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Results on the Existence of Solutions Associated with Some Weak Vector Variational Inequalities

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Abstract: In this paper, by considering the notions of the invex set, Fréchet differentiability, invexity and pseudoinvexity for the involved functionals of curvilinear integral type, we establish some relations between the solutions of a class of weak vector variational inequalities and (weak) efficient solutions of the associated control problem.

Keywords: Fréchet differentiability; weak efficient solution; curvilinear integral; invex set

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1. Introduction

In order to study and investigate the multiple-objective optimization problems, over time, several concepts of *efficient solutions* have been considered. In this sense, Geoffrion [1] defined *proper efficiency*. Furthermore, Klinger [2] proposed *improper solutions* associated with a class of vector optimization problems. By using vector variational-like inequalities, Kazmi [3] established some existence results of a *weak minimum* in constrained multiple-objective optimization problems. In addition, Ghaznavi-ghosoni and Khorram [4] introduced *approximate solutions* to the state conditions of efficiency in general multiple-objective optimization problems.

On the other hand, as it is well-known, the concept of *convexity* plays an important role in optimization theory. However, since convexity does not cover certain concrete problems, its generalization became a real necessity. In this direction, Hanson [5] introduced the notion of *invex functions*. Of course, over time, a lot of various extensions have been defined (for example, preinvexity, pseudoinvexity, univexity, quasi-invexity, approximate convexity) by authors such as Antczak [6], Ahmad et al. [7]) and Mishra et al. [8]. Moreover, some of these notions have been transposed in a multidimensional framework involving multiple or curvilinear integral functionals; see, for instance, Treanţă [9], Mititelu and Treanţă [10].

Notable results associated with variational inequalities, having important applications in engineering or traffic analysis, have been formulated by Giannessi [11]. We all know that vector variational inequalities provide results for the existence of solutions in multiple-objective optimization problems. In this regard, the reader is directed to the research work by Ruiz-Garzón et al. [12]. Treanţă [13] studied a class of variational inequalities involving curvilinear integrals. Kim [14] established some connections between multiple-objective continuous-time problems and vector variational-type inequalities. As a natural extension of continuous-time variational problems, optimal control problems have been used to study different engineering problems or processes in game theory, operations research and economics. In this sense, Jha et al. [15] and Treanţă [16,17] studied and established optimality conditions of efficiency, well-posedness, saddle-point optimality criteria and modified the objective function method in multitime variational control problems with multiple or curvilinear integral-type functionals.

Very recently, Treanţă [18] established relations between the solutions of a class of vector variational inequalities and (proper) efficient solutions of the associated multiple-objective control problem. As a natural sequel of the advances mentioned above, in this paper, we introduce weak vector variational inequalities and the associated multiple-objective optimization problems generated by curvilinear integral-type functionals, which are path-independent. By using the concepts of the invex set, Fréchet differentiability, invexity and pseudoinvexity for the considered curvilinear integral functionals, we state some relations between the solutions of a class of weak vector variational inequalities and (weak) efficient solutions of the associated optimization problem.

Further, the paper is structured as follows. In Section 2, we present the preliminaries/auxiliary results and problem description. In Section 3, we formulate and prove some characterization results associated with the solutions of the considered control problems. In Section 4, we state the conclusions for this study.

2. Preliminaries and Problem Description

We consider $\mathcal{T}$ a compact set included in $\mathbb{R}^b$ and $\mathcal{T} \ni \mu = (\mu^\zeta)$, $\zeta = \overline{1,b}$, is a multivariable. Let $\mathcal{T} \supset \mathsf{C} : \mu = \mu(\varsigma), \varsigma \in [t_0, t_1]$ be a piecewise smooth (differentiable) curve joining the following two fixed points $\mu_1 = (\mu_1^1, \dots, \mu_1^b)$, $\mu_2 = (\mu_2^1, \dots, \mu_2^b)$ in $\mathcal{T}$. In addition, we denote by Λ the space consisting in all piecewise smooth *state* functions $\sigma : \mathcal{T} \to \mathbb{R}^a$, and by Ω the space consisting of all piecewise continuous *control* functions $\eta : \mathcal{T} \to \mathbb{R}^k$. Moreover, on $\Lambda \times \Omega$, we consider the inner product

$$\begin{aligned}
\langle(\sigma,\eta),(\xi,x)\rangle &= \int_{\mathsf{C}} \Big[\sigma(\mu)\cdot\xi(\mu) + \eta(\mu)\cdot x(\mu)\Big] d\mu^\zeta \\
&= \int_{\mathsf{C}} \left[\sum_{i=1}^{a} \sigma^i(\mu)\xi^i(\mu) + \sum_{j=1}^{k} \eta^j(\mu)x^j(\mu)\right] d\mu^1 \\
&\quad + \cdots + \Big[\sum_{i=1}^{a} \sigma^i(\mu)\xi^i(\mu) + \sum_{j=1}^{k} \eta^j(\mu)x^j(\mu)\Big] d\mu^b,
\end{aligned}$$

for all $(\sigma,\eta),(\xi,x) \in \Lambda \times \Omega$, together with the norm induced by it.

