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Article

Radially Symmetric Solution for Fractional Laplacian Systems with Different Negative Powers

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Abstract: By developing the direct method of moving planes, we study the radial symmetry of nonnegative solutions for a fractional Laplacian system with different negative powers: $(-\Delta)^{\frac{\alpha}{2}}u(x) + u^{-\gamma}(x) + v^{-q}(x) = 0, x \in R^N$, $(-\Delta)^{\frac{\beta}{2}}v(x) + v^{-\sigma}(x) + u^{-p}(x) = 0, x \in R^N$, $u(x) \gtrsim |x|^a, v(x) \gtrsim |x|^b$as $|x| \to \infty$, where $\alpha, \beta \in (0,2)$, and $a, b > 0$ are constants. We study the decay at infinity and narrow region principle for the fractional Laplacian system with different negative powers. The same results hold for nonlinear Hénon-type fractional Laplacian systems with different negative powers.

Keywords: fractional Laplacian systems; different negative powers; radial symmetry; the direct method of moving planes

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1. Introduction

Equations with negative powers appear in various models, such as Micro-Electro-Mechanical system (MEMS) devices [1], which have become a key part of many commercial systems, dynamics of thin-film of viscous fluids [2,3], reaction–diffusion processes [4], singular minimal hypersurfaces related to chemical catalyst kinetics [5], prescribed curvature equations in conformal geometry [6], and the Lichnerowicz equation on closed Riemannian- or Liouville-type manifolds [7], etc.

In view of the wide application of a fractional Laplacian in physical sciences, probability, and finance, etc. [8–11], many scholars have shown a great interest in various semi-linear equations and systems with fractional Laplacian operators; for instance, see [12–15]. A fractional Laplacian is a kind of nonlocal linear pseudo-differential operator, which generally adopts the form:

$$(-\Delta)^{\frac{\alpha}{2}}u(x) = C_{N,\alpha}PV\int_{R^N}\frac{u(x)-u(y)}{|x-y|^{N+\alpha}}dy,$$

where $\alpha \in (0,2)$, $C_{N,\alpha} > 0$ is a constant, and PV represents the Cauchy principal value. Here, $u(x) \in C^{1,1}_{loc}(R^N) \cap L_\alpha$, $(-\Delta)^{\frac{\alpha}{2}}u(x)$ is well-defined, where

$$L_\alpha = \left\{u \in L^1_{loc}(R^N) \mid \int_{R^N}\frac{|u(x)|}{1+|x|^{N+\alpha}}dx < \infty\right\}.$$

Notice that the fractional Laplacian is a nonlocal operator, and its non-locality poses certain difficulties for research. Caffarelli and Silvestre [16] recommended the extension method to conquer the difficulties caused by non-locality. The main idea of the extension method is to transform the nonlocal problem into a higher-order, one-dimensional local one. The details of this method can be found in [17–19]. In addition, there is an integral equation method such as the moving plane of integral form and regular lifting, in which the moving plane of the integral form is applied to the corresponding equivalent integral problems [20–22]. However, for the application of these two methods, additional conditions need to be added to the equation, or the corresponding equivalent integral equation needs to be obtained, which accounts for the drawbacks of these techniques. In [23], the author pointed out that the extension method or the integral equation approach cannot be applied to the fully nonlinear nonlocal operator

$$H_\beta(u(x)) = C_{N,\beta} PV \int\limits_{R^N} \frac{G(u(x)-u(y))}{|x-y|^{N+\beta}} dy.$$

Thanks to the direct moving planes method proposed by Chen, Li and Li in 2017 [24], which solves nonlocal problems and makes amends for the shortcoming of the fore-mentioned two methods, the method of moving planes has been used widely in the recent years. In [23], the symmetry and monotonicity of positive solutions for fully nonlinear nonlocal equations are discussed by means of the direct method of moving planes. To overcome the degeneracy of fractional p-Laplacian operator, in 2018, Chen, Li [25] proposed boundary estimation instead of the narrow region principle to obtain the properties of nonnegative solutions to fractional p-Laplacian problems. We can refer to [26–37] for the fruitful application of the direct method of moving planes.

Inspired by the above discussion, our main focus in the present article is to apply the direct method of moving planes to study the radial symmetry of the following fractional Laplacian system with different negative powers:

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) + u^{-\gamma}(x) + v^{-q}(x) = 0, & x \in R^N, \\ (-\Delta)^{\frac{\beta}{2}} v(x) + v^{-\sigma}(x) + u^{-p}(x) = 0, & x \in R^N. \end{cases} \tag{1}$$

One of our main tasks is described below.

Theorem 1. *Let $u \in L_\alpha \cap C^{1,1}_{loc}(R^N), v \in L_\beta \cap C^{1,1}_{loc}(R^N)$ be a pair of nonnegative solutions for a fractional Laplacian system with different negative powers (1), and suppose that*

$$u(x) = m|x|^a + o(1), \quad v(x) = n|x|^b + o(1) \quad \text{as } |x| \to \infty, \tag{2}$$

where $1 > a, b > 0$. If $\alpha, \beta \in (0,2)$ satisfy

$$\alpha = \min\{a(\gamma+1), b(q+1)\}, \quad \beta = \min\{b(\sigma+1), a(p+1)\},$$

then $u(x), v(x)$ are radially symmetric about some points on R^N.

The growth/decay condition (2) in the above result was proposed for the fractional Laplacian equation with negative powers in [35]. The purpose of this article is to weaken the growth/decay condition (2) in Theorem 1, so that the conclusion of the theorem remains valid for better growth/decay conditions. We reformulate Theorem 1 by introducing the weaker form of the growth/decay conditions as follows.

