f(t,z(\eta(t))) + b(t) = 0, & t \in [c,\eta(d)]_{\mathbb{T}}, \\ \alpha z(c) - \beta z^{\triangle}(c) = 0, \\ \gamma z(\eta(d)) + \delta z^{\triangle}(\eta(d)) = 0, \end{cases} \tag{1}$$

where $f \in C([c,\eta(d)]_{\mathbb{T}} \times [0,+\infty), (-\infty,+\infty))$, and

$$\alpha, \gamma \geq 0, \beta, \delta > 0,\ \rho = \beta\gamma + \alpha\delta + \alpha\gamma(\eta(d) - c) > 0. \tag{2}$$

$a : L^1((c,\eta(d))_{\mathbb{T}}, [0,+\infty)$, $b : L^1((c,\eta(d))_{\mathbb{T}}, (-\infty,+\infty))$, which implies that a, b can have finitely many singularities in the time scales interval $[c,\eta(d)]_{\mathbb{T}}$.

The dynamic equations on time scales arise from the modeling of many natural phenomena, such as insect population models, epidemic models, heat transfer, and neural networks—for details the reader is referred to [1–11] and the references therein. Some related definitions and properties on time scales can be found in [12–15]. In particular, Hao, Xiao, and Liang [16] discussed the following nonlinear dynamic equation on time scales

$$\begin{cases} (\varphi z^{\triangle})^{\nabla}(t) + \lambda m(t)f(t,z(\eta(t))) = 0, & t \in [c,d]_{\mathbb{T}}, \\ \alpha z(c) - \beta z^{\triangle}(c) = 0, \\ \gamma z(\eta(d)) + \delta z^{\triangle}(\eta(d)) = 0, \end{cases}$$

where $f \in C([c,\eta(d)]_{\mathbb{T}} \times [0,\infty), (0,\infty))$, $m : ((c,\eta(d))_{\mathbb{T}} \to [0,+\infty)$ is rd-continuous. By using the Krasnosel'skii fixed-point theorem, an existence theorem of positive solutions was established. In [17], Sang and Meng studied the following Sturm–Liouville boundary value problem on time scales

$$\begin{cases}(pz^{\triangle})^{\nabla}(t) + \lambda f(t, z(t)) = 0, \ \ t \in (c, d]_{\mathbb{T}} \\ \alpha z(c) - \beta(pz^{\triangle})(c) = 0, \\ \gamma z(\eta(d)) + \delta(pz^{\triangle})(\eta(d)) = 0, \end{cases}$$

where $\lambda > 0, \beta, \delta \in (0, +\infty), \alpha, \gamma \in [0, +\infty), \beta\gamma + \alpha\delta + \alpha\gamma \int_c^{\eta(d)} \frac{d\tau}{p(\tau)} > 0, \ \ p : [c, \eta(d)]_{\mathbb{T}} \to (0, +\infty)$. For other work regarding the Sturm–Liouville boundary value problems, see [18,19].

On the other hand, in Equation (1), the nonlinearity $f(t, z) \geq -M, M > 0$ and disturbance term q may be a changing sign on the time scales interval. In particular, in the literature, the former was called a semipositone problem [20–23]. In a recent work [23], Yao considered the solutions and positive solutions for the following nonlinear boundary value problem involving the second derivative:

$$\begin{cases} z^{(4)}(t) = f(t, z(t), z''(t)), \ \ t \in [0, 1], \\ z(0) = z(1) = z''(0) = z''(1) = 0, \end{cases} \tag{3}$$

where $f : C[0, 1] \times [-\frac{5}{384}M, +\infty) \times (-\infty, \frac{1}{8}M] \to [-M, +\infty)$ is continuous, in which $M \geq 0$ is a constant, so the problem (3) is semipositone. In [24], Yang and Meng considered the following singular semipositone boundary value problem on time scales

$$\begin{cases}(z^{\triangle})^{\nabla}(t) + f(t, z(\eta(t))) = 0, \ \ t \in (t_1, t_2), \\ \alpha z(t_1) - \beta z^{\Delta}(t_1) = 0, \\ \gamma z(\eta(t_2)) + \delta z^{\Delta}(\eta(t_2)) = 0, \end{cases}$$

where $f \in C((t_1, t_2) \times (0, +\infty), \mathbb{R})$ and there exists $M > 0$, such that $f(t, z) > -M$. $\alpha, \beta, \gamma, \delta$ are nonnegative with $\rho = \beta\gamma + \alpha\delta + \alpha\gamma(\eta(b) - a) > 0$. In addition, Zhang and Liu [21] studied a singular changing-sign Dirichlet boundary value problem on a finite interval

$$\begin{cases} z''(t) + f(t, z(t)) + q(t) = 0, \ \ t \in [0, 1], \\ z(0) = z(1) = 0, \end{cases} \tag{4}$$

where $f : C(0, 1) \times [0, +\infty) \to [0, +\infty)$ is continuous, $q(t) : (0, 1) \to (-\infty, +\infty)$ is Lebesgue integrable. By constructing a special cone, the existence of positive solutions for the singular changing-sign problem (4) was established.

However, to the best of our knowledge, the existing work either only studied semipositone problems, or only focused on perturbation problems; fewer papers considered the case in which the nonlinearity f is semipositone and the disturbance term b can be a changing sign. Thus, inspired by [21,23,24], we shall focus on the existence of positive solutions for the changing-sign boundary value problem (1). By using the properties of solutions of the corresponding nonhomogeneous boundary value problem and constructing a suitable translation transformation, we transfer the changing-sign problem to a positone problem, then by means of the known fixed-point theorem, several sufficient conditions for the existence of positive solutions are derived. Compared to the existing work, our results have some new features. Firstly, the nonlinearity term in Equation (1) contains two parts—the semipositone term and the disturbance term. Secondly, the semipositone term has a negative lower bound and the disturbance term is a changing sign and can tend to negative infinity. Thirdly, the weight function a and the disturbance term b only belong to Lebesgue integrable functions, which implies that a, b can have finitely many singularities in the time scales interval $[c, \eta(d)]_{\mathbb{T}}$.

