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Mild Solution for the Time-Fractional Navier–Stokes Equation Incorporating MHD Effects

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Abstract: The Navier–Stokes (NS) equations involving MHD effects with time-fractional derivatives are discussed in this paper. This paper investigates the local and global existence and uniqueness of the mild solution to the NS equations for the time fractional differential operator. In addition, we work on the regularity effects of such types of equations which are caused by MHD flow.

Keywords: Navier–Stokes equations; mild solution; existence and uniqueness; Caputo fractional derivative; Mittag–Leffler functions; regularity

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1. Introduction

Applied Mathematics is a sub-branch of fractional calculus with ordinary derivatives and integrals of arbitrary orders. It has become increasingly popular thanks to demonstrated applications in science [1–3]. These types of equations are widely used in fluid flow [4], diffusion, anomalous diffusion [5], transmutation of distribution [6], turbulence, rheology, and many other physical processes. To explain the existence and uniqueness of boundary conditions, we consider the entire summary of mathematics.

Electromagnetic influencers or Magnetohydrodynamics (MHD) deal with the electronic conduction of conductive liquids in a magnetic field. A magnetic field carries currents in a moving liquid. A current passing through a a carrier can create forces on the liquid and affect the magnetic flux. Similar to electrokinetics, the effects of MHD represent multiple physics problems, which require the different domains to be connected. The effects of MHD can be explained by the NS equations of mobile dynamics and Maxwell's equations of Electromagnetism [7].

The full form of MHD is Magnetohydrodynamics. MHD is an analysis of the characteristics and magnetic properties of electroconductive fluids. Liquefied metals, plasma, salt water, and electrolytes all involve magnetic–liquid properties.

The term Magnetohydrodynamics is derived from magneto, meaning a magnetic field, hydro, meaning water, and dynamics, meaning fluctuation or flux. Hannes Alfvén, a Swedish electrical engineer, inaugurated the field of MHD, receiving the Nobel Prize in Physics because of his work on MHD. The basic concept of MHD involves magnetic fields that can produce currents in movable conductive liquids, which successively generate forces on the fluids and convert the entire field. Magnetohydrodynamics is described by a set of equations that are a combination of the NS equations of fluid dynamics and Maxwell's equations for electromagnetism. These differential equations (DE) must be resolved at the same time, either analytically or numerically. Abbas et al. [8] solved ordinary differential equations. Shafqat et al. [9], Alnahdi et al. [10], and Abuasbeh et al. [11,12] investigated the existence and uniqueness of the fuzzy fractional evolution equations.

Euler's original equation is as follows:

$$\rho \frac{\partial w}{\partial \varsigma} + (w.\nabla)w = -\nabla P, \quad (1)$$

where w is the fluid velocity vector, P is the fluid pressure, ρ is the fluid density, and ∇ indicates the gradient differential operator.

The Navier–Stokes equation of Magnetohydrodynamic flow in modern notation is

$$\rho\left(\frac{\partial w}{\partial \varsigma} + (w.\nabla)w\right) = -\nabla P + \mu\nabla^2 w - \sigma {B_0}^2 v, \quad (2)$$

where w is the velocity vector, P is the fluid pressure, ρ is the fluid density, σ is the electrical conductivity, μ is the dynamic viscosity, and ∇^2 is the Laplacian operator.

The Magnetohydrodynamic (MHD) Effect is a physical phenomenon that explains the motion of a conductive fluid flowing under the impact of an exterior magnetic field.

The Cauchy problem for solving the incompressible NS equation incorporating MHD effects is provided by

$$\begin{cases} \partial_\varsigma^\gamma v - w\triangle v + (v.\nabla)v = -\nabla p + (-\sigma {B_0}^2 \frac{v}{\rho}), & \varsigma > 0, \\ \nabla.v = 0, \\ v|_{\partial\Omega} = 0, \\ v(0,x) = a, \end{cases} \quad (3)$$

where $\partial_\varsigma^\gamma$ denotes the fractional order Caputo derivative at $x \in \Omega$, where Ω is the smooth boundary and time $\varsigma > 0$, $v = (v_1(\varsigma,x), v_2(\varsigma,x), \ldots, v_n(\varsigma,x))$ shows the velocity field, the pressure is $\rho = \rho(\varsigma,x)$, σ is the electrical conductivity, and B_0 is the magnetic field strength. Thus, MHD is the body force and the initial velocity is defined by a [13].