Theorem 2. *Let $u \in L_\alpha \cap C^{1,1}_{loc}(R^N), v \in L_\beta \cap C^{1,1}_{loc}(R^N)$ be a pair of nonnegative solutions for a fractional Laplacian system with different negative powers (1). Suppose that*

$$u(x) \gtrsim |x|^a, \quad v(x) \gtrsim |x|^b \quad \text{as } |x| \to \infty,$$

where $a, b > 0$, and $u(x), v(x)$ are monotone increasing in $|x|$ about the origin. If $\alpha, \beta \in (0,2)$ satisfy

$$\alpha = \min\left\{a(\gamma+1), b(q+1)\right\}, \quad \beta = \min\left\{b(\sigma+1), a(p+1)\right\},$$

then $u(x), v(x)$ are radially symmetric about some points on R^N.

In order to prove Theorem 2, we first obtain the decay at infinity and narrow region principle for the fractional Laplacian system (1). Here, our approach is different from the one employed for fractional Laplacian systems without negative powers in [27].

If $u(x)$ and $v(x)$ in system (1) do not have a decay condition at infinity, then we will rely on Kelvin transform to apply the direct method of moving planes. Let x^ι be any point in R^N, and

$$\begin{aligned} \hat{u}(x) &= \frac{1}{|x-x^\iota|^{N-\alpha}} u\Big(\frac{x-x^\iota}{|x-x^\iota|^2} + x^\iota\Big), \quad x \in R^N \backslash \{x^\iota\}, \\ \hat{v}(x) &= \frac{1}{|x-x^\iota|^{N-\beta}} v\Big(\frac{x-x^\iota}{|x-x^\iota|^2} + x^\iota\Big), \quad x \in R^N \backslash \{x^\iota\}, \end{aligned}$$

be the Kelvin transform of $u(x)$ and $v(x)$ centered at x^ι, respectively. As in [27], without loss of generality, let $x^\iota = 0$, then the Kelvin transform of the fractional Laplacian system

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = \mathcal{F}(v(x)), & x \in R^N, \\ (-\Delta)^{\frac{\beta}{2}} v(x) = \mathcal{H}(u(x)), & x \in R^N, \end{cases}$$

yields

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} \hat{u}(x) = \dfrac{1}{|x|^{N+\alpha}} \mathcal{F}(|x|^{N-\beta}\hat{v}(x)), & x \in R^N \backslash \{0\}, \\ (-\Delta)^{\frac{\beta}{2}} \hat{v}(x) = \dfrac{1}{|x|^{N+\beta}} \mathcal{H}(|x|^{N-\alpha}\hat{u}(x)), & x \in R^N \backslash \{0\}. \end{cases}$$

Therefore, the Kelvin transform of system (1) takes the form:

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} \hat{u}(x) + \dfrac{\hat{u}^{-\gamma}(x)}{|x|^{\tau_1}} + \dfrac{\hat{v}^{-q}(x)}{|x|^{o_1}} = 0, & x \in R^N \backslash \{0\}, \\ (-\Delta)^{\frac{\alpha}{2}} \hat{v}(x) + \dfrac{\hat{v}^{-\sigma}(x)}{|x|^{o_2}} + \dfrac{\hat{u}^{-p}(x)}{|x|^{\tau_2}} = 0, & x \in R^N \backslash \{0\}, \end{cases}$$

where

$$\tau_1 = N + \alpha + \gamma(N-\alpha), \quad o_1 = N + \alpha + q(N-\beta),$$

$$\tau_2 = N + \beta + p(N-\alpha), \quad o_2 = N + \beta + \sigma(N-\beta),$$

which is similar to the classic Hénon type system. Thus, one can also consider a Hénon type system with different negative powers as

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) + |x|^{a_1} u^{-\gamma}(x) + |x|^{b_1} v^{-q}(x) = 0, & x \in R^N \backslash \{0\}, \\ (-\Delta)^{\frac{\beta}{2}} v(x) + |x|^{b_2} v^{-\sigma}(x) + |x|^{a_2} u^{-p}(x) = 0, & x \in R^N \backslash \{0\}, \end{cases}$$

where $\gamma, q, \sigma, p > 0$ and

$$0 < a_1 < N + \alpha + \gamma(N-\alpha), \quad 0 < b_1 < N + \alpha + q(N-\beta),$$

$$0 < a_2 < N + \beta + p(N-\alpha), \quad 0 < b_2 < N + \beta + \sigma(N-\beta).$$

More generally, we can explore the Hénon type system:

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) + |x|^{a_1} u^{-\gamma}(x) + |x|^{b_1} v^{-q}(x) = 0, & x \in R^N \backslash \{0\}, \\ (-\Delta)^{\frac{\beta}{2}} v(x) + |x|^{b_2} v^{-\sigma}(x) + |x|^{a_2} u^{-p}(x) = 0, & x \in R^N \backslash \{0\}, \end{cases} \tag{3}$$

where $\gamma, q, \sigma, p > 0$ and

$$-\infty < a_1 < N + \alpha + \gamma(N-\alpha), \quad -\infty < b_1 < N + \alpha + q(N-\beta),$$

$$-\infty < a_2 < N + \beta + p(N-\alpha), \quad -\infty < b_2 < N + \beta + \sigma(N-\beta).$$

In the second part of this paper, we will study the radial symmetry of the nonnegative solutions for the Hénon type nonlinear fractional Laplacian system (3) with different negative powers as follows.

Theorem 3. *Let $u \in L_\alpha \cap C^{1,1}_{loc}(R^N \backslash \{0\}), v \in L_\beta \cap C^{1,1}_{loc}(R^N \backslash \{0\})$ be a pair of nonnegative solutions for the Hénon type system (3). Suppose that*

$$u(x) \gtrsim |x|^a, \quad v(x) \gtrsim |x|^b \quad as \ |x| \to \infty,$$

where $a, b > 0$ and $u(x), v(x)$ are monotone increasing in $|x|$ about the origin. If $\alpha, \beta \in (0,2)$ satisfy

$$\alpha = \min\left\{a(\gamma+1) - a_1, \ b(q+1) - b_1\right\}, \quad \beta = \min\left\{b(\sigma+1) - b_2, \ a(p+1) - a_2\right\},$$

then $u(x), v(x)$ are radially symmetric about the origin.