In the following, we consider a vector functional given by curvilinear integrals $\Psi : \Lambda \times \Omega \to \mathbb{R}^\nu$,

$$\begin{aligned}
\Psi(\sigma,\eta) &= \int_{\mathsf{C}} \psi_\zeta(\mu,\sigma(\mu),\sigma_\alpha(\mu),\eta(\mu))d\mu^\zeta \\
&= \left(\int_{\mathsf{C}} \psi_\zeta^1(\mu,\sigma(\mu),\sigma_\alpha(\mu),\eta(\mu))d\mu^\zeta, \dots, \int_{\mathsf{C}} \psi_\zeta^\nu(\mu,\sigma(\mu),\sigma_\alpha(\mu),\eta(\mu))d\mu^\zeta\right),
\end{aligned}$$

where $\psi_\zeta = (\psi_\zeta^l) : \mathcal{T} \times \mathbb{R}^a \times \mathbb{R}^{ab} \times \mathbb{R}^k \to \mathbb{R}^\nu$, $\zeta = \overline{1,b}$, $l = \overline{1,\nu}$ are assumed to be vector-valued C^2-class functions. Furthermore, by D_α, $\alpha \in \{1,\dots,b\}$, we denote the operator of total derivative, and assume that the aforementioned 1-forms

$$\psi_\zeta = \left(\psi_\zeta^1, \dots, \psi_\zeta^\nu\right) : \mathcal{T} \times \mathbb{R}^a \times \mathbb{R}^{ab} \times \mathbb{R}^k \to \mathbb{R}^\nu, \quad \zeta = \overline{1,b}$$

are closed ($D_\alpha\psi_\zeta^l = D_\zeta\psi_\alpha^l$, $\zeta,\alpha = \overline{1,b}$, $\zeta \neq \alpha$, $l = \overline{1,\nu}$).

Throughout this paper, the next rules will be considered for equalities and inequalities:

$$m = n \Leftrightarrow m^l = n^l, \quad m \le n \Leftrightarrow m^l \le n^l,$$

$$m < n \Leftrightarrow m^l < n^l, \quad m \preceq n \Leftrightarrow m \le n,\ m \neq n, \quad l = \overline{1,\nu}$$

for any ν-tuples $m = \left(m^1, \cdots, m^\nu\right)$, $n = \left(n^1, \cdots, n^\nu\right)$ in $\mathbb{R}^\nu$.

Next, we state the following PDE/PDI constrained control problem

$$(P) \quad \min_{(\sigma,\eta)}\left\{\Psi(\sigma,\eta)=\int_C \psi_\zeta(\mu,\sigma(\mu),\sigma_\alpha(\mu),\eta(\mu))d\mu^\zeta\right\} \text{ subject to } (\sigma,\eta)\in\mathcal{S},$$

where

$$\begin{aligned}\Psi(\sigma,\eta) &= \int_C \psi_\zeta(\mu,\sigma(\mu),\sigma_\alpha(\mu),\eta(\mu))d\mu^\zeta\\ &= \left(\int_C \psi_\zeta^1(\mu,\sigma(\mu),\sigma_\alpha(\mu),\eta(\mu))d\mu^\zeta,\ldots,\int_C \psi_\zeta^\nu(\mu,\sigma(\mu),\sigma_\alpha(\mu),\eta(\mu))d\mu^\zeta\right)\\ &= \left(\Psi^1(\sigma,\eta),\ldots,\Psi^\nu(\sigma,\eta)\right)\end{aligned}$$

and

$$\mathcal{S}=\left\{(\sigma,\eta)\in\Lambda\times\Omega \mid Z(\mu,\sigma(\mu),\sigma_\alpha(\mu),\eta(\mu))=0,\ Y(\mu,\sigma(\mu),\sigma_\alpha(\mu),\eta(\mu))\le 0,\ \sigma|_{\mu=\mu_1,\mu_2}=\text{given}\right\}.$$

In the above mathematical context, we consider $Z=(Z^\iota):\mathcal{T}\times\mathbb{R}^a\times\mathbb{R}^{ab}\times\mathbb{R}^k\to\mathbb{R}^t$, $\iota=\overline{1,t}$, $Y=(Y^r):\mathcal{T}\times\mathbb{R}^a\times\mathbb{R}^{ab}\times\mathbb{R}^k\to\mathbb{R}^q$, $r=\overline{1,q}$, are functions of C^2-class.