First, by applying the Helmholtz–Leray projector P to Equation (3), we can remove the pressure term, which converts Equation (3) to

$$\begin{cases} \partial_\varsigma^\gamma v - wP\triangle v + P(v.\nabla)v = \left(-P\sigma {B_0}^2 \frac{v}{\rho}\right), & \varsigma > 0, \\ \nabla.v = 0, \\ v|_{\partial\Omega} = 0, \\ v(0,x) = a. \end{cases} \quad (4)$$

The term $-wP\triangle$, having Dirichlet boundary conditions, refers to the Stokes operator A, which is evaluated in divergence-free function space. Thus, the abstract form of Equation (3) is

$$\begin{cases} {}^cD_\varsigma^\gamma v(\varsigma) = -Av + F(v,w) - P\sigma {B_0}^2 \frac{v}{\rho}, & \varsigma > 0, \\ v(0) = a, \end{cases} \quad (5)$$

whereas $(v,w) = -P(v.\nabla)w$. If someone making sense to the Helmholtz–Leray projector P and Stokes operator A are sensible, then the result of Equation (5) is the result of Equation (2). The main purpose of this paper is to demonstrate the existence and uniqueness of local and global mild solutions to problem (5) in $H^{\gamma,r}$.

Additionally, we determine the regularity outcomes, which express significantly that if $\sigma {B_0}^2 \frac{v}{\rho}(\varsigma)$ is Hölder continuous, at that point $v(\varsigma)$ is a unique classical solution in order for Av and ${}^cD_\varsigma^\gamma v(\varsigma)$ to be Hölder continuous in J_r.

The basic idea behind the MHD is that magnetic fields in a movable conductive fluid can initiate currents, which results in the liquid being polarized and changes the magnetic field by itself. A combination of the NS equations of fluid dynamics and Maxwell's equations for electromagnetism provide the mathematical explanation of MHD. There have been several productive studies related to MHD effects and fluid dynamics [4,14–18].

2. Preliminaries

In this section, we define the Gamma function, fractional order integral, Riemann-Liouville fractional derivative, Caputo fractional derivative, and additional definitions, lemmas, and theorems. For a brief review of fractional calculus definitions and properties, see [19,20].

Let the half space in $\mathbb{R}^n$ as $\Omega = \mathcal{H} = (x_1, \ldots, x_n) : x_n > 0$ be the open subset of $\mathbb{R}^n$, whereas $n \geq 3$. Let $1 < r < \infty$. Then, we have the Hödge-projection, which is a bounded projection P on $(L^r(\Omega)^n)$, of which the range is the conclusion of:=

$$C_\sigma^\infty(\mathcal{H}) = \Big(v \in (C^\infty(\mathcal{H}))^n : \nabla.v = 0\Big), \tag{6}$$

to which null space is the conclusion of

$$v \in (C^\infty(\mathcal{H}))^n : v = \nabla.\phi, \;\; \phi \in C^\infty(\mathcal{H}). \tag{7}$$

For a suitable approach, let $J_r = \overline{C_\sigma^\infty(\mathcal{H})}^{|.|_r}$, which is a closed subspace of $(L^r(\mathcal{H}))^n$, with $A = -vP\Delta$ the Stokes operator in the J_r-containing domain $D_r(A) = D_r(\Delta) \cap J_r$. Stokes, an Irish-English physicist and mathematician, defined the unbounded linear operator, named the Stokes operator, which is used in the theory of partial differential equations and specifically in the fields of fluid dynamics and electromagnetics.

$$D_r(\Delta) = v \in (W^{2,r}(\mathcal{H})^n) : v|_{\partial\mathcal{H}} = 0.$$

Now, we have to introduce the definitions of fractional power spaces that are related to $-A$. For $\gamma > 0$ and $v \in J_r$, we define

$$A^{-\gamma}v = \frac{1}{\Gamma(\gamma)} \int_0^\infty \varsigma^{\gamma-1} e^{-\varsigma A} u d\varsigma.$$

