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Results for Fuzzy Mappings and Stability of Fuzzy Sets with Applications

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Abstract: The purpose of this paper is to develop some fuzzy fixed point results for the sequence of locally fuzzy mappings satisfying rational type almost contractions in complete dislocated metric spaces. We apply our results to obtain new results for set-valued and single-valued mappings. We also study the stability of fuzzy fixed point γ-level sets. An example is presented in favor of these results.

Keywords: fuzzy mapping; complete dislocated metric spaces; hausdorff metric; fixed point; rational type almost contraction

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1. Introduction

The idea of a fuzzy set was given by Lotfi Zadeh for the first time in 1965 [1]. This concept has been extended in fuzzy functional analysis, fuzzy topology, fuzzy control theory and decision making. One of the significant developments of fuzzy sets in fuzzy functional analysis is fuzzy mapping presented by Weiss [2] and Butnariu [3]. One of the branches of functional analysis is fixed point theory. Fixed point theory plays a key role in finding solutions to mathematical and engineering problems. The fixed point results for multivalued mappings generalizes the results for single valued mappings. Heilpern [4] established a result to obtain fixed point for fuzzy mappings and generalized Nadler's fixed point result [5] for multivalued mappings. Since then a lot of work has been done by various authors in this field, see [6,7].

Stability is an idea to obtain an approximate solution of such equations which cannot have an exact solution. It has applications in nonlinear continuous and discrete dynamical systems [8,9]. The stability of fixed points is a study about the relationship between the fixed points of certain mappings and the limit of the sequence of those mappings. It has been extensively studied in various aspects [10–17]. Since the set valued mappings usually give more than one fixed points than the self mappings [5,18,19], so the set of fixed points for set valued mappings becomes more interesting for the study of stability. The sequence of sets $\{F(A_j)\}_{j\in\mathbb{N}}$ containing fixed points of a sequence of multivalued mappings $\{A_j\}_{j\in\mathbb{N}}$ are called stable if $F(A_j) \to F(A)$ in the Hausdorff metric, where the mapping A is the limit of the sequence $\{A_j\}_{j\in\mathbb{N}}$ and $F(A)$ is the set of fixed points of A.

Recently, Alansari et al. [20] initiate the study of stability and well-posedness of functional inclusions involving fuzzy set-valued maps. In this sequel, we establish fixed point results for fuzzy mappings in complete dislocated metric space satisfying a rational type of almost contractions only for the elements in a closed ball. An example is also given which supports the proved results. We also discuss the stability of fuzzy fixed point sets of above mentioned multivalued contractions. We present some definitions and results which will be helpful in the article.

2. Preliminaries

In this section, we will recall some specific notations, definitions and results which are needed in the article. All of these preliminaries are taken from Nawab et al. [21], Azam [6], and Shoaib et al. [22]. Throughout this paper, $\mathbb{N}$ and $\mathbb{R}$ represent the sets of natural and real numbers, respectively. Let S be a universe of discourse of all parameters and δ_l is called dislocated metric over the set S.

Definition 1 ([21]). *Let S be a nonempty set and $\delta_l : S \times S \to [0, \infty)$, be a real valued function. Then, the function δ_l called dislocated metric (or simply δ_l-metric), if for any $l_1, l_2, l_3 \in S$, the following hold:*
(i) If $\delta_l(l_1, l_2) = 0$, then $l_1 = l_2$;
(ii) $\delta_l(l_1, l_2) = \delta_l(l_2, l_1)$;
(iii) $\delta_l(l_1, l_2) \leq \delta_l(l_1, l_3) + \delta_l(l_3, l_2)$
The pair (S, δ_l) is called a δ_l metric space. It can be seen that if $\delta_l(l_1, l_2) = 0$, then by (i) $l_1 = l_2$. But if $l_1 = l_2$, then $\delta_l(l_1, l_2)$ is not necessarily 0.

Example 1 ([21]). *If $S = \mathbb{Q}^+ \cup \{0\}$, and $\delta_l : S \times S \to [0, \infty)$ then $\delta_l(l_1, l_2) = l_1 + l_2$ is a δ_l-metric on S.*

Definition 2 ([23]). *Let $CB(S)$ denotes the collection of all nonempty closed and bounded subsets of a set S. The function $H_{\delta_l} : CB(S) \times CB(S) \to \mathbb{R}^+$, defined by*

$$H_{\delta_l}(C, E) = \max\left\{ \sup_{c \in C} D_l(c, E), \sup_{e \in E} D_l(C, e) \right\}$$

is called δ_l Hausdorff metric on $CB(S)$, where

$$D_l(c, E) = \inf\{\delta_l(c, e) : e \in E\}.$$

Definition 3 ([22]). *A fuzzy set T is a function from S to $[0, 1]$, $F_l(S)$ is the set of all fuzzy sets in S. The function values $T(l)$ is called the grade of membership of l in T if T is a fuzzy set and $l \in S$. The γ-level set of fuzzy set T, is denoted by $[T]_\gamma$, and defined as:*

$$\begin{aligned} [T]_\gamma &= \{l : T(l) \geq \gamma\} \quad \textit{where} \quad \gamma \in (0, 1], \\ [T]_0 &= \overline{\{l : T(l) > 0\}}. \end{aligned}$$

Suppose that S is a nonempty set and Z be a δ_l metric, then $A : S \to F_l(Z)$ is a fuzzy mapping. A fuzzy mapping A is a fuzzy subset on $S \times Z$ with membership function $A(l)(z)$. The function $A(l)(z)$ is the grade of membership of z in $A(l)$. For convenience, we denote the γ-level set of $A(l)$ by $[Al]_\gamma$ instead of $[A(l)]_\gamma$.