Definition 1 (Mititelu and Treanţă [10]). *A pair $(\sigma^0,\eta^0)\in\mathcal{S}$ is an efficient solution of (P) if there exists no other $(\sigma,\eta)\in\mathcal{S}$ such that $\Psi(\sigma,\eta)\preceq\Psi(\sigma^0,\eta^0)$, or equivalently, $\Psi^l(\sigma,\eta)-\Psi^l(\sigma^0,\eta^0)\le 0$, $(\forall)l=\overline{1,\nu}$, with strict inequality for at least one l.*

Definition 2 (Geoffrion [1]). *A pair $(\sigma^0,\eta^0)\in\mathcal{S}$ is a proper efficient solution of (P) if $(\sigma^0,\eta^0)\in\mathcal{S}$ is an efficient solution in (P) and there exists a positive real number M such that, for all $l=\overline{1,\nu}$, we have*

$$\Psi^l(\sigma^0,\eta^0)-\Psi^l(\sigma,\eta)\le M\left(\Psi^s(\sigma,\eta)-\Psi^s(\sigma^0,\eta^0)\right)$$

for some $s\in\{1,\ldots,\nu\}$ such that

$$\Psi^s(\sigma,\eta)>\Psi^s(\sigma^0,\eta^0),$$

whenever $(\sigma,\eta)\in\mathcal{S}$ and

$$\Psi^l(\sigma,\eta)<\Psi^l(\sigma^0,\eta^0).$$

Definition 3. *A pair $(\sigma^0,\eta^0)\in\mathcal{S}$ is a weak efficient solution of (P) if there exists no other $(\sigma,\eta)\in\mathcal{S}$ such that $\Psi(\sigma,\eta)<\Psi(\sigma^0,\eta^0)$, or equivalently, $\Psi^l(\sigma,\eta)-\Psi^l(\sigma^0,\eta^0)<0$, $(\forall)l=\overline{1,\nu}$.*

Taking into account Treanţă's works [18,19], we consider the next vector functional defined by the curvilinear integral (independent of the path)

$$K:\Lambda\times\Omega\to\mathbb{R}^\nu,\quad K(\sigma,\eta)=\int_C \kappa_\zeta(\mu,\sigma(\mu),\sigma_\alpha(\mu),\eta(\mu))d\mu^\zeta$$

and formulate the concepts of invexity and pseudoinvexity for K.

Definition 4 (Treanţă [18]). *K is said to be invex at $\left(\sigma^0,\eta^0\right)\in\Lambda\times\Omega$ with respect to π and v if there exist*

$$\pi:\mathcal{T}\times\mathbb{R}^a\times\mathbb{R}^k\times\mathbb{R}^a\times\mathbb{R}^k\to\mathbb{R}^a,$$

$$\pi=\pi\left(\mu,\sigma(\mu),\eta(\mu),\sigma^0(\mu),\eta^0(\mu)\right)=\left(\pi^i\left(\mu,\sigma(\mu),\eta(\mu),\sigma^0(\mu),\eta^0(\mu)\right)\right),\quad i=\overline{1,a},$$

of C^1-class with $\pi\left(\mu,\sigma^0(\mu),\eta^0(\mu),\sigma^0(\mu),\eta^0(\mu)\right)=0$, $(\forall)\mu\in\mathcal{T}$, $\pi(\mu_1)=\pi(\mu_2)=0$, and

$$v:\mathcal{T}\times\mathbb{R}^a\times\mathbb{R}^k\times\mathbb{R}^a\times\mathbb{R}^k\to\mathbb{R}^k,$$

$$v = v\left(\mu, \sigma(\mu), \eta(\mu), \sigma^0(\mu), \eta^0(\mu)\right) = \left(v^j\left(\mu, \sigma(\mu), \eta(\mu), \sigma^0(\mu), \eta^0(\mu)\right)\right), \quad j = \overline{1,k},$$

of C^0*-class with* $v\left(\mu, \sigma^0(\mu), \eta^0(\mu), \sigma^0(\mu), \eta^0(\mu)\right) = 0$, $(\forall)\mu \in \mathcal{T}$, $v(\mu_1) = v(\mu_2) = 0$, *such that*

$$K(\sigma, \eta) - K\left(\sigma^0, \eta^0\right) \geq \int_C \left[\frac{\partial \kappa_\zeta}{\partial \sigma}\left(\mu, \sigma^0(\mu), \sigma^0_\alpha(\mu), \eta^0(\mu)\right)\pi + \frac{\partial \kappa_\zeta}{\partial \sigma_\alpha}\left(\mu, \sigma^0(\mu), \sigma^0_\alpha(\mu), \eta^0(\mu)\right)D_\alpha \pi\right] d\mu^\zeta$$
$$+ \int_C \left[\frac{\partial \kappa_\zeta}{\partial \eta}\left(\mu, \sigma^0(\mu), \sigma^0_\alpha(\mu), \eta^0(\mu)\right)v\right] d\mu^\zeta,$$

for any $(\sigma, \eta) \in \Lambda \times \Omega$.

Definition 5 (Treanţă [18]). *In the above definition, we say that K is strictly invex at* $\left(\sigma^0, \eta^0\right) \in \Lambda \times \Omega$ *with respect to* π *and* v *if we replace* $\geq$ *with* $>$, *with* $(\sigma, \eta) \neq \left(\sigma^0, \eta^0\right)$.