The rest of the paper is organized as follows. In Section 2, we start with some preliminaries and lemmas. In Section 3, we present our main results and give the proofs of the results. In Section 4, several examples are given to illustrate the main results.

2. Preliminaries and Lemmas

Note that $[c, \eta(d)]_{\mathbb{T}}$ in Equation (1) denotes the time scale interval, where the time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$; thus, in order to understand the boundary value problem on time scales, some preliminary and some lemmas are given below.

Definition 1. *Define the forward jump operator η with $t < \sup \mathbb{T}$ and backward jump operator λ with $t > \inf \mathbb{T}$ at a point t, respectively:*

$$\eta(t) := \inf\{s > t : s \in \mathbb{T}\}, \ \ t \in \mathbb{T}$$

$$\lambda(t) := \sup\{s < t : s \in \mathbb{T}\}, \ \ t \in \mathbb{T}.$$

If $\lambda(t) = t, \lambda(t) < t, \eta(t) = t, \eta(t) > t$, we say the point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered, respectively. If $\mathbb{T}$ has a left-scattered maximum M, then we define $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

Definition 2. *Assume that $z : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. Then we define $z^{\triangle}(t)$ to be the number with the property that, given any $\varepsilon > 0$, there is a neighborhood U of t such that*

$$\left| z(\eta(t)) - z(s) - z^{\triangle}(t)[\eta(t) - s] \right| \leq \varepsilon |\eta(t) - s|,$$

for all $s \in U, t \in \mathbb{T}$. The second derivative of $z(t)$ is defined by $z^{\triangle\triangle}(t) = (z^{\triangle})^{\nabla}(t)$.

In the following, we shall find the related Green's function for Equation (1). By [25], we have the following Lemma:

Lemma 1. *Let $h \in L^1[c, \eta(d)]_{\mathbb{T}}$ and assume that the condition (2) holds, then the nonhomogeneous boundary value problem*

$$\begin{cases} -(z^{\triangle})^{\nabla}(t) = h(t), \ \ t \in [c, \eta(d)]_{\mathbb{T}}, \\ \alpha z(c) - \beta z^{\triangle}(c) = 0, \\ \gamma z(\eta(d)) + \delta z^{\triangle}(\eta(d)) = 0, \end{cases} \tag{5}$$

has unique solution z in the form

$$z(t) = \int_c^{\eta(d)} G(t,s)h(s)\nabla s, \ t, s \in [c, \eta(d)]_{\mathbb{T}},$$

and for all $t, s \in [c, \eta(d)]_{\mathbb{T}}$,

$$G(t,s) = \frac{1}{\rho} \begin{cases} [\alpha(s-c) + \beta][\gamma(\eta(d) - t) + \delta], \ \ c \leq s \leq t \leq \eta(d), \\ [\alpha(t-c) + \beta][\gamma(\eta(d) - s) + \delta], \ \ c \leq t \leq s \leq \eta(d), \end{cases}$$

is the Green's function of the nonhomogeneous boundary value problem (5) with $\rho = \beta\gamma + \alpha\delta + \alpha\gamma(\eta(d) - c)$.

Let

$$\omega(t) = \frac{1}{\rho}[\alpha(t-c) + \beta][\gamma(\eta(d) - t) + \delta], \ \ H = \frac{\rho}{[\alpha(\eta(d) - c) + \beta][\gamma(\eta(d) - c) + \delta]},$$

then

$$\omega(t) \geq \frac{\beta\delta}{\rho} > 0.$$

Lemma 2. *Suppose that the condition (2) is satisfied, then the Green's function possesses the following properties:*

$$H\omega(t)\omega(s) \leq G(t,s) \leq \omega(s) \ \ or \ \ \omega(t), \ \ t,s \in [c,\eta(d)]_{\mathbb{T}}.$$

Proof. It is clear that the right of the inequality holds. In the following, we prove the left of the inequality.

In fact, since

$$\frac{[\alpha(t-c)+\beta][\gamma(\eta(d)-s)+\delta]}{[\alpha(\eta(d)-c)+\beta][\gamma(\eta(d)-c)+\delta]} \leq 1, \quad \frac{[\alpha(s-c)+\beta][\gamma(\eta(d)-t)+\delta]}{[\alpha(\eta(d)-c)+\beta][\gamma(\eta(d)-c)+\delta]} \leq 1,$$

we have

$$G(t,s) \geq H\omega(t)\omega(s).$$

In this paper, we use the space $E = C([c,\eta(d)]_{\mathbb{T}}, \mathbb{R})$. Clearly, the space E is a Banach space if it is endowed with the following norm

$$||u|| = \sup_{t \in [c,\eta(d)]_{\mathbb{T}}} |u(t)|.$$

Define a cone $P \subset E$ by $P := \{u \in E \mid u(t) \geq H\omega(t)\|u\|\}$.

In order to establish the results of existence of positive solutions for Equation (1), we adopt the following basic assumptions:

(G1) There exists a constant $M > 0$ such that $f : [c,\eta(d)]_{\mathbb{T}} \times [0,+\infty) \to [-M,+\infty)$ is continuous.

(G2) For any $t \in [c,\eta(d)]_{\mathbb{T}}$, $\int_c^{\eta(d)} b_-(t)\nabla t = r > 0$ and

$$0 < \int_c^{\eta(d)} \omega(t)[a(t) + b_+(t) + M]\nabla t = L_1 < +\infty,$$

where

$$b_+(t) := \max\{b(t), 0\}, \ \ b_-(t) = \max\{-b(t), 0\}$$

and M is given by **(G1)**.

