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Split-Plot Designs with Few Whole Plot Factors Containing Clear Effects

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Abstract: Fractional factorial split-plot designs are widely used when it is impractical to perform fractional factorial experiments in a completely random order. When there are too many subplots per whole plot, or too few whole plots, fractional factorial split-plot designs with replicated settings of the whole plot factors are preferred. However, such an important study is undeveloped in the literature. This paper considers fractional factorial split-plot designs with replicated settings of the WP factors from the viewpoint of clear effects. We investigate the sufficient and necessary conditions for this class of designs to have clear effects. An algorithm is proposed to generate the desirable designs which have the most clear effects of interest. The fractional factorial split-plot design with replicated settings of the WP factors is analysed and the results are discussed.

Keywords: clear effect; fractional factorial design; split-plot design

1. Introduction

Fractional factorial (FF) designs are often used in various experiments. Randomization is one of the fundamental principles in the design of an experiment which usually results in frequent changes in the levels of the factors. However, in certain experiments there are factors with levels that are difficult to change. In these cases, fractional factorial split-plot (FFSP) designs can be used. In an FFSP design, the factors are divided into two types: WP factors, with levels that are difficult to change, and SP factors, with levels that are easy to change.

To choose FF designs, [1] proposed the maximum resolution criterion and [2] proposed the minimum aberration (MA) criterion. Rapid growth has subsequently been witnessed in the literature on MA FF designs. In [3], the authors have provided a good summary of the development of MA designs. The authors of [4] extended the concept of MA to FFSP designs and presented two methods for constructing MA FFSP designs. In [5], the authors discussed the differences between FF and FFSP designs and developed theoretical results on MA FFSP designs. The authors of [6] introduced an algorithm to construct MA FFSP designs. In [7], the authors discussed how to choose non-isomorphic MA FFSP designs. The authors of [8] provided a finite projective geometric formulation for FFSP designs and studied such designs under the MA criterion. In [9], the authors considered the construction of FFSP designs in terms of consulting designs. The authors of [10] extended the MA criterion to multi-level FFSP designs. In [11], the authors proposed an approach for constructing MA orthogonal split-plot designs. The authors of [12] considered MA FFSP designs where the WP factors were more important than the SP factors, and [13] constructed such designs via complementary designs.

In [14], the authors introduced the notation of clear effect, which was called the clear effect criterion for choosing designs. A main effect or a two-factor interaction (2FI) is said to be clear if it is not aliased with any other main effect or 2FI. The least squares estimate of a clear main effect or 2FI is unbiased if all the interactions involving three or more factors are ignorable. In [15], the authors studied the conditions of an FF design containing clear main effects. The authors of [16] determined the bounds on the maximum number of
clear 2FIs by constructing two-level regular FF designs. In [17], the authors provided the conditions of an FFSP design containing clear main effects and 2FIs. The authors of [18] derived the bounds on the maximum number of clear effects by constructing FFSP designs. In [19,20], the authors studied mixed-level FFSP designs with a four-level factor in WP or SP section. The authors of [21] derived the conditions of a mixed-level FFSP design with two-level factors and an eight-level factor containing various clear effects. In [22], the authors studied the conditions of FFSP designs with two-level factors and a 2^s-level factor containing various clear effects. The authors of [23] studied the conditions of FFSP designs with s-level factors and an s'-level factor containing various clear effects.

In [24], the authors discussed a category of FFSP experiments which have only a few WP factors and in which the number of subplot treatments in a whole plot is larger than the whole plot can accommodate. For such cases, FFSP designs in which the settings of the WP factors are replicated are sometimes preferable. A new type of factors called splitting factors can be introduced to create such replications (see [24]). As a result, the subplot treatments for a fixed setting of the WP factors can be consequently split into groups. In other words, there is an increase in the number of whole plots and a decrease in the number of subplots per whole plot. As mentioned, FFSP designs with splitting factors are more suitable than those without splitting factors, as these have too few WP factors or too many subplots for a fixed setting of the WP factors. With the importance of the FFSP designs with splitting factors in mind, however, there has been only limited study on constructing optimal FFSP designs with splitting factors beyond [24], in which an algorithm was proposed for constructing designs of this class under MA criterion.

As mentioned, the least squares estimate of a clear main effect or 2FI is unbiased. This desirable property has inspired a large body of work, such as those in the literature reviewed in the third paragraph in Section 1, on choosing FF designs and FFSP designs to have the maximum number of clear effects of interest. However, the issue of seeking FFSP designs with splitting factors which have the maximum number of clear effects has not been studied yet. With a different perspective from that of [24], this paper considers the construction of optimal FFSP designs with splitting factors under the clear effect criterion. The contributions of this work are fourfold: (1) we investigate the necessary and sufficient conditions for 2^{(n_1+n_2)-(0+k_2)} FFSP designs with resolution III and at least IV having various types of clear effects, and introduce the notation 2^{(n_1+n_2)-(0+k_2)} and the definition of resolution in Section 2; (2) an algorithm for constructing optimal 2^{(n_1+n_2)-(0+k_2)} FFSP designs with various numbers of splitting factors is proposed; (3) useful design tables are provided for practical use; (4) FFSP designs with splitting factors are analysed and discussed.

The rest of the paper is organized as follows: Section 2 provides the notation and definitions; Sections 3 and 4 provide the conditions for 2^{(n_1+n_2)-(0+k_2)} FFSP designs with resolution III and at least IV to contain clear effects; Section 5 provides an algorithm for 2^{(n_1+n_2+r)-(0+k_2+r)} FFSP designs with the maximum number of clear effects; an analysis of FFSP designs with splitting factors is discussed using examples in Section 6; finally, Section 7 provides our conclusions.

2. Notation and Definitions

Consider the construction of a 2^{(n_1+n_2)-(k_1+k_2)} FFSP design with n_1 WP factors and n_2 SP factors. Let p_1 = n_1 - k_1, p_2 = n_2 - k_2 and p = p_1 + p_2. Let a_1, a_2, ..., a_{p_1}, b_1, b_2, ..., b_{p_2} be p independent 2^p × 1 columns with entries +1 and -1 which generate the saturated design H_p = H_p(a_1, a_2, ..., a_{p_1}, b_1, b_2, ..., b_{p_2}) by taking their possible component-wise products. Let H_a = H(a_1, a_2, ..., a_{p_1}) be the subset of H_p generated by taking the possible component-wise products of the p_1 columns a_1, a_2, ..., a_{p_1}. Clearly, H_p and H_a have 2^p - 1 and 2^{p_1} - 1 columns, respectively. As the factors are assigned to the columns of the design when running an experiment, we do not differentiate between factors and columns in the following. A 2^{(n_1+n_2)-(k_1+k_2)} FFSP design can be obtained by taking n_1 WP columns/factors from H_a with p_1 independent ones and n_2 SP columns/factors from...
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In this paper, we focus on the $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP designs with few WP factors, i.e., $n_1$ is small. Note that the WP section of a $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP design constitutes a $2^{n-k_1}$ FF design. When $n_1$ is small, the experimenter usually chooses the WP section as a full design, i.e., $k_1 = 0$. For an example, please see the cheese-making experiment in [24]. Thus, we suppose $k_1 = 0$ and $n_1 = p_1$ in the following. Then, without loss of generality, suppose $B_1 = \{a_1, a_2, \ldots, a_{n_1}\}$. Here, we would like to emphasize that $a_1, a_2, \ldots, a_{n_1}$ are just the $p_1$ independent columns generating $H_a$. For the SP section, without loss of generality, suppose $B_2 = \{b_1, b_2, \ldots, b_{p_2}, c_{p_2+1}, c_{p_2+2}, \ldots, c_{n_2}\}$, i.e., the $p_2$ independent columns $b_1, b_2, \ldots, b_{p_2}$ are selected into $B_2$ as SP factors. The other $k_2$ subplot dependent columns $c_{p_2+1}, c_{p_2+2}, \ldots, c_{n_2}$ are selected from $H_p \setminus H_a$. With the above in mind, a $2^{(n_1+n_2)-(0+k_2)}$ FFSP design can be determined by a set-pair