Definition 6. *K is said to be pseudoinvex at* $\left(\sigma^0, \eta^0\right) \in \Lambda \times \Omega$ *with respect to* π *and* v *if there exist*

$$\pi : \mathcal{T} \times \mathbb{R}^a \times \mathbb{R}^k \times \mathbb{R}^a \times \mathbb{R}^k \to \mathbb{R}^a,$$

$$\pi = \pi\left(\mu, \sigma(\mu), \eta(\mu), \sigma^0(\mu), \eta^0(\mu)\right) = \left(\pi^i\left(\mu, \sigma(\mu), \eta(\mu), \sigma^0(\mu), \eta^0(\mu)\right)\right), \quad i = \overline{1,a}$$

of C^1*-class with* $\pi\left(\mu, \sigma^0(\mu), \eta^0(\mu), \sigma^0(\mu), \eta^0(\mu)\right) = 0$, $(\forall)\mu \in \mathcal{T}$, $\pi(\mu_1) = \pi(\mu_2) = 0$ *and*

$$v : \mathcal{T} \times \mathbb{R}^a \times \mathbb{R}^k \times \mathbb{R}^a \times \mathbb{R}^k \to \mathbb{R}^k,$$

$$v = v\left(\mu, \sigma(\mu), \eta(\mu), \sigma^0(\mu), \eta^0(\mu)\right) = \left(v^j\left(\mu, \sigma(\mu), \eta(\mu), \sigma^0(\mu), \eta^0(\mu)\right)\right), \quad j = \overline{1,k}$$

of C^0*-class with* $v\left(\mu, \sigma^0(\mu), \eta^0(\mu), \sigma^0(\mu), \eta^0(\mu)\right) = 0$, $(\forall)\mu \in \mathcal{T}$, $v(\mu_1) = v(\mu_2) = 0$, *such that*

$$K(\sigma, \eta) - K\left(\sigma^0, \eta^0\right) < 0$$

implies

$$\int_C \left[\frac{\partial \kappa_\zeta}{\partial \sigma}\left(\mu, \sigma^0(\mu), \sigma^0_\alpha(\mu), \eta^0(\mu)\right)\pi + \frac{\partial \kappa_\zeta}{\partial \sigma_\alpha}\left(\mu, \sigma^0(\mu), \sigma^0_\alpha(\mu), \eta^0(\mu)\right)D_\alpha \pi\right] d\mu^\zeta$$
$$+ \int_C \left[\frac{\partial \kappa_\zeta}{\partial \eta}\left(\mu, \sigma^0(\mu), \sigma^0_\alpha(\mu), \eta^0(\mu)\right)v\right] d\mu^\zeta < 0,$$

or equivalently

$$\int_C \left[\frac{\partial \kappa_\zeta}{\partial \sigma}\left(\mu, \sigma^0(\mu), \sigma^0_\alpha(\mu), \eta^0(\mu)\right)\pi + \frac{\partial \kappa_\zeta}{\partial \sigma_\alpha}\left(\mu, \sigma^0(\mu), \sigma^0_\alpha(\mu), \eta^0(\mu)\right)D_\alpha \pi\right] d\mu^\zeta$$
$$+ \int_C \left[\frac{\partial \kappa_\zeta}{\partial \eta}\left(\mu, \sigma^0(\mu), \sigma^0_\alpha(\mu), \eta^0(\mu)\right)v\right] d\mu^\zeta \geq 0 \Rightarrow K(\sigma, \eta) - K\left(\sigma^0, \eta^0\right) \geq 0,$$

for any $(\sigma, \eta) \in \Lambda \times \Omega$.

Some examples for invex or pseudoinvex curvilinear-type integral functionals can be found in Treanţă [19]. For other points of view regarding vector/scalar optimization problems, the reader can consult Lee et al. [20], Kazmi et al. [21] and Treanţă [22].

Definition 7 (Treanţă [18])**.** *The subset $\emptyset \neq \mathsf{X} \times \mathsf{Q} \subset \Lambda \times \Omega$ is called invex with respect to π and v if*

$$(\sigma^0, \eta^0) + \lambda\Big(\pi\Big(\mu, \sigma, \eta, \sigma^0, \eta^0\Big), v\Big(\mu, \sigma, \eta, \sigma^0, \eta^0\Big)\Big) \in \mathsf{X} \times \mathsf{Q},$$

for all (σ, η), $(\sigma^0, \eta^0) \in \mathsf{X} \times \mathsf{Q}$ and $\lambda \in [0,1]$.

In order to establish some existence results of solutions for a control problem (P), we consider the following *(weak) variational inequalities*:

(I) *Find $(\sigma^0, \eta^0) \in \mathcal{S}$ such that there exists no $(\sigma, \eta) \in \mathcal{S}$ fulfilling*