Definition 4 ([6]). *A point $l \in S$ is called a fuzzy fixed point of a fuzzy mapping $A : S \to F_l(S)$ if there exists $\gamma \in (0, 1]$ such that $l \in [Al]_\gamma$.*

Lemma 1 ([22]). *Let U and V be nonempty closed and bounded subsets of a δ_l metric space (S, δ_l). If $u \in U$, then*

$$\delta_l(u, V) \leq H_{\delta_l}(U, V).$$

Lemma 2 ([22]). *Let (S, δ_l) be a δ_l metric space. Let $(CB(S), H_{\delta_l})$ be a δ_l Hausdorff metric space. Then, for all $U, V \in CB(S)$ and for each $u \in U$, there exists $v_u \in V$ satisfies*

$$\delta_l(u, V) = \delta_l(u, v_u),$$

then,

$$H_{\delta_l}(U, V) \geq \delta_l(u, v_u).$$

3. Main Results

Theorem 1. *Let (S, δ_l) be a complete δ_l metric space and $A : S \to F_l(S)$ be a fuzzy mapping. Suppose $\psi : \mathbb{R}^+ \cup \{0\} \to [0, \infty)$ is a continuous and nondecreasing function with $\sum_{n=1}^{\infty} \psi^n(s) < \infty$ and $\psi(s) < s$ for each $s > 0$. Assume that l_0 be any point in S, $\gamma : S \to (0, 1]$ be a mapping and there exists a real number $M \geq 0$ satisfying the following:*

$$\begin{aligned} H_{\delta_l}([Al_1]_{\gamma(l_1)}, [Al_2]_{\gamma(l_2)}) \quad &\leq \quad \psi\left(\max\left\{\begin{array}{c} \delta_l(l_1, l_2), D_l(l_1, [Al_1]_{\gamma(l_1)}), D_l(l_2, [Al_2]_{\gamma(l_2)}), \\ \frac{D_l(l_2, [Al_1]_{\gamma(l_1)}) + D_l(l_1, [Al_2]_{\gamma(l_2)})}{2}, \\ \frac{D_l(l_2, [Al_2]_{\gamma(l_2)})[1 + D_l(l_1, [Al_1]_{\gamma(l_1)})]}{1 + \delta_l(l_1, l_2)}, \\ \frac{D_l(l_2, [Al_1]_{\gamma(l_1)})[1 + D_l(l_1, [Al_2]_{\gamma(l_2)})]}{1 + \delta_l(l_1, l_2)} \end{array}\right\}\right) \\ &\quad + M \min\left\{\begin{array}{c} D_l(l_1, [Al_1]_{\gamma(l_1)}), D_l(l_2, [Al_2]_{\gamma(l_2)}), \\ D_l(l_1, [Al_2]_{\gamma(l_2)}), D_l(l_2, [Al_1]_{\gamma(l_1)}) \end{array}\right\}. \end{aligned} \tag{1}$$

for all $l_1, l_2 \in \overline{B_{\delta_l}(l_0, \sigma)}$, $\sigma > 0$ and

$$\sum_{i=0}^{n} \psi^i\Big(D_l(l_0, [Al_0]_{\gamma(l_0)})\Big) \leq \sigma \quad \text{for } n \in \mathbb{N}. \tag{2}$$

Then, there exists z in $\overline{B_{\delta_l}(l_0, \sigma)}$ such that $z \in [Az]_{\gamma(z)}$.

Proof. Let $l_0 \in S$ and $l_1 \in [Al_0]_{\gamma(l_0)}$. Consider a sequence $\{l_r\}$ of points in Z such that $l_r \in [Al_{r-1}]_{\gamma(l_{r-1})}$. First we will show that $l_r \in \overline{B_{\delta_l}(l_0, \sigma)}$. By (2), we have

$$\begin{aligned} \delta_l(l_0, l_1) \quad &= \quad D_l(l_0, [Al_0]_{\gamma(l_0)}) \leq \sum_{i=0}^{n} \psi^i\Big(D_l(l_0, [Al_0]_{\gamma(l_0)})\Big) \leq \sigma, \\ \delta_l(l_0, l_1) \quad &\leq \quad \sigma, \end{aligned}$$

implies $l_1 \in \overline{B_{\delta_l}(l_0, \sigma)}$. Consider $l_2, l_3, \cdots, l_n \in \overline{B_{\delta_l}(l_0, \sigma)}$ for $n \in \mathbb{N}$. By Lemma 2 and (1), we have

$$\begin{aligned}
\delta_l(l_n, l_{n+1}) &\leq H_{\delta_l}([Al_{n-1}]_{\gamma(l_{n-1})}, [Al_n]_{\gamma(l_n)}) \\
&\leq \psi\left(\max\left\{\begin{array}{c} \delta_l(l_{n-1}, l_n), D_l(l_{n-1}, [Al_{n-1}]_{\gamma(l_{n-1})}), D_l(l_n, [Al_n]_{\gamma(l_n)}), \\ \frac{D_l(l_n, [Al_{n-1}]_{\gamma(l_{n-1})}) + D_l(l_{n-1}, [Al_n]_{\gamma(l_n)})}{2}, \\ \frac{D_l(l_n, [Al_n]_{\gamma(l_n)})[1 + D_l(l_{n-1}, [Al_{n-1}]_{\gamma(l_{n-1})})]}{1 + \delta_l(l_{n-1}, l_n)}, \\ \frac{D_l(l_n, [Al_{n-1}]_{\gamma(l_{n-1})})[1 + D_l(l_{n-1}, [Al_n]_{\gamma(l_n)})]}{1 + \delta_l(l_{n-1}, l_n)} \end{array}\right\}\right) \\
&\quad + M \min\left\{\begin{array}{c} D_l(l_{n-1}, [Al_{n-1}]_{\gamma(l_{n-1})}), D_l(l_n, [Al_n]_{\gamma(l_n)}), \\ D_l(l_{n-1}, [Al_n]_{\gamma(l_n)}), D_l(l_n, [Al_{n-1}]_{\gamma(l_{n-1})}) \end{array}\right\} \\
&\leq \psi\left(\max\left\{\begin{array}{c} \delta_l(l_{n-1}, l_n), \delta_l(l_{n-1}, l_n), \delta_l(l_n, l_{n+1}), \\ \frac{\delta_l(l_n, l_n) + \delta_l(l_{n-1}, l_{n+1})}{2}, \\ \frac{\delta_l(l_n, l_{n+1})[1 + \delta_l(l_{n-1}, l_n)]}{1 + \delta_l(l_{n-1}, l_n)}, \\ \frac{\delta_l(l_n, l_n)[1 + \delta_l(l_{n-1}, l_{n+1})]}{1 + \delta_l(l_{n-1}, l_n)} \end{array}\right\}\right) \\
&\quad + M \min\{\delta_l(l_{n-1}, l_n), \delta_l(l_n, l_{n+1}), \delta_l(l_{n-1}, l_{n+1}), \delta_l(l_n, l_n)\} \\
\delta_l(l_n, l_{n+1}) &\leq \psi\left(\max\left\{\delta_l(l_{n-1}, l_n), \delta_l(l_n, l_{n+1}), \frac{\delta_l(l_{n-1}, l_{n+1})}{2}, \right\}\right) \qquad (3)
\end{aligned}$$