To the best of our knowledge, a few authors studied the radial symmetry of the nonnegative solutions for fractional Laplacian systems (1) and (3). In comparing with system (1), we point out that system (3) contains a singularity at $x = 0$.

Notation.

For unification of the symbols in the forthcoming analysis, we set

$$P_l = \left\{x \in R^N \mid x_1 = l, \text{ for some } l \in R\right\}$$

as the moving plane,

$$\Sigma_l = \left\{x \in R^N \mid x_1 < l\right\}$$

denotes the left region to P_l,

$$x^l = (2l - x_1, x_2, \cdots, x_n)$$

is the reflection point of x about P_l, and

$$u_l(x) = u(x^l), \quad v_l(x) = v(x^l).$$

In order to compare $u_l(x)$ and $u(x)$, $v_l(x)$ and $v(x)$, and show that $u(x)$ and $v(x)$ are symmetric, we define

$$\varphi_l(x) = u(x) - u_l(x), \quad \psi_l(x) = v(x) - v_l(x),$$

and

$$\Sigma^-_{\varphi_l(x)} = \{x \in \Sigma_l \mid \varphi_l(x) < 0\}, \quad \Sigma^-_{\psi_l(x)} = \{x \in \Sigma_l \mid \psi_l(x) < 0\}.$$

In system (1), by applying the mean value theorem, for $x \in \Sigma_l$, we have

$$\begin{aligned}(-\Delta)^{\frac{\alpha}{2}}\varphi_l(x) &= u_l^{-\gamma}(x) - u^{-\gamma}(x) + v_l^{-q}(x) - v^{-q}(x) \\ &= \gamma\xi_1^{-\gamma-1}(x,l)\varphi_l(x) + q\eta_1^{-q-1}(x,l)\psi_l(x),\end{aligned} \tag{4}$$

and

$$(-\Delta)^{\frac{\beta}{2}}\psi_l(x) = \sigma\eta_2^{-\sigma-1}(x,l)\psi_l(x) + p\xi_2^{-p-1}(x,l)\varphi_l(x), \tag{5}$$

where $\xi_i(x,l)$ lies between $u_l(x)$ and $u(x)$, while $\eta_i(x,l)$ lies between $v_l(x)$ and $v(x)$, $i = 1,2$.

The structure of the remainder of the article is as follows. In Section 2, our main task is to prove the radial symmetry of the nonnegative solutions for a fractional Laplacian system with different negative powers (1), namely Theorem 2. As preparation for applying the direct method of moving planes, the decay at infinity and narrow region principle are proved for (1). In Section 3, we generalize these two principles for the Hénon type system (3) and then prove Theorem 3.

2. Fractional Laplacian System with Different Negative Powers

Theorem 4. *(Decay at infinity) Let $u \in L_\alpha \cap C^{1,1}_{loc}(R^N)$, $v \in L_\beta \cap C^{1,1}_{loc}(R^N)$ be a pair of positive solutions for system (1), and suppose that*

$$u(x) \gtrsim |x|^a, \quad v(x) \gtrsim |x|^b, \quad \text{as } |x| \to \infty,$$

where $a, b > 0$, $\gamma, \sigma, p, q \geq 1$. If $\alpha, \beta \in (0,2)$ satisfy

$$\alpha = \min\{a(\gamma+1), b(q+1)\}, \quad \beta = \min\{b(\sigma+1), a(p+1)\},$$

and

$$\varphi_l(\bar\theta) = \min_{\bar\Sigma_l}\varphi_l(x) < 0, \quad \psi_l(\tilde\theta) = \min_{\bar\Sigma_l}\psi_l(x) < 0,$$

then there exists $R_0 > 0$ such that there is at least one of $\bar\theta$ or $\tilde\theta$ satisfying

$$|x| \leq R_0.$$

Proof. From the condition of the theorem, it is known that there is $\bar\theta \in \bar\Sigma_l$ that satifies

$$\varphi_l(\bar\theta) = \min_{\bar\Sigma_l}\varphi_l(x) < 0.$$

By the anti-symmetry of $\varphi_l(x)$, we can obtain

$$\begin{aligned}(-\Delta)^{\frac{\alpha}{2}}\varphi_l(\bar{\theta}) &= C_{N,\alpha}PV\int_{R^N}\frac{\varphi_l(\bar{\theta})-\varphi_l(y)}{|\bar{\theta}-y|^{N+\alpha}}dy \\ &= C_{N,\alpha}PV\left\{\int_{\Sigma_l}\frac{\varphi_l(\bar{\theta})-\varphi_l(y)}{|\bar{\theta}-y|^{N+\alpha}}dy+\int_{R^N\setminus\Sigma_l}\frac{\varphi_l(\bar{\theta})-\varphi_l(y)}{|\bar{\theta}-y|^{N+\alpha}}dy\right\} \\ &= C_{N,\alpha}PV\left\{\int_{\Sigma_l}\frac{\varphi_l(\bar{\theta})-\varphi_l(y)}{|\bar{\theta}-y|^{N+\alpha}}dy+\int_{\Sigma_l}\frac{\varphi_l(\bar{\theta})-\varphi_l(y^l)}{|\bar{\theta}-y^l|^{N+\alpha}}dy\right\} \\ &\leq C_{N,\alpha}\left\{\int_{\Sigma_l}\frac{\varphi_l(\bar{\theta})-\varphi_l(y)}{|\bar{\theta}-y^l|^{N+\alpha}}dy+\int_{\Sigma_l}\frac{\varphi_l(\bar{\theta})+\varphi_l(y)}{|\bar{\theta}-y^l|^{N+\alpha}}dy\right\} \\ &= 2C_{N,\alpha}\int_{\Sigma_l}\frac{1}{|\bar{\theta}-y^l|}dy\cdot\varphi_l(\bar{\theta}).\end{aligned} \tag{6}$$

