*fractal and fractional*

*Article*

# Multivalent Functions and Differential Operator Extended by the Quantum Calculus

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**Abstract:** We used the concept of quantum calculus (Jackson's calculus) in a recent note to develop an extended class of multivalent functions on the open unit disk. Convexity and star-likeness properties are obtained by establishing conditions for this class. The most common inequalities of the proposed functions are geometrically investigated. Our approach was influenced by the theory of differential subordination. As a result, we called attention to a few well-known corollaries of our main conclusions.

**Keywords:** quantum calculus; analytic function; subordination and superordination; differential subordination; univalent function; open unit disk; fractional calculus; multivalent functions; meromorphic functions



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## 1. Introduction

The (Jackson's calculus) (Quantum calculus (QC)) is a brand-new branch of mathematics that has applications in both physics and mathematics [1,2]. However, Ismail et al. [3] presented QC in the geometric function theory. Based on this investigation, many Ma and Minda classes of analytic functions are being proposed and developed on the open unit disk (the classes of analytic functions that are defined by the subordination notion). For example, quantum star-like function sub-classes were formulated in the effort of Seoudy and Aouf [4] employing the notion of $q$-derivatives. Zainab et al. [5] developed acceptable $q$-star-likeness criteria applying a unique curve. Furthermore, $q$-star-like functions dominating the 2D-Julia set were examined by Samir et al. [6]. This calculus proved its efficiency and accuracy to generalize the families of differential and integral operators in a complex domain. In addition, special functions (see [7,8]) have associated with this calculus, especially the queen of special functions: Mittag–Leffler function (see [9–12]). The quantum calculus ($q$-calculus) has tremendous applications in different fields, for example, integral inequalities [13], summability [14], approximation and polynomials [15], and sequence spaces [16].

In a complex domain $\Omega$, $p$-valent is an ordinary simplification of the concept of a univalent function (normalized or meromorphic) in the complex plane. The number of zeros of the equation $\phi(\xi) = \varsigma$ in $\Omega$ does not go above $p$ for any $\varsigma$. This result highlights there are at most $p$ points of the Riemann surface and the $\varsigma$-plane into which $\varsigma = \phi(\xi)$ maps $\Omega$. Note that in $\Omega, \phi(\xi)$ is univalent for $p = 1$. Based on the QC, this class of analytic functions has been widely generalized by many researchers. Srivastava [17] utilized the QC to study the existence of numerous arrangements of function theory. Arif et al. [18] and Khan et al. [19] added investigations on the integral operator theory for holomorphic and $p$-valent functions. Wang et al. [20] presented a generalization of Janowski $p$-valent functions in view of QC.

We propose a set of functional formulas connected with the class of multivalent functions on the open unit disk in this work using QC. The proposed quantum formulas will be geometrically investigated. For previous works, a collection of ramifications is presented. Our technique is indicated by the theory of differential subordination and superordination.

## 2. Preliminaries

In this section, we provide necessary definitions, lemmas and corollaries for explaining the proofs of our Theorems.

### *2.1. Geometric Approaches*

Let us begin with the fundamentals of geometric function theory which are covered in this book [21].

**Definition 1.** *The set $\mathbb{U} := \{\xi \in \mathbb{C} : |\xi| < 1\}$ is specified in the open unit disk. The analytic functions $v_1, v_2$ in $\mathbb{U}$ are subordinated $v_1 \prec v_2$ or $v_1(\xi) \prec v_2(\xi), \xi \in \mathbb{U}$ if, for an analytic function $v, |v| \leq |\xi| < 1$ satisfies*

$$v_1(\xi) = v_2(v(\xi)), \quad \xi \in \mathbb{U}.$$

In the open unit disk, Ma and Minda [22] introduced two significant classes of star-like and convex functions demarcated by the definition of the subordination, respectively:

$$\frac{\xi v'(\xi)}{v(\xi)} \prec p(\xi), \quad p(0) = 1;$$

$$1 + \frac{\xi v''(\xi)}{v'(\xi)} \prec p(\xi), \quad p(0) = 1,$$

Numerous investigations have prolonged and generalized these classes. Furthermore, the investigators utilized differential and integral processes to generate advanced classes of analytic functions.

**Definition 2.** *Let $\Lambda_p$ be the class of p-valent analytic functions defined as follows:*

$$v(\xi) = \xi^p + \sum_{n=p+1}^{\infty} a_n \xi^n, \quad \xi \in \mathbb{U},$$

*where $p \in \mathbb{N}$. Additionally, two functions are defined for $v \in \Lambda_p$ as follows:*

$$T_v(\xi) := \frac{\xi v'(\xi)}{v(\xi)}, \quad \xi \in \mathbb{U};$$

*and*

$$V_v(\xi) := 1 + \frac{\xi v''(\xi)}{v'(\xi)}, \quad \xi \in \mathbb{U}.$$

*Based on the preceding functions, there are two classes of p-valent functions, the star-like $\mathbb{T}_v$ class and convex $\mathbb{V}_v$, which meet the following conditions, respectively:*

$$\Re(T_v(\xi)) > 0, \quad \Re(V_v(\xi)) > 0, \quad \xi \in \mathbb{U}.$$

**Definition 3.** *Two functions*

$$v(\xi) = \xi^p + \sum_{n=p+1}^{\infty} a_n \xi^n$$

*and*

$$\vartheta(\xi) = \xi^p + \sum_{n=p+1}^{\infty} b_n \, \xi^n$$

*are convoluted if they satisfy the product*

$$v(\xi) * \vartheta(\xi) = \xi^p + \sum_{n=p+1}^{\infty} a_n b_n \, \xi^n, \quad \xi \in \mathbb{U}.$$