Since

$$\frac{\delta_l(l_{n-1}, l_{n+1})}{2} \leq \frac{\delta_l(l_{n-1}, l_n) + \delta_l(l_n, l_{n+1})}{2} \leq \max\{\delta_l(l_{n-1}, l_n), \delta_l(l_n, l_{n+1})\}$$

By (3)

$$\delta_l(l_n, l_{n+1}) \leq \psi(\max\{\delta_l(l_{n-1}, l_n), \delta_l(l_n, l_{n+1})\}). \qquad (4)$$

Suppose that

$$\delta_l(l_{n-1}, l_n) < \delta_l(l_n, l_{n+1})$$

Then, $\delta_l(l_n, l_{n+1}) \neq 0$, and it follows from (4) and a property of ψ that

$$\delta_l(l_n, l_{n+1}) \leq \psi(\delta_l(l_n, l_{n+1})) < \delta_l(l_n, l_{n+1}),$$

which is not possible. So,

$$\delta_l(l_n, l_{n+1}) \leq \psi(\delta_l(l_{n-1}, l_n)).$$

In this way, we get

$$\delta_l(l_n, l_{n+1}) \leq \psi^n(\delta_l(l_0, l_1)). \qquad (5)$$

Now, by (5) and by triangular inequality, we get

$$\begin{aligned}
\delta_l(l_0, l_{n+1}) &\leq \delta_l(l_0, l_1) + \delta_l(l_1, l_2) + \cdots + \delta_l(l_n, l_{n+1}) \\
&\leq \sum_{m=0}^{n} \psi^m(\delta_l(l_0, l_1)) \leq \sigma \\
\delta_l(l_0, l_{n+1}) &\leq \sigma.
\end{aligned}$$

So, we get $l_{n+1} \in \overline{B_{\delta_l}(l_0, \sigma)}$. Hence, $l_r \in \overline{B_{\delta_l}(l_0, \sigma)}$ for all $n \in \mathbb{N}$. Now, we prove that $\{l_r\}$ is a Cauchy sequence. Fix $\eta > 0$ and let $q(\eta) \in \mathbb{N}$ such that $\sum \psi^q(\delta_l(l_0, l_1)) < \eta$. Let for any integer $q, r \in \mathbb{N}$ $(r > q > m(\eta))$. Now by triangular inequality and the property of ψ, we get

$$\begin{aligned}\delta_l(l_q, l_r) &\leq \delta_l(l_q, l_{q+1}) + \delta_l(l_{q+1}, l_{q+2}) + \cdots + \delta_l(l_{r-1}, l_r) \\ &\leq \sum_{m=q}^{r-1} \delta_l(l_m, l_{m+1}) \leq \sum_{m=q}^{r-1} \psi^m(\delta_l(l_0, l_1)) \\ &\leq \sum_{q \geq q(\eta)} \psi^m(\delta_l(l_0, l_1)) < \eta.\end{aligned}$$

Hence, $\{l_r\}$ is a Cauchy sequence in $\overline{B_{\delta_l}(l_0, \sigma)}$. As $\overline{B_{\delta_l}(l_0, \sigma)}$ is complete, so $l_r \to z \in \overline{B_{\delta_l}(l_0, \sigma)}$ such that as $r \to \infty$. Since $l_{r+1} \in [Al_r]_{\gamma(l_r)}$, for all $r \geq 1$, using Lemma 2 and inequality (1), we get

$$\begin{aligned}D_l(l_{r+1}, [Az]_{\gamma(z)}) &\leq H_{\delta_l}([Al_r]_{\gamma(l_r)}, [Az]_{\gamma(z)}) \\ &\leq \psi\left(\max\left\{\begin{array}{c}\delta_l(l_r, z), D_l(l_r, [Al_r]_{\gamma(l_r)}), D_l(z, [Az]_{\gamma(z)}), \\ \frac{D_l(z, [Al_r]_{\gamma(l_r)}) + D_l(l_r, [Az]_{\gamma(z)})}{2}, \\ \frac{D_l(z, [Az]_{\gamma(z)})[1 + D_l(l_r, [Al_r]_{\gamma(l_r)}]}{1 + \delta_l(l_r, z)}, \\ \frac{D_l(z, [Al_r]_{\gamma(l_r)})[1 + D_l(l_r, [Az]_{\gamma(z)})]}{1 + \delta_l(l_r, z)}\end{array}\right\}\right) \\ &\quad + M \min\left\{\begin{array}{c}D_{lb}(l_r, [Al_r]_{\gamma(l_r)}), D_{lb}(z, [Az]_{\gamma(z)}), \\ D_{lb}(l_r, [Az]_{\gamma(z)}), D_{lb}(z, [Al_r]_{\gamma(l_r)})\end{array}\right\} \\ &\leq \psi\left(\max\left\{\begin{array}{c}\delta_l(l_r, z), \delta_l(l_r, l_{r+1}), D_l(z, [Az]_{\gamma(z)}), \\ \frac{\delta_l(z, l_{r+1}) + D_l(l_r, [Az]_{\gamma(z)})}{2}, \\ \frac{D_l(z, [Az]_{\gamma(z)})[1 + \delta_l(l_r, l_{r+1})]}{1 + \delta_l(l_r, z)}, \\ \frac{\delta_l(z, l_{r+1})[1 + D_l(l_r, [Az]_{\gamma(z)})]}{1 + \delta_l(l_r, z)}\end{array}\right\}\right) \\ &\quad + M \min\left\{\begin{array}{c}\delta_l(l_r, l_{r+1}), D_l(z, [Az]_{\gamma(z)}), \\ D_l(l_r, [Az]_{\gamma(z)}), \delta_l(z, l_{r+1})\end{array}\right\}.\end{aligned}$$