Therefore, $A^{-\gamma}$ is bounded [21], just as the injective operator on J_r. Suppose $A^{-\gamma}$ is the inverse of $A^{-\gamma}$; for $\gamma > 0$, we symbolize the space $H^{\gamma,r}$ by the extent of $A^{-\gamma}$ with the following norm:

$$|v|_{H^{\gamma,r}} = |A^\gamma v|_r.$$

Here, we consider K, L, M, and N as four Banach spaces with norms $|\ |_K, |\ |_L, |\ |_M$, and $|\ |_N$. All these spaces are continuously inserted in common topological vector space; here, $e^{\varsigma A}$ denotes semigroup C_0 on X, with the following properties.

$SG_1{}^*$: for each $\varsigma > 0$, $e^{\varsigma A}$ is a bounded map $K \to L$. For certain $\alpha > 0$, there are positive constants C^* and T^* such that

$$|e^{\varsigma A} f|_L \leq C^* \varsigma^{-\alpha} |f|_K \forall \;\; f \in K \;\;\; and \;\;\; \varsigma \in (0, \Im].$$

$SG_2{}^*$: for each $\varsigma > 0$, $e^{\varsigma A}$ extends to a bounded map $L \to M$. For certain $\beta > 0$, there are positive constants C^* and T^* such that

$$|e^{\varsigma A} f|_M \leq C^* \varsigma^{-\beta} |f|_L \forall \;\; f \in L \;\;\; and \;\;\; \varsigma \in (0, \Im].$$

Moreover, $\varsigma \to e^{\varsigma A} f$ is continuous into M for $\varsigma > 0$ and $\lim_{\varsigma \to 0} \varsigma^\beta |e^{\varsigma A} f|_M = 0 \; \forall f \in L$.

$SG_3{}^*$: for each $\varsigma > 0$, $e^{\varsigma A}$ extends to a bounded map $L \to N$. For certain $\gamma > 0$, there are positive constants C^* and $\Im^*$ such that

$$|e^{\varsigma A} f|_N \leq C^* \varsigma^{-\gamma} |f|_L$$

$\forall\ f \in L$ and $\varsigma \in (0,T]$.
Besides, $\varsigma \to e^{\varsigma A} f$ is continuous into N for $\varsigma > 0$ and $\lim_{\varsigma\to 0} \varsigma^{\gamma}|e^{\varsigma A} f|_N = 0\ \forall f \in L$.

Definition 1. *The fractional integration of order $\gamma > 0$ for a function f is defined as*

$$I_0^{\gamma} f(\varsigma) = \frac{1}{\Gamma(\gamma)} \int_0^{\varsigma} (\varsigma - s)^{\gamma-1} f(s) ds, \ \ \varsigma > 0.$$

The Riemann-Liouville (RL) [22] fractional derivative for a function $v : [0,\infty) \to \mathbb{R}$ of order $\gamma \in \mathbb{R}$ is defined by

$${}_0^L D_{\varsigma}^{\gamma} v(\varsigma) = \frac{d^n}{d\varsigma^n}(g_{n-\gamma} * v)\varsigma, \quad \varsigma \geq 0, \quad n-1 < \gamma < n.$$

The RL fractional order integral is defined as

$$J_{\varsigma}^{\gamma} v(\varsigma) := g_{\gamma} * v(\varsigma) = \frac{1}{\Gamma(\gamma)} \int_0^{\varsigma} (\varsigma - s)^{\gamma-1} v(s) ds, \ \ \varsigma \in [0, \Im].$$

Thus, by derivation from the definitions of the RL fractional integral, we can construct the Caputo fractional order differential operator.

Definition 2 ([22])**.** *The Caputo fractional order derivative is defined as follows:*

$${}_0^c D_{\varsigma}^{\gamma} v(\varsigma) = \frac{d}{d\varsigma}\left(J_{\varsigma}^{1-\gamma}[v(\varsigma) - v(0)] \right) = \frac{d}{d\varsigma}\left(\frac{1}{\Gamma(1-\gamma)} \int_0^{\varsigma} (\varsigma - s)^{-\gamma}[v(s) - v(0)] ds \right), \ \ \varsigma > 0.$$