$$ (B_1, B_2) = \{a_1, a_2, \ldots, a_{n_1}; b_1, b_2, \ldots, b_{p_2}, c_{p_2+1}, c_{p_2+2}, \ldots, c_{n_2}\}. \tag{1} $$

A $2^{(n_1+n_2)-(0+k_2)}$ FFSP design has $2^{n_1}$ whole plots and $2^{n_2-k_2}$ subplots in each whole plot. When $2^{n_2-k_2}$ is too large such that a whole plot cannot accommodate so many subplots, we need to split the $2^{n_2-k_2}$ subplots into $2^r$ groups and arrange each group into a whole plot. This requires us to select $r$ independent splitting factors from $H_p \setminus H_a$. Denote the set of $r$ splitting factors by $B_3 = \{d_1, d_2, \ldots, d_r\}$. Let $2^{(n_1+n_2+r)-(0+k_2+r)}$ denote an FFSP design with $r$ splitting factors. Then, a $2^{(n_1+n_2+r)-(0+k_2+r)}$ FFSP design is determined by a set-triple

$$ (B_1, B_2, B_3) = \{a_1, a_2, \ldots, a_{n_1}; b_1, b_2, \ldots, b_{p_2}, c_{p_2+1}, c_{p_2+2}, \ldots, c_{n_2}; d_1, d_2, \ldots, d_r\}. \tag{2} $$

In Definition 1, we define the isomorphism for the FFSP designs without splitting factors (see [3] for details) as well as the isomorphism for the FFSP designs with splitting factors, first mentioned in [24].

**Definition 1.** Two $2^{(n_1+n_2)-(0+k_2)}$ FFSP designs are said to be isomorphic if one can be obtained from the other by relabelling between the WP factors and between the SP factors. Two $2^{(n_1+n_2+r)-(0+k_2+r)}$ FFSP designs are said to be isomorphic if one can be obtained from the other by relabelling between the WP factors, between the SP factors, and between the splitting factors.

In ranking and selecting $2^{(n_1+n_2+r)-(0+k_2+r)}$ FFSP designs, the isomorphic designs are treated as the same, as they are equivalent when used in real experiments.

Suppose $D$ is a $2^{(n_1+n_2)-(0+k_2)}$ FFSP design determined by a set-pair $(B_1, B_2)$ as in (1). Each $c_i$ is a component-wise product of $a_1, a_2, \ldots, a_{n_1}, b_1, b_2, \ldots, b_{p_2}$ which determines a defining word of $D$. The number of letters in a defining word is referred to as its word length. For example, if $c_1 = a_1 a_2 b_1 b_2$, then $a_1 a_2 b_1 b_2 c_1$ is a defining word with length five. A $2^{(n_1+n_2)-(0+k_2)}$ FFSP design $D$ has $k_2$ independent defining words which generate a group called the defining contrast subgroup of $D$. Clearly, a $2^{(n_1+n_2)-(0+k_2)}$ FFSP design has $2^{k_2} - 1$ defining words. We use $2^{[1]}$ and $2^{[2]}$ to denote the $2^{(n_1+n_2)-(0+k_2)}$ FFSP design of resolution III and at least IV, respectively. While the definition of resolution was originally proposed for FF designs (see [1]), it is applicable to FFSP designs as well. A $2^{(n_1+n_2)-(0+k_2)}$ FFSP design is said to have resolution $R$ if no $c$-factor interaction is aliased with any other interaction involving fewer than $R$-factors. For the $2^{(n_1+n_2)-(0+k_2)}$ FFSP designs, resolution III implies that there must be at least one main effect which is aliased with 2Fls; resolution IV implies that there is no main effect which is aliased with any 2Fl, while there must be at least one 2Fl which is aliased with other 2Fls. Note that the splitting factors are not real factors. When calculating the resolution and alias structures of a $2^{(n_1+n_2+r)-(0+k_2+r)}$ FFSP design, the splitting factors are not counted. This implies that

$H_p \setminus H_a$ with $p_2$ independent ones. Denote the set of $n_1$ WP columns as $B_1$ and the set of $n_2$ SP columns as $B_2$, respectively. Then, a $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP design is determined by a set-pair $(B_1, B_2)$. The other $k_2$ subplot dependent columns $c_{p_2+1}, c_{p_2+2}, \ldots, c_{n_2}$ are selected from $H_p \setminus H_a$. With the above in mind, a $2^{(n_1+n_2)-(0+k_2)}$ FFSP design can be determined by a set-pair
adding splitting factors to a $2^{(n_1+n_2)-(0+k_2)}$ FFSP design does not change the resolution or alias structures of this design. This point is explained in more detail in Section 6. Throughout the paper, without special statement, an effect/interaction which contains only WP factors is called a WP effect/interaction, an effect/interaction which contains at least one SP factor (but no splitting factor) is called an SP effect/interaction, and effects mean the WP or SP effects. Thus, a $2^{(n_1+n_2+r)-(0+k_2+r)}$ FFSP design determined by (2) has the same clear main effects and 2FIs as that of the corresponding $2^{(n_1+n_2)-(0+k_2)}$ FFSP design determined by (1). The 2FIs of a $2^{(n_1+n_2)-(0+k_2)}$ FFSP design can be divided into three types: WP2FI, SP2FI, and WS2FI. A WP2FI/SP2FI is a 2FI of two WP/SP factors, and a WS2FI is a 2FI of a WP factor and an SP factor. In the next section, we study the conditions for a $2^{(n_1+n_2)-(0+k_2)}$ FFSP design to have clear main effects or 2FIs for given $p_1 = n_1$, $p_2 = n_2 - l_2$, and $p = p_1 + p_2$.