When $r \to \infty$ in the above inequality, we have

$$D_l(z, [Az]_{\gamma(z)}) \leq \psi(D_l(z, [Az]_{\gamma(z)})).$$

Suppose that $D_l(z, [Az]_{\alpha(z)}) \neq 0$. As $\psi(t) < t$ for $t > 0$, so

$$D_l(z, [Az]_{\gamma(z)}) \leq \psi(D_l(z, [Az]_{\gamma(z)})) < D_l(z, [Az]_{\gamma(z)}),$$

which is a contradiction. Hence $D_l(z, [Az]_{\gamma(z)}) = 0$. So, we get $z \in [Az]_{\gamma(z)}$; that is, z is a fixed point of A. □

Remark 1. *In the above Theorem 1, δ_l Hausdorff metric [23] is used for nonempty set $[Ax]_{\gamma(x)}$. In fact fuzzy mappings comes as a generalization of single valued mapping $T : X \to X$. Here Tx must be a point (element) of $X\dot{I}$f for some $x \in X$, Tx is undefined then we say that T is not a mapping on $X\dot{T}$o pursue this definition we assume that $[Ax]_{\gamma(x)}$ is non empty for using δ_l Hausdorff metric H_{δ_l} on nonempty sets $[Ax]_{\gamma(x)}$. Indeed, the validity of the assumption of inequality 1 of Theorem 1 and the validity of H_{δ_l} for family of nonempty subsets of X make the set $[Ax]_{\gamma(x)}$ nonempty, see [4,6,23].*

Example 2. *Let* $S = \mathbb{Q}^+ \cup \{0\}$ *and* $\delta_l(l_1, l_2) = l_1 + l_2$*, whenever* $l_1, l_2 \in S$*, then* (S, δ_l) *is a complete dislocated metric space. Define a fuzzy mapping* $A : S \to F_l(S)$ *by*

$$A(l)(s) = \begin{cases} 1 & 0 \leq s \leq l/6 \\ 1/3 & l/6 < s \leq l/4 \\ 1/6 & l/4 < s \leq l/2 \\ 0 & l/2 < s \leq 1 \end{cases}$$

For all $l \in S$*, there exists* $\gamma(l) = \dfrac{1}{3}$*, such that*

$$[Al]_{\gamma(l)} = \left[0, \frac{l}{4}\right].$$

Consider $l_0 = 2$ *and* $\sigma = 5$*, then* $\overline{B_{\delta_l}(l_0, \sigma)} = [0, 3]$*. Also,* $\psi : \mathbb{R}^+ \cup \{0\} \to [0, \infty)$ *defined by*

$$\psi(k) = pk \quad with \quad 0 < p < 1.$$

Let $M \geq 0$ *be any real number. Then,*

$$\begin{aligned} H_{\delta_l}([Al_1]_{\gamma(l)}, [Al_2]_{\gamma(l_2)}) &\leq \psi\left(\max\left\{\begin{array}{c} \delta_l(l_1, l_2), D_l(l_1, [Al_1]_{\gamma(l_1)}), D_l(l_2, [Al_2]_{\gamma(l_2)}), \\ \dfrac{D_l(l_2, [Al_1]_{\gamma(l)}) + D_l(l_1, [Al_2]_{\gamma(l_2)})}{2}, \\ \dfrac{D_l(l_2, [Al_2]_{\gamma(l_2)})[1 + D_l(l_1, [Al_1]_{\gamma(l)})]}{1 + \delta_l(l_1, l_2)}, \\ \dfrac{D_l(l_2, [Al_1]_{\gamma(l)})[1 + D_l(l_1, [Al_2]_{\gamma(l_2)})]}{1 + \delta_l(l_1, l_2)} \end{array}\right\}\right) \\ &\quad + M \min\left\{\begin{array}{c} D_l(l_1, [Al_1]_{\gamma(l)}), D_l(l_2, [Al_2]_{\gamma(l_2)}), \\ D_l(l_1, [Al_2]_{\gamma(l_2)}), D_l(l_2, [Al_1]_{\gamma(l)}) \end{array}\right\}. \\ \frac{l_1}{16} &\leq \psi\left(\max\left\{(l_1 + l_2), l_1, l_2, \frac{l_1 + l_2}{2}, \frac{l_2(1 + l_1)}{1 + l_1 + l_2}\right\}\right) \\ &\quad + M \min\{l_1, l_2\}. \end{aligned}$$

This satisfies the conditions of Theorem 1. So, we get $0 \in \overline{B_{\delta_l}(l_0, \sigma)}$ *is a fuzzy fixed point of A. If we have* $\psi(k) = pk$*, where* $0 < p < 1$*, in Theorem 1, we have the following result.*