Definition 3 ([23])**.** *The Mittag-Leffler function was introduced by the Swedish mathematician Magnus Gustaf (Gösta) Mittag-Leffler in 1902. It is a simple conclusion of the exponential function. Recently, researchers have been attracted to the study of the Mittag-Leffler function because of its use in the analysis of fractional differential equations (FDE). It occurs often in the solutions of FDE and fractional integral equations. The Mittag-Leffler function with one parameter $E_{\gamma}(\varsigma)$ is defined as follows:*

$$E_{\gamma}(\varsigma) = \sum_{k=0}^{\infty} \frac{\varsigma^k}{\Gamma(\gamma k + 1)}, \ \varsigma \in \mathbb{C}, \ \ \Re(\gamma) > 0.$$

Now, let us consider the generalized Mittag-Leffler functions

$$E_{\gamma}(-\varsigma^{\gamma} A) = \int_0^{\infty} M_{\gamma}(s) e^{-s\varsigma^{\gamma} A} ds,$$

and

$$E_{\gamma,\gamma}(-\varsigma^{\gamma} A) = \int_0^{\infty} \gamma s M_{\gamma}(s) e^{-s\varsigma^{\gamma} A} ds,$$

where

$$M_{\gamma}(\varsigma) := \sum_{n=0}^{\infty} \frac{-\varsigma^n}{n!(\Gamma)[-\gamma(n) + (1-\gamma)]}.$$

The function M_{γ} is known as the Mainardi function. To distinguish between the fundamental solutions for certain standard boundary value problems, Mainardi introduced a type of functions which are a special type of Wright-type functions. The Mainardi function is impressively adept at playing the role of a bridge between classical abstract theories and fractional theories.

Proposition 1. (i) $\quad E_{\gamma,\gamma}(-\varsigma^{\gamma} A) = \frac{1}{2\pi\imath} \int_{\Gamma_{\theta}} E_{\gamma,\gamma}(-\mu\varsigma^{\gamma})(\mu I + A)^{-1} d\mu;$

(ii) $\quad A^{\gamma} E_{\gamma,\gamma}(-\varsigma^{\gamma} A) = \frac{1}{2\pi\imath} \int_{\Gamma_{\theta}} \mu^{\gamma} E_{\gamma,\gamma}(-\mu\varsigma^{\gamma})(\mu I + A)^{-1} d\mu.$

Proof. See results [24]. □

Proposition 2. *Let $\gamma \in (0,1)$ and $-1 < r < \infty$, $\lambda > 0$; then, the Mainardi function possesses the following properties:*

(i) $M_\gamma(\varsigma) \geq 0$ *for all* $\varsigma \geq 0$;
(ii) $\int_0^\infty \varsigma^r M_\gamma(\varsigma) d\varsigma = \dfrac{\Gamma(r+1)}{\Gamma(\gamma r+1)}$;
(iii) $\mathcal{L}\{\gamma\varsigma M_\gamma(\varsigma)\}(z) = E_{\gamma,\gamma}(-z)$;
(iv) $\mathcal{L}\{M_\gamma(\varsigma)\}(z) = E_\gamma(-z)$;
(v) $\mathcal{L}\{\gamma\varsigma^{-(1+\gamma)} M_\gamma(\varsigma^{-\gamma})\}(\lambda) = e^{-\lambda^\gamma}$.

Proof. The proof of this proposition can be found in [25,26]. □

Lemma 1. *For $\varsigma > 0$, the operators $E_\gamma(-\varsigma^\gamma A)$ and $E_{\gamma,\gamma}(-\varsigma^\gamma A)$ in the uniform operator topology are continuous and well defined from $\mathbf{X}$ to $\mathbf{X}$. Then, continuity is uniform on $[r,\infty)$ for every $r > 0$.*

Lemma 2 ([27])**.** *Let $0 < \gamma < 1$. Then,*

(i) $\forall\, v \in X, lim_{\varsigma\to 0^+} E_\gamma(-\varsigma^\gamma A)v = v$;
(ii) $\forall\, v \in D(A)$ *and* $\varsigma > 0$, ${}^C D_\varsigma^\gamma E_\gamma(-\varsigma^\gamma A)v = -A\, E_\gamma(-\varsigma^\gamma A)v$;
(iii) $\forall\, v \in X$, $E'_\gamma(-\varsigma^\gamma A)v = -\varsigma^{\gamma-1} A E_{\gamma,\gamma}(-\varsigma^\gamma A)v$;
(iv) *for* $\varsigma > 0$, $E_\gamma(-\varsigma^\gamma A)v = I_\varsigma^{1-\gamma}\{(\varsigma^{\gamma-1} E_{\gamma,\gamma}(-\varsigma^\gamma A)u)\}$.

