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Some New Inequalities and Extremal Solutions of a Caputo–Fabrizio Fractional Bagley–Torvik Differential Equation

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Abstract: This paper studies the existence of extremal solutions for a nonlinear boundary value problem of Bagley–Torvik differential equations involving the Caputo–Fabrizio-type fractional differential operator with a non-singular kernel. With the help of a new inequality with a Caputo–Fabrizio fractional differential operator, the main result is obtained by applying a monotone iterative technique coupled with upper and lower solutions. This paper concludes with an illustrative example.

Keywords: Bagley–Torvik differential equation; Caputo–Fabrizio fractional differential operator; extremal solutions; monotone iterative technique

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1. Introduction

Fractional calculus, which deals with fractional-order differential and integral operators, has developed into a popular branch of mathematical analysis in light of its extensive applications in a variety of disciplines of natural and social sciences. A distinguished feature of the fractional-order model is that it is capable of tracing the past history of the phenomena involved in the model. For details and examples in different disciplines of engineering and applied sciences, see [1–3] and the references cited therein. A fractional-order model consists of fractional differential or integro-differential equations, which are nonlinear in nature, and it is not possible to find the exact solutions of these equations in general. Thus, many researchers focused on developing the approximate analytical and numerical methods for solving the initial and boundary value problems of fractional differential equations. One can find the details and examples in [4–12] and the references cited therein.

In this paper, we formulate and solve a boundary value problem of nonlinear generalized Bagley–Torvik differential equations involving a fractional differential operator with a non-singular kernel due to Caputo and Fabrizio as well as nonlinear boundary conditions. One can find details about the Caputo–Fabrizio fractional operator in [13–16]. We make use of a monotone iterative technique coupled with upper and lower solutions to obtain the extremal solutions for the problem at hand. It is imperative to note that the monotone method deals with the existence as well as the construction of solutions as well as the comparison of results for nonlinear differential equations (for examples, see [17–25]). In precise terms, we study the following:

$$\begin{cases} [{}^{CF}D^{\delta}g(\tau)]' = G(\tau, g(\tau), {}^{CF}D^{\delta}g(\tau)), & \Im = [0, T], \\ g(0) = H(g(\eta)), \quad \eta \in (0, T], \end{cases} \tag{1}$$

where $G \in C(\mathfrak{I} \times \mathbb{R}^2, \mathbb{R})$, $H \in C(\mathbb{R}, \mathbb{R})$ and ${}^{CF}D^{\delta}$ denotes a Caputo–Fabrizio operator of a fractional order $0 < \delta < 1$ with a non-singular kernel defined by

$$ {}^{CF}D^{\delta} g(\tau) = \frac{(2-\delta)\rho(\delta)}{2(1-\delta)} \int_0^{\tau} \exp(-\frac{\delta}{1-\delta}(\tau - s)) g'(s) ds, \quad \tau \geq 0, \tag{2} $$

where $\rho(\delta)$ is a normalization constant depending on δ.

In the long history of the development of fractional derivatives, many types of fractional derivatives have appeared. The more popular and mentioned fractional derivatives are the Riemann–Liouville and Caputo fractional derivatives, which are particularly suitable for describing physical phenomena related to fatigue, damage and electromagnetic hysteresis. Unfortunately, they are not applicable to describing and simulating some behavior observed in materials with huge heterogeneities and structures with different scales. In this case, Caputo and Fabrizio developed and proposed a new type of fractional order derivatives without a singular kernel, which was named the Caputo–Fabrizio fractional derivative by later scholars. As pointed out in [26], the Caputo–Fabrizio fractional derivative has an exponential function kernel, which is more realistic that the one with a power function due to the fact that the singularity does not occur at the end of the interval within which the fractional derivative of a given function is taken. In addition, the fractional derivative with an exponential function kernel is generally considered to be better than the one with a power kernel, since the exponent function is a better filter than the power function. In fact, the Caputo–Fabrizio fractional derivative has been used extensively as a filter regulator [27]. For more details on fractional derivatives without singular kernels, see [28–32].

Motivated by the above, this paper attempts to investigate some new inequalities and extremal solutions of a Caputo–Fabrizio fractional Bagley–Torvik differential equation, given in Equation (1).