The next result can be found in [21] (Corollary 3.4h.1 p.135)

**Lemma 1.** *Let $f$ be analytic and $g$ be univalent in $\mathbb{U}$ with $f(0) = g(0)$, and let $\psi$ be analytic in a domain containing $g(\mathbb{U})$ and $g(\mathbb{U})$. If $\xi g'(\xi)\psi(g(\xi))$ is star-like, then the inequality*

$$\xi f'(\xi)\psi(f(\xi)) \prec \xi g'(\xi)\psi(g(\xi))$$

*implies that $f(\xi) \prec g(\xi)$ and $g$ is the best dominant.*

**Lemma 2** ([22])**.** *Let $P(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ be analytic in $\mathbb{U}$ satisfying $\Re(P(z)) > 0$. Then*

$$|p_2 - \Bbbk p_1^2| \le 2\max\{1, |2\Bbbk - 1|\}, \quad \Bbbk \in \mathbb{C}.$$

*2.2. Quantum Calculus*

**Definition 4.** *The Jackson's derivative, which contains the difference operator, can be shown as follows:*

$$(\eth_q\,)f(\xi) = \frac{f(\xi) - f(q\,\xi)}{\xi(1-q)}, \quad q \in (0,1) \tag{1}$$

*where*

$$\eth_q\,(\xi^p) = \left(\frac{1-q^p}{1-q}\right)\xi^{p-1}, \quad \kappa \in \mathbb{R}.$$

*In Maclaurin's series representation, the total of the numbers is also included as follows:*

$$(\eth_q\, f)\;(\xi) = \sum_{n=0}^{\infty} \varphi_n\, [n]_q\, \xi^{n-1}, \tag{2}$$

*where*

$$[n]_q := \frac{1-q^n}{1-q}.$$

Note that

$$\eth_q\, K = 0, \quad \lim_{q \to 1^-} (\eth_q\, f)\;(\xi) = f'(\xi),$$

where $K$ is a constant function.

**Definition 5.** *For a function $v \in \Lambda_p$, let $\Upsilon_q : \mathbb{U} \to \mathbb{C}$ be formulated by*

$$\Upsilon_q(\xi) := (1-\delta)v(\xi) + \frac{\delta}{[p]_q}\left(\xi\eth_q\, v(\xi)\right), \quad \xi \in \mathbb{U}, \delta \in [0,1].$$

**Remark 1.** *Clearly, for $v \in \Lambda_p$, we obtain*

$$\begin{aligned}
\Upsilon_q(\xi) &= (1-\delta)v(\xi) + \frac{\delta}{[p]_q}\big(\xi\eth_q\, v(\xi)\big) \\
&= (1-\delta)\left(\xi^p + \sum_{n=p+1}^{\infty} a_n\,\xi^n\right) + \frac{\delta}{[p]_q}\left(\xi\eth_q\left(\xi^p + \sum_{n=p+1}^{\infty} a_n\,\xi^n\right)\right) \\
&= (1-\delta)\left(\xi^p + \sum_{n=p+1}^{\infty} a_n\,\xi^n\right) + \frac{\delta}{[p]_q}\left(\xi\left(\eth_q\xi^p + \sum_{n=p+1}^{\infty} a_n\,\eth_q\xi^n\right)\right) \\
&= (1-\delta)\left(\xi^p + \sum_{n=p+1}^{\infty} a_n\,\xi^n\right) + \frac{\delta}{[p]_q}\left(\xi\left([p]_q\xi^{p-1} + \sum_{n=p+1}^{\infty} a_n\,[n]_q\xi^{n-1}\right)\right) \\
&= (1-\delta)\left(\xi^p + \sum_{n=p+1}^{\infty} a_n\,\xi^n\right) + \frac{\delta}{[p]_q}\left([p]_q\xi^p + \sum_{n=p+1}^{\infty} a_n\,[n]_q\xi^n\right) \\
&= \left(\xi^p + \sum_{n=p+1}^{\infty}\left((1-\delta) + \frac{\delta[n]_q}{[p]_q}\right)a_n\,\xi^n\right) \\
&:= \xi^p + \sum_{n=p+1}^{\infty}\chi_n(q,\delta,p)a_n\,\xi^n, \quad \chi_n(q,\delta,p) = \left((1-\delta) + \frac{\delta[n]_q}{[p]_q}\right).
\end{aligned}$$

*Thus,* $\Upsilon_q(\xi) \in \Lambda_p$.

## 3. Results

We have the following results.

### 3.1. Properties of $\Upsilon_q(\xi)$

This part deals with the geometric properties of the functional $\Upsilon_q(\xi)$.