$$(VI) \quad \Bigg(\int_{\mathsf{C}}\left[\frac{\partial \psi^1_\zeta}{\partial \sigma}\Big(\mu, \sigma^0(\mu), \sigma^0_\alpha(\mu), \eta^0(\mu)\Big)\pi + \frac{\partial \psi^1_\zeta}{\partial \eta}\Big(\mu, \sigma^0(\mu), \sigma^0_\alpha(\mu), \eta^0(\mu)\Big)v\right]d\mu^\zeta$$
$$+ \int_{\mathsf{C}}\left[\frac{\partial \psi^1_\zeta}{\partial \sigma_\alpha}\Big(\mu, \sigma^0(\mu), \sigma^0_\alpha(\mu), \eta^0(\mu)\Big)D_\alpha \pi\right]d\mu^\zeta, \dots,$$
$$\int_{\mathsf{C}}\left[\frac{\partial \psi^\nu_\zeta}{\partial \sigma}\Big(\mu, \sigma^0(\mu), \sigma^0_\alpha(\mu), \eta^0(\mu)\Big)\pi + \frac{\partial \psi^\nu_\zeta}{\partial \eta}\Big(\mu, \sigma^0(\mu), \sigma^0_\alpha(\mu), \eta^0(\mu)\Big)v\right]d\mu^\zeta$$
$$+ \int_{\mathsf{C}}\left[\frac{\partial \psi^\nu_\zeta}{\partial \sigma_\alpha}\Big(\mu, \sigma^0(\mu), \sigma^0_\alpha(\mu), \eta^0(\mu)\Big)D_\alpha \pi\right]d\mu^\zeta\Bigg) \leq 0;$$

(II) *Find $(\sigma^0, \eta^0) \in \mathcal{S}$ such that there exists no $(\sigma, \eta) \in \mathcal{S}$ fulfilling*

$$(WVI) \quad \Bigg(\int_{\mathsf{C}}\left[\frac{\partial \psi^1_\zeta}{\partial \sigma}\Big(\mu, \sigma^0(\mu), \sigma^0_\alpha(\mu), \eta^0(\mu)\Big)\pi + \frac{\partial \psi^1_\zeta}{\partial \eta}\Big(\mu, \sigma^0(\mu), \sigma^0_\alpha(\mu), \eta^0(\mu)\Big)v\right]d\mu^\zeta$$
$$+ \int_{\mathsf{C}}\left[\frac{\partial \psi^1_\zeta}{\partial \sigma_\alpha}\Big(\mu, \sigma^0(\mu), \sigma^0_\alpha(\mu), \eta^0(\mu)\Big)D_\alpha \pi\right]d\mu^\zeta, \dots,$$
$$\int_{\mathsf{C}}\left[\frac{\partial \psi^\nu_\zeta}{\partial \sigma}\Big(\mu, \sigma^0(\mu), \sigma^0_\alpha(\mu), \eta^0(\mu)\Big)\pi + \frac{\partial \psi^\nu_\zeta}{\partial \eta}\Big(\mu, \sigma^0(\mu), \sigma^0_\alpha(\mu), \eta^0(\mu)\Big)v\right]d\mu^\zeta$$
$$+ \int_{\mathsf{C}}\left[\frac{\partial \psi^\nu_\zeta}{\partial \sigma_\alpha}\Big(\mu, \sigma^0(\mu), \sigma^0_\alpha(\mu), \eta^0(\mu)\Big)D_\alpha \pi\right]d\mu^\zeta\Bigg) < 0.$$

Next, we present an illustrative example to verify that the above-mentioned class of vector-controlled variational inequalities is solvable at a given point.

Example 1. *Let us consider $\mathcal{T} = [0,1] \times [0,1]$ and $\mathsf{C} \subset \mathcal{T}$ is a differentiable curve that links $(0,0)$ and $(1,1)$. Furthermore, we assume that $\sigma, \eta : \mathcal{T} \to \mathbb{R}$ are piecewise differentiable functions, $\pi, v : \mathcal{T} \times \mathbb{R}^4 \to \mathbb{R}$ are given by: $\pi = 0$, $v = e^{\sigma^0(\mu)} - e^{\sigma(\mu)}$, $(\forall)\mu \in \mathcal{T} \setminus \{\mu_1, \mu_2\}$ and $v = 0$ for $\mu \in \{\mu_1, \mu_2\}$. Define the following 1-forms of Lagrange type*

$$\psi_\zeta = \Big(\psi^1_\zeta, \psi^2_\zeta\Big) : \mathcal{T} \times \mathbb{R}^2 \to \mathbb{R}^2, \quad \zeta = \overline{1,2},$$

as

$$\psi^1_\zeta(\mu, \sigma(\mu), \eta(\mu)) = \left(-\eta(\mu) - \frac{1}{2},\ -\sigma(\mu)\right),$$
$$\psi^2_\zeta(\mu, \sigma(\mu), \eta(\mu)) = \left(e^{\eta(\mu)} + \frac{1}{2},\ e^{\eta(\mu)}\right).$$

Further, we can easily notice that $(\sigma^0, \eta^0) = (0,0)$ is a solution for the associated vector-controlled variational inequality (VI). Indeed, we have

$$\Big(\int_C \left[\frac{\partial \psi^1_\zeta}{\partial \sigma}\left(\mu, \sigma^0(\mu), \eta^0(\mu)\right)\pi + \frac{\partial \psi^1_\zeta}{\partial \eta}\left(\mu, \sigma^0(\mu), \eta^0(\mu)\right)v \right] d\mu^\zeta,$$

$$\int_C \left[\frac{\partial \psi^2_\zeta}{\partial \sigma}\left(\mu, \sigma^0(\mu), \eta^0(\mu)\right)\pi + \frac{\partial \psi^2_\zeta}{\partial \eta}\left(\mu, \sigma^0(\mu), \eta^0(\mu)\right)v \right] d\mu^\zeta \Big)$$

$$= \Big(\int_C \left(e^{\sigma(\mu)} - 1\right) d\mu^1, \int_C e^{\eta(\mu)}\left(1 - e^{\sigma(\mu)}\right)\left(d\mu^1 + d\mu^2\right) \Big) \nleq (0,0),$$

for all piecewise differentiable functions $\sigma, \eta : \mathcal{T} \to \mathbb{R}$.