2. Auxiliary Material

In this section, we present the preliminary concepts related to the given problem and comparison principles:

Definition 1. *$g \in C^1(\mathfrak{I})$ is called a lower solution of a Caputo–Fabrizio fractional Bagley–Torvik differential equation (Equation (1)) if*

$$ \begin{cases} [{}^{CF}D^{\delta} g(\tau)]' \leq G(\tau, g(\tau), {}^{CF}D^{\delta} g(\tau)), \\ g(0) \leq H(g(\eta)), \end{cases} $$

which, on reversing the inequalities, defines an upper solution for Equation (1).

Lemma 1. *Suppose $N \geq 0$, $\rho(\delta) > 0$ and $\nu \in C^1(\mathfrak{I})$ satisfy*

$$ \begin{cases} [{}^{CF}D^{\delta} \nu(\tau)]' \geq -\dfrac{\delta}{1-\delta} {}^{CF}D^{\delta} \nu(\tau) - N\nu(\tau), \\ \nu(0) \geq 0. \end{cases} \tag{3} $$

Then, one has $\nu(\tau) \geq 0$, $\forall \tau \in \mathfrak{I}$.

Proof. By differentiating the Caputo–Fabrizio operator ${}^{CF}D^{\delta}\nu(\tau)$ with respect to τ, we obtain

$$ [{}^{CF}D^{\delta} \nu(\tau)]' = \beta \nu'(\tau) - \frac{\delta}{1-\delta} {}^{CF}D^{\delta} \nu(\tau), \tag{4} $$

where $\beta = \dfrac{(2-\delta)\rho(\delta)}{2(1-\delta)}$. Using Equation (4) in (3), we obtain

$$ \beta \nu'(\tau) + N\nu(\tau) \geq 0, $$

which can alternatively be expressed as

$$\beta e^{-\frac{N}{\beta}\tau}[e^{\frac{N}{\beta}\tau}\nu(\tau)]' \geq 0.$$

Since $\beta \geq 0$, we have $[e^{\frac{N}{\beta}\tau}\nu(\tau)]' \geq 0$, which leads to

$$e^{\frac{N}{\beta}\tau}\nu(\tau) \geq \nu(0) \geq 0, \quad \forall\tau \in \mathfrak{I}.$$

Thus, one can easily come to the conclusion that $\nu(\tau) \geq 0, \forall\tau \in \mathfrak{I}$. □

A result analogous to Lemma 1 can be formulated as follows:

Lemma 2. *With $N \geq 0$ and $\rho(\delta) > 0$, as given in Lemma 1, if a function $\nu \in C^1(\mathfrak{I})$ satisfies the following problem:*

$$\begin{cases} [{}^{CF}D^{\delta}\nu(\tau)]' \leq -\dfrac{\delta}{1-\delta}\,{}^{CF}D^{\delta}\nu(\tau) - N\nu(\tau), \\ \nu(0) \leq 0. \end{cases} \tag{5}$$

Then, one has $\nu(\tau) \leq 0$, $\forall\tau \in \mathfrak{I}$.

3. The Linear Caputo–Fabrizio Fractional Bagley–Torvik Differential Equation

For $\psi \in C[0, T]$ and $N, h \in \mathbb{R}$, let us consider the following linear Caputo–Fabrizio fractional Bagley–Torvik differential equation:

$$\begin{cases} [{}^{CF}D^{\delta}g(\tau)]' + \dfrac{\delta}{1-\delta}\,{}^{CF}D^{\delta}g(\tau) + Ng(\tau) = \psi(\tau), \\ g(0) = h. \end{cases} \tag{6}$$

Definition 2. *$g \in C^1(\mathfrak{I})$ is said to be a lower solution of the above linear Caputo–Fabrizio fractional Bagley–Torvik differential equation (Equation (6)) if*

$$\begin{cases} [{}^{CF}D^{\delta}g(\tau)]' + \dfrac{\delta}{1-\delta}\,{}^{CF}D^{\delta}g(\tau) + Ng(\tau) \leq \psi(\tau), \\ g(0) \leq h, \end{cases} \tag{7}$$

The above definition takes the form of an upper solution of the above linear Caputo–Fabrizio fractional Bagley–Torvik differential equation (Equation (6)) if we reverse the inequalities in it:

Theorem 1. *Assume that u_0 and v_0, which satisfy the inequality $u_0(\tau) \leq v_0(\tau)$, $\forall\tau \in \mathfrak{I}$, are the lower and upper solutions of the above linear Caputo–Fabrizio fractional Bagley–Torvik differential equation (Equation (6)), respectively. Then, there must exist a unique solution $g \in [u_0, v_0]$ for the linear Caputo–Fabrizio fractional Bagley–Torvik differential equation (Equation (6)), given by*

$$g(\tau) = he^{-\frac{N}{\beta}\tau} + \frac{1}{\beta}\int_0^{\tau} e^{-\frac{N}{\beta}(\tau-s)}\psi(s)ds, \quad \forall\tau \in \mathfrak{I}, \tag{8}$$

where $\beta = \dfrac{(2-\delta)\rho(\delta)}{2(1-\delta)}$.

Proof. Clearly, Equation (6) can be rewritten as

$$[{}^{CF}D^{\delta}g(\tau)]' + \frac{\delta}{1-\delta}\,{}^{CF}D^{\delta}g(\tau) + Ng(\tau) = \beta g'(\tau) + Ng(\tau) = \beta e^{-\frac{N}{\beta}t}[e^{\frac{N}{\beta}t}g(\tau)]' = \psi(\tau),$$

which leads to

$$g(\tau) = g(0)e^{-\frac{N}{\beta}\tau} + \frac{1}{\beta}\int_0^\tau e^{-\frac{N}{\beta}(\tau - s)}\psi(s)ds, \ \ \forall \tau \in \mathfrak{I}. \tag{9}$$

Using the condition $g(0) = h$ in Equation (9), we obtain Equation (8). This shows that the linear Caputo–Fabrizio fractional Bagley–Torvik differential equation (Equation (6)) has a unique solution given by Equation (8).

Next, if g is a solution of Equation (6), then we can show that $u_0 \leq g \leq v_0$. By letting $\nu = g - u_0$, we find

$$\begin{cases} [{}^{CF}D^\delta \nu(\tau)]' \geq -\dfrac{\delta}{1-\delta}\, {}^{CF}D^\delta \nu(\tau) - N\nu(\tau), \\ \nu(0) \geq \ 0. \end{cases}$$

Then, a straightforward application of Lemma 1 implies that $\nu(\tau) \geq 0$, $\forall \tau \in \mathfrak{I}$; that is, $g \geq u_0$. Similarly, by taking $\mu = v_0 - g$, it can be shown that $g \leq v_0$. Therefore, we deduce that $g \in [u_0, v_0]$. □

4. The Nonlinear Bagley–Torvik Differential Equation

Theorem 2. *Assume the following:*

- **(H_1)** *u_0, $v_0 \in C^1(\mathfrak{I})$, which satisfy the inequality $u_0(\tau) \leq v_0(\tau)$, $\forall \tau \in \mathfrak{I}$, are the lower and upper solutions of the nonlinear Caputo–Fabrizio fractional Bagley–Torvik differential equation (Equation (1)), respectively;*
- **(H_2)** *There exists a constant $N \geq 0$ such that*

$$G(\tau, \Im, \pounds) - G(\tau, \overline{\Im}, \overline{\pounds}) \geq -N(\Im - \overline{\Im}) - \frac{\delta}{1-\delta}(\pounds - \overline{\pounds}),$$

for $u_0 \leq \overline{\Im} \leq \Im \leq v_0$, ${}^{CF}D^\delta u_0 \leq \overline{\pounds} \leq \pounds \leq {}^{CF}D^\delta v_0$;

- **(H_3)** *The function H is nondecreasing on $[u_0, v_0]$.*

Then, one must be able to construct two explicit monotonic iterative sequences $\{u_n\}$ and $\{v_n\}$ without difficulty, and they converge uniformly to the extremal solutions of the nonlinear Caputo–Fabrizio fractional Bagley–Torvik differential equaiton (Equation (1)) in $[u_0, v_0]$ on $\mathfrak{I}$.

Proof. We consider the linear Caputo–Fabrizio fractional Bagley–Torvik differential equation:

$$\begin{cases} [{}^{CF}D^\delta g(\tau)]' = \psi_h(\tau) - \dfrac{\delta}{1-\delta}\, {}^{CF}D^\delta g(\tau) - Ng(\tau), \\ g(0) = H(h(\eta)), \end{cases} \tag{10}$$

where $\psi_h(\tau) = G(\tau, h(\tau), {}^{CF}D^\delta h(\tau)) + \dfrac{\delta}{1-\delta}\, {}^{CF}D^\delta h(\tau) + Nh(\tau)$ and $h \in [u_0, v_0]$.