3. Results for $2^{(n_1+n_2)-(0+k_2)}$ FFSP Designs

In this section, we discuss the conditions for $2^{(n_1+n_2)-(0+k_2)}$ FFSP designs to contain various clear effects. Theorem 1 provides the necessary and sufficient conditions for a $2^{(n_1+n_2)-(0+k_2)}$ FFSP design to have clear WP main effects and WP2FIs.

**Theorem 1.** The necessary and sufficient condition for a $2^{(n_1+n_2)-(0+k_2)}$ FFSP design to have clear WP main effects or WP2FIs is $n_2 \leq 2^{p-1} - 2^{p_1-1}$.

**Proof.** Suppose $D$ is a $2^{(n_1+n_2)-(0+k_2)}$ FFSP design determined by (1) containing at least a clear WP main effect. Without loss of generality, suppose the main effect $a_1$ is clear. Then, the columns $a_1 b_i, a_1 c_j \in (H_p \setminus H_b) \setminus B_2$ are different from each other for $i = 1, \ldots, p_2, j = 1, \ldots, k_2$. Thus, we obtain $p_2 + k_2 \leq 2^{p-1} - 2^{p_1-1} - 2^{p_2-1} - 2^{p_1-1} - 2^{k_2-1}$ by $n_2 = p_2 + k_2$. If $D$ has a clear WP2FI, say, $a_1 b_2$, then the columns $a_1 a_2 b_i, a_1 a_2 c_j \in (H_p \setminus H_b) \setminus B_2$ and are different from each other for $i = 1, \ldots, p_2, j = 1, \ldots, k_2$. Then, we can obtain $n_2 \leq 2^{p-1} - 2^{p_1-1} - 2^{k_2-1}$, which proves the necessary condition.

Now, we construct a $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP design containing a clear WP main effect and WP2FIs. Let $B_2$ be any $n_2$-subset of $(B_1 \setminus B_2) \setminus H(a_2, \ldots, a_{p_1}, b_1, \ldots, b_{p_2})$ such that $b_i \in B_2$, $i = 1, \ldots, p_2$. Then, the main effect $a_1$ and WP2FIs $a_1 b_1, a_1 a_2 b_2$, $a_1 a_2 c_j$ are clear in the $2^{(n_1+n_2)-(0+k_2)}$ FFSP design $D$ determined by $(B_1, B_2)$, where $B_1 = \{a_1, \ldots, a_{p_1}\}$. This completes the proof. □

Theorem 2 below entertain SP main effects, SP2FIs, and WS2FIs of $2^{(n_1+n_2)-(0+k_2)}$ FFSP designs.

**Theorem 2.** The necessary and sufficient condition for a $2^{(n_1+n_2)-(0+k_2)}$ FFSP design to have clear SP main effects, SP2FIs, or WS2FIs is $n_2 \leq 2^{p-1} - p_1$.

**Proof.** Suppose $D$ is a $2^{(n_1+n_2)-(0+k_2)}$ FFSP design determined by (1) containing at least a clear SP main effect. Without loss of generality, suppose that the main effect of the SP factor $b_1$ is clear. Then, the columns $b_1 a_i, b_1 b_j, b_1 c_j \in H_p \setminus (B_1 \cup B_2)$ and are different from each other for $i = 1, \ldots, p_1, j = 2, \ldots, p_2, l = 1, \ldots, k_2$. Thus, we obtain $p_1 + p_2 - 1 + k_2 \leq 2^{p-1} - 2^{p_1-1} - 2^{p_2-1} - 2^{k_2-1}$ by $n_2 = p_2 + k_2$. If $D$ has a clear SP2FI, say, $b_1 b_2$, then the columns $b_1 b_2, b_1 b_2 a_i, b_1 b_2 b_j, b_1 b_2 c_j \in H_p \setminus (B_1 \cup B_2)$ and are different from each other for $i = 1, \ldots, p_1, j = 3, \ldots, p_2, l = 1, \ldots, k_2$. Similarly, we can obtain $n_2 \leq 2^{p-1} - p_1$. If $D$ has a clear WS2FI, say, $a_1 b_1$, then the columns $a_1 b_1 a_i, a_1 b_1 b_j, a_1 b_1 c_j \in H_p \setminus (B_1 \cup B_2)$ and are different from each other for $i = 2, \ldots, p_1, j = 2, \ldots, p_2, l = 1, \ldots, k_2$. Then, we can obtain $n_2 \leq 2^{p-1} - p_1$, which proves the necessary condition.

To show the sufficiency of the condition, we construct $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP designs containing clear effects. First, we construct a $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP design containing an SP
main effect. Choose $B'_2$ as any $(n_2 - 1)$-subset of \( \{ b_1 : e \in H(a_1, ..., a_{p_1}, b_2, ..., b_{p_2}) \setminus B_1 \} \), where $B_1 = \{ a_1, ..., a_{p_1} \}$. Let $B_2 = \{ b_1 \} \cup B'_2$. Then, the SP main effect $b_1$ is clear in the 2 $W_{IV}$ FFSP design $D$ determined by $(B_1, B_2)$.

Now, we construct a 2 $W_{IV}$ FFSP design containing a clear SP2FI. The column $b_1b_2$ divides the $2^p - 4$ columns of $H_{p} \setminus \{ b_1, b_2, b_1b_2 \}$ into $2^{p-1} - 2$ disjoint pairs, denoted by $(\epsilon_i, f_j)$, such that $\epsilon_if_1 = b_1b_2$, $i = 1, ..., 2^{p-1} - 2$. Select one column from each pair $(\epsilon_i, f_j)$ to constitute a set $B$ such that $a_i, b_j \in B$, $i = 1, ..., p_1$, $j = 3, ..., p_2$. Let $B_1 = \{ a_1, ..., a_{p_1} \}$ and $B_2$ be any $(n_2 - 2)$-subset of $B \setminus B_1$ such that $b_2 \in B_2$, $j = 3, ..., p_2$. Then, the SP2FI $b_1b_2$ is clear in the design $D$ determined by $(B_1, B_2)$, where $B_2 = \{ b_1 \} \cup B'_2$.

To construct a 2 $W_{IV}$ FFSP design containing a clear WS2FI, we choose the column $a_1b_1$, which divides the $2^p - 4$ columns of $H_{p} \setminus \{ a_1, b_1, a_1b_1 \}$ into $2^{p-1} - 2$ disjoint pairs, denoted by $(\epsilon_i, f_j)$, such that $\epsilon_if_1 = a_1b_1$, $i = 1, ..., 2^{p-1} - 2$. Select one column from each pair $(\epsilon_i, f_j)$ to constitute a set $B$ such that $a_i, b_j \in B$, $i = 1, ..., p_2$, $j = 2, ..., p_2$. Let $B'_1 = \{ a_2, ..., a_{p_1} \}$ and $B'_2$ be any $(n_2 - 2)$-subset of $B \setminus B'_1$ such that $b_2 \in B'_2$, $j = 2, ..., p_2$. Then, the WS2FI $ba_1b_1$ is clear in the design $D$ determined by $(B_1, B_2)$, where $B_1 = \{ a_1 \} \cup B'_1$ and $B_2 = \{ b_1 \} \cup B'_2$. This completes the proof.  