Now let $\Gamma := (\eta(d) - c)M + r$, where M and r are defined by **(G1)** and **(G2)**, then we list some additional assumptions:

(G3) There exists a constant $r_1 > \max\{L_1, \frac{2\Gamma}{H}\}$ such that for $(t,z) \in [c,\eta(d)]_{\mathbb{T}} \times [0,r_1]$,

$$f(t,z) \leq \frac{r_1}{L_1} - 1.$$

(G4) There exist t_1, t_2 such that $c \leq t_1 < \eta(t_2) \leq \eta(d)$ satisfying

$$\lim_{z \to +\infty} \min_{t \in [t_1,\eta(t_2)]_{\mathbb{T}}} \frac{f(t,z)}{z} = +\infty.$$

(**G5**) There exists a constant $R > \frac{2\Gamma}{H}$ such that, for any $(t,z) \in [c,\eta(d)]_{\mathbb{T}} \times [\frac{H\beta\delta R}{2\rho}, R]$

$$f(t,z) \geq \frac{R}{l},$$

where

$$l = \frac{H\beta\delta}{\rho}\int_{c}^{\eta(d)} \omega(t)a(t)\nabla t.$$

(**G6**)

$$\lim_{z\to+\infty}\max_{t\in[c,\eta(d)]_{\mathbb{T}}}\frac{f(t,z)}{z} = 0.$$

□

Lemma 3. *Assume that $\kappa(t)$ is a positive solution for the following boundary value problem*

$$\begin{cases} -(z^{\triangle})^{\nabla}(t) = b_{-}(t) + M, \ \ t\in[c,\eta(d)]_{\mathbb{T}}, \\ \alpha z(c) - \beta z^{\triangle}(c) = 0, \\ \gamma z(\eta(d)) + \delta z^{\triangle}(\eta(d)) = 0, \end{cases} \tag{6}$$

then $\kappa(t)\le\Gamma\omega(t), t\in[c,\eta(d)]_{\mathbb{T}}$.

Proof. For $t\in[c,\eta(d)]_{\mathbb{T}}$, one has

$$\begin{aligned} \kappa(t) &= \int_{c}^{\eta(d)} G(t,s)(b_{-}(s)+M)\nabla s \\ &\le \omega(t)\int_{c}^{\eta(d)}(b_{-}(s)+M)\nabla s \\ &\le ((\eta(d)-c)M + r)\omega(t) = \Gamma\omega(t). \end{aligned}$$

Thus, we complete the proof of Lemma 3. □

Next, for $y\in[c,\eta(d)]_{\mathbb{T}}$, define a star function

$$y^{*}(t) = \begin{cases} y(t), \ \ y(t)\ge 0, \\ 0\,, \ \ \ y(t) < 0, \end{cases}$$

then we state the following lemma:

Lemma 4. *Let $\kappa(t)$ be the positive solution of the boundary value problem (6), then z is a positive solution of the boundary value problem (1) if and only if $u = z+\kappa$ satisfying $u(t) > \kappa(t)$, $t\in[c,\eta(d)]_{\mathbb{T}}$ is a solution of the following boundary value problem:*

$$\begin{cases} (u^{\triangle})^{\nabla}(t) + a(t)f(t,(u-\kappa)^{*}(\eta(t))) + b_{+}(t) + M = 0, \ \ t\in[c,\eta(d)]_{\mathbb{T}} \\ \alpha u(c) - \beta u^{\triangle}(c) = 0 \\ \gamma u(\eta(d)) + \delta u^{\triangle}(\eta(d)) = 0 \end{cases} \tag{7}$$

Proof. Firstly, we shall show the necessary condition of Lemma 4. It follows from Lemma 3 that $(\kappa^{\triangle})^{\nabla}(t) = -M - b_{-}(t)$. If z is a positive solution of problem (1), then we have $z(t) = u(t)-\kappa(t) > 0$, $t\in(c,\eta(d))_{\mathbb{T}}$, i.e., $u(t)>\kappa(t)$, for any $t\in(c,\eta(d))_{\mathbb{T}}$. Thus, one obtains

$$\begin{aligned} &(u^{\triangle})^{\nabla}(t) + a(t)f(t,(u-\kappa)^{*}(\eta(t))) + b_{+}(t) + M \\ &= (u^{\triangle})^{\nabla}(t) + a(t)f(t,(u-\kappa)(\eta(t))) + b_{+}(t) + M \\ &= (z^{\triangle})^{\nabla}(t) + (\kappa^{\triangle})^{\nabla}(t) + a(t)f(t,z(\eta(t))) + b_{+}(t) + M \\ &= (z^{\triangle})^{\nabla}(t) + a(t)f(t,z(\eta(t))) + b_{+}(t) - b_{-}(t) \\ &= (z^{\triangle})^{\nabla}(t) + a(t)f(t,z(\eta(t))) + b(t) = 0. \end{aligned}$$

Due to boundary conditions, we clearly have $\alpha u(c) - \beta u^{\triangle}(c) = 0$. Notice that

$$\begin{aligned}
&\gamma u(\eta(d)) + \delta u^{\triangle}(\eta(d)) \\
&= \gamma[z(\eta(d)) + \kappa(\eta(d))] + \delta[z^{\triangle}(\eta(d)) + \kappa^{\triangle}(\eta(d))] \\
&= [\gamma z(\eta(d)) + \delta z^{\triangle}(\eta(d))] + [\gamma\kappa(\eta(d)) + \delta\kappa^{\triangle}(\eta(d))] \\
&= 0.
\end{aligned}$$

Therefore, the boundary conditions are satisfied, which implies that u is a solution of the boundary value problem (7).