It follows from (H_1) that u_0, $v_0 \in C^1(\mathfrak{I})$ are the lower and upper solutions of the nonlinear Caputo–Fabrizio fractional Bagley–Torvik differential equation (Equation (1)), respectively. Furthermore, by conditions (H_2) and (H_3), we obtain

$$\begin{cases} [\,{}^{CF}D^\delta u_0(\tau)]' & \leq G(\tau, u_0(\tau),\, {}^{CF}D^\delta u_0(\tau)) \\ & \leq G(\tau, h(\tau),\, {}^{CF}D^\delta h(\tau)) + \dfrac{\delta}{1-\delta}\, {}^{CF}D^\delta h(\tau) + Nh(\tau) \\ & \quad - \dfrac{\delta}{1-\delta}\, {}^{CF}D^\delta u_0(\tau) - Nu_0(\tau) \\ & = \psi_h(\tau) - \dfrac{\delta}{1-\delta}\, {}^{CF}D^\delta u_0(\tau) - Nu_0(\tau), \\ u_0(0) & \leq H(u_0(\eta)) \leq H(h(\eta)), \end{cases}$$

and

$$\begin{cases} [\,^{CF}D^{\delta}v_0(\tau)]' & \geq G(\tau, v_0(\tau),\ ^{CF}D^{\delta}v_0(\tau)) \\ & \geq G(\tau, h(\tau),\ ^{CF}D^{\delta}h(\tau)) + \dfrac{\delta}{1-\delta}\ ^{CF}D^{\delta}h(\tau) + Nh(\tau) \\ & \quad - \dfrac{\delta}{1-\delta}\ ^{CF}D^{\delta}v_0(\tau) - Nv_0(\tau) \\ & = \psi_h(\tau) - \dfrac{\delta}{1-\delta}\ ^{CF}D^{\delta}v_0(\tau) - Nv_0(\tau), \\ v_0(0) & \geq H(v_0(\eta)) \geq H(h(\eta)), \end{cases}$$

The above inequalities ensure that the linear Caputo–Fabrizio fractional Bagley–Torvik differential equation (Equation (10)) has the lower solution u_0 and the upper solution v_0. Furthermore, by Theorem 1, we know that the Caputo–Fabrizio fractional Bagley–Torvik differential equation (Equation (10)) has a unique solution $w \in [u_0, v_0]$. Now, we introduce an operator $Q : [u_0, v_0] \to [u_0, v_0]$ by $g = Qh$ and verify the operator Q is nondecreasing. Let $h_1, h_2 \in [u_0, v_0]$ be such that the relation $h_1 \leq h_2$ holds. By letting $x = u_2 - u_1$, $u_1 = Qh_1$ and $u_2 = Qh_2$ and using the assumptions (H_2) and (H_3), one can find that

$$\begin{aligned} [\,^{CF}D^{\delta}x(\tau)]' &= G(\tau, h_2(\tau),\ ^{CF}D^{\delta}h_2(\tau)) + \frac{\delta}{1-\delta}\ ^{CF}D^{\delta}h_2(\tau) + Nh_2(\tau) \\ &\quad - \frac{\delta}{1-\delta}\ ^{CF}D^{\delta}u_2(\tau) - Nu_2(\tau) - G(\tau, h_1(\tau),\ ^{CF}D^{\delta}h_1(\tau)) \\ &\quad - \frac{\delta}{1-\delta}\ ^{CF}D^{\delta}h_1(\tau) - Nh_1(\tau) + \frac{\delta}{1-\delta}\ ^{CF}D^{\delta}u_1(\tau) + Nu_1(\tau) \\ &\geq -N(h_2 - h_1)(\tau) - \frac{\delta}{1-\delta}\ ^{CF}D^{\delta}(h_2 - h_1)(\tau) + N(h_2 - h_1)(\tau) \\ &\quad + \frac{\delta}{1-\delta}\ ^{CF}D^{\delta}(h_2 - h_1)(\tau) - \frac{\delta}{1-\delta}\ ^{CF}D^{\delta}(u_2 - u_1)(\tau) - N(u_2 - u_1)(\tau) \\ &= -\frac{\delta}{1-\delta}\ ^{CF}D^{\delta}x(\tau) - Nx(\tau), \\ x(0) &= H(h_2(\eta)) - H(h_1(\eta)) \geq 0. \end{aligned}$$

It follows from Lemma 1 (comparison principle) that $x(\tau) \geq 0$ and $\tau \in [0, T]$, which shows that the operator Q is nondecreasing.