4. Results for 2 $W_{IV}$ FFSP Designs

In this section, we study the conditions for 2 $W_{IV}$ FFSP designs to contain various clear effects. Because 2 $W_{IV}$ FFSP designs have at least resolution IV, all of their main effects are clear. Theorem 3 provides the necessary and sufficient conditions for a 2 $W_{IV}$ FFSP design to have clear WP2FIs.

\textbf{Theorem 3.} The necessary and sufficient condition for a 2 $W_{IV}$ FFSP design to have clear WP2FIs is $n_2 \leq 2^{p-2} - 2^{n_1-2}$.

\textbf{Proof.} Suppose $D$ is a 2 $W_{IV}$ FFSP design determined by (1) containing clear WP2FI $p_2$. Then, the columns $a_1b_1, a_1a_2, a_2b_1, a_2c_1, a_1a_2b_1, a_1a_2c_1 \in H_{p} \setminus H_{p}$ and $a_1a_2a_2 \in H_{p} \setminus H_{p}$ are different from each other for $i = 1, ..., p_1, j = 1, ..., k_2$. Thus, we obtain $3(p_2 + k_2) \leq 2^p - 2^{p_1} - 2^{n_2}$, which leads to $n_2 \leq 2^{p-2} - 2^{n_1-2}$ by $n_2 = p_2 + k_2$. This proves the necessary condition.

We can now construct a 2 $W_{IV}$ FFSP design containing clear WP2FI to show the sufficiency of the condition. Let $B_1 = \{ a_1, ..., a_{p_1} \}$. The column $a_2$ divides the $2^{p-1} - 2^{n_1-1}$ columns of $H_{p} \setminus H_{p}$ into $2^{p-2} - 2^{n_1-2}$ disjoint pairs, denoted by $(\epsilon_i, f_j)$, such that $\epsilon_if_1 = a_2b_1$, $i = 1, ..., 2^{p-2} - 2^{n_1-2}$. Select $n_2$ columns from the $2^{p-2} - 2^{n_1-2}$ pairs to constitute $B'_2$ such that at most one column is selected from each pair. Then, the WP2FI $a_1a_2$ is clear in the 2 $W_{IV}$ FFSP design determined by $(B_1, B_2)$, where $B_2 = \{ a_1 : e \in B'_2 \}$. This completes the proof. 

Theorem 4 below entertains the SP2FIs and WS2FIs of 2 $W_{IV}$ FFSP designs.

\textbf{Theorem 4.} The necessary and sufficient conditions for a 2 $W_{IV}$ FFSP design to have clear SP2FIs or WS2FIs is $n_2 \leq 2^{p-2} - p_1 + 1$.

\textbf{Proof.} Suppose $D$ is a 2 $W_{IV}$ FFSP design determined by (1) containing clear SP2FI $b_1b_2$. Then, the columns $b_1b_2, b_1a_1, b_1b_1, b_1b_1c_1, b_2b_1, b_2b_1c_2, b_1b_2a_1, b_1b_2b_1, b_2b_1c_1 \in H_{p} \setminus B_2$ and are different from each other for $i = 1, ..., p_1, j = 1, ..., k_2$. Thus, we obtain $1 + 2^3(p_1 + p_2 - 2 + k_2) \leq 2^p - 1 - (p_1 + p_2 + k_2)$, which leads to $n_2 \leq 2^{p-2} - p_1 + 1$ by $n_2 = p_2 + k_2$. This proves the necessary condition.

We construct 2 $W_{IV}$ FFSP designs containing clear SP2FI or WS2FI to show the sufficiency of the condition. First, we construct a 2 $W_{IV}$ FFSP design containing a clear SP2FI. Let $B_1 = \{ a_1, ..., a_{p_1} \}$. The column $b_2$ divides the $2^{p-2} - 2$ columns of
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$H(a_1, \ldots, a_p, b_2, \ldots, b_p) \setminus \{b_2\}$ into $2^{p-2} - 1$ disjoint pairs, denoted by $(e_i, f_i)$, such that $e_i f_i = b_2$, $i = 1, \ldots, 2^{p-2} - 1$. Among the $2^{p-2} - 1$ pairs there are $p_1$ pairs, each of which contains one of $a_1, \ldots, a_p$. We select $n_2 - 2$ columns from the other $2^{p-2} - 1 - p_1$ pairs to constitute $B'_2$ such that at most one column is selected from each of these pairs. Then, the SP2FI $b_1b_2$ is clear in the $2^{(n_1+n_2)-(0+k_2)}$ FFSP design determined by $(B_1, B_2)$, where $B_2 = \{b_1, b_2\} \cup \{b\ e \in B'_2\}.

Now, we must construct $2^{(n_1+n_2)-(0+k_2)}$ FFSP designs containing a clear WS2FI. The column $b_1$ divides the $2^{p-1} - 2$ columns of $H(a_2, \ldots, a_p, b_1, \ldots, b_p) \setminus \{b_1\}$ into $2^{p-2} - 1$ disjoint pairs, denoted by $(e_i, f_i)$, such that $e_i f_i = b_1$, $i = 1, \ldots, 2^{p-2} - 1$. Among the $2^{p-2} - 1$ pairs there are $p_1 - 1$ pairs, each of which contains one of $a_2, \ldots, a_p$. We select $n_2 - 1$ columns from the other $2^{p-2} - p_1$ pairs to constitute $B'_2$ such that at most one column is selected from each of these pairs. Then, the WS2FI $p_1b_1$ is clear in the $2^{(n_1+n_2)-(0+k_2)}$ FFSP design determined by $(B_1, B_2)$, where $B_1 = \{a_1, \ldots, a_p\}$ and $B_2 = \{b_1\} \cup \{a\ e \in B'_2\}$. This completes the proof. \hfill $\square$

5. Construction of $2^{(n_1+n_2)-(0+k_2)}$ FFSP Designs with the Maximum Number of Clear Effects

In Sections 3 and 4 we have investigated the conditions for $2^{(n_1+n_2)-(0+k_2)}$ FFSP designs to have various clear effects. In this section, we provide an algorithm to construct $2^{(n_1+n_2)-(0+k_2)}$ and $2^{(n_1+n_2)-(0+k_2)}$ FFSP designs with the most clear main effects, whether 2FIs, WP2FIs, WS2FIs, or SP2FIs.

Based on the algorithms of Bingham and Sitter [6] and Bingham et al. [24] to construct MA $2^{(n_1+n_2)-(0+k_2)}$ and $2^{(n_1+n_2+r)-(0+k_2+r)}$ FFSP designs, we propose the following algorithm to construct $2^{(n_1+n_2+r)-(0+k_2+r)}$ FFSP designs with the most clear effects.

To illustrate the applications of Algorithm 1, we use it here to find $2^{(1+5+2)-(0+2+2)}$ and $2^{(1+5+2)-(0+2+2)}$ FFSP designs which contain the maximum number of clear 2FIs, with the results shown in Example 1.