Next, we prove the sufficiency. If $u > \kappa$ is a solution of the boundary value problem (7), then we assert that $z = u - \kappa$ is a positive solution of the boundary value problem (1). In fact,

$$\begin{aligned}
&(z^{\triangle})^{\nabla}(t) + a(t)f(t, z(\eta(t))) + b(t) \\
&= (u^{\triangle})^{\nabla}(t) - (\kappa^{\triangle})^{\nabla}(t) + a(t)f(t, (u-\kappa)(\eta(t))) + b(t) \\
&= (u^{\triangle})^{\nabla}(t) + a(t)f(t, (u-\kappa)^*(\eta(t))) + b_+(t) - b_-(t) - (\kappa^{\triangle})^{\nabla}(t) \\
&= (u^{\triangle})^{\nabla}(t) + a(t)f(t, (u-\kappa)^*(\eta(t))) + b_+(t) + M = 0.
\end{aligned}$$

Similarly, we obtain $\alpha z(c) - \beta z^{\triangle}(c) = 0$, $\gamma z(\eta(d)) + \delta z^{\triangle}(\eta(d)) = 0$, thus z is a positive solution of the boundary value problem (1). □

Lemma 5 ([26,27]). *Let E be a Banach space, and $P \subset E$ be a cone in E. Let Ω_1, Ω_2 be bounded open subsets of E with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and $S : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that, either*

$$\begin{aligned}
&(1)||Sz|| \le ||z||, z \in P \cap \partial\Omega_1; ||Sz|| \ge ||z||, z \in P \cap \partial\Omega_2, or \\
&(2)||Sz|| \ge ||z||, z \in P \cap \partial\Omega_1; ||Sz|| \le ||z||, z \in P \cap \partial\Omega_2.
\end{aligned}$$

Then S has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main Results

Before presenting the main results and the proofs, we give the following remark:

Remark 1. *From the definition of y^*, for $u \in [c, \eta(d)]_{\mathbb{T}}$, we have*

$$[u(s) - \kappa(s)]^* \le u(s) \le ||u||.$$

Theorem 1. *Assume **(G1) – (G4)** hold, then the boundary value problem (1) has at least one positive solution $z(t)$, and there exists a constant $\mathcal{M} > 0$, such that $z(t) \ge \mathcal{M}\omega(t)$ for any $t \in [c, \eta(d)]_{\mathbb{T}}$.*

Proof. Let $\kappa(t)$ be the solution of the boundary value problem (6). It follows from Lemma 3 that it is sufficient to prove that the following boundary value problem

$$\begin{cases}
(u^{\triangle})^{\nabla}(t) + a(t)f(t, (u-\kappa)^*(\eta(t))) + b_+(t) + M = 0, \quad t \in [c, \eta(d)]_{\mathbb{T}} \\
\alpha u(c) - \beta u^{\triangle}(c) = 0, \\
\gamma u(\eta(d)) + \delta u^{\triangle}(\eta(d)) = 0.
\end{cases} \tag{8}$$

has at least a solution u satisfying $u(t) > \kappa(t)$, $t \in [c, \eta(d)]_{\mathbb{T}}$.

To do this, define an operator $S : P \to E$ given by

$$(Su)(t) = \int_c^{\eta(d)} G(t,s)[a(s)f(s, (u-\kappa)^*(\eta(s))) + b_+(s) + M]\nabla s, \quad t \in [c, \eta(d)]_{\mathbb{T}}.$$

In the following, we show that the operator S is well defined and $S(P) \subset P$ is completely continuous.

In fact, for any fixed $u \in P$, we can find a positive constant L such that $||u|| \leq L$. It follows from Remark 1 that

$$(u-\kappa)^*(\eta(s)) \leq u(\eta(s)) \leq L,$$

thus, for any $t \in [c,\eta(d)]_{\mathbb{T}}$, by letting $\widetilde{N} = \max_{[c,\eta(d)]_{\mathbb{T}}\times[0,L]} f(t,u)$, by Lemma 2 and **(G2)**, one has

$$\begin{aligned}(Su)(t) &= \int_c^{\eta(d)} G(t,s)[a(s)f(s,(u-\kappa)^*(\eta(s))) + b_+(s) + M]\nabla s \\ &\leq \int_c^{\eta(d)} \omega(s)[\widetilde{N}a(s) + b_+(s) + M]\nabla s \\ &\leq (\widetilde{N}+1)\int_c^{\eta(d)} \omega(s)[a(s) + b_+(s) + M]\nabla s < +\infty,\end{aligned}$$

which implies that S is well defined.

Next we prove that $S(P) \subset P$. In the view of Lemma 2, for any $u \in P, t \in [c,\eta(d)]_{\mathbb{T}}$, one has

$$\begin{aligned}(Su)(t) &= \int_c^{\eta(d)} G(t,s)[a(s)f(s,(u-\kappa)^*(\eta(s))) + b_+(s) + M]\nabla s \\ &\leq \int_c^{\eta(d)} \omega(s)[a(s)f(s,(u-\kappa)^*(\eta(s))) + b_+(s) + M]\nabla s,\end{aligned}$$

and then

$$||Su|| \leq \int_c^{\eta(d)} \omega(s)[a(s)f(s,(u-\kappa)^*(\eta(s))) + b_+(s) + M]\nabla s.$$

Similarly, we have

$$(Su)(t) \geq H\omega(t)\int_c^{\eta(d)} \omega(s)[a(s)f(s,(u-\kappa)^*(\eta(s))) + b_+(s) + M]\nabla s \geq H\omega(t)||Su||,$$

which yields $S(P) \subset P$.

According to the strategy of [21] and the Ascoli–Arzela theorem, we know that $S(P) \subset P$ is a completely continuous operator.