On account of the fact that Q is a nondecreasing operator, we put $u_n = Qu_{n-1}$, $v_n = Qv_{n-1}$, $n = 1, 2, \ldots$. Then, the following conclusion is tenable:

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0,\ n = 1, 2, \ldots. \tag{11}$$

By employing the standard arguments, with the aid of the Arzela–Ascoli theorem, one can easily show that, uniformly, we have

$$\lim_{n\to\infty} u_n = u^*, \quad \lim_{n\to\infty} v_n = v^*$$

In addition, $u^*, v^* \in [u_0, v_0]$ solve the nonlinear Caputo–Fabrizio fractional Bagley–Torvik differential equation (Equation (1)).

In the last step of this proof, we show $u^*, v^* \in [u_0, v_0]$ are the extremal solutions of the nonlinear Caputo–Fabrizio fractional Bagley–Torvik differential equation (Equation (1)). Suppose that $w \in [u_0, v_0]$ is any solution of the nonlinear Caputo–Fabrizio fractional Bagley–Torvik differential equation (Equation (1)). Thus, $Qw = w$ holds. It follows from $u_0 \leq w \leq v_0$ and the nondecreasing property of operator Q that

$$u_n \leq w \leq v_n,\ n = 1, 2, \ldots, \tag{12}$$

which, when $n \to +\infty$, yields $u^* \leq w \leq v^*$. As a consequence, we deduce that $u^*, v^* \in [u_0, v_0]$ are the extremal solutions of the nonlinear Caputo–Fabrizio fractional Bagley–Torvik differential equation (Equation (1)). □

5. An Ancillary Example

We present an example session to aid in the interpretation of the main results.

Consider the nonlinear Bagley–Torvik differential equation involving a Caputo–Fabrizio fractional operator supplemented with the nonlocal condition given by

$$\begin{cases} [{}^{CF}D^{\delta}g(\tau)]' = -\frac{\tau^3}{2} + \frac{[\tau - g(\tau)]^3}{2} + 3[\tau - g(\tau)]^5 - \frac{\delta}{10(1-\delta)}[{}^{CF}D^{\delta}g(\tau)]^2, \\ g(0) = 8g^3(\eta), \end{cases} \tag{13}$$

where $\tau \in [0,1],\ \eta \in (0,1],\ H(g) = 8g^3, (2-\delta)\rho(\delta) \leq 2\delta, 0 < \delta < 1$ and

$$G(\tau, \Im, \pounds) = -\frac{\tau^3}{2} + \frac{(\tau - \Im)^3}{2} + 3(\tau - \Im)^5 - \frac{\delta}{10(1-\delta)}\pounds^2.$$

By letting $u_0(\tau) = 0$ and $v_0(\tau) = \tau$, we can easily verify that $u_0,\ v_0 \in C^1([0,1])$ are the lower and upper solutions of the nonlinear Caputo–Fabrizio fractional Bagley–Torvik differential equation (Equation (13)), respectively. Thus, condition (H_1) holds.

By a simple computation, for $0 \leq \overline{\Im} \leq \Im \leq \tau,\ 0 \leq \overline{\pounds} \leq \pounds \leq \frac{(2-\delta)\rho(\delta)}{2\delta}(1 - e^{-\frac{\delta\tau}{1-\delta}})$, we have

$$\begin{aligned} & G(\tau, \Im, \pounds) - G(\tau, \overline{\Im}, \overline{\pounds}) \\ & = \frac{1}{2}[(\tau - \Im)^3 - (\tau - \Im)^3] + 3[(\tau - \Im)^5 - (\tau - \overline{\Im})^5] - \frac{\delta}{10(1-\delta)}[\pounds^2 - \overline{\pounds}^2] \\ & \geq -(\frac{3}{2} + 15)(\Im - \overline{\Im}) - \frac{\delta}{5(1-\delta)}(\pounds - \overline{\pounds}) \\ & \geq -16.5(\Im - \overline{\Im}) - \frac{\delta}{(1-\delta)}(\pounds - \overline{\pounds}). \end{aligned}$$