Very recently, Treanţă [18] established the following two results:

Theorem 1 (Treanţă [18]). *Consider* $\mathcal{S} \subset \Lambda \times \Omega$ *is an invex set with respect to* π *and* v *and let* $(\sigma^0, \eta^0) \in \mathcal{S}$ *be a proper efficient solution of* (P)*. If each curvilinear integral*

$$\Psi^l(\sigma, \eta), \quad l = \overline{1, \nu}$$

is Fréchet differentiable at $(\sigma^0, \eta^0) \in \mathcal{S}$*, then the pair* (σ^0, η^0) *solves* (VI)*.*

By considering the vector variational inequality (VI), the next theorem provides a characterization of efficient solutions in (P).

Theorem 2 (Treanţă [18]). *Consider* $(\sigma^0, \eta^0) \in \mathcal{S}$ *is a solution of* (VI) *and each curvilinear integral* $\Psi^l(\sigma, \eta)$, $l = \overline{1, \nu}$ *is Fréchet differentiable and invex at* $(\sigma^0, \eta^0) \in \mathcal{S}$ *with respect to* π *and* v*. Then, the pair* (σ^0, η^0) *is an efficient solution for* (P)*.*

3. Main Results

In this section, we formulate some connections between the solutions of the considered weak vector variational inequalities and (weak, proper) efficient solutions of the associated control problem (P).

The next result formulates a sufficient condition for a pair $(\sigma^0, \eta^0) \in \mathcal{S}$ to be a solution of (WVI).

Theorem 3. *Let* $\mathcal{S} \subset \Lambda \times \Omega$ *be an invex set with respect to* π *and* v *and let* $(\sigma^0, \eta^0) \in \mathcal{S}$ *be a weak efficient solution of* (P)*. If each curvilinear integral* $\Psi^l(\sigma, \eta), l = \overline{1, \nu}$ *is Fréchet differentiable at* $(\sigma^0, \eta^0) \in \mathcal{S}$*, then the pair* (σ^0, η^0) *solves* (WVI)*.*

Proof. Since $(\sigma^0, \eta^0) \in \mathcal{S}$ is a weak efficient solution of (P), the results show that there exists no other feasible solution $(\sigma, \eta) \in \mathcal{S}$ such that $\Psi(\sigma, \eta) < \Psi(\sigma^0, \eta^0)$, or equivalently

$$\Psi^l(\sigma, \eta) - \Psi^l(\sigma^0, \eta^0) < 0, \quad (\forall) l = \overline{1, \nu}. \tag{1}$$

By hypothesis, we have that $\mathcal{S} \subset \Lambda \times \Omega$ is an invex set with respect to π and v. Thus, for $\lambda \in [0,1]$, we have $(z, w) = (\sigma^0, \eta^0) + \lambda\left(\pi(\mu, \sigma, \eta, \sigma^0, \eta^0), v(\mu, \sigma, \eta, \sigma^0, \eta^0)\right) \in \mathcal{S}$. Thus, by using (1), we get that there exists no other feasible solution $(\sigma, \eta) \in \mathcal{S}$ such that $\Psi(z, w) < \Psi(\sigma^0, \eta^0)$, or equivalently

$$\Psi^l(z, w) - \Psi^l(\sigma^0, \eta^0) < 0, \quad (\forall) l = \overline{1, \nu}. \tag{2}$$

Further, we apply that each curvilinear integral $\Psi^l(\sigma, \eta)$, $l = \overline{1, \nu}$, is Fréchet differentiable at $(\sigma^0, \eta^0) \in \mathcal{S}$ and proceeding as in the proof of Theorem 1, by (2), we obtain that there exists no other feasible solution $(\sigma, \eta) \in \mathcal{S}$ such that

$$\int_C \left[\frac{\partial \psi_\zeta^l}{\partial \sigma} \Big(\mu, \sigma^0(\mu), \sigma_\alpha^0(\mu), \eta^0(\mu) \Big) \pi + \frac{\partial \psi_\zeta^l}{\partial \eta} \Big(\mu, \sigma^0(\mu), \sigma_\alpha^0(\mu), \eta^0(\mu) \Big) v \right] d\mu^\zeta$$
$$+ \int_C \left[\frac{\partial \psi_\zeta^l}{\partial \sigma_\alpha} \Big(\mu, \sigma^0(\mu), \sigma_\alpha^0(\mu), \eta^0(\mu) \Big) D_\alpha \pi \right] d\mu^\zeta < 0$$

for all $l = \overline{1,\nu}$, and this ends the proof. □

The next theorem provides a characterization of weak efficient solutions for (P) by using the weak-vector-controlled variational inequality (WVI).