Corollary 1. *Let* (S, δ_l) *be a complete* δ_l *metric space with* $A : S \to F_l(S)$ *be a fuzzy mapping. Assume that* l_0 *be any point in S,* $\gamma : S \to (0, 1]$ *be a mapping and there exists a real number* $M \geq 0$ *satisfying the following:*

$$\begin{aligned} H_{\delta_l}([Al_1]_{\gamma(l_1)}, [Al_2]_{\gamma(l_2)}) &\leq p\left(\max\left\{\begin{array}{c} \delta_l(l_1, l_2), D_l(l_1, [Al_1]_{\gamma(l_1)}), D_l(l_2, [Al_2]_{\gamma(l_2)}), \\ \dfrac{D_l(l_2, [Al_1]_{\gamma(l_1)}) + D_l(l_1, [Al_2]_{\gamma(l_2)})}{2}, \\ \dfrac{D_l(l_2, [Al_2]_{\gamma(l_2)})[1 + D_l(l_1, [Al_1]_{\gamma(l_1)})]}{1 + \delta_l(l_1, l_2)}, \\ \dfrac{D_l(l_2, [Al_1]_{\gamma(l_1)})[1 + D_l(l_1, [Al_2]_{\gamma(l_2)})]}{1 + \delta_l(l_1, l_2)} \end{array}\right\}\right) \\ &\quad + M \min\left\{\begin{array}{c} D_l(l_1, [Al_1]_{\gamma(l_1)}), D_l(l_2, [Al_2]_{\gamma(l_2)}), \\ D_l(l_1, [Al_2]_{\gamma(l_2)}), D_l(l_2, [Al_1]_{\gamma(l_1)}) \end{array}\right\}. \end{aligned}$$

for all $l_1, l_2 \in \overline{B_{\delta_l}(l_0, \sigma)}$*,* $\sigma > 0$ *and*

$$\sum_{i=0}^{n} \psi^i\left(D_l(l_0, [Al_0]_{\gamma(l_0)})\right) \leq \sigma \quad \text{for } n \in \mathbb{N}.$$

Then, there exists z in $\overline{B_{\delta_l}(l_0, \sigma)}$ *such that* $z \in [Az]_{\gamma(z)}$*.*

If we have $M = 0$ and $\psi(k) = pk$, where $0 < p < 1$, in Theorem 1, we have the following result.

Corollary 2. *Let (S, δ_l) be a complete δ_l metric space with $A : S \to F_l(S)$ be a fuzzy mapping. Assume that l_0 be any point in S, $\gamma : S \to (0, 1]$ be a mapping and there exists a real number $M \geq 0$ satisfying the following:*

$$H_{\delta_l}([Al_1]_{\gamma(l_1)}, [Al_2]_{\gamma(l_2)}) \leq p\left(\max\left\{\begin{array}{c} \delta_l(l_1, l_2), D_l(l_1, [Al_1]_{\gamma(l_1)}), D_l(l_2, [Al_2]_{\gamma(l_2)}), \\ \dfrac{D_l(l_2, [Al_1]_{\gamma(l_1)}) + D_l(l_1, [Al_2]_{\gamma(l_2)})}{2}, \\ \dfrac{D_l(l_2, [Al_2]_{\gamma(l_2)})[1 + D_l(l_1, [Al_1]_{\gamma(l_1)})]}{1 + \delta_l(l_1, l_2)}, \\ \dfrac{D_l(l_2, [Al_1]_{\gamma(l_1)})[1 + D_l(l_1, [Al_2]_{\gamma(l_2)})]}{1 + \delta_l(l_1, l_2)} \end{array}\right\}\right)$$

for all $l_1, l_2 \in \overline{B_{\delta_l}(l_0, \sigma)}$, $\sigma > 0$ and

$$\sum_{i=0}^{n} \psi^i\left(D_l(l_0, [Al_0]_{\gamma(l_0)})\right) \leq \sigma \quad \text{for } n \in \mathbb{N}.$$

Then, there exists z in $\overline{B_{\delta_l}(l_0, \sigma)}$ such that $z \in [Az]_{\gamma(z)}$.

4. Stability of Fuzzy Fixed Point α-Level Sets

Theorem 2. *Suppose (S, δ_l) is a complete δ_l metric space, $A_1, A_2 : S \to F_l(S)$ are two fuzzy mappings and $\psi : \mathbb{R}^+ \cup \{0\} \to [0, \infty)$ be a continuous and nondecreasing mapping with $\Psi(s) = \sum_{n=1}^{\infty} \psi^n(s) < \infty$. Also $\Psi(s) \to 0$ as $s \to 0$ and $\psi(s) < s$ for each $s > 0$. Suppose l_0 be any point in S and also there exists a real number $M \geq 0$, with $\gamma(l) \in (0, 1]$ such that, A_j for $j = 1, 2$ satisfies (1) and (2) for all $l \in S$. Then,*

$$H_{\delta_l}(F(A_1), F(A_2)) \leq \Psi(p),$$

where

$$p = \sup_{l \in S} H_{\delta_l}([A_1 l]_{\alpha(l)}, [A_2 l]_{\alpha(l)}).$$

Proof. As by the above Theorem 1, the set of fuzzy fixed point is non-empty. Suppose $l_0 \in F(A_1)$, it means $l_0 \in [A_1 l_0]_{\gamma(l_0)}$, then by Lemma 2 there exists $l_1 \in [A_2 l_0]_{\gamma(l_0)}$ such that

$$\delta_l(l_0, l_1) \leq H_{\delta_l}([A_1 l_0]_{\gamma(l_0)}, [A_2 l_0]_{\gamma(l_0)}). \tag{6}$$

As $l_1 \in [A_2 l_0]_{\gamma(l_0)}$, so by Lemma 2 there exists $l_2 \in [A_2 l_1]_{\gamma(l_1)}$ such that

$$\delta_l(l_1, l_2) \leq H_{\delta_l}([A_2 l_0]_{\gamma(l_0)}, [A_2 l_1]_{\gamma(l_1)}).$$

Following Theorem 1, we have for all $n \in \mathbb{N} \cup \{0\}$

$$\begin{array}{rcl} l_{n+1} & \in & [A_2 l_n]_{\gamma(l_n)} \\ \delta_l(l_{n+1}, l_{n+2}) & \leq & \psi(\delta_l(l_n, l_{n+1})) \end{array}$$

and

$$\delta_l(l_{n+1}, l_{n+2}) \leq \psi(\delta_l(l_n, l_{n+1})) \leq \ldots \leq \psi^{n+1}(\delta_l(l_0, l_1)). \tag{7}$$

Following similar steps as done in the proof of Theorem 1, we can obtain that the sequence $\{l_n\}$ is Cauchy in S and $l_n \to z \in S$. Also, $z \in [A_2 z]_{\gamma(z)}$. Now, by (6) and the definition of p, we have

$$\delta_l(l_0, l_1) \leq H_{\delta_l}([A_1 l_0]_{\gamma(l_0)}, [A_2 l_0]_{\gamma(l_0)}) \leq p = \sup_{l \in S} H_{\delta_l}([A_1 l]_{\gamma(l)}, [A_2 l]_{\gamma(l)}).$$