Therefore, condition (H_2) holds true for $N = 16.5$. In addition, the condition (H_3) is clearly satisfied. In summation, all assumptions of Theorem 2 are satisfied. Therefore, based on Theorem 2, one must be able to construct two explicit monotonic iterative sequences $\{u_n\}$ and $\{v_n\}$ without difficulty, and these sequences converge uniformly to the extremal solutions of the nonlinear Caputo–Fabrizio fractional Bagley–Torvik differential equation (Equation (13)) on $[0,1]$. Here, we have

$$\begin{aligned} u_n(\tau) &= 8u_{n-1}^3(\eta)e^{-\frac{N}{\beta}\tau} + \frac{1}{\beta}\int_0^{\tau} e^{-\frac{N}{\beta}(\tau-s)}\Big[G(s, u_{n-1}(s), {}^{CF}D^{\delta}u_{n-1}(s)) \\ &\qquad + \frac{\delta}{1-\delta}\,{}^{CF}D^{\delta}u_{n-1}(s) + Nu_{n-1}(s)\Big]ds \\ &= 8u_{n-1}^3(\eta)e^{-\frac{N}{\beta}\tau} + \frac{1}{\beta}\int_0^{\tau} e^{-\frac{N}{\beta}(\tau-s)}\Big[-\frac{s^3}{2} + \frac{[s - u_{n-1}(s)]^3}{2} + 3[s - u_{n-1}(s)]^5 \\ &\qquad - \frac{\delta}{10(1-\delta)}[{}^{CF}D^{\delta}u_{n-1}(s)]^2 + \frac{\delta}{1-\delta}\,{}^{CF}D^{\delta}u_{n-1}(s) \\ &\qquad + Nu_{n-1}(s)\Big]ds, \end{aligned} \tag{14}$$

$$\begin{aligned} v_n(\tau) &= 8v_{n-1}^3(\eta)e^{-\frac{N}{\beta}\tau} + \frac{1}{\beta}\int_0^\tau e^{-\frac{N}{\beta}(\tau-s)}\Big[G(s, v_{n-1}(s),\ {}^{CF}D^\delta v_{n-1}(s)) \\ &\qquad + \frac{\delta}{1-\delta}\,{}^{CF}D^\delta v_{n-1}(s) + Nv_{n-1}(s)\Big]ds \\ &= 8v_{n-1}^3(\eta)e^{-\frac{N}{\beta}\tau} + \frac{1}{\beta}\int_0^\tau e^{-\frac{N}{\beta}(\tau-s)}\Big[-\frac{s^3}{2} + \frac{[s-v_{n-1}(s)]^3}{2} + 3[s-v_{n-1}(s)]^5 \\ &\qquad - \frac{\delta}{10(1-\delta)}[{}^{CF}D^\delta v_{n-1}(s)]^2 + \frac{\delta}{1-\delta}\,{}^{CF}D^\delta v_{n-1}(s) \\ &\qquad + Nv_{n-1}(s)\Big]ds, \end{aligned} \tag{15}$$

where $N = 16.5,\ \beta = \dfrac{(2-\delta)\rho(\delta)}{2(1-\delta)}$.

6. Conclusions

Fractional calculus and differential equations are a present line of research. Currently, hundreds of mathematicians and engineers are working on this topic. As one of the new research directions of fractional calculus, Caputo–Fabrizio fractional calculus and differential equations are continuously gaining attention. In this context, this paper studied some new inequalities and extremal solutions of a Caputo–Fabrizio fractional Bagley–Torvik differential equation. In order to achieve this goal, we developed a comparison principle involving Caputo–Fabrizio derivatives and the monotonic iterative method combining upper and lower solutions. We not only proved the existence of extremal solutions, but also obtained some explicit monotone iterative sequences that converged uniformly to extremal solutions.

In the future, as a good extension of the current work, one can carry out the following related research:

(1) Asymptotic stability of some Caputo–Fabrizio fractional-related systems;

(2) Some nonlinear problems involving new fractional operators, such as the generalized fractional Hilfer operator [33];

(3) Some control problems with qualitative property controllability, optimal control, etc. See [34,35].

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