For each fixed $l\leq 0$, since $\bar{\theta}\in\Sigma_l$, let $B_{|\bar{\theta}|}(\theta')\subset R^N\setminus\Sigma_l$, where $\theta'=(3|\bar{\theta}|+\bar{\theta}_1,\bar{\theta}_2,\bar{\theta}_3,\cdots,\bar{\theta}_n)$. Then, we have

$$\begin{aligned}\int_{\Sigma_l}\frac{1}{|\bar{\theta}-y^l|^{N+\alpha}}dy &= \int_{R_N\setminus\Sigma_l}\frac{1}{|\bar{\theta}-y|^{N+\alpha}}dy \\ &\geq \int_{B_{|\bar{\theta}|}(\theta')}\frac{1}{|\bar{\theta}-y|^{N+\alpha}}dy \\ &\geq \int_{B_{|\bar{\theta}|}(\theta')}\frac{1}{4^{N+\alpha}|\bar{\theta}|^{N+\alpha}}dy \\ &= \frac{\varpi_N}{4^{N+\alpha}|\bar{\theta}|^{\alpha}},\end{aligned}$$

where $\varpi_N=|B_1(0)|$ in R^N. Hence, it follows that

$$(-\Delta)^{\frac{\alpha}{2}}\varphi_l(\bar{\theta})\leq\frac{2C_{N,\alpha}\varpi_N}{4^{N+\alpha}|\bar{\theta}|^{\alpha}}\varphi_l(\bar{\theta}). \tag{7}$$

Combining (4) and (7), we obtain

$$q\eta_1^{-q-1}(\bar{\theta},l)\psi_l(\bar{\theta})\leq\left[\frac{2C_{N,\alpha}\varpi_N}{4^{N+\alpha}|\bar{\theta}|^{\alpha}}-\gamma\xi_1^{-\gamma-1}(\bar{\theta},l)\right]\varphi_l(\bar{\theta}). \tag{8}$$

Owing to $\bar{\theta}\in\Sigma^-_{\varphi_l(x)}$, $u(x)\gtrsim|x|^a$ as $|x|\to\infty$, and $\alpha\leq a(\gamma+1)$,

$$\left[\frac{2C_{N,\alpha}\varpi_N}{4^{N+\alpha}|\bar{\theta}|^{\alpha}}-\gamma\xi_1^{-\gamma-1}(\bar{\theta},l)\right]\varphi_l(\bar{\theta})\leq\left[\frac{2C_{N,\alpha}\varpi_N}{4^{N+\alpha}|\bar{\theta}|^{\alpha}}-\frac{\gamma}{|\bar{\theta}|^{a(\gamma+1)}}\right]\varphi_l(\bar{\theta})<0, \tag{9}$$

and thus

$$\psi_l(\bar{\theta})<0. \tag{10}$$

In consequence, from (8), we obtain

$$qv^{-q-1}(\bar{\theta})\psi_l(\bar{\theta})\leq\left[\frac{2C_{N,\alpha}\varpi_N}{4^{N+\alpha}|\bar{\theta}|^{\alpha}}-\frac{\gamma}{|\bar{\theta}|^{a(\gamma+1)}}\right]\varphi_l(\bar{\theta}).$$

Since $\alpha = \min\{a(r+1), b(q+1)\}$, and $v(x) \gtrsim |\bar{\theta}|^b$ as $|\bar{\theta}| \to \infty$, there is a constant c_1, such that

$$b_1 \geq c_1 \cdot |\bar{\theta}|^{b(q+1)-\alpha} \to \infty, \quad \text{as } |\bar{\theta}| \to \infty,$$

where

$$b_1 = \left[\frac{2C_{N,\alpha}\varpi_N}{4^{N+\alpha}|\bar{\theta}|^{\alpha}} - \frac{\gamma}{|\bar{\theta}|^{a(\gamma+1)}}\right]\frac{v^{q+1}}{q}.$$

Therefore, there is $|\bar{\theta}| > R_1$, such that $b_1 \geq 2$, and hence

$$\psi_l(\bar{\theta}) < 2\varphi_l(\bar{\theta}).$$

From the condition of the theorem, we can see that there exists $\tilde{\theta}$ such that

$$\psi_l(\tilde{\theta}) = \min_{x \in \bar{\Sigma}_l} \psi_l(x) < 0.$$

Arguing in the same manner for (7), we can obtain

$$(-\Delta)^{\frac{\beta}{2}}\psi_l(\tilde{\theta}) \leq \frac{2C_{N,\beta}\varpi_N}{4^{N+\beta}|\tilde{\theta}|^{\beta}}\psi_l(\tilde{\theta}). \tag{11}$$

Combining (5) and (11), we obtain

$$p\xi_2^{-p-1}(\tilde{\theta}, l)\varphi_l(\tilde{\theta}) \leq \left[\frac{2C_{N,\beta}\varpi_N}{4^{N+\beta}|\tilde{\theta}|^{\beta}} - \sigma\eta_2^{-\sigma-1}(\tilde{\theta}, l)\right]\psi_l(\tilde{\theta}). \tag{12}$$

Since $\tilde{\theta} \in \Sigma^-_{\psi_l(x)}$, $v(x) \gtrsim |x|^b$ as $|x| \to \infty$, and $\beta \leq b(\sigma+1)$, therefore,

$$\varphi_l(\tilde{\theta}) < 0. \tag{13}$$

Hence

$$pu^{-p-1}(\tilde{\theta})\varphi_l(\tilde{\theta}) \leq \left[\frac{2C_{N,\beta}\varpi_N}{4^{N+\beta}|\tilde{\theta}|^{\beta}} - \frac{\sigma}{|\tilde{\theta}|^{b(\sigma+1)}}\right]\psi_l(\tilde{\theta}).$$