**Theorem 1.** *Consider the following assumptions:*

*(i)* $\rho$ *is univalent in* $\mathbb{U}$;

*(ii)* $\dfrac{\xi\rho'(\xi)}{\rho(\xi)(\rho(\xi)-1)}$ *is star-like in* $\mathbb{U}$;

*(iii)* *the subordination*

$$\frac{V_{\Upsilon_q}(\xi) - p}{T_{\Upsilon_q}(\xi) - p} \prec 1 + \frac{1}{p}\frac{\xi\rho'(\xi)}{\rho(\xi)(\rho(\xi)-1)}$$

*holds.*

*Then*

$$\frac{T_{\Upsilon_q}(\xi)}{p} \prec \rho(\xi), \quad \xi \in \mathbb{U}$$

*and* $\rho$ *is the best dominant.*

**Proof.** Define function $P$ as follows:

$$P(\xi) := \frac{T_{\Upsilon_q}(\xi)}{p}, \quad \xi \in \mathbb{U}.$$

A computation implies

$$T_P(\xi) = V_{\Upsilon_q}(\xi) - pP(\xi).$$

Substitution implies that

$$\frac{V_{\Upsilon_q}(\xi)-p}{T_{\Upsilon_q}(\xi)-p}=\frac{T_P(\xi)+pP(\xi)-p}{pP(\xi)-p}$$
$$=1+\frac{1}{p}\frac{\xi P'(\xi)}{P(\xi)(P(\xi)-1)}.$$

Hence,

$$\frac{\xi P'(\xi)}{P(\xi)(P(\xi)-1)}\prec\frac{\xi\rho'(\xi)}{\rho(\xi)(\rho(\xi)-1)},\quad \xi\in\mathbb{U}.$$

According to Lemma 1, we obtain the result. □

**Theorem 2.** *Consider the following assumptions:*

*(i)* *Let $\rho$ be univalent in $\mathbb{U}$;*
*(ii)* *Let $\frac{\xi\rho'(\xi)}{\rho(\xi)-1}$ be star-like in $\mathbb{U}$;*
*(iii)* *Let the subordination*

$$T_{\Upsilon_q}(\xi)\left(\frac{V_{\Upsilon_q}(\xi)-p}{T_{\Upsilon_q}(\xi)-p}-1\right)\prec\frac{\xi\rho'(\xi)}{\rho(\xi)-1}$$

*hold.*

*Then*

$$\frac{T_{\Upsilon_q}(\xi)}{p}\prec\rho(\xi),\quad \xi\in\mathbb{U}$$

*and $\rho$ is the best dominant.*

**Proof.** We formulate the function $P$ as follows:

$$P(\xi):=\frac{T_{\Upsilon_q}(\xi)}{p},\quad \xi\in\mathbb{U}.$$

Consequently, we have

$$T_P(\xi)+pP(\xi)=V_{\Upsilon_q}(\xi).$$

Substitution yields that

$$T_{\Upsilon_q}(\xi)\left(\frac{V_{\Upsilon_q}(\xi)-p}{T_{\Upsilon_q}(\xi)-p}-1\right)=\frac{\xi P'(\xi)}{P(\xi)-1}.$$

Hence,

$$\frac{\xi P'(\xi)}{P(\xi)-1}\prec\frac{\xi\rho'(\xi)}{\rho(\xi)-1},\quad \xi\in\mathbb{U}.$$

According to Lemma 1, we obtain the result. □

**Theorem 3.** *Consider the following assumptions:*

*(i)* *$\rho$ is univalent in $\mathbb{U}$;*
*(ii)* *$T_\rho$ is star-like in $\mathbb{U}$;*
*(iii)* *The subordination*

$$V_{\Upsilon_q}(\xi)-T_{\Upsilon_q}(\xi)\prec T_\rho(\xi)$$

*holds.*

*Then*

$$\frac{T_{\Upsilon_q}(\xi)}{p} \prec \rho(\xi), \quad \xi \in \mathbb{U}$$

*and ρ is the best dominant.*

**Proof.** We present the function $P$ as follows:

$$P(\xi) := \frac{T_{\Upsilon_q}(\xi)}{p}, \quad \xi \in \mathbb{U}.$$

Thus, we have

$$T_P(\xi) + pP(\xi) = V_{\Upsilon_q}(\xi).$$

Substitution yields that

$$V_{\Upsilon_q}(\xi) - T_{\Upsilon_q}(\xi) = T_P(\xi).$$

Hence,

$$T_P(\xi) \prec T_\rho(\xi), \quad \xi \in \mathbb{U}.$$

According to Lemma 1, we have the result $P(\xi) \prec \rho(\xi)$. □

**Theorem 4.** *Consider the following assumptions:*

*(i)* $\rho$ *is univalent in* $\mathbb{U}$*;*
*(ii)* $\xi\rho'(\xi)$ *is star-like in* $\mathbb{U}$*;*
*(iii) The subordination*

$$T_{\Upsilon_q}(\xi)\Big(V_{\Upsilon_q}(\xi) - T_{\Upsilon_q}(\xi)\Big) \prec p\xi\rho'(\xi)$$

*holds.*

*Then*

$$\frac{T_{\Upsilon_q}(\xi)}{p} \prec \rho(\xi), \quad \xi \in \mathbb{U}$$

*and ρ is the best dominant.*

**Proof.** We define the function $P$ as follows:

$$P(\xi) := \frac{T_{\Upsilon_q}(\xi)}{p}, \quad \xi \in \mathbb{U}.$$

Consequently, we have

$$T_P(\xi) + pP(\xi) = V_{\Upsilon_q}(\xi).$$

Substitution yields that

$$T_{\Upsilon_q}(\xi)\Big(V_{\Upsilon_q}(\xi) - T_{\Upsilon_q}(\xi)\Big) = p\xi P'(\xi).$$

Hence,

$$p\xi P'(\xi) \prec p\xi\rho'(\xi), \quad \xi \in \mathbb{U}.$$

According to Lemma 1, we obtain the result $P(\xi) \prec \rho(\xi)$. □

Then, we consider the multivalued function $\rho(\xi) = 1 + \sinh^{-1}(\xi)$ (as can be seen in Figure 1).