Theorem 4. *Let $(\sigma^0, \eta^0) \in \mathcal{S}$ be a solution of (WVI). If each curvilinear integral $\Psi^l(\sigma, \eta),\ l = \overline{1,\nu}$ is Fréchet differentiable and pseudoinvex at $(\sigma^0, \eta^0) \in \mathcal{S}$ with respect to π and v, then the pair (σ^0, η^0) is a weak efficient solution of (P).*

Proof. By reductio ad absurdum, consider that $(\sigma^0, \eta^0) \in \mathcal{S}$ is a solution of (WVI) but it is not a weak efficient solution of (P). In consequence, there exists $(\sigma, \eta) \in \mathcal{S}$ such that, for all $l = \overline{1,\nu}$,

$$\Psi^l(\sigma, \eta) - \Psi^l(\sigma^0, \eta^0) < 0.$$

By hypothesis, each curvilinear integral $\Psi^l(\sigma, \eta),\ l = \overline{1,\nu}$, is Fréchet differentiable and pseudoinvex at $(\sigma^0, \eta^0) \in \mathcal{S}$ with respect to π and v. In consequence, we have

$$\int_C \left[\frac{\partial \psi_\zeta^l}{\partial \sigma} \Big(\mu, \sigma^0(\mu), \sigma_\alpha^0(\mu), \eta^0(\mu) \Big) \pi + \frac{\partial \psi_\zeta^l}{\partial \eta} \Big(\mu, \sigma^0(\mu), \sigma_\alpha^0(\mu), \eta^0(\mu) \Big) v \right] d\mu^\zeta$$
$$+ \int_C \left[\frac{\partial \psi_\zeta^l}{\partial \sigma_\alpha} \Big(\mu, \sigma^0(\mu), \sigma_\alpha^0(\mu), \eta^0(\mu) \Big) D_\alpha \pi \right] d\mu^\zeta < 0$$

for any $(\sigma, \eta) \in \mathcal{S}$ and $l = \overline{1,\nu}$. This contradicts that $(\sigma^0, \eta^0) \in \mathcal{S}$ is a solution of (WVI) and the proof is complete. □

The next result formulates a sufficient condition for a weak efficient solution $(\sigma^0, \eta^0) \in \mathcal{S}$ of (P) to be an efficient solution $(\sigma^0, \eta^0) \in \mathcal{S}$ of (P).

Theorem 5. *Let $(\sigma^0, \eta^0) \in \mathcal{S}$ be a weak efficient solution of (P). If each curvilinear integral $\Psi^l(\sigma, \eta),\ l = \overline{1,\nu}$ is Fréchet differentiable and strictly invex at $(\sigma^0, \eta^0) \in \mathcal{S}$ with respect to π and v and $\mathcal{S}$ is an invex set with respect to π and v, then the pair (σ^0, η^0) is an efficient solution for (P).*

Proof. By contradiction, assume that $(\sigma^0, \eta^0) \in \mathcal{S}$ is a weak efficient solution of (P) but not an efficient solution of (P). It results there exists a feasible solution $(\sigma, \eta) \in \mathcal{S}$ satisfying $\Psi(\sigma, \eta) \preceq \Psi(\sigma^0, \eta^0)$, or equivalently

$$\Psi^l(\sigma, \eta) - \Psi^l(\sigma^0, \eta^0) \le 0,\ (\forall) l = \overline{1,\nu} \tag{3}$$

with strict inequality for at least one l.

By hypothesis, each curvilinear integral $\Psi^l(\sigma, \eta),\ l = \overline{1,\nu}$ is Fréchet differentiable and strictly invex at $(\sigma^0, \eta^0) \in \mathcal{S}$ with respect to π and v. In consequence, we have

$$\Psi^l(\sigma, \eta) - \Psi^l(\sigma^0, \eta^0) > \int_C \left[\frac{\partial \psi_\zeta^l}{\partial \sigma} \Big(\mu, \sigma^0(\mu), \sigma_\alpha^0(\mu), \eta^0(\mu) \Big) \pi + \frac{\partial \psi_\zeta^l}{\partial \eta} \Big(\mu, \sigma^0(\mu), \sigma_\alpha^0(\mu), \eta^0(\mu) \Big) v \right] d\mu^\zeta$$
$$+ \int_C \left[\frac{\partial \psi_\zeta^l}{\partial \sigma_\alpha} \Big(\mu, \sigma^0(\mu), \sigma_\alpha^0(\mu), \eta^0(\mu) \Big) D_\alpha \pi \right] d\mu^\zeta \tag{4}$$

for any $(\sigma, \eta) \neq (\sigma^0, \eta^0) \in \mathcal{S}$ and $l = \overline{1,\nu}$.