Let us now set

$$b_2 = \left[\frac{2C_{N,\beta}\varpi_N}{4^{N+\beta}|\tilde{\theta}|^{\beta}} - \frac{\sigma}{|\tilde{\theta}|^{b(\sigma+1)}}\right]\frac{u^{p+1}}{p}.$$

Since $\beta = \min\{a(p+1), b(\sigma+1)\}$, and $u(x) \gtrsim |x|^a$, as $|\tilde{\theta}| \to \infty$, there is a constant c_2, such that

$$b_2 \geq c_2 \cdot |\tilde{\theta}|^{a(p+1)-\beta} \to \infty, \quad \text{as } |\tilde{\theta}| \to \infty.$$

Therefore, we can find $|\tilde{\theta}| > R_2$, such that $b_2 \geq 2$. Consequently, we obtain

$$\varphi_l(\tilde{\theta}) < 2\psi_l(\tilde{\theta}).$$

When $|\bar{\theta}|, |\tilde{\theta}| > R_0 > \max\{R_1, R_2\}$, we have

$$\varphi_l(\bar{\theta}) < \varphi_l(\tilde{\theta}) < 2\psi_l(\tilde{\theta}) < 2\psi_l(\bar{\theta}) < 4\varphi_l(\bar{\theta}),$$

which leads to contradiction. Thus, there exists at least one of $\bar{\theta}$ or $\tilde{\theta}$ satisfying $|x| < R_0$. $\square$

Next, we give an important concept, namely the Narrow region principle, which plays a crucial role in the subsequent proof.

Theorem 5. *Let* $u \in L_\alpha \cap C^{1,1}_{loc}(R^N), v \in L_\beta \cap C^{1,1}_{loc}(R^N)$ *be a pair of positive solutions of (1). In addition, assume that there exists* $r_0 > 0$, *such that* $\Omega_r \subset \{x | l - r < x_1 < l\}$ *is a bounded region in* Σ_l *for any* $r \in (0, r_0]$. *Then*

$$\varphi_l(x),\ \psi_l(x) \geq 0 \quad in\ \Omega_r. \tag{14}$$

Proof. If (14) is not satisfied, then there exists $\bar{\zeta} \in \Omega_r$ such that

$$\varphi_l(\bar{\zeta}) = \min_{\Omega_r} \varphi_l(x) < 0.$$

Then, from (6), we have

$$(-\Delta)^{\frac{\alpha}{2}} \varphi_l(\bar{\zeta}) \leq 2C_{N,\alpha} \int\limits_{\Sigma_l} \frac{1}{|\bar{\zeta} - y^l|} dy \cdot \varphi_l(\bar{\zeta}) < 0. \tag{15}$$

Combining (4) and (15), we obtain

$$q\eta_1^{-q-1}(\bar{\zeta}, l)\psi_l(\bar{\zeta}) \leq \left[2C_{N,\alpha} \int\limits_{\Sigma_l} \frac{1}{|\bar{\zeta} - y^l|} dy - \gamma \bar{\zeta}_1^{-\gamma-1}(\bar{\zeta}, l)\right] \varphi_l(\bar{\zeta}). \tag{16}$$

According to the analysis of the integral of the right side of the above inequality presented in [24], let

$$B = \{y | r < y_1 - \bar{\zeta}_1 < 1, |y' - \bar{\zeta}'| < 1\}, \quad \varrho = y_1 - \bar{\zeta}_1, \quad \tau = |y' - \bar{\zeta}'| \tag{17}$$

and $\varpi_{N-2} = |B_1(0)|$ in R^{N-2}, where $\zeta' = (\zeta_2, \zeta_3, \cdots, \zeta_N)$. Then, through direct calculation, we have

$$\begin{aligned}
\int\limits_{\Sigma_l} \frac{1}{|\bar{\zeta} - y^l|^{N+\alpha}} dy \geq& \int\limits_{B} \frac{1}{|\bar{\zeta} - y|^{N+\alpha}} dy \\
=& \int\limits_{\xi}^{1} \int\limits_{0}^{1} \frac{\varpi_{N-2}\tau^{N-2} d\tau}{(\rho^2 + \tau^2)^{\frac{N+\alpha}{2}}} d\rho \\
=& \int\limits_{\xi}^{1} \int\limits_{0}^{\frac{1}{\rho}} \frac{\varpi_{N-2}(\rho t)^{N-2} \rho dt}{\rho^{N+\alpha}(1 + t^2)^{\frac{N+\alpha}{2}}} d\rho \\
=& \int\limits_{\xi}^{1} \frac{1}{\rho^{1+\alpha}} \int\limits_{0}^{\frac{1}{\rho}} \frac{\varpi_{N-2} t^{N-2} dt}{(1 + t^2)^{\frac{N+\alpha}{2}}} d\rho \\
\geq& \int\limits_{\xi}^{1} \frac{1}{\rho^{1+\alpha}} \int\limits_{0}^{1} \frac{\varpi_{N-2} t^{N-2} dt}{(1 + t^2)^{\frac{N+\alpha}{2}}} d\rho \\
\geq& C \int_{\xi}^{1} \frac{1}{\rho^{1+s}} d\rho \geq \frac{C}{\xi^s} \to \infty,
\end{aligned}$$

as $r \to 0^+$. Since Ω_r is a bounded region and $u \in L_\alpha \cap C^{1,1}_{loc}(R^N), v \in L_\beta \cap C^{1,1}_{loc}(R^N)$, therefore, $u(x), v(x)$ are bounded functions on Ω_r, and hence $\xi_i(x,l)$ and $\eta_i(x,l), i = 1,2$, are bounded. Letting

$$b_1 = \left[2C_{N,\alpha} \int\limits_{\Sigma_l} \frac{1}{|\bar{\zeta} - y^l|} dy - \gamma \xi_1^{-\gamma-1}(\bar{\zeta}, l)\right] \frac{\eta_1^{q+1}(\bar{\zeta}, l)}{q},$$