**Algorithm 1: Constructing $2^{(n_1+n_2+r)-(0+k_2+r)}$ FFSP Designs**

**Step 1.** Generate the set $S_{n_1, n_2, k_0}$ of the non-isomorphic $2^{(n_1+n_2)-(0+k_2)}$ FFSP designs using the algorithm of Bingham and Sitter [6].

**Step 2.** Regard the splitting factor $d_1$ in (2) as an SP factor. Generate the set $S_{n_1, n_2, k_1}$ of non-isomorphic $2^{(n_1+n_2+1)-(0+k_2+1)}$ FFSP designs using the algorithm of Bingham and Sitter [6], where the brackets are used in the notation $2^{(n_1+n_2+1)-(0+k_2+1)}$ to emphasize that the splitting factor $d_1$ is regarded as an additional SP factor at this point. Eliminate the ineligible designs in $S_{n_1, n_2, k_1}$ which has at least one defining word containing less than two SP factors, and redenote the resulting set as $S_{n_1, n_2, k_2}$ which contains all the non-isomorphic $2^{(n_1+n_2+1)-(0+k_2+1)}$ FFSP designs.

**Step 3.** Regard the splitting factor $d_2$ in (2) as an SP factor. Generate the set $S_{n_1, n_2, k_2}$ of non-isomorphic $2^{(n_1+n_2+2)-(0+k_2+2)}$ FFSP designs using the algorithm of Bingham and Sitter [6], where the brackets are used in the notation $2^{(n_1+n_2+2)-(0+k_2+2)}$ to emphasize that the splitting factor $d_2$ is regarded as an additional SP factor at this point. Eliminate the ineligible designs in $S_{n_1, n_2, k_2}$ which has at least one defining word containing less than two SP factors, and redenote the resulting set as $S_{n_1, n_2, k_3}$ which contains all the non-isomorphic $2^{(n_1+n_2+1)-(0+k_2+1)}$ FFSP designs. Continue this procedure until we obtain $S_{n_1, n_2, k_3}$, the set of the non-isomorphic $2^{(n_1+n_2+r)-(0+k_2+r)}$ FFSP designs.

**Example 1.** Consider the construction of a $2^{(1+5+2)-(0+2+2)}$ and a $2^{(1+5+2)-(0+2+2)}$ FFSP design which contain the maximum number of clear 2FIs. We denote the whole plot column (factor)
as \( a_1 \), the SP columns (factors) as \( b_1, b_2, b_3, c_4, c_5 \), and the splitting columns (factors) as \( d_1, d_2 \), where \( b_1, b_2, b_3 \) are independent SP columns (factors). Note again that we do not differentiate between columns and factors. The four independent columns \( a_1, b_1, b_2 \) and \( b_3 \) generate the saturated design \( H_4 \), in which the columns are in Yates order.

1. Generate the catalog \( C_{1.4.1.0} \) of non-isomorphic \( 2^{(1+4+0)-(0+1+0)} \) FFSP designs. The first generator we consider is \( c_4 = a_1 b_1 \), i.e., assigning \( c_4 \) to the third column of \( H_4 \). At this point, \( C_{1.4.1.0} \) is empty, then \( 3 \) (indicating that the third column in \( H_4 \) is used to assign \( c_4 \) is included in \( C_{1.4.1.0} \). We next consider the generator \( c_4 = a_1 b_2 \). As the generator \( c_4 = a_1 b_2 \) is isomorphic to the generator \( c_4 = a_1 b_2 \), which can be easily verified by relabeling \( b_2 \) as \( b_1 \), the generator \( c_4 = a_1 b_2 \) is discarded. Continue until all the interaction columns in \( H_4 \) are considered to assign \( c_4 \). The resulting \( C_{1.4.1.0} \) is displayed in Table 1.

2. Generate the catalog \( C_{1.5.2.0} \) of non-isomorphic \( 2^{(1+5+0)-(0+2+0)} \) FFSP designs by adding factor \( c_5 \) to every design in \( C_{1.4.1.0} \). For the first design in \( C_{1.4.1.0} \) with generator \( c_4 = a_1 b_1 \), we first consider generator \( c_5 = a_1 b_2 \). Assigning \( c_5 \) to the fifth column of \( H_4 \), At this point, \( C_{1.4.2.0} \) is empty, and \( (3, 5) \), representing the design \( (c_4 = a_1 b_1, c_5 = a_1 b_2) \), is included in \( C_{1.5.2.0} \). Next, we consider adding the generator \( c_5 = b_1 b_2 \) (which means that \( c_5 \) is assigned to the sixth column in \( H_4 \)) to the first design in \( C_{1.4.1.0} \) with generator \( c_4 = a_1 b_1 \). Because \( (c_4 = a_1 b_1, c_5 = b_1 b_2) \) is non-isomorphic to \( (c_4 = a_1 b_1, c_5 = a_1 b_2) \), we include \( (3, 6) \), representing the design \( (c_4 = a_1 b_1, c_5 = b_1 b_2) \), in \( C_{1.5.2.0} \). Continue until all the interaction columns behind the third column in \( H_4 \) have been considered to assign \( c_5 \) for the first design (with generator \( c_4 = a_1 b_1 \) ) in \( C_{1.4.1.0} \). The final \( C_{1.5.2.0} \) cannot be obtained until this procedure is conducted for all of the remainder designs in \( C_{1.4.1.0} \). The resulting \( C_{1.5.2.0} \) is displayed in Table 1.

3. Generate the catalog \( C_{1.5.2.1} \) of non-isomorphic \( 2^{(1+5+1)-(0+2+1)} \) FFSP designs by adding factor \( d_1 \) to every design in \( C_{1.5.2.0} \). When assigning the splitting factor \( d_1 \), \( d_1 \) is regarded as an SP factor. When performing the isomorphism test, \( d_1 \) is regarded as a splitting factor. For the first design, \( (3, 5) \), in \( C_{1.5.2.0} \), representing \( (c_4 = a_1 b_1, c_5 = a_1 b_2) \), we consider adding generator \( d_1 = b_1 b_2 \) to it, i.e., assigning \( d_1 \) to the 6th column, \( b_1 b_2 \). At this point, \( C_{1.5.2.1} \) is empty and \( (3, 5, 6) \) is included in \( C_{1.5.2.1} \). Continue until all of the interaction columns except for the third and fifth in \( H_4 \) are considered, after which \( d_1 \) can be assigned. This procedure is repeated until all of the remainder designs in \( C_{1.5.2.0} \) have been considered in order to be added \( d_1 \). Then, we obtain the catalog \( C_{1.5.2.1} \) of non-isomorphic \( 2^{(1+5+1)-(0+2+1)} \) FFSP designs. At this point, certain designs in \( C_{1.5.2.1} \) are ineligible because they may contain defining words, as mentioned in Step 2 of Algorithm 1. We can eliminate the ineligible designs from \( C_{1.5.2.1} \) and redenote the resulting set as \( C_{1.5.2.1} \), which is displayed in Table 1.