Now let $\Omega_1 = \{u \in E \mid ||u|| < r_1\}$ and $\partial\Omega_1 = \{u \in E \mid ||u|| = r_1\}$. In the following, we show that $||Su|| \leq ||u||$, for $u \in P \cap \partial\Omega_1$. In fact, for any $u \in P \cap \partial\Omega_1$, by Remark 1, we have

$$0 \leq [u(\eta(t)) - \kappa(\eta(t))]^* \leq u(\eta(t)) \leq ||u|| \leq r_1.$$

Thus by means of (**G3**) and Lemma 2, one has

$$\begin{aligned}||Su|| &= \sup_{t\in[c,\eta(d)]_{\mathbb{T}}} (Su)(t) \\ &= \sup_{t\in[c,\eta(d)]_{\mathbb{T}}} \int_c^{\eta(d)} G(t,s)[a(s)f(s,(u-\kappa)^*(\eta(s))) + b_+(s) + M]\nabla s \\ &\leq \int_c^{\eta(d)} \omega(s)[a(s)f(s,(u-\kappa)^*(\eta(s))) + b_+(s) + M]\nabla s \\ &\leq \int_c^{\eta(d)} \omega(s)\left[\left(\frac{r_1}{L_1} - 1\right)a(s) + b_+(s) + M\right]\nabla s \\ &\leq \frac{r_1}{L_1}\int_c^{\eta(d)} \omega(s)[a(s) + b_+(s) + M]\nabla s \\ &\leq r_1 = ||u||,\end{aligned}$$

i.e., $||Su|| \le ||u||, \ u \in \partial\Omega_1 \cap P$.

On the other hand, by taking

$$\eta = \left\{ \frac{H^2\beta^2\delta^2}{2\rho^2} \int_{t_1}^{\eta(t_2)} \omega(s)a(s)\nabla s \right\}^{-1}, \tag{9}$$

it follows from **(G4)** that there exists $N > 0$ such that for any $u > N$

$$f(t,u) > \eta u. \tag{10}$$

Let

$$r_2 = r_1 + \frac{2\rho N}{H\beta\delta},$$

and $\Omega_2 = \{u \in E \mid ||u|| < r_2\}$, $\partial\Omega_2 = \{u \in E \mid ||u|| = r_2\}$. We shall prove $||Su|| \ge ||u||$, $u \in P \cap \partial\Omega_2$.

By Lemma 3 and the definition of P, for any $t \in [c, \eta(d)]_{\mathbb{T}}, u \in P \cap \partial\Omega_2$, we obtain

$$\kappa(t) \le \Gamma\omega(t) \le \Gamma \frac{u(t)}{H||u||} = \frac{\Gamma}{Hr_2} u(t).$$

Since $r_2 > r_1 > 2\frac{\Gamma}{H}$, we have $\frac{\Gamma}{Hr_2} < \frac{1}{2}$, thus for any $t \in [c, \eta(d)]_{\mathbb{T}}$ and $u \in P \cap \partial\Omega_2$, one has

$$\begin{aligned} u(t) - \kappa(t) &\ge \left(1 - \frac{\Gamma}{Hr_2}\right) u(t) \ge \frac{1}{2} u(t) \\ &\ge \frac{1}{2} Hr_2\omega(t) \ge \frac{Hr_2\beta\delta}{2\rho} > N > 0. \end{aligned} \tag{11}$$

It follows from (8)–(11) that

$$\begin{aligned} ||Su|| &= \sup_{t \in [c,\eta(d)]_{\mathbb{T}}} |\,(Su)(t)\,| \\ &= \sup_{t \in [c,\eta(d)]_{\mathbb{T}}} \int_c^{\eta(d)} G(t,s)[a(s)f(s,(u-\kappa)^*(\eta(s))) + b_+(s) + M]\nabla s \\ &\ge \sup_{t \in [c,\eta(d)]_{\mathbb{T}}} \int_{t_1}^{\eta(t_2)} G(t,s)[a(s)f(s,(u-\kappa)^*(\eta(s))) + b_+(s) + M]\nabla s \\ &\ge \sup_{t \in [c,\eta(d)]_{\mathbb{T}}} \int_{t_1}^{\eta(t_2)} G(t,s)a(s)f(s,(u-\kappa)(\eta(s)))\nabla s \\ &\ge \sup_{t \in [c,\eta(d)]_{\mathbb{T}}} \int_{t_1}^{\eta(t_2)} G(t,s)a(s)\eta(u-\kappa)(\eta(s))\nabla s \\ &\ge \sup_{t \in [c,\eta(d)]_{\mathbb{T}}} \int_{t_1}^{\eta(t_2)} G(t,s)a(s)\eta\frac{Hr_2\beta\delta}{2\rho}\nabla s \\ &\ge \left(\frac{H^2\eta\beta\delta}{2\rho} \sup_{t \in [c,\eta(d)]_{\mathbb{T}}} \omega(t) \int_{t_1}^{\eta(t_2)} \omega(s)a(s)\nabla s \right) r_2 \\ &\ge \left(\frac{H^2\eta\beta^2\delta^2}{2\rho^2} \int_{t_1}^{\eta(t_2)} \omega(s)a(s)\nabla s \right) r_2 \\ &\ge r_2 = ||u||, \end{aligned}$$

i.e., $||Su|| \ge ||u||$, $u \in P \cap \partial\Omega_2$.

Thus, it follows from Lemma 5 that S has a fixed point $u \in P$ such that $u = Su$ and $r_1 \le ||u|| \le r_2$, which also implies that u is a solution of the problem (8).

Finally, in order to prove that $z = u - \kappa$ is a positive solution of (1), we only need to show $u > \kappa,\ t \in [c, \eta(d)]_{\mathbb{T}}$. In fact, for any $t \in [c, \eta(d)]_{\mathbb{T}}$, as $u \in P$ and $r_1 > \frac{2\Gamma}{H} > \frac{\Gamma}{H}$, we have

$$u(t) \geq H\omega(t)||u|| \geq Hr_1\omega(t) \geq \frac{Hr_1}{\Gamma}\kappa(t) > \kappa(t) > 0.$$

Let $z(t) = u(t) - \kappa(t)$, then it follows from (11) that

$$z(t) \geq \frac{1}{2}Hr_2\omega(t) = \mathcal{M}\omega(t), \tag{12}$$

where $\mathcal{M} = \frac{Hr_2}{2}$.