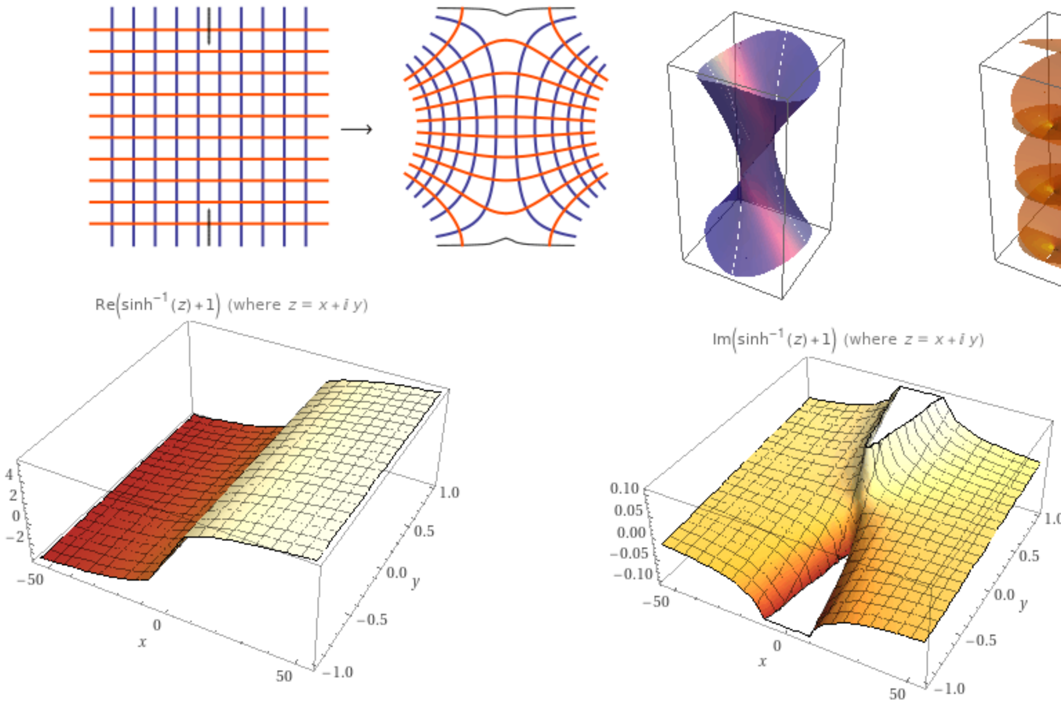


**Figure 1.** Plotting of the function $1 + \sinh^{-1}(\xi)$, which is analytic in the open unit disk and maps it onto the petal-shape domain. The graphs presents the complex map, Riemann surface, real part, and imaginary part of the function.

**Theorem 5.** *Consider the functional* $\Upsilon_q$ *such that*

$$\frac{T_{\Upsilon_q}(\xi)}{p} \prec 1 + \sinh^{-1}(\xi), \quad \xi \in \mathbb{U}.$$

*Then*

$$\left|\chi_{p+2}(q,\delta,p)a_{p+2} - \Bbbk\left(\chi_{p+1}(q,\delta,p)a_{p+1}\right)^2\right| \leq \frac{p}{2}\max\{1, |2\Bbbk - 1|\}, \quad \Bbbk \in \mathbb{C}.$$

*Furthermore,*

$$|\chi_{p+2}(q,\delta,p)a_{p+2}| \leq \frac{p}{2}.$$

**Proof.** By the assumption, we have

$$\frac{T_{\Upsilon_q}(\xi)}{p} = 1 + \sinh^{-1}(\omega(\xi)),$$

where $\omega(\xi)$ is analytic in $\mathbb{U}$ such that $\omega(0) = 0$ and $|\omega| < |\xi| < 1$. It is clear that

$$\begin{aligned}\frac{T_{\Upsilon_q}(\xi)}{p} &= 1 + \frac{1}{p}\chi_{p+1}(q,\delta,p)a_{p+1}\xi \\ &+ \left(\frac{2}{p}\chi_{p+2}(q,\delta,p)a_{p+2} - \frac{1}{p}\left(\chi_{p+1}(q,\delta,p)a_{p+1}\right)^2\right)\xi^2 \\ &+ \dots\end{aligned}$$

and that

$$1 + \sinh^{-1}(\omega(\xi)) = 1 + \varsigma_1\xi + \varsigma^2\xi^2 + \dots$$

then, we obtain

$$\chi_{p+1}(q,\delta,p)a_{p+1} = p\varsigma_1$$
$$\chi_{p+2}(q,\delta,p)a_{p+2} = \frac{p}{2}\varsigma_2 + \frac{p^2}{2}\varsigma_1^2.$$

Combining the above equations, we obtain that

$$\left|\chi_{p+2}(q,\delta,p)a_{p+2} - \Bbbk\left(\chi_{p+1}(q,\delta,p)a_{p+1}\right)^2\right| = \frac{p}{2}|\varsigma_2 - (2\Bbbk - 1)\varsigma_1^2|.$$

Hence, Lemma 2 implies the requested result. For the second result, let $\Bbbk = 0$, then we obtain the inequality. □

**Corollary 1** ([23], Theorem 1)**.** *Consider the functional $\Upsilon_q$ such that $\delta = 0$ and*

$$\frac{T_{\Upsilon_q}(\xi)}{p} \prec 1 + \sinh^{-1}(\xi), \quad \xi \in \mathbb{U},$$

*then*

$$\left|a_{p+2} - \Bbbk\left(a_{p+1}\right)^2\right| \le \frac{p}{2}\max\{1, |2\Bbbk - 1|\}, \quad \Bbbk \in \mathbb{C}.$$