Now, again by triangular inequality and (7) we have

$$\begin{aligned} \delta_l(l_0, z) &\leq \sum_{i=0}^{n} \delta_l(l_i, l_{i+1}) + \delta_l(l_i, z) \\ &\leq \sum_{i=0}^{n} \psi^i(\delta_l(l_0, l_1)) + \delta_l(l_i, z). \end{aligned}$$

Applying $n \to \infty$, in above inequality and the property of ψ, we get

$$\delta_l(l_0, z) \leq \sum_{i=0}^{\infty} \psi^i(\delta_l(l_0, l_1)) \leq \sum_{i=0}^{\infty} \psi^i(p) = \Psi(p).$$

So, for an arbitrary $l_0 \in F(A_1)$, we have find $z \in F(A_2)$, such that

$$\delta_l(l_0, z) \leq \Psi(p).$$

Similarly, for an arbitrary $w_0 \in F(A_2)$ we can find $u \in F(A_1)$, such that

$$\delta_l(w_0, u) \leq \Psi(p).$$

So, we get

$$H_{\delta_l}(F(A_1), F(A_2)) \leq \Psi(p).$$

□

Theorem 3. *Suppose (S, δ_l) is a complete δ_l metric space and $\{A_j : S \to F_l(S) \text{ for } j \in \mathbb{N}\}$ be a sequence of fuzzy mappings, which is uniformly convergent to a fuzzy mapping $A : S \to F_l(S)$. Suppose l_0 be any point in S and $\gamma(l) \in (0, 1]$. If A_j satisfies (1) and (2) for each $j \in \mathbb{N}$, then A also satisfies (1) and (2).*

Proof. As A_j satisfies (1) and (2) for every $j \in \mathbb{N}$, we have

$$\begin{aligned} H_{\delta_l}([A_j l_1]_{\gamma(l_1)}, [A_j l_2]_{\gamma(l_2)}) &\leq \psi\left(\max\left\{\begin{array}{c} \delta_l(l_1, l_2), D_l(l_1, [A_j l_1]_{\gamma(l_1)}), D_l(l_2, [A_j l_2]_{\gamma(l_2)}), \\ \frac{D_l(l_2, [A_j l_1]_{\gamma(l_1)}) + D_l(l_1, [A_j l_2]_{\gamma(l_2)})}{2}, \\ \frac{D_l(l_2, [A_j l_2]_{\gamma(l_2)})[1 + D_l(l_1, [A_j l_1]_{\gamma(l_1)})]}{1 + \delta_l(l_1, l_2)}, \\ \frac{D_l(l_2, [A_j l_1]_{\gamma(l_1)})[1 + D_l(l_1, [A_j l_2]_{\gamma(l_2)})]}{1 + \delta_l(l_1, l_2)} \end{array}\right\}\right) \\ &\quad + M \min\left\{\begin{array}{c} D_l(l_1, [A_j l_1]_{\gamma(l_1)}), D_l(l_2, [A_j l_2]_{\gamma(l_2)}), \\ D_l(l_1, [A_j l_2]_{\gamma(l_2)}), D_l(l_2, [A_j l_1]_{\gamma(l_1)}) \end{array}\right\}. \end{aligned}$$

and

$$\sum_{i=0}^{n} \psi^i\Big(D_l(l_0, [A_j l_0]_{\gamma(l_0)})\Big) \leq \sigma \quad \text{ for } \sigma \in N.$$

As $\{A_j\}$ is uniformly convergent to A with ψ continuous. By applying limit $j \to \infty$ in the above inequalities, we have

$$H_{\delta_l}([Al_1]_{\gamma(l_1)},[Al_2]_{\gamma(l_2)}) \leq \psi\left(\max\left\{\begin{array}{c} \delta_l(l_1,l_2), D_l(l_1,[Al_1]_{\gamma(l_1)}), D_l(l_2,[Al_2]_{\gamma(l_2)}), \\ \dfrac{D_l(l_2,[Al_1]_{\gamma(l_1)})+D_l(l_1,[Al_2]_{\gamma(l_2)})}{2}, \\ \dfrac{D_l(l_2,[Al_2]_{\gamma(l_2)})[1+D_l(l_1,[Al_1]_{\gamma(l_1)})]}{1+\delta_l(l_1,l_2)}, \\ \dfrac{D_l(l_2,[Al_1]_{\gamma(l_1)})[1+D_l(l_1,[Al_2]_{\gamma(l_2)})]}{1+\delta_l(l_1,l_2)} \end{array}\right\}\right) + M\min\left\{\begin{array}{c} D_l(l_1,[Al_1]_{\gamma(l_1)}), D_l(l_2,[Al_2]_{\gamma(l_2)}), \\ D_l(l_1,[Al_2]_{\gamma(l_2)}), D_l(l_2,[Al_1]_{\gamma(l_1)}) \end{array}\right\}.$$

and

$$\sum_{i=0}^{n}\psi^i\Big(D_l(l_0,[Al_0]_{\gamma(l_0)})\Big) \leq \sigma \quad \text{for } \sigma \in N.$$

which implies that A satisfies (1) and (2). □

Theorem 4. *Let (S,δ_l) be a complete δ_l metric space and $\{A_j : S \to F_l(S) \text{ for } j \in \mathbb{N}\}$ be a sequence of fuzzy mappings, which is uniformly convergent to a fuzzy mapping $A : S \to F_l(S)$. Suppose l_0 be any point in S and with $\gamma(l) \in (0,1]$. If A_j satisfies (1) and (2) for each $j \in \mathbb{N}$. Then,*

$$\lim_{j\to\infty} H_{\delta_l}(F(A_j),F(A)) = 0,$$

that is, the sequence of sets $\{F(A_j)\}_{j\in\mathbb{N}}$ containing fuzzy fixed points of $\{A_j\}_{j\in\mathbb{N}}$ are stable.