we note that

$$\lim_{r \to 0^+} b_1 = \infty.$$

Thus, we can find $r_1 > 0$ with $b_1 > 2$ so that

$$\psi_l(\bar{\zeta}) < 0,$$

and

$$\psi_l(\bar{\zeta}) < 2\varphi_l(\bar{\zeta}).$$

Therefore, there is $\tilde{\zeta} \in \Sigma_l$ such that

$$\psi_l(\tilde{\zeta}) = \min_{\Sigma_l} \psi_l(x) < 0.$$

Following the preceding argument, we can set

$$b_2 = \left[2C_{N,\beta} \int\limits_{\Sigma_l} \frac{1}{|\tilde{\zeta} - y^l|} dy - \sigma \eta_2^{-\sigma-1}(\tilde{\zeta}, l)\right] \frac{\xi_2^{p+1}(\tilde{\zeta}, l)}{p},$$

and find $r_2 > 0$ with $b_2 > 2$, such that

$$\varphi_l(\tilde{\zeta}) < 2\psi_l(\tilde{\zeta}).$$

Therefore, when $\bar{\zeta}, \tilde{\zeta} \in \Omega_r$, for any $r \in (0, r_0]$, $r_0 = min\{r_1, r_2\}$, we have

$$\varphi_l(\bar{\zeta}) < \varphi_l(\tilde{\zeta}) < 2\psi_l(\tilde{\zeta}) < 2\psi_l(\bar{\zeta}) < 4\varphi_l(\bar{\zeta}),$$

which is a contradiction. Therefore, in the narrow region Ω_r, we have

$$\varphi_l(x), \psi_l(x) \geq 0, \quad x \in \Omega_r.$$

□

Proof of Theorem 2. *Step 1.* For l to be negative enough, it holds that

$$\varphi_l(x), \psi_l(x) \geq 0, \quad x \in \Sigma_l. \tag{18}$$

Thanks to sufficiently negative l and monotone increasing character of $u(x)$ and $v(x)$ in $|x|$ with respect to the origin, we have

$$\lim_{|x|\to\infty} \varphi_l(x) = \lim_{|x|\to\infty} [u(x) - u(x^l)] \geq 0,$$

and

$$\lim_{|x|\to\infty} \psi_l(x) = \lim_{|x|\to\infty} [v(x) - v(x^l)] \geq 0.$$

This shows that the minimum negative values are obtained in the interior of Σ_l when $\varphi_l(x)$ and $\psi_l(x)$ have negative values in Σ_l. Using the decay at infinity, we infer that there exists $R_0 > 0$, such that for $l \leq -R_0$, we have

$$\varphi_l(x),\ \psi_l \geq 0, \quad x \in \Sigma_l.$$

This completes the proof of Step 1 and provides a starting point for the moving plane.

Step 2. Now, if (18) holds, let us move the plane P_l to its limiting position from the starting point provided in Step 1. More specifically, let

$$l_0 = sup\{l \leq x_1 \mid \varphi_\mu,\ \psi_\mu \geq 0, x \in \Sigma_\mu; \mu \leq l\}.$$

We will establish that

$$\varphi_{l_0}(x) \equiv \psi_{l_0}(x) \equiv 0, \quad x \in \Sigma_{l_0}. \tag{19}$$

Before proving (19), for fixed l, we first explore the relationship and properties of functions $\varphi_l(x)$ and $\psi_l(x)$ as follows.

(1): *(a)* If $\varphi_l(x) \equiv 0$, then $\psi_l(x) \equiv 0$; *(b)* If $\psi_l(x) \equiv 0$, then $\varphi_l(x) \equiv 0$.

Proof *(a)* If $\varphi_l(x) \equiv 0$, then it follows from (4) that

$$0 = (-\Delta)^{\frac{\alpha}{2}} \varphi_l(x) = q\eta_1^{-q-1}(x, l)\psi_l(x).$$

As the values of $\eta_1(x, l)$ remain between $v_l(x)$ and $v(x)$, we obtain

$$\psi_l \equiv 0.$$

Following the same argument, we can establish the proof of *(b)*. □

(2): If $\varphi_l(x) \not\equiv 0$ or $\psi_l(x) \not\equiv 0$ for $x \in \Sigma_l$, then $\varphi_l(x) > 0$ and $\psi_l(x) > 0, x \in \Sigma_l$.

Proof. Without loss of generality, let us assume that $\varphi_l(x) \not\equiv 0$. We already know that

$$\varphi_l(x) \geq 0, \quad \forall x \in \Sigma_l.$$

In order to show that $\varphi_l(x) > 0,\ \forall x \in \Sigma_l$, we assume that there exists $\breve{\theta} \in \Sigma_l$ such that $\varphi_l(\breve{\theta}) = 0$, and

$$\begin{aligned}(-\Delta)^{\frac{\alpha}{2}} \varphi_l(\check{x}) =& C_{N,\alpha} PV \int_{R^N} \frac{-\varphi_l(y)}{|\breve{\theta} - y|^{N+\alpha}} dy \\ =& C_{N,\alpha} PV \left(\int_{\Sigma_l} \frac{-\varphi_l(y)}{|\breve{\theta} - y|^{N+\alpha}} dy + \int_{R^N \setminus \Sigma_l} \frac{-\varphi_l(y)}{|\breve{\theta} - y|^{N+\alpha}} dy \right) \\ =& C_{N,\alpha} PV \left(\int_{\Sigma_l} \frac{-\varphi_l(y)}{|\breve{\theta} - y|^{N+\alpha}} dy + \int_{\Sigma_l} \frac{-\varphi_l(y^l)}{|\breve{\theta} - y^l|^{N+\alpha}} dy \right) \\ =& C_{N,\alpha} PV \int_{\Sigma_l} \left(\frac{1}{|\breve{\theta} - y^l|^{N+\alpha}} - \frac{1}{|\breve{\theta} - y|^{N+\alpha}} \right) dy \cdot \varphi_l(y) \\ &< 0.\end{aligned} \tag{20}$$