*Furthermore,*

$$|a_{p+2}| \le \frac{p}{2}.$$

**Corollary 2** ([23], Corollary 1)**.** *Consider the functional $\Upsilon_q$ such that $\delta = 0$ and*

$$\frac{T_{\Upsilon_q}(\xi)}{p} \prec 1 + \sinh^{-1}(\xi), \quad \xi \in \mathbb{U},$$

*then*

$$\left|a_{p+2} - \left(a_{p+1}\right)^2\right| \le \frac{p}{2}.$$

*Furthermore,*

$$|a_{p+2}| \le \frac{p}{2}.$$

**Corollary 3** ([23], Corollary 2)**.** *Consider the functional $\Upsilon_q$ such that $\delta = 0$ and*

$$\frac{T_{\Upsilon_q}(\xi)}{p} \prec 1 + \sinh^{-1}(\xi), \quad \xi \in \mathbb{U}.$$

*Then*

$$\left|a_3 - \Bbbk(a_2)^2\right| \le \frac{p}{2}\max\{1, |2\Bbbk - 1|\}, \quad \Bbbk \in \mathbb{C},$$

$$|a_3| \le \frac{1}{2}$$

*and*

$$\left|a_3 - a_2^2\right| \le \frac{1}{2}.$$

**Theorem 6.** *If $\rho$ is convex univalent in $\mathbb{U}$ such that*

$$\frac{T_{\Upsilon_q}(\xi)}{p} \prec \rho(\xi), \quad \xi \in \mathbb{U},$$

*where* $\rho(0) = 1$. *Then*

$$\Upsilon_q(\xi) \prec \xi \exp\left(\int_0^\xi \frac{p\rho(\omega(\chi))}{\chi} d\chi\right),$$

*where* $\omega$ *satisfies* $\omega(0) = 0$ *and* $|\omega(\xi)| < 1$. *Furthermore, the inequality* $|\xi| := \varrho < 1$ *implies*

$$\exp\left(\int_0^1 \frac{p\rho(-\varrho)}{\varrho} d\varrho\right) \leq \left|\frac{\Upsilon_q(\xi)}{\xi}\right| \leq \exp\left(\int_0^1 \frac{p\rho(\varrho)}{\varrho} d\varrho\right).$$

**Proof.** By the assumption of the theorem, we obtain

$$\frac{\left(\Upsilon_q(\xi)\right)'}{\Upsilon_q(\xi)} - \frac{1}{\xi} = \frac{p\rho(\omega(\xi)) - 1}{\xi}.$$

Integration implies that

$$\Upsilon_q(\xi) \prec \xi \exp\left(\int_0^\xi \frac{p\rho(\omega(\chi))}{\chi} d\chi\right),$$

which is equivalent to

$$\frac{\Upsilon_q(\xi)}{\xi} \prec \exp\left(\int_0^\xi \frac{p\rho(\omega(\chi))}{\chi} d\chi\right).$$

Since

$$\rho(-\varrho|\xi|) \leq \Re(\rho(\omega(\xi\varrho))) \leq \rho(\varrho|\xi|)$$

then, consequently this yields

$$\int_0^1 \frac{\rho(-\varrho|\xi|)}{\varrho} d\varrho \leq \int_0^1 \frac{\Re(\rho(\omega(\xi\varrho)))}{\varrho} d\varrho \leq \int_0^1 \frac{\rho(\varrho|\xi|)}{\varrho} d\varrho.$$

Combining the above inequalities, we obtain

$$p\int_0^1 \frac{\rho(-\varrho|\xi|)}{\varrho} d\varrho \leq \log\left|\frac{\Upsilon_q(\xi)}{\xi}\right| \leq p\int_0^1 \frac{\rho(\varrho|\xi|)}{\varrho} d\varrho.$$

This leads to

$$\exp\left(\int_0^1 \frac{p\rho(-\varrho)}{\varrho} d\varrho\right) \leq \left|\frac{\Upsilon_q(\xi)}{\xi}\right| \leq \exp\left(\int_0^1 \frac{p\rho(\varrho)}{\varrho} d\varrho\right).$$

□

**Corollary 4.** *Let* $\delta = 0$. *If* $\rho$ *is convex univalent in* $\mathbb{U}$ *such that*

$$\frac{T_\upsilon(\xi)}{p} \prec \rho(\xi), \quad \xi \in \mathbb{U},$$

*where* $\rho(0) = 1$ *then*

$$\upsilon(\xi) \prec \xi \exp\left(\int_0^\xi \frac{p\rho(\omega(\chi))}{\chi} d\chi\right),$$

*where* $\omega$ *satisfies* $\omega(0) = 0$ *and* $|\omega(\xi)| < 1$. *Furthermore, the inequality* $|\xi| := \varrho < 1$ *implies*

$$\exp\left(\int_0^1 \frac{p\rho(-\varrho)}{\varrho} d\varrho\right) \leq \left|\frac{\upsilon(\xi)}{\xi}\right| \leq \exp\left(\int_0^1 \frac{p\rho(\varrho)}{\varrho} d\varrho\right).$$

Following that, we consider the functional $\Upsilon_q(\xi)$ as a $q$-differential operator, which is a generalization of the Salagean $q$-differential operator.