Proof. By Theorem 3, A satisfies (1) and (2). Suppose $p_j = \sup_{l\in S} H_{\delta_l}([A_jl]_{\gamma(l)},[Al]_{\gamma(l)}))$. As $\{A_j\} \to A$ on S, so

$$\lim_{j\to\infty} p_j = \lim_{j\to\infty} H_{\delta_l}([A_jl]_{\gamma(l)},[Al]_{\gamma(l)})) = 0.$$

By applying Theorem 2, we have

$$H_{\delta_l}(F(A_j),F(A)) \leq \Psi(p_j) \text{ for each } j \in \mathbb{N}.$$

As $\Psi(s) \to 0$ as $s \to 0$ and ψ is continuous, we get

$$\lim_{j\to\infty} H_{\delta_l}(F(A_j),F(A)) \leq \lim_{j\to\infty}\Psi(p_j) = 0,$$

that is,

$$\lim_{j\to\infty} H_{\delta_l}(F(A_j),F(A)) = 0.$$

Hence, the sequence of sets $\{F(A_j)\}_{j\in\mathbb{N}}$ containing fuzzy fixed points of $\{A_j\}_{j\in\mathbb{N}}$ are stable. □

5. Application

Now, we indicate that by using Theorem 1, we can derive a fixed point for a multivalued mapping in a complete δ_l metric space.

Theorem 5. *Let (Z, δ_l) be a complete δ_l metric space and $S : Z \to CB(Z)$ be a set-valued mapping. Suppose $\psi : \mathbb{R}^+ \cup \{0\} \to [0, \infty)$ is a continuous and nondecreasing function with $\sum_{n=1}^{\infty} \psi^n(s) < \infty$ and $\psi(r) < r$ for each $r > 0$. Suppose for a real number $M \geq 0$, satisfying the following:*

$$H_{\delta_l}(Sl_1, Sl_2) \leq \psi\left(\max\left\{\begin{array}{c} \delta_l(l_1, l_2), D_l(l_1, Sl_1), D_l(l_2, Sl_2), \\ \frac{D_l(l_2, Sl_1) + D_l(l_1, Sl_2)}{2}, \\ \frac{D_l(l_2, Sl_2)[1 + D_l(l_1, Sl_1)]}{1 + \delta_l(l_1, l_2)}, \\ \frac{D_l(l_2, Sl_1)[1 + D(l_1, Sl_2)]}{1 + \delta_l(l_1, l_2)} \end{array}\right\}\right) + M \min\{D_l(l_1, Sl_1), D_l(l_2, Sl_2), D_l(l_1, Sl_2), D_l(l_2, Sl_1)\}. \tag{8}$$

for all $l_1, l_2 \in \overline{B_{\delta_l}(l_0, \sigma)}$, $\sigma > 0$ and

$$\sum_{i=0}^{n} \psi^i(D_l(l_0, Sl_0)) \leq \sigma \quad \text{for } n \in \mathbb{N}. \tag{9}$$

Then, there exists w in $\overline{B_{\delta_l}(l_0, \sigma)}$ such that $w \in Sw$.

Proof. Let $\theta : Z \to (0, 1]$ be any mapping. If we consider a fuzzy mapping $A : Z \to F_l(Z)$ as

$$A(l)(p) = \begin{cases} \theta(l), & p \in Sl \\ 0, & p \notin Sl. \end{cases}$$

So, we get

$$[Al]_{\theta(l)} = \{p : A(l)(p) \geq \theta = Sl\}.$$

In this way the (8) and (9) becomes the (1) and (2) of Theorem 1. So, we get $w \in Z$ such that $w \in [Aw]_{\theta(w)} = Sw$. □

Now, we present our result for single-valued mappings.

Theorem 6. *Let (Z, δ_l) be a complete δ_l metric space and $S : Z \to Z$ be a single-valued mapping. Suppose $\psi : \mathbb{R}^+ \cup \{0\} \to [0, \infty)$ is a continuous and nondecreasing function with $\sum_{n=1}^{\infty} \psi^n(s) < \infty$ and $\psi(r) < r$ for each $r > 0$. Suppose for a real number $M \geq 0$, satisfying the following:*

$$\delta_l(Sl_1, Sl_2) \leq \psi\left(\max\left\{\begin{array}{c} \delta_l(l_1, l_2), \delta_l(l_1, Sl_1), \delta_l(l_2, Sl_2), \\ \frac{\delta_l(l_2, Sl_1) + \delta_l(l_1, Sl_2)}{2}, \\ \frac{\delta_l(l_2, Sl_2)[1 + \delta_l(l_1, Sl_1)]}{1 + \delta_l(l_1, l_2)}, \\ \frac{\delta_l(l_2, Sl_1)[1 + \delta_l(l_1, Sl_2)]}{1 + \delta_l(l_1, l_2)} \end{array}\right\}\right) + M \min\{\delta_l(l_1, Sl_1), \delta_l(l_2, Sl_2), \delta_l(l_1, Sl_2), \delta_l(l_2, Sl_1)\}.$$

for all $l_1, l_2 \in \overline{B_{\delta_l}(l_0, \sigma)}$, $\sigma > 0$ and

$$\sum_{i=0}^{n} \psi^i(\delta_l(l_0, Sl_0)) \leq \sigma \quad \text{for } n \in \mathbb{N}.$$

Then, there exists w in $\overline{B_{\delta_l}(l_0, \sigma)}$ such that $w = Sw$.

Now, we present the results for sequence of set-valued mappings.