Thus Lemma 4 guarantees that $z = u - \kappa$ is a positive solution of the problem (1). The proof of Theorem 1 is completed. □

Theorem 2. *Assume that* **(G1)–(G2)** *and* **(G5)–(G6)** *are satisfied, then the problem (1) has at least one positive solution $z(t)$, and there exists a positive constant $\mathcal{N}$ such that $z(t) \geq \mathcal{N}\omega(t)$ for any $t \in [c, \eta(d)]_{\mathbb{T}}$.*

Proof. We still define the operator $S : P \to E$ as follows:

$$(Su)(t) = \int_c^{\eta(d)} G(t,s)[a(s)f(s,(u-\kappa)^*(\eta(s))) + b_+(s) + M]\nabla s, \quad t \in [c, \eta(d)]_{\mathbb{T}}.$$

By Theorem 1, we know that $S(P) \subset P$ is a completely continuous operator.

Let $\Omega_3 = \{u \in P : ||u|| < R\}$ and $\partial\Omega_3 = \{u \in P : ||u|| = R\}$. Then, for any $u \in \partial\Omega_3, t \in [c, \eta(d)]_{\mathbb{T}}$, noticing $R > \frac{2\Gamma}{H}$ and Lemma 3, we have

$$\begin{aligned} u(t) - \kappa(t) &\geq u(t) - \Gamma\omega(t) \geq u(t) - \frac{\Gamma u(t)}{HR} \\ &\geq \frac{1}{2}u(t) \geq \frac{H}{2}\omega(t)R \geq 0. \end{aligned} \tag{13}$$

Thus, for any $u \in \partial\Omega_3, t \in [c, \eta(d)]_{\mathbb{T}}$, noticing that $\omega(t) \geq \frac{\beta\delta}{\rho}$, we have

$$\frac{H\beta\delta R}{2\rho} \leq u(t) - \kappa(t) \leq R. \tag{14}$$

It follows from (**G5**), (13), (15) and Lemma 2 that, for any $u \in \partial\Omega_3, t \in [c, \eta(d)]_{\mathbb{T}}$,

$$\begin{aligned} ||Su|| &\geq \int_c^{\eta(d)} G(t,s)[a(s)f(s,(u-\kappa)^*(\eta(s))) + b_+(s) + M]\nabla s \\ &\geq \int_c^{\eta(d)} G(t,s)a(s)f(s,(u-\kappa)(\eta(s)))\nabla s \\ &\geq \int_c^{\eta(d)} H\omega(t)\omega(s)a(s)\frac{R}{l}\nabla s \\ &\geq \frac{\beta\delta HR}{\rho l}\int_c^{\eta(d)} \omega(s)a(s)\nabla s = R = ||u||. \end{aligned}$$

So, we have

$$||Su|| \geq ||u||, u \in P \cap \partial\Omega_3.$$

Next, choose a small $\varepsilon > 0$ such that

$$\varepsilon\int_c^{\eta(d)} \omega(s)a(s)\nabla s < 1.$$

Then, for the above ε, it follows from (**G6**) that there exists $N > R > 0$ such that, for any $t \in [c, \eta(d)]_{\mathbb{T}}$,

$$f(t,u) \leq \varepsilon|u|, \;\text{ if }\; |u| > N. \tag{15}$$

Take

$$R^* = \frac{(\chi+1)L_1}{1 - \varepsilon \int_c^{\eta(d)} \omega(s)a(s)\nabla s} + N,$$

where

$$\chi = \max_{(t,u)\in[c,\eta(d)]_{\mathbb{T}}\times[0,N]} f(t,u) + 1.$$

Then $R^* > N > R$.

Now let $\Omega_4 = \{u \in P : ||u|| < R^*\}$ and $\partial\Omega_4 = \{u \in P : ||u|| = R^*\}$. Then, for any $u \in P \cap \partial\Omega_4$, one has

$$\begin{aligned}
||Su|| &= \sup_{t\in[c,\eta(d)]_{\mathbb{T}}} (Su)(t) \\
&= \sup_{t\in[c,\eta(d)]_{\mathbb{T}}} \int_c^{\eta(d)} G(t,s)[a(s)f(s,(u-\kappa)^*(\eta(s))) + b_+(s) + M]\nabla s \\
&\leq \int_c^{\eta(d)} \omega(s)[a(s)f(s,(u-\kappa)^*(\eta(s))) + b_+(s) + M]\nabla s \\
&\leq \int_c^{\eta(d)} \omega(s)\left[\left(\max_{(s,u)\in[c,\eta(d)]_{\mathbb{T}}\times[0,N]} f(s,u)\right)a(s) + b_+(s) + M\right]\nabla s \\
&\quad + \int_c^{\eta(d)} \omega(s)[\varepsilon||u||a(s) + b_+(s) + M]\nabla s \\
&\leq \chi \int_c^{\eta(d)} \omega(s)[a(s) + b_+(s) + M]\nabla s + \varepsilon R^* \int_c^{\eta(d)} \omega(s)a(s)\nabla s \\
&\quad + \int_c^{\eta(d)} \omega(s)[b_+(s) + M]\nabla s \\
&\leq (\chi+1)L_1 + \varepsilon R^* \int_c^{\eta(d)} \omega(s)a(s)\nabla s < R^* = ||u||,
\end{aligned}$$

which implies that

$$||Su|| \leq ||u||, u \in P \cap \partial\Omega_4.$$

By Lemma 5, S has one fixed point u such that $R \leq ||u|| \leq R^*$.