### 3.2. Quantum Differential Operator

In this part, we present a quantum differential operator generated by the functional $\Upsilon_q(\xi)$ as follows:

$$
\begin{aligned}
&Q_q^1[\Upsilon_q(\xi)] = (1-\delta)v(\xi) + \frac{\delta}{[p]_q}\left(\xi\eth_q\, v(\xi)\right)\\
&= \xi^p + \sum_{n=p+1}^{\infty} \chi_n(q,\delta,p)a_n\,\xi^n\\
&Q_q^2[\Upsilon_q(\xi)] = (1-\delta)[\Upsilon_q(\xi)] + \frac{\delta}{[p]_q}\left(\xi\eth_q\,[\Upsilon_q(\xi)]\right)\\
&= (1-\delta)[\xi^p + \sum_{n=p+1}^{\infty} \chi_n(q,\delta,p)a_n\,\xi^n] + \frac{\delta}{[p]_q}\left([p]_q\xi^p + \sum_{n=p+1}^{\infty} \chi_n(q,\delta,p)a_n[n]_q\,\xi^n\right)\\
&= \xi^p + \sum_{n=p+1}^{\infty}\left((1-\delta) + \frac{\delta[n]_q}{[p]_q}\right)\chi_n(q,\delta,p)a_n\,\xi^n\\
&= \xi^p + \sum_{n=p+1}^{\infty} [\chi_n(q,\delta,p)]^2 a_n\,\xi^n\\
&\vdots\\
&Q_q^m[\Upsilon_q(\xi)] = Q_q^1[Q_q^{m-1}[\Upsilon_q(\xi)]]\\
&= \xi^p + \sum_{n=p+1}^{\infty} [\chi_n(q,\delta,p)]^m a_n\,\xi^n.
\end{aligned}
$$

$$\left(\xi \in \mathbb{U}\ \ \chi_n(q,\delta,p) = \left((1-\delta) + \frac{\delta[n]_q}{[p]_q}\right)\right)$$

It is clear that

$$Q_q^m[\Upsilon_q(\xi)] = \xi^p + \sum_{n=p+1}^{\infty} [\chi_n(q,\delta,p)]^m a_n\,\xi^n \in \Lambda_p. \tag{3}$$

Note that when $p = 1$ and $\delta = 1$, we obtain the Salagean $q$-differential operator [24]. Furthermore, for $\delta = 1$, we obtain the $q$-multivalent case [25].

**Theorem 7.** *Consider the following assumptions:*

*(i)* $\Theta$ *is univalent in* $\mathbb{U}$;

*(ii)* $\dfrac{\xi\Theta'(\xi)}{\Theta(\xi)(\Theta(\xi)-1)}$ *is star-like in* $\mathbb{U}$;

*(iii) The subordination*

$$\frac{V_{Q_q^m[\Upsilon_q]}(\xi) - p}{T_{Q_q^m[\Upsilon_q]}(\xi) - p} \prec 1 + \frac{1}{p}\frac{\xi\Theta'(\xi)}{\Theta(\xi)(\Theta(\xi)-1)}$$

*holds.*

*Then*

$$\frac{T_{Q_q^m[\Upsilon_q]}(\xi)}{p} \prec \Theta(\xi), \quad \xi \in \mathbb{U}$$

*and* $\Theta$ *is the best dominant.*

**Proof.** Define the function $\Sigma$ as follows:

$$\Sigma(\xi) := \frac{T_{Q_q^m[\Upsilon_q]}(\xi)}{p}, \quad \xi \in \mathbb{U}.$$

Accordingly, we have

$$T_{\Sigma}(\xi) = V_{Q_q^m[\Upsilon_q]}(\xi) - p\Sigma(\xi).$$

Substitution implies that

$$\begin{aligned}\frac{V_{Q_q^m[\Upsilon_q]}(\xi) - p}{T_{Q_q^m[\Upsilon_q]}(\xi) - p} &= \frac{T_{\Sigma}(\xi) + p\Sigma(\xi) - p}{p\Sigma(\xi) - p} \\ &= 1 + \frac{1}{p}\frac{\xi\Sigma'(\xi)}{\Sigma(\xi)(\Sigma(\xi) - 1)}.\end{aligned}$$

Hence,

$$\frac{\xi\Sigma'(\xi)}{\Sigma(\xi)(\Sigma(\xi) - 1)} \prec \frac{\xi\Theta'(\xi)}{\Theta(\xi)(\Theta(\xi) - 1)}, \quad \xi \in \mathbb{U}.$$

According to Lemma 1, we obtain the result. □

**Theorem 8.** *Consider the following assumptions:*

*(i)* $\Theta$ *is univalent in* $\mathbb{U}$;

*(ii)* $\dfrac{\xi\Theta'(\xi)}{\Theta(\xi) - 1}$ *is star-like in* $\mathbb{U}$;

*(iii) The subordination*

$$T_{Q_q^m[\Upsilon_q]}(\xi)\left(\frac{V_{Q_q^m[\Upsilon_q]}(\xi) - p}{T_{Q_q^m[\Upsilon_q]}(\xi) - p} - 1\right) \prec \frac{\xi\Theta'(\xi)}{\Theta(\xi) - 1}$$

*occurs.*

*Then*

$$\frac{T_{Q_q^m[\Upsilon_q]}(\xi)}{p} \prec \Theta(\xi), \quad \xi \in \mathbb{U}$$

*and* $\rho$ *is the best dominant.*

**Proof.** Formulate the function $P$ as follows:

$$\Sigma(\xi) := \frac{T_{Q_q^m[\Upsilon_q]}(\xi)}{p}, \quad \xi \in \mathbb{U}.$$