4. Generate the catalog \( C_{1.5.2.2} \) of non-isomorphic \( 2^{(1+5+2)-(0+2+2)} \) FFSP designs by adding factor \( d_2 \) to every design in \( C_{1.5.2.2} \). When assigning the splitting factor \( d_2 \), \( d_2 \) is regarded as an SP factor. When performing the isomorphism test, on the other hand, \( d_2 \) is regarded as a splitting factor. For the first design, \( (3, 5, 6) \), in \( C_{1.5.2.2} \), representing \( (c_4 = a_1 b_1, c_5 = a_1 b_2, d_1 = b_1 b_2) \), we consider adding generator \( d_1 = a_1 b_1 b_2 \), i.e., assigning \( d_2 \) to the 7th column, \( a_1 b_1 b_2 \). At this point, \( C_{1.5.2.2} \) is empty and \( (3, 5, 6, 7) \) is included in \( C_{1.5.2.2} \). We continue until all the interaction columns behind the sixth in \( H_4 \) have been considered in order to assign \( d_2 \). This procedure is repeated until all the designs in \( C_{1.5.2.2} \) have been considered to add the splitting factor \( d_2 \). Then, we obtain the catalog \( C_{1.5.2.2} \) of non-isomorphic \( 2^{(1+5+2)-(0+2+2)} \) FFSP designs. At this point, certain designs in \( C_{1.5.2.2} \) are ineligible, because they may contain the defining words mentioned in Step 2 of Algorithm 1. We eliminate the ineligible designs from \( C_{1.5.2.2} \) and redenote the resulting set as \( C_{1.5.2.2} \), which is displayed in Table 1.

5. After calculating the resolution and the number of clear 2FIs of every design in \( C_{1.5.2.2} \), we find that all the designs in \( C_{1.5.2.2} \) have resolution III, and that the design represented by \( (3, 14, 6, 10) \) (displayed in the third row of Table 2) has the maximum number of clear 2FIs.
Table 1. Catalogs of the non-isomorphic FFSP designs in Example 1.

<table>
<thead>
<tr>
<th>C_{1.4.1.0}</th>
<th>C_{1.5.2.0}</th>
<th>C_{1.5.2.1}</th>
<th>C_{1.5.2.2}</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3, 5</td>
<td>3, 5, 6</td>
<td>3, 5, 6, 10</td>
</tr>
<tr>
<td>6</td>
<td>3, 6</td>
<td>3, 5, 14</td>
<td>3, 14, 10</td>
</tr>
<tr>
<td>7</td>
<td>3, 12</td>
<td>3, 6, 14</td>
<td>3, 14, 11</td>
</tr>
<tr>
<td>14</td>
<td>3, 13</td>
<td>3, 6, 14</td>
<td>3, 14, 11</td>
</tr>
<tr>
<td>15</td>
<td>3, 14</td>
<td>3, 6, 15</td>
<td></td>
</tr>
<tr>
<td></td>
<td>6, 10</td>
<td>3, 12, 6</td>
<td></td>
</tr>
<tr>
<td></td>
<td>6, 11</td>
<td>3, 13, 6</td>
<td></td>
</tr>
<tr>
<td></td>
<td>7, 11</td>
<td>3, 14, 6</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>3, 14, 7</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>6, 10, 15</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>6, 11, 14</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>7, 11, 14</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>7, 11, 15</td>
<td></td>
</tr>
</tbody>
</table>

Researchers can apply Algorithm 1 to choose the \(2^{(n_1+n_2+r)-(0+k_2+r)}\) FFSP designs with the most main effects, 2FIs, WP2FIs, WS2FIs, or SP2FIs according to the experimental situation under consideration. As an application of Algorithm 1, in Tables 2–4, a number of 16-run, 32-run, and 64-run \(2^{(n_1+n_2+r)-(0+k_2+r)}\) FFSP designs with the most clear 2FIs are tabulated in this section. In these tables, the notation \(n_1.n_2.k_2.r.R\) denotes the FFSP design with \(n_1\) WP factors, \(n_2\) SP factors, \(k_2\) independent defining words, \(r\) splitting factors, and resolution \(R\) equal to III or at least IV. The notation † in the second column means that the corresponding FFSP design does not exist, while the numbers in the second and third columns indicate which columns of \(H_p\) are used to assign the splitting factors and dependent SP factors, respectively. In each of these designs, the WP factors and independent SP factors are assigned to the first, second, fourth, \(2^3\)-th, \(2^4\)-th, and \(2^p\)-th columns in \(H_p\) in Yates order. These designs are desirable when there is no prior information about which type of 2FI is more important. With such information, researchers can apply Algorithm 1 to choose those \(2^{(n_1+n_2+r)-(0+k_2+r)}\) FFSP designs containing the most relevant clear effects.

Table 2. Example 16-run FFSP designs with splitting factors which have the most clear 2FIs.

<table>
<thead>
<tr>
<th>(n_1.n_2.k_2.r.R)†</th>
<th>Splitting Factors</th>
<th>Dependent SP Columns</th>
<th>The Number of Clear 2FIs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.4.1.2.III</td>
<td>6, 10</td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>1.4.1.2.IV</td>
<td>6, 10</td>
<td>15</td>
<td>10</td>
</tr>
<tr>
<td>1.5.2.2.III</td>
<td>6, 10</td>
<td>3, 14</td>
<td>6</td>
</tr>
<tr>
<td>1.5.2.2.IV†</td>
<td></td>
<td>3, 5, 9</td>
<td>2</td>
</tr>
<tr>
<td>1.6.3.2.III†</td>
<td></td>
<td>3, 5, 9, 14</td>
<td>1</td>
</tr>
<tr>
<td>1.7.4.2.III†</td>
<td></td>
<td>3, 5, 9, 14, 15</td>
<td>0</td>
</tr>
<tr>
<td>1.8.5.2.III†</td>
<td></td>
<td>3, 5, 9, 14, 15</td>
<td>0</td>
</tr>
<tr>
<td>1.8.5.2.IV†</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.3.1.1.III</td>
<td>12</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>2.3.1.1.IV</td>
<td>12</td>
<td>7</td>
<td>4</td>
</tr>
<tr>
<td>2.4.2.1.III</td>
<td>12</td>
<td>5, 10</td>
<td>9</td>
</tr>
<tr>
<td>2.4.2.1.IV</td>
<td>15</td>
<td>7, 11</td>
<td>0</td>
</tr>
<tr>
<td>2.5.3.1.III</td>
<td>15</td>
<td>5, 6, 7</td>
<td>6</td>
</tr>
<tr>
<td>2.5.3.1.IV†</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.6.4.1.III</td>
<td>15</td>
<td>5, 9, 12, 13</td>
<td>7</td>
</tr>
<tr>
<td>2.6.4.1.IV†</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.7.5.1.III†</td>
<td></td>
<td>5, 6, 7, 9, 10</td>
<td>0</td>
</tr>
<tr>
<td>2.7.5.1.IV†</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.8.6.1.III†</td>
<td></td>
<td>5, 6, 7, 9, 10, 11</td>
<td>0</td>
</tr>
</tbody>
</table>