On the other hand, from (13), we have

$$u(t) - \kappa(t) \geq \frac{H}{2}\omega(t)R \geq \frac{H\beta\delta}{2\rho}R > 0, \tag{16}$$

which implies that $u(t) > \kappa(t), t \in [c, \eta(d)]_{\mathbb{T}}$.

Let $z(t) = u(t) - \kappa(t)$, then

$$z(t) \geq \frac{RH}{2}\omega(t) = \mathcal{N}\omega(t), t \in [c,\eta(d)]_{\mathbb{T}},$$

where $\mathcal{N} = \frac{RH}{2}$.

Thus by Lemma 4, $z = u - \kappa$ is a positive solution of the problem (1) satisfying $z(t) \geq \mathcal{N}\omega(t)$. □

4. Examples

Example 1. *Let* $\mathbb{T} = \left\{\frac{1}{2^n}\right\}_{n=0}^{\infty} \cup \{0,1\}$. *Consider the following boundary value problem*

$$\begin{cases} (z^{\triangle})^{\nabla}(t) + \dfrac{z^2-2}{8(2-t)} - \dfrac{1}{\sqrt{t}} = 0\,, \quad t \in [0,1]_{\mathbb{T}} \\ z(0) - z^{\Delta}(0) = 0, \\ z(1) + z^{\Delta}(1) = 0. \end{cases} \tag{17}$$

From Theorem 1, it can be established that the boundary value problem (17) has at least one positive solution and there exists a positive constant $\mathcal{M}$, *such that* $z(t) \geq \frac{\mathcal{M}}{3}(1+t)(2-t)$ *for any* $t \in [0,1]_{\mathbb{T}}$.

To obtain the above result, first take

$$c = 0, \;\; \eta(d) = 1, \;\; \alpha = \gamma = \beta = \delta = 1, \; f(t,z) = \frac{1}{4}(z^2-2), b(t) = -\frac{1}{\sqrt{t}}, \;\; a(t) = \frac{1}{2(2-t)},$$

then we have $b_+(t) = 0, \;\; b_-(t) = \frac{1}{\sqrt{t}}$ *and*

$$f(t,z) = \frac{1}{4}(z^2-2) \geq -\frac{1}{2},$$

i.e., $M = \frac{1}{2}$, *which implies* ***(G1)*** *holds.*

Consequently, we also have $\rho = \beta\gamma + \alpha\delta + \alpha\gamma(\eta(d)-c) = 3 > 0$ *and*

$$\omega(t) = \frac{1}{\rho}[\alpha(t-c)+\beta][\gamma(\eta(d)-t)+\delta] = \frac{1}{3}(1+t)(2-t),$$

$$r = \int_c^{\eta(d)} b_-(t)\nabla t = \int_0^1 \frac{1}{\sqrt{t}}\nabla t = 1 + \frac{1}{\sqrt{2}} \approx 1.71 > 0$$

and

$$\begin{aligned} L_1 &= \int_c^{\eta(d)} \omega(t)[a(t)+b_+(t)+M]\nabla t \\ &= \int_0^1 \frac{1}{3}(1+t)(2-t)[\frac{1}{2(2-t)} + 0 + \frac{1}{2}]\nabla t \\ &= \frac{1}{3}\left(\int_0^1 t\nabla t - \frac{1}{2}\int_0^1 t^2\nabla t + \int_0^1 \frac{3}{2}\nabla t\right) \\ &= \frac{1}{3}\left[1\cdot\frac{1}{2} + \frac{1}{2}\cdot(\frac{1}{2}-\frac{1}{4}) + \frac{1}{4}\cdot(\frac{1}{4}-\frac{1}{8}) + \dots\right] \\ &\quad - \frac{1}{6}\left[1\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{16}\cdot\frac{1}{8} + \dots\right] + \frac{1}{2} \\ &= \frac{2}{9} - \frac{1}{6}\cdot\frac{4}{7} + \frac{1}{2} = \frac{79}{126} < +\infty. \end{aligned}$$

Thus, ***(G2)*** *is satisfied.*

For ***(G3)***, *since* $\Gamma := (\eta(d)-c)M + r = 2.21$ *and* $H = \frac{\rho}{[\alpha(\eta(d)-c)+\beta][\gamma(\eta(d)-c)+\delta]} = \frac{3}{4}$, *we obtain that* $r_1 := 50 > \max\{L_1, \frac{2\Gamma}{H}\} = 5.89$, *for any* $(t,z) \in [0,1]_{\mathbb{T}} \times [0,50]$, *then we have*

$$f(t,z) \leq 75 < 78.75 = \frac{r_1}{L_1} - 1,$$

which implies that ***(G3)*** *is satisfied.*

Finally, let $t_1 = \frac{1}{4},\ t_2 = \frac{3}{4}$*, then we have*

$$\lim_{z\to+\infty} \min_{t\in[t_1,\eta(t_2)]_{\mathbb{T}}} \frac{f(t,z)}{z} = \lim_{z\to+\infty} \min_{t\in[\frac{1}{4},\eta(\frac{3}{4})]_{\mathbb{T}}} \frac{(z^2-2)}{4z} = +\infty.$$

Thus, ***(G4)*** *is also satisfied.*

Consequently, Theorem 1 guarantees that the problem (17) has at least one positive solution $z(t)$*, and there exists a positive constant* $\mathcal{M}$*, such that* $z(t) \geq \frac{\mathcal{M}}{3}(1+t)(2-t)$ *for any* $t \in [0,1]_{\mathbb{T}}$*.*