Accordingly, we have

$$T_{\Sigma}(\xi) + p\Sigma(\xi) = V_{Q_q^m[\Upsilon_q]}(\xi).$$

Substitution yields that

$$T_{Q_q^m[\Upsilon_q]}(\xi)\left(\frac{V_{Q_q^m[\Upsilon_q]}(\xi) - p}{T_{Q_q^m[\Upsilon_q]}(\xi) - p} - 1\right) = \frac{\xi\Sigma'(\xi)}{\Sigma(\xi) - 1}.$$

Hence,

$$\frac{\xi\Sigma'(\xi)}{\Sigma(\xi) - 1} \prec \frac{\xi\Theta'(\xi)}{\Theta(\xi) - 1}, \quad \xi \in \mathbb{U}.$$

According to Lemma 1, we obtain the result. □

**Theorem 9.** *Consider the following assumptions:*

*(i)* $\Theta$ *is univalent in* $\mathbb{U}$;

*(ii)* $T_{\Theta}$ *is star-like in* $\mathbb{U}$;

*(iii) The subordination*

$$V_{Q_q^m[\Upsilon_q]}(\xi) - T_{Q_q^m[\Upsilon_q]}(\xi) \prec T_\Theta(\xi)$$

*holds.*

*Then*

$$\frac{T_{Q_q^m[\Upsilon_q]}(\xi)}{p} \prec \Theta(\xi), \quad \xi \in \mathbb{U}$$

*and* $\Theta$ *is the best dominant.*

**Proof.** We present the function $\Sigma$ as follows:

$$\Sigma(\xi) := \frac{T_{Q_q^m[\Upsilon_q]}(\xi)}{p}, \quad \xi \in \mathbb{U}.$$

Thus, we have

$$T_\Sigma(\xi) + p\Sigma(\xi) = V_{Q_q^m[\Upsilon_q]}(\xi).$$

Substitution yields that

$$V_{Q_q^m[\Upsilon_q]}(\xi) - T_{Q_q^m[\Upsilon_q]}(\xi) = T_\Sigma(\xi).$$

Hence,

$$T_\Sigma(\xi) \prec T_\Theta(\xi), \quad \xi \in \mathbb{U}.$$

According to Lemma 1, we obtain the result $\Sigma(\xi) \prec \Theta(\xi)$. □

**Theorem 10.** *Consider the q-differential operator* $Q_q^m[\Upsilon_q](\xi), \xi \in \mathbb{U}$. *Then*

$$\frac{1}{[p]_q} \frac{\xi \eth_q\Big(Q_q^m[\Upsilon_q](\xi)\Big)}{Q_q^m[\Upsilon_q](\xi)} \prec \frac{1+\sigma_1\xi}{1+\sigma_2\xi}, \quad -1 \le \sigma_2 < \sigma_1 \le 1$$

*if and only if*

$$\left(Q_q^m[\Upsilon_q](\xi) * \left(\frac{\xi^p(1+\sigma_2\exp(i\theta))}{(1-\xi)(1-q\xi)} - \frac{\xi^p[p]_q(1+\sigma_1\exp(i\theta))}{1-\xi}\right)\right) \neq 0.$$

**Proof.** The first direction (necessary) yields

$$\frac{1}{[p]_q} \frac{\xi \eth_q\Big(Q_q^m[\Upsilon_q](\xi)\Big)}{Q_q^m[\Upsilon_q](\xi)} = \frac{1+\sigma_1\varpi(\xi)}{1+\sigma_2\varpi(\xi)},$$

where $|\varpi(\xi)| < |\xi| < 1$ and $\varpi(0) = 0$. That is

$$\frac{1}{[p]_q} \frac{\xi \eth_q\Big(Q_q^m[\Upsilon_q](\xi)\Big)}{Q_q^m[\Upsilon_q](\xi)} \neq \frac{1+\sigma_1\exp(i\theta)}{1+\sigma_2\exp(i\theta)}, \quad \theta \in [0, 2\pi],$$

which is equivalent to

$$\left(\xi\eth_q\Big(Q_q^m[\Upsilon_q](\xi)\Big)(1+\sigma_2\exp(i\theta)) - [p]_q Q_q^m[\Upsilon_q](\xi)(1+\sigma_1\exp(i\theta))\right) \neq 0. \tag{4}$$

Convolution properties imply

$$\begin{aligned} Q_q^m[\Upsilon_q](\xi) &= Q_q^m[\Upsilon_q](\xi) * \frac{\xi^p}{1-\xi} \\ \xi\eth_q\Big(Q_q^m[\Upsilon_q](\xi)\Big) &= Q_q^m[\Upsilon_q](\xi) * \frac{\xi^p}{(1-\xi)(1-q\xi)}. \end{aligned}$$

Thus, in terms of convolution properties, Formula (4) becomes

$$\begin{aligned} &\Big(\xi\eth_q\Big(Q_q^m[\Upsilon_q](\xi)\Big)(1+\sigma_2\exp(i\theta)) - [p]_q Q_q^m[\Upsilon_q](\xi)(1+\sigma_1\exp(i\theta))\Big) \\ &= \left(Q_q^m[\Upsilon_q](\xi) * \frac{\xi^p}{(1-\xi)(1-q\xi)}\right)(1+\sigma_2\exp(i\theta)) \\ &- \left(Q_q^m[\Upsilon_q](\xi) * \frac{\xi^p}{1-\xi}\right)[p]_q(1+\sigma_1\exp(i\theta)) \\ &= Q_q^m[\Upsilon_q](\xi) * \left(\frac{\xi^p(1+\sigma_2\exp(i\theta))}{(1-\xi)(1-q\xi)} - \frac{\xi^p[p]_q(1+\sigma_1\exp(i\theta))}{1-\xi}\right) \\ &\neq 0, \end{aligned}$$