Note: † in the second column means the corresponding design does not exist. ‡ \(n_1.n_2.k_2.r.R\) denotes an FFSP design with \(n_1\) WP factors, \(n_2\) SP factors, \(k_2\) independent defining words and resolution \(R\) equals to III or at least IV.
In [24], several example MA FFSP designs with splitting factors were provided. It is well known that an optimal FFSP design or a fractional factorial design chosen under a
clear effect criterion must have more or at least an equal number of clear effects of interest than MA designs, considering that these two types of designs have the same resolution. For example, the MA $2^{(1+5+2−(0+2+2))}$ FFSP design $(5, 6, 11, 12)$ (meaning that the three dependent SP factors are assigned to the 5th, 6th, and 11th columns in $H_4$ and the splitting factor is assigned to the 12th column in $H_4$) displayed in Table 2 of [24] has resolution III and 2 clear 2FIs, while the $2^{(1+5+2−(0+2+2))}$ FFSP design displayed in the third row of Table 2 has 6 clear 2FIs. As another example, the MA $2^{(1+8+2−(0+4+2))}$ FFSP design $(15, 21, 27, 28, 10, 18)$ displayed in Table 2 of [24] has resolution IV and 8 clear 2FIs, while the $2^{(1+8+2−(0+4+2))}$ FFSP design displayed in 8th row of Table 3 has 15 clear 2FIs.

6. Analysis of FFSP Designs with Splitting Factors

In this section, we provide an analysis of $2^{(n_1+n_2+r)−(k_1+k_2+r)}$ FFSP designs through a number of examples. In order to provide a general instruction which is applicable to any $2^{(n_1+n_2+r)−(k_1+k_2+r)}$ FFSP design as well as those in Section 5 and [24], we assume that interactions involving more than two factors are negligible. To clearly present the work in this section, we use capital letters to denote WP factors, lowercase letters to denote SP factors, and Greek letters to denote splitting factors.

Let $A$ denote a WP factor and $p, q$ and $t$ denote three SP factors. We first consider the $2^{(1+3)−(0+1)}$ FFSP design $D_1$ (without splitting factors) with defining relation $I = A pt$, where $t$ denotes the dependent SP factor. The alias set of $D_1$ is

$$ I = A pt, $$
$$ A = pt, $$
$$ p = At, $$
$$ Ap = t, $$
$$ q = A pq t, $$
$$ Aq = pq t, $$
$$ pq = Aqt and $$
$$ Apq = qt. $$

From the alias set of $D_1$, it can be seen that $D_1$ has resolution III and that some of the main effects are aliased with 2FIs such as $A = pt$. The runs of $D_1$ are displayed in Table 5 (see the first four columns).

Now, we can consider adding a splitting factor $\rho$ to $D_1$ with the defining relation $I = Apq \rho$; we denote the resulting $2^{(1+3+1)−(0+1+1)}$ FFSP design as $D_2$. The design $D_2$ has the alias set

$$ I = A pt = Apq \rho = qt \rho, $$
$$ A = pt = pq \rho = Aqp \rho, $$
$$ p = At = Aqp = pq t \rho, $$
$$ Ap = t = q \rho = Apq t \rho, $$
$$ q = A pq t = A pp = t \rho, $$
$$ Aq = pq t = pp = A t \rho, $$
$$ pq = Aqt = A \rho = pt \rho and $$
$$ Apq = qt = \rho = A pt \rho. (3) $$

As previously mentioned, the splitting factors are not real factors. This implies that when calculating the resolution and alias structures of $D_2$, the generators with splitting factors do not count, i.e., the last two generators in each of the eight formulas of the alias set of $D_2$ do not count. Therefore, $D_1$ and $D_2$ have the same resolution, III, and the same alias structures. The runs of $D_2$ are displayed in Table 6 (see the first five columns). Comparing
the runs of $D_2$ with those of $D_1$, the splitting factor $\rho$ partitions the four subplots within each of the two whole plots of $D_1$ into two groups, resulting in two subplots within each of the four whole plots of $D_2$.

**Table 5. The runs of $D_1$.**

<table>
<thead>
<tr>
<th>Whole Plot</th>
<th>Sub Plot</th>
<th>Observation</th>
<th>Error Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$p$</td>
<td>$q$</td>
<td>$r(=Apb)$</td>
</tr>
<tr>
<td>$+,$</td>
<td>$+,$</td>
<td>$+,$</td>
<td>$+,$</td>
</tr>
<tr>
<td>$-,$</td>
<td>$-,$</td>
<td>$-,$</td>
<td>$-,$</td>
</tr>
<tr>
<td>$+,$</td>
<td>$-,$</td>
<td>$+,$</td>
<td>$+,$</td>
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<tr>
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<td>$-,$</td>
<td>$-,$</td>
</tr>
<tr>
<td>$-,$</td>
<td>$-,$</td>
<td>$+,$</td>
<td>$-,$</td>
</tr>
</tbody>
</table>

**Table 6. The runs of $D_2$.**

<table>
<thead>
<tr>
<th>Whole Plot</th>
<th>Sub Plot</th>
<th>Observation</th>
<th>Error Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$p$</td>
<td>$q$</td>
<td>$t(=Ap)$</td>
</tr>
<tr>
<td>$+,$</td>
<td>$+,$</td>
<td>$+,$</td>
<td>$+,$</td>
</tr>
<tr>
<td>$-,$</td>
<td>$-,$</td>
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<td>$-,$</td>
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<td>$+,$</td>
<td>$-,$</td>
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</tr>
<tr>
<td>$-,$</td>
<td>$-,$</td>
<td>$+,$</td>
<td>$-,$</td>
</tr>
</tbody>
</table>