On the other hand, it follows from (4) that

$$\begin{aligned}(-\Delta)^{\frac{\alpha}{2}} \varphi_l(\breve{\theta}) &= \gamma \xi_1^{-\gamma-1}(x,l) \varphi_l(\breve{\theta}) + q \eta_1^{-q-1}(x,l) \psi_l(\breve{\theta}) \\ &= q \eta_1^{-q-1}(x,l) \psi_l(\breve{\theta}) \\ &\geq 0,\end{aligned}$$

which is contrary to the formula (20). Therefore, we have

$$\varphi_l(x) > 0, \quad \forall x \in \Sigma_l.$$

Also, we have from (1) that $\psi_l(x) \not\equiv 0$. In the same way, we have

$$\psi_l(x) > 0, \quad \forall x \in \Sigma_l.$$

Next, we will prove that (19) is true. If not, then we have

$$\varphi_{l_0}(x),\ \psi_{l_0}(x) > 0, \quad x \in \Sigma_{l_0}. \tag{21}$$

Thus, for very small δ and constant d_0, we obtain

$$\varphi_{l_0}(x),\ \psi_{l_0}(x) > d_0, \quad x \in \overline{\Sigma_{l_0-\delta} \cap B_{R_0}(0)}.$$

As $\varphi_l(x), \psi_l(x)$ depend on l continuously, for $l \in (l_0, l_0 + \epsilon)$, there must exist a $\epsilon > 0$ such that

$$\varphi_l(x), \psi_l(x) \geq 0, \quad x \in \overline{\Sigma_{l_0-\delta} \cap B_{R_0}(0)}. \tag{22}$$

Let $\Sigma_l^- = \Sigma_{\varphi_l(x)}^- \cup \Sigma_{\psi_l(x)}^-$. By virtue of the boundedness of the narrow region

$$\overline{\Sigma_l^- \setminus \Sigma_{l_0-\delta} \cap B_{R_0}(0)},$$

it follows from Theorem 2 that

$$\varphi_{l_0}(x),\ \psi_{l_0}(x) \geq 0, \quad x \in \Sigma_l \setminus \Sigma_{l_0-\delta}, \tag{23}$$

which together with (21) and (22) leads to

$$\varphi_{l_0}(x),\ \psi_{l_0}(x) \geq 0, \quad \forall x \in \Sigma_l,$$

for $l \in (l_0, l_0 + \epsilon)$, which does not agree with the definition of l_0. In consequence, we obtain

$$\varphi_{l_0}(x) \equiv \psi_{l_0}(x) \equiv 0, \quad x \in \Sigma_{l_0}.$$

Thus, $u(x)$ and $v(x)$ are symmetric about plane P_{l_0}. Therefore, we conclude that the arbitrariness of x_1-direction ensures the radial symmetry of $u(x)$ and $v(x)$ about some points on R^N. This completes the proof. □

3. Hénon Type Fractional Laplacian System with Different Negative Powers

Theorem 6. *(Decay at infinity) Let $u \in L_\alpha \cap C^{1,1}_{loc}(R^N \backslash \{0\}), v \in L_\beta \cap C^{1,1}_{loc}(R^N \backslash \{0\})$ be a pair of positive solutions for the Hénon type system (3) with*

$$u(x) \gtrsim |x|^a, \quad v(x) \gtrsim |x|^b, \quad as\ |x| \to \infty,$$

where $a, b > 0, \gamma, \sigma, p, q \geq 1$. If $\alpha, \beta \in (0, 2)$ satisfy

$$\alpha = \min\{a(\gamma+1) - a_1, b(q+1) - b_1\}, \quad \beta = \min\{b(\sigma+1) - b_2, a(p+1) - a_2\}$$

and

$$\varphi_l(\bar{\theta}) = \min_{\tilde{\Sigma}_l} \varphi(x) < 0, \quad \psi_l(\tilde{\theta}) = \min_{\tilde{\Sigma}_l} \psi(x) < 0,$$

then there exists $R_0 > 0$ such that there is at least one of $\bar{\theta}$ and $\tilde{\theta}$ satisfying

$$|x| \leq R_0.$$

Theorem 7. *(Narrow region principle) Let $u \in L_\alpha \cap C^{1,1}_{loc}(R^N \backslash \{0\}), v \in L_\beta \cap C^{1,1}_{loc}(R^N \backslash \{0\})$ be a pair of positive solutions for the Hénon type system (3), and suppose that there exists $r_0 > 0$, for any $r \in (0, r_0]$, $\Omega_r \subset \{x | l - r < x_1 < l\}$ is a bounded region in Σ_l. Then*

$$\varphi_l(x),\ \psi_l(x) \geq 0 \quad in\ \Omega_r. \tag{24}$$

Since the proofs of Theorems 6 and 7 are similar to the ones for Theorems 4 and 5 with a slight modification, therefore, we omit them.

Proof of Theorem 3. Let us follow the symbols introduced in the first section.
Step 1. For sufficiently negative l, we show that

$$\varphi_l(x),\ \psi_l(x) \geq 0, \quad x \in \Sigma_l \backslash \{0^l\}. \tag{25}$$

To better apply the decay at infinity, one needs to analyze:

(*i*) $\lim_{|x| \to \infty} \varphi_l(x) \geq 0, \quad \lim_{|x| \to \infty} \psi_l(x) \geq 0;$

(*ii*) $\lim_{x \to 0^l} \varphi_l(x) \geq 0, \quad \lim_{x \to 0^l} \psi_l(x) \geq 0.$

The assumption of monotone character of $u(x)$ and $v(x)$ immediately implies that (i) is correct. When l is negative enough, 0^l is the same. As $u(x)$ is monotone increasing in $|x|$ about the origin, we obtain

$$\lim_{x\to 0^l} \varphi_l(x) = \lim_{x\to 0^l} \left[u(x) - u_l(x)\right] = \lim_{x\to 0^l} u(x) - u(0) > 0.$$

In a similar manner, we can obtain

$$\lim_{x\to 0^l} \psi_l(x) = \lim_{x\to 0^l} \left[v(x) - v_l(x)\right] = \lim_{x\to 0^l} v(x) - v(0) > 0.$$

From the decay at infinity, when $\bar{\theta}$ and $\tilde{\theta}$ are negative minimum values of $\varphi_l(x)$ and $\psi_l(x)$ in Σ_l, there exists at least one of $\bar{\theta}$ and $\tilde{\theta}$ such that

$$|x| \leq R_0.$$

Thus, for sufficiently negative l, (25) is valid.