which proves the necessary direction. Conversely, since

$$\frac{1}{[p]_q}\frac{\xi\eth_q\Big(Q_q^m[\Upsilon_q](\xi)\Big)}{Q_q^m[\Upsilon_q](\xi)} \neq \frac{1+\sigma_1\exp(i\theta)}{1+\sigma_2\exp(i\theta)}, \quad \theta \in [0,2\pi], \tag{5}$$

where

$$f(\xi) := \frac{1}{[p]_q}\frac{\xi\eth_q\Big(Q_q^m[\Upsilon_q](\xi)\Big)}{Q_q^m[\Upsilon_q](\xi)}$$

is analytic in $\mathbb{U}$.

Let

$$h(\xi) = \frac{1+\sigma_1\xi}{1+\sigma_2\xi}, \quad \xi \in \mathbb{U}.$$

Relation (5) indicates that

$$f(\mathbb{U})\bigcap h(\mathbb{U}) = \varnothing.$$

As a result, a connected component of $\mathbb{C}(U)\backslash\{h(\partial\mathbb{U})\}$ includes the simply connected domain $f(\mathbb{U})$. The fact that

$$f(0) = h(0) = 1,$$

together with the function's univalence leads to the conclusion that

$$\frac{1}{[p]_q}\frac{\xi\eth_q\Big(Q_q^m[\Upsilon_q](\xi)\Big)}{Q_q^m[\Upsilon_q](\xi)} \prec \frac{1+\sigma_1\xi}{1+\sigma_2\xi}, \quad -1 \le \sigma_2 < \sigma_1 \le 1.$$

This completes the proof. □

**Corollary 5** ([26])**.** *Consider the q-differential operator* $Q_q^m[\Upsilon_q](\xi), \xi \in \mathbb{U}$*, with* $\delta = 0$*. Then*

$$\frac{1}{[p]_q}\frac{\xi\eth_q v_q(\xi)}{v_q(\xi)} \prec \frac{1+\sigma_1\xi}{1+\sigma_2\xi}, \quad -1 \le \sigma_2 < \sigma_1 \le 1$$

*if and only if*

$$\left(v_q(\xi) * \left(\frac{\xi^p(1+\sigma_2\exp(i\theta))}{(1-\xi)(1-q\xi)} - \frac{\xi^p[p]_q(1+\sigma_1\exp(i\theta))}{1-\xi}\right)\right) \neq 0.$$

**Corollary 6** (Theorem 5-[26])**.** *Consider the q-differential operator* $Q_q^m[\Upsilon_q](\xi), \xi \in \mathbb{U}$, *with* $\delta = 0$. *Then*

$$\frac{1}{[p]_q}\frac{\xi\eth_q v_q(\xi)}{v_q(\xi)} \prec \frac{1+\sigma_1\xi}{1+\sigma_2\xi}, \quad -1 \le \sigma_2 < \sigma_1 \le 1$$

*if and only if*

$$\left(v_q(\xi) * \left(\frac{\xi^p(1+\sigma_2\exp(i\theta))}{(1-\xi)(1-q\xi)} - \frac{\xi^p[p]_q(1+\sigma_1\exp(i\theta))}{1-\xi}\right)\right) \neq 0.$$

**Corollary 7** (Corollary 6-[26])**.** *Consider the q-differential operator* $Q_q^m[\Upsilon_q](\xi), \xi \in \mathbb{U}$, *with* $\delta = 0$. *Furthermore, let* $\sigma_1 = 1-2\alpha, \alpha \in [0,1), \sigma_2 = -1$. *Then*

$$\frac{1}{[p]_q}\frac{\xi\eth_q v_q(\xi)}{v_q(\xi)} \prec \frac{1+(1-2\alpha)\xi}{1-\xi}$$

*if and only if*

$$\left(v_q(\xi) * \left(\frac{\xi^p(1-\exp(i\theta))}{(1-\xi)(1-q\xi)} - \frac{\xi^p[p]_q(1+(1-2\alpha)\exp(i\theta))}{1-\xi}\right)\right) \neq 0.$$

*Moreover, when* $q \to 1^-$, *we have received the result in [Theorem 2-[27]].*

## 4. Conclusions

In view of Jackson's calculus, a formula of multivalent functions on the open unit disk was presented ($\Upsilon_q(\xi)$). We introduced the sufficient conditions on $\Upsilon_q(\xi)$ to satisfy the star-like inequality

$$\frac{T_{\Upsilon_q(\xi)}}{p} \prec \rho(\xi).$$

The upper and lower bounds are determined in Theorem 6. Subsequently, we prepared $\Upsilon_q(\xi)$ as a $q$-differential operator (3), which is a generalization of the Salagean $q$-differential operator. We studied the main $q$-star-like formula by giving the sufficient conditions. The operator can be viewed as a conformable differential operator of constant coefficients of convex frame. The consequences are provided for earlier efforts.

In future works, one can employ the $q$-differential operator (3) in different classes of multivalent types such as the convex, uniform and symmetry styles. Furthermore, it can be used to generalize classes of differential equations of a complex variable in the open unit disk and investigate the geometric properties of the solutions. For example, one can investigate the following Briot–Bouquet differential Equation [21]

$$\frac{T_{\Upsilon_q(\xi)}}{p} = \rho(\xi), \quad \xi \in \mathbb{U},\ \rho(0) = 1.$$

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