The models for the $2^{(n_1+n_2)-(k_1+k_2)}$ and $2^{(n_1+n_2+r)-(k_1+k_2+r)}$ FFSP designs have the same form as

$$y_{ijk} = u + f(WP \text{ effects}) + \epsilon_{ij} + g(SP \text{ effects}) + \epsilon_{ijk},$$

where the brackets in subscripts are used to emphasize that $i$ is associated with the WP plots, $y_{ijk}$ is the observation for the $j$th replicate of the treatment combination in the $k$th subplot of the $i$th whole plot (with $i = 1, 2, \ldots, I$, $j = 1, 2, \ldots, J$ and $k = 1, 2, \ldots, K$), $I$ is the number of whole plots, $J$ is the number of replications, $K$ is the number of subplots, $u$ denotes the overall mean effect, $f(\cdot)$ is a linear function of the WP main effects and interactions, $g(\cdot)$ is a linear function of the SP main effects and interactions, and $\epsilon$ and $\epsilon$ denote the WP plot error and SP plot error, respectively. It is assumed that $\epsilon_{ij}$ and $\epsilon_{ijk}$ are mutually independent random variables such that $\epsilon \sim N(0, \sigma^2_W)$ and $\epsilon \sim N(0, \sigma^2_S)$. Despite the same model form, there are slight differences when modeling FFSP designs with and without splitting factors as a result of the increased number of whole plots and decreased number of subplots. For example, to analyze designs $D_1$ and $D_2$, we can build the following model:

$$y_{ijk} = u + A(ij) + e_{ij}(i) + p(ij) + q(ij) + t(ijk) + Apq(ijk) + Aq(ijk) + At(ijk) + pq(ijk) + pt(ijk) + qt(ijk) + Apq(ijk) + \ldots + Apq(ijk) + e_{ijk},$$

where $i = 1, 2, \ldots, I$ with $I = 2$ for $D_1$ and $I = 4$ for $D_2$; $k = 1, 2, \ldots, K$ with $K = 4$ for $D_1$ and $K = 2$ for $D_2$; and $j = 1, 2, \ldots, J$. The different values of $I$ and $K$ for $D_1$ and $D_2$, respectively, indicate the slight difference between modeling the $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP designs and modeling the $2^{(n_1+n_2+r)-(k_1+k_2+r)}$ FFSP designs. For more details on modeling FFSP experiments without splitting factors, readers are referred to [25]. In the second-last
columns of Tables 5 and 6, the observations for a single-replication (i.e., $J = 1$) experiment are illustrated for $D_1$ and $D_2$, respectively, where the subscript $j = 1$ in $y_{(i)jk}$ is omitted for brevity. Following these observations, the corresponding error terms are displayed in Tables 5 and 6 for $D_1$ and $D_2$, respectively, with the subscript $j = 1$ in $e_{(i)jk}$ omitted for brevity. The methods in the subsequent analysis, including estimations of effects and calculation of the variances of estimated effects for FFSP designs without splitting factors (see [3,7,26] for more details), are also applicable to FFSP designs with splitting factors.

As mentioned previously, the splitting factors can move SP effects to the whole plot level, which is important when performing significance testing of effects. The Analysis of Variance (ANOVA) test is a widely used technique for testing the significance of effects in FFSP experiments without splitting factors; due to two different types of errors, such tests have the following rules:

(i) the WP effects are tested against the WP error;
(ii) the SP effects that are aliased with WP effects are tested against the WP error;
(iii) the SP effects that are not aliased with any WP effects are tested against the SP error.

For example, in $D_1$, the effects $A$ and $pq$ need to be tested against the WP error, and the effects $p, q, t, Ap, Aq, At, pt, qt, Apq, Apt, Aqt, pqt$, and $Apqt$ need to be tested against the SP error.

As a result of adding splitting factors to a $2^{n_1+n_2-(k_1+k_2)}$ FFSP design, SP effects in (iii) may be moved to the whole plot level, which changes the previous rules for testing the significance of effects. For FFSP designs with splitting factors:

(iv) WP effects are tested against the WP error;
(v) SP effects that are aliased with WP effects, splitting factors, interactions which contain only splitting factors, or interactions which contain at least one splitting factor, at least one WP factor, and no SP factor are tested against the WP error;
(vi) SP effects that are not aliased with WP effects, splitting factors, interactions which contain only splitting factors, or interactions which contain at least one splitting factor, at least one WP factor, and no SP factor are tested against the SP error.

For example, in $D_2$, the SP effects $pq, Aqt, Apq, and qt$ are moved to the whole plot level as they are aliased with $p$ or $Ap$. Therefore, the effects $A, pq, qt, Apq, and Aqt$ need to be tested against the WP error, and the effects $p, q, t, Ap, Aq, At, pt, Aqt, pqt$, and $Apqt$ need to be tested against the SP error. For more details on estimating these two types of errors in unreplicated or replicated FFSP designs without splitting factors, readers are referred to [25]. The two types of errors in FFSP designs with splitting factors can be estimated with the same methods used for designs without splitting factors, with two noteworthy points in mind:

1. there are differences in the numbers of whole plots and sub plots;
2. the splitting factors can move SP effects to the whole plot level.

As an example associated with the second point above, for unreplicated experiments, if there is prior information suggesting that the effects $qt$ and $Apq$ are negligible, then the sum of the square of the effect $qt (= Apq)$ can be pooled to estimate the SP error in design $D_1$, while the sum of the square of $qt (= Apq)$ can be pooled to estimate the WP error in design $D_2$, as $qt (= Apq)$ is aliased with the splitting factor $p$.

The half-normal (normal) plot (see [27]) is a popular method for identifying significant effects when there are no extra degrees of freedom for estimating the two types of errors. When applying a half-normal plot to FFSP designs without splitting factors, it is necessary to separately apply this method first to the effects in (i) and (ii) together, and then to the effects in (iii) together due to the different error terms. Similarly, when applying a half-normal plot to FFSP designs with splitting factors, it is necessary to separately apply this method first to the effects in (iv) and (v) together and then to the effects in (vi) together. For more details on half-normal plot for FFSP designs, readers are referred to [7].
From the above analysis, it is clear that methods of analysis for FFSP designs without splitting factors are applicable to FFSP designs with splitting factors as long as the two important points we have noted are kept in mind.

7. Conclusions

FFSP designs enjoy wide application when it is impractical to perform an FF design in a completely random order. The study of FFSP designs in the literature has mainly focused on FFSP designs without replicating the setting of WP factors. However, when there are too many subplots per whole plot or too few whole plots, FFSP designs without replicated settings for the WP factors are not suitable. In [24], the authors discussed a class of FFSP designs in which splitting factors are introduced to replicate settings of WP factors. This results in an increase in the number of whole plots and a decrease in the number of subplots per whole plot.

Despite the rich development in FFSP designs, the study of FFSP designs with splitting factors is undeveloped. In this paper, we have focused on the study of FFSP designs with splitting factors and investigated the conditions of \(2^{(n_1 + n_2) - (0 + k_2)}\) FFSP designs with resolution III or at least IV containing various clear effects. We propose an algorithm for constructing \(2^{(n_1 + n_2 + r) - (0 + k_2 + r)}\) FFSP designs with the highest possible number of different clear effects. As an application of the proposed algorithm, we tabulated \(2^{(n_1 + n_2 + r) - (0 + k_2 + r)}\) FFSP designs with the most clear 2FIs. In Section 6, our analysis of FFSP designs with splitting factors is discussed. We conclude that methods of analysis for FFSP designs without splitting factors are applicable to FFSP designs with splitting factors as long as suitable attention is paid to the possibility that SP effects may be moved to the whole plot level, which matters when testing the significance of effects.

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**Abbreviations**
The following abbreviations are used in this manuscript:

- FF: Fractional factorial
- MA: Minimum aberration
- FFSP: Fractional factorial split-plot
- WP: Whole plot
- SP: Subplot
- 2FI: Two-factor interaction

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