Step 2. Now, if (25) holds, let us move the plane P_l to its limiting position from $-\infty$, which shows that u, v are symmetrical about the limiting plane. More specifically, put

$$l_0 = sup\{l \leq 0 \mid \varphi_\mu,\ \psi_\mu \geq 0, x \in \Sigma_\mu \backslash \{0^\mu\}; \mu \leq l\},$$

and claim that

$$l_0 = 0, \quad \varphi_{l_0}(x) \equiv 0, \quad \psi_{l_0}(x) \equiv 0, \quad x \in \Sigma_{l_0} \backslash \{0^{l_0}\}. \tag{26}$$

Let us assume that $l_0 < 0 \; \forall \; l_0 < 0$, because $u(x)$ is monotonically increasing in $|x|$ about the origin. Then we have

$$\lim_{x\to 0^{l_0}} \varphi_{l_0}(x) = \lim_{x\to 0^{l_0}} \left[u(x) - u_{l_0}(x)\right] = \lim_{x\to 0^{l_0}} u(x) - u(0) > 0.$$

Analogously, we can obtain

$$\lim_{x\to 0^{l_0}} \psi_{l_0}(x) = \lim_{x\to 0^{l_0}} \left[v(x) - v_{l_0}(x)\right] = \lim_{x\to 0^{l_0}} v(x) - v(0) > 0.$$

Thus, there exist $\epsilon > 0$ and $d_i > 0, i = 1, 2$, such that

$$\varphi_{l_0}(x) \geq d_1, \quad \psi_{l_0}(x) \geq d_2, \quad \forall \; x \in B_\epsilon(0^{l_0}) \backslash \{0^{l_0}\}.$$

It follows from the decay at infinity that $\varphi_{l_0}(x)$ and $\psi_{l_0}(x)$ have no negative minimum in $B^c_{R_0}(0)$. Here, we will show that $\varphi_{l_0}(x)$ and $\psi_{l_0}(x)$ have no negative minimum in the interior of $B_{R_0}(0)$. It means that

$$\varphi_l(x) \geq 0, \quad \psi_l(x) \geq 0, \quad \forall x \in (\Sigma_l \cap B_{R_0}(0)) \backslash \{0^l\}),$$

when l is close enough to l_0.

When $l_0 < 0$, we have

$$\varphi_{l_0}(x) \geq 0, \quad \psi_{l_0}(x) \geq 0, \quad x \in \Sigma_{l_0} \backslash \{0^{l_0}\}.$$

If (26) does not hold, then

$$\varphi_{l_0}(x) > 0, \quad \psi_{l_0}(x) > 0, \quad x \in \Sigma_{l_0} \backslash \{0^{l_0}\}. \tag{27}$$

The proof of this fact is similar to that of the propertied (1), (2) of function $\varphi_l(x)$ and $\psi_l(x)$ established in Theorem 2. From (27), we know that there must exist $d_3 > 0$ such that

$$\varphi_{l_0}(x), \psi_{l_0}(x) \geq d_3, \quad x \in \overline{(\Sigma_{l_0-\delta} \cap B_{R_0}(0)) \backslash \{0^{l_0}\}}.$$

Since $\varphi_l(x)$ and $\psi_l(x)$ are continuous about l, therefore, for $l \in (l, l_0 + \epsilon)$, there must exist $\epsilon > 0$, such that

$$\varphi_l(x),\ \psi_l(x) \geq 0, \quad x \in \overline{(\Sigma_{l_0-\delta} \cap B_{R_0}(0)) \backslash \{0^l\}}.$$

From the boundedness of narrow region $\overline{\Sigma_l^- \backslash \Sigma_{l_0-\delta} \cap B_{R_0}(0)}$, it follows from Theorem 7 that

$$\varphi_l(x),\ \psi_l(x) \geq 0, \quad x \in \overline{(\Sigma_l^- \backslash \Sigma_{l_0-\delta} \cap B_{R_0}(0)) \backslash \{0^l\}}.$$

Summarizing the above discussion, for any $l \in (l_0, l_0 + \epsilon)$, we obtain

$$\varphi_l(x),\ \psi_l(x) \geq 0, \quad x \in \Sigma_l \backslash \{0^l\},$$

which is inconsistent with the definition of l_0. Therefore, we must have

$$l_0 = 0, \quad \varphi_{l_0}(x) \equiv \psi_{l_0}(x) \equiv 0, \quad x \in \Sigma_{l_0} \backslash \{0^{l_0}\}. \tag{28}$$

Thus, $u(x)$ and $v(x)$ are symmetric about plane P_{l_0}. Furthermore, the arbitrariness of x_1-direction ensures the radial symmetry of $u(x)$ and $v(x)$ about the origin. The proof is complete. □

4. Conclusions

We discussed the radial symmetry of nonnegative solutions for a fractional Laplacian system with different negative powers supplemented with a weaker form of growth/decay conditions by applying the direct method of moving planes. We have also investigated a nonlinear Hénon type fractional Laplacian system with different negative powers subject to weaker growth/decay conditions. We emphasize that our results improve the ones presented in [35] in the sense that we have weakened the growth/decay condition considered in [35].

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