Example 2. *Let* $\mathbb{T} = \left\{\frac{1}{2^n}\right\}_{n=0}^{\infty} \cup \{0,1\}$*. Consider the following boundary value problem (18)*

$$\begin{cases} (z^{\triangle})^{\nabla}(t) + \dfrac{\sqrt{z}-4}{2(2-t)} - \dfrac{1}{\sqrt{t}} = 0\,, \quad t \in [0,1]_{\mathbb{T}} \\ z(0) - 2z^{\Delta}(0) = 0, \\ z(1) + z^{\Delta}(1) = 0. \end{cases} \tag{18}$$

Then, from Theorem 2, we can establish that the boundary value problem (18) has at least one positive solution and there exists a positive constants $\mathcal{N}$*, such that* $z(t) \geq \frac{\mathcal{N}}{4}(2+t)(2-t)$ *for any* $t \in [0,1]_{\mathbb{T}}$*.*

To derive the above result, first take

$$c = 0,\ \ \eta(d) = 1,\ \ \alpha = \gamma = \delta = 1,\ \beta = 2\, f(t,z) = \sqrt{z} - 4, b(t) = -\frac{1}{\sqrt{t}}\ \ a(t) = \frac{1}{2(2-t)},$$

and we obtain $f(t,z) = \sqrt{z} - 4 \geq -4$*; therefore, we can take* $M = 4$*.*

Moreover, one has $\rho = \beta\gamma + \alpha\delta + \alpha\gamma(\eta(d) - c) = 4 > 0$*,*

$$\omega(t) = \frac{1}{\rho}[\alpha(t-c) + \beta][\gamma(\eta(d) - t) + \delta] = \frac{1}{4}(2+t)(2-t),$$

and

$$r = \int_c^{\eta(d)} b_-(t)\nabla t = \int_0^1 \frac{1}{\sqrt{t}}\nabla t = 1 + \frac{1}{\sqrt{2}} \approx 1.71 > 0,$$

$$\begin{aligned} L_1 &= \int_c^{\eta(d)} \omega(t)[a(t) + b_+(t) + M]\nabla t \\ &= \int_0^1 \frac{1}{4}(2+t)(2-t)[\frac{1}{2(2-t)} + 0 + 4]\nabla t \\ &= \frac{1}{4}\left(\frac{1}{2}\int_0^1 t + 2\nabla t + \int_0^1 4(4-t^2)\nabla t\right) \\ &= \frac{1}{8}\int_0^1 t\nabla t - \int_0^1 t^2\nabla t + \frac{17}{4} \\ &= \frac{1}{8}\left[1\cdot\frac{1}{2} + \frac{1}{2}\cdot(\frac{1}{2} - \frac{1}{4}) + \frac{1}{4}\cdot(\frac{1}{4} - \frac{1}{8}) + \dots\right] \\ &\quad - \left[1\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{16}\cdot\frac{1}{8} + \dots\right] + \frac{17}{4} \\ &= \frac{1}{12} - \frac{1}{7} + \frac{17}{4} = \frac{79}{21} < +\infty. \end{aligned}$$

Clearly, ***(G1)***, ***(G2)*** *are satisfied.*

Take

$$H = \frac{\rho}{[\alpha(\eta(d) - c) + \beta][\gamma(\eta(d) - c) + \delta]} = \frac{2}{3},\ \Gamma := (\eta(d) - c)M + r = 5.71.$$

Let $R := 30 > \frac{2\Gamma}{H} = 17.13$, *for any* $(t,z) \in [0,1]_{\mathbb{T}} \times [5,30]$, *we have*

$$\begin{aligned} l &= \frac{H\beta\delta}{\rho}\int_c^{\eta(d)} \omega(t)a(t)\nabla t \\ &= \frac{1}{24}\int_0^1 (t+2)\nabla t \\ &= \frac{1}{24}\left[\int_0^1 t\nabla t + \int_0^1 2\nabla t\right] \\ &= \frac{1}{24}\left[1\cdot\frac{1}{2} + \frac{1}{2}\cdot(\frac{1}{2}-\frac{1}{4}) + \frac{1}{4}\cdot(\frac{1}{4}-\frac{1}{8}) + \dots\right] + \frac{1}{12} \\ &= \frac{1}{9} \end{aligned}.$$

and

$$f(t,z) \geq 396 > 270 = \frac{R}{l}.$$

Then, ***(G5)*** *is satisfied. Furthermore, we also have*

$$\lim_{z\to+\infty}\max_{t\in[c,\eta(d)]_{\mathbb{T}}} \frac{f(t.z)}{z} = \lim_{z\to+\infty}\max_{t\in[0,1]_{\mathbb{T}}} \frac{\sqrt{z}-4}{z} = 0.$$

Thus, ***(G6)*** *is also satisfied.*

Therefore, the hypotheses ***(G1)*** $-$ ***(G2)*** *and* ***(G5)*** $-$ ***(G6)*** *are satisfied, and thus Theorem 2 guarantees the boundary value problem (18) has at least one positive solution* $z(t)$*, and there exists a positive constants* $\mathcal{N}$*, such that* $z(t) \geq \frac{\mathcal{N}}{4}(2+t)(2-t)$ *for any* $t \in [0,1]_{\mathbb{T}}$*.*

5. Conclusions

In this paper, we derived the existence results of positive solution for a singular Sturm–Liouville boundary value problem with semipositone and the disturbance term on time scales. By using a translation transformation and combining with the properties of solution of the nonhomogeneous boundary value problem, several sufficient conditions for the existence of positive solutions are established. Compared to the previous work, we not only focus on the case in which the nonlinearity is mixed and the disturbance term tends to negative infinity, but also in the case where the weight function a and the disturbance term b are singular and can have finitely many singularities in the time scales interval $[c,\eta(d)]_{\mathbb{T}}$.

Author Contributions: Investigation and formal analysis, X.Z. and H.T.; writing-original draft, H.T.; writing-review and editing, X.Z., Y.W. and B.W. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the National Natural Science Foundation of China (11871302).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors are grateful to the anonymous referees for his or her constructive comments which led to improvement of the original manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

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