Theorem 7. *Suppose* (Z, δ_l) *is a complete* δ_l *metric space and* $\{T_j : Z \to CB(Z)$ *for* $j \in \mathbb{N}\}$ *be a sequence of set-valued mappings, which is uniformly convergent to a set-valued mapping* $T : Z \to CB(Z)$*. Suppose* T_j *satisfies* (8) *and* (9) *for each* $j \in \mathbb{N}$*, then* T *also satisfies* (8) *and* (9)*.*

Theorem 8. *Suppose* (Z, δ_l) *is a complete* δ_l *metric space and* $\{T_j : Z \to CB(Z)$ *for* $j \in \mathbb{N}\}$ *be a sequence of set-valued mappings, which is uniformly convergent to a set-valued mapping* $T : Z \to CB(Z)$*. Suppose* T_j *satisfies* (8) *and* (9) *for each* $j \in \mathbb{N}$*. Then,*

$$\lim_{j\to\infty} H_{\delta_l}(F(T_j), F(T)) = 0,$$

that is, the sequence of sets $\{F(T_j)\}_{j\in\mathbb{N}}$ *containing fixed points of* $\{T_j\}_{j\in\mathbb{N}}$ *are stable.*

6. Conclusions

In this article we established some fuzzy fixed point results in a closed ball for fuzzy mappings satisfying rational type almost contractions in a complete dislocated metric spaces. We also study about stability of fuzzy fixed point γ-level sets. We also obtained fixed point results for set-valued mappings. Hausdorff distance is used and an example is presented to support these results. The proposed operators can be extended to Fermatean fuzzy sets see [24,25].

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References

1. Zadeh, L.A. Fuzzy Sets. *Inf. Control.* **1965**, *8*, 338–353. [CrossRef]
2. Weiss, M.D. Fixed points and induced fuzzy topologies for fuzzy sets. *J. Math. Anal. Appl.* **1975**, *50*, 142–150. [CrossRef]
3. Butnariu, D. Fixed points for fuzzy mappings. *Fuzzy Sets Syst.* **1982**, *7*, 191–207. [CrossRef]
4. Heilpern, S. Fuzzy mappings and fixed point theorem. *J. Math. Anal. Appl.* **1981**, *83*, 566–569. [CrossRef]
5. Nadler, B.S. Multivalued contraction mappings. *Pacific J. Math.* **1969**, *30*, 475–488. [CrossRef]
6. Azam, A. Fuzzy Fixed Points Of Fuzzy Mappings via A Rational Inequality. *Hacet. J. Math. Stat.* **2011**, *40*, 421–431.
7. Rashid, M.; Shahzad, A.; Azam, A. Fixed point theorems for L-fuzzy mappings in quasi-pseudo metric spaces. *J. Intell. Fuzzy Syst.* **2017**, *32*, 499–507. [CrossRef]
8. Robinson, C. *Dynamical Systems, Stability, Symbolic Dynamics, and Chaos*, 2nd ed.; CRC Press: Boca Raton, FL, USA, 1998.
9. Strogatz, S. *Nonlinear Dynamics and Chaos, with Applications to Physics, Biology, Chemistry, and Engineering*; Westview Press: Boulder, CO, USA, 2001.
10. Bose, R.K.; Mukherjee, R.N. Stability of fixed point sets and common fixed points of families of mappings. *Indian J. Pure Appl. Math.* **1980**, *9*, 1130–1138.
11. Brzdek, J.; Cadariu, L.; Cieplinski, K. Fixed Point Theory and the Ulam Stability. *J. Funct. Spaces* **2014**, *2014*, 829419. [CrossRef]
12. Brzdek, J.; Cieplinski, K. A fixed point approach to the stability of functional equations in non-Archimedean metric spaces. *Nonlinear Anal.* **2011**, *74*, 6861–6867. [CrossRef]
13. Brzdek, J. Banach limit, fixed points and Ulam stability. *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat.* **2022**, *116*, 79. [CrossRef]
14. Choudhury, B.S.; Metiya, N.; Bandyopadhyay, C. Fixed points of multivalued α-admissible mappings and stability of fixed point sets in metric spaces. *Rend. Circ. Mat. Palermo* **2015**, *64*, 43–55. [CrossRef]
15. Choudhury, B.S.; Metiya, N.; Som, T.; Bandyopadhyay, C. Multivalued Fixed Point Results and Stability of Fixed Point Sets in Metric Spaces. *Facta Univ. Ser. Math. Inform.* **2015**, *30*, 501–512.
16. El-Hady, E.-S.; El-Fassi, I.-I. Stability of the Equation of q-Wright Affine Functions in Non-Archimedean (n, β) -Banach Spaces. *Symmetry* **2022**, *14*, 633. [CrossRef]

17. Markin, J.T. A fixed point stability theorem for nonexpansive set valued mappings. *J. Math. Anal. Appl.* **1976**, *54*, 441–443. [CrossRef]
18. Shen, M.; Hong, S. Common fixed points for generalized contractive multivalued operators in complete metric spaces. *Appl. Math. Lett.* **2009**, *22*, 1864–1869. [CrossRef]
19. Hitzler, P.; Seda, A.K. Dislocated Topologies. *J. Electr. Eng.* **2000**, *51*, 3–7.
20. Alansari, M.; Shagari, M.S.; Azam, A. Fuzzy fixed point theorems and Ulam-Hyers stability of fuzzy set-valued maps. *Math. Slovaca* **2022**, *72*, 459–482. [CrossRef]
21. Hussain, N.; Arshad, M.; Shoaib, A.; Fahimuddin. Common Fixed Point results for α-ψ-contractions on a metric space endowed with graph. *J. Inequal. Appl.* **2014**, *2014*, 136. [CrossRef]
22. Shoaib, A.; Kumam, P.; Shahzad, A.; Phiangsungnoen, S.; Mahmood, Q. Fixed point results for fuzzy mappings in a b-metric space. *Fixed Point Theory Appl.* **2018**, *2018*, 1–12. [CrossRef]
23. Shahzad, A.; Rasham, T.; Marino, G.; Shoaib, A. On fixed point result for $\alpha_* - \psi$ Dominated Fuzzy Contractive Mapping with Graph. *J. Intell. Fuzzy Syst.* **2020**, *38*, 3093–31033. [CrossRef]
24. Deng, Z.; Wang, J. New distance measure for Fermatean fuzzy sets and its applications. *Int. J. Intell. Syst.* **2021**, *37*, 1903–1930. [CrossRef]
25. Jeevaraj, S. Ordering of interval-valued Fermatean fuzzy sets and its applications. *Expetr Syst. Appl.* **2021**, *185*, 115613.