By using (3) and (4), we obtain that there exists $(\sigma, \eta) \in \mathcal{S}$ such that

$$\int_{C} \left[\frac{\partial \psi_{\zeta}^{l}}{\partial \sigma} \left(\mu, \sigma^{0}(\mu), \sigma_{\alpha}^{0}(\mu), \eta^{0}(\mu) \right) \pi + \frac{\partial \psi_{\zeta}^{l}}{\partial \eta} \left(\mu, \sigma^{0}(\mu), \sigma_{\alpha}^{0}(\mu), \eta^{0}(\mu) \right) v \right] d\mu^{\zeta}$$
$$+ \int_{C} \left[\frac{\partial \psi_{\zeta}^{l}}{\partial \sigma_{\alpha}} \left(\mu, \sigma^{0}(\mu), \sigma_{\alpha}^{0}(\mu), \eta^{0}(\mu) \right) D_{\alpha} \pi \right] d\mu^{\zeta} < 0$$

for all $l = \overline{1, \nu}$. In consequence, $(\sigma^0, \eta^0) \in \mathcal{S}$ is not a solution for (WVI) and, in accordance with Theorem 3, it follows that $(\sigma^0, \eta^0) \in \mathcal{S}$ is not a weak efficient solution of (P). Therefore, we obtain a contradiction, and the proof is complete. □

Example 2. *Let us extremize the mechanical work accomplished by the variable forces* $\bar{V}_1 \left(e^{-\eta(\mu)} + \frac{1}{2}, e^{-\eta(\mu)} \right)$ *and* $\bar{V}_2 \left(e^{\sigma(\mu)} + \frac{1}{2}, e^{\sigma(\mu)} \right)$ *to move the application point along the piecewise differentiable curve* C*, contained in* $\mathcal{T} = [0,1]^2 = [0,1] \times [0,1]$ *and linking* $\mu_1 = (0,0)$ *and* $\mu_2 = (1,1)$*, such that the following controlled dynamic system*

$$\frac{\partial \sigma}{\partial \mu^1}(\mu) = \frac{\partial \sigma}{\partial \mu^2}(\mu) = \eta(\mu),$$

$$1 - e^{\sigma(\mu) + \sigma^2(\mu)} \leq 0, \quad e^{\eta(\mu)} - 2 + e^{\eta^2(\mu)} \leq 0, \quad \sigma|_{\mu=(0,0),(1,1)} = 0$$

is satisfied with respect to $\pi = e^{(\sigma^0)^2(\mu)} - e^{\sigma^2(\mu)}$, $(\forall)\mu \in \mathcal{T} \setminus \{\mu_1, \mu_2\}$ *and* $\pi = 0$ *for* $\mu \in \{\mu_1, \mu_2\}$*, and* $v = e^{(\eta^0)^2(\mu)} - e^{\eta^2(\mu)}$, $(\forall)\mu \in \mathcal{T} \setminus \{\mu_1, \mu_2\}$ *and* $v = 0$ *for* $\mu \in \{\mu_1, \mu_2\}$.

Define the following closed 1-forms of Lagrange type

$$\psi_{\zeta} = \left(\psi_{\zeta}^{1}, \psi_{\zeta}^{2} \right) : \mathcal{T} \times \mathbb{R}^2 \to \mathbb{R}^2, \quad \zeta = \overline{1,2},$$

as below

$$\psi_{\zeta}^{1}(\mu, \sigma(\mu), \eta(\mu)) = \left(e^{-\eta(\mu)} + \frac{1}{2}, e^{-\eta(\mu)} \right),$$
$$\psi_{\zeta}^{2}(\mu, \sigma(\mu), \eta(\mu)) = \left(e^{\sigma(\mu)} + \frac{1}{2}, e^{\sigma(\mu)} \right).$$

Obviously, the vector functional

$$\Psi(\sigma, \eta) = \int_{C} \psi_{\zeta}(\mu, \sigma(\mu), \eta(\mu)) d\mu^{\zeta} = \int_{C} \psi_{1}(\mu, \sigma(\mu), \eta(\mu)) d\mu^{1} + \int_{C} \psi_{2}(\mu, \sigma(\mu), \eta(\mu)) d\mu^{2}$$

is Fréchet differentiable at $(\sigma^0, \eta^0) = (0,0)$*. Moreover, it can be easily verified that each curvilinear integral* $\int_{C} \psi_{\zeta}^{l}(\mu, \sigma(\mu), \eta(\mu)) d\mu^{\zeta}$, $l = \overline{1,2}$*, is invex/pseudoinvex at* $(\sigma^0, \eta^0) = (0,0)$ *with respect to* π *and* v*. Further, we can easily see that* $(\sigma^0, \eta^0) = (0,0)$ *is a solution for* (WVI)*. Therefore, by Theorem 4, we get that* (σ^0, η^0) *is a weak efficient solution of the associated optimization problem.*

4. Conclusions

In this paper, by using the notions of the invex set, Fréchet differentiability, invexity and pseudoinvexity for the involved curvilinear integral functionals, we established some relations between the solutions of a class of weak vector variational inequalities and (weak) efficient solutions of the associated control problem. The results derived in this paper, taking into account the notion of variational derivative for curvilinear-type integral functionals (see Treanţă [13]), can be rediscovered in a new form.

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