Lemma 3. *Suppose $1 < r < \infty$ and $\gamma_1 \leq \gamma_2$. Then, there exists a constant $C = C(\gamma_1,\gamma_2)$ such that*

$$|e^{-\varsigma A} v|_{H^{\gamma_2,r}} \leq C\varsigma^{-(\gamma_2-\gamma_1)} |v|_{H^{\gamma_1,r}}, \ \text{as}\ \varsigma > 0,\ \ for\ v \in H^{\gamma_1,r}.$$

Moreover, $\lim\limits_{\varsigma\to 0} \varsigma^{(\gamma_2-\gamma_1)} |e^{-\varsigma A} v|_{H^{\gamma_2,r}} = 0$.

Lemma 4. *Suppose $1 < r < \infty$ and $\gamma_1 \leq \gamma_2$. For any $\Im > 0$, there exists a constant $C_1 = C_1(\gamma_1,\gamma_2)$ such that*

$$|E_\gamma(-\varsigma^\gamma A)|_{H^{\gamma_2,r}} \leq C_1 \varsigma^{-\alpha(\gamma_2-\gamma_1)} |v|_{H^{\gamma_1,r}};$$

and

$$|E_{\gamma,\gamma}(-\varsigma^\gamma A)|_{H^{\gamma_2,r}} \leq C_1 \varsigma^{-\gamma(\gamma_2-\gamma_1)} |v|_{H^{\gamma_1,r}},$$

for all $v \in H^{\gamma_1,r}$ and $\varsigma \in (0,\Im]$. Therefore,

$$\lim_{\varsigma\to 0} \varsigma^{\alpha(\gamma_2-\gamma_1)} ||E_\gamma(-\varsigma^\gamma A)v|_{H^{\gamma_2,r}} = 0.$$

Proof. The proof of this lemma can be found in [24]. □

Theorem 1. *If $f(\varsigma)$ defined on the interval $[c,d]$ is Riemann-integrable, then $|f(\varsigma)|$ is Riemann-integrable defined by the interval $[c,d]$, and*

$$\left| \int_c^d f(\varsigma) d\varsigma \right| \leq \int_c^d |f(\varsigma)| d\varsigma.$$

Theorem 2. *Suppose that $f : [a,b] \to \mathbb{R}$ is continuous and $g : I \to \mathbb{R}$ is continuously differentiable with image $g(I) \subset [a,b]$, where $I \subset \mathbb{R}$ is some open interval showing that the function*

$$F(s) = -\int_a^{g(s)} f(\varsigma) d\varsigma.$$

is continuously differentiable on I.

Theorem 3 (Theorem 1.17 of [28]). *Let $\Im(\varsigma) : \varsigma \geq 0 \subset \mathcal{X}$ be a C_0 semigroup on X. Then,*

(i) *The infinitesimal generator of $\Im(\varsigma) : \varsigma \geq 0$; if $C : D(G) \subset X \to X$, then G is said to be dense and close and is defined by a linear operator. Therefore, $\varsigma \in [0, \infty) \to \Im(\varsigma)x \in X$ is continuously differentiable for any $x \in D(G)$.*

$$\frac{d}{d\varsigma}\Im(\varsigma)x = G\Im(\varsigma)x = \Im(\varsigma)Gx, \ \ for \ \ \varsigma > 0.$$

(ii) *Then, there exists $\sigma > 0$ such that $Re(\lambda) > 0$, meaning that $\lambda \in \rho(C)$, and we have*

$$(\lambda - IC)^{-1}x = \int_0^\infty e^{-\lambda\varsigma}\Im(\varsigma)\ x\, d\varsigma \ \ for\ all\ x \in X.$$

Theorem 4 ([29], Lemma 9). *Let $\gamma \in (0, 1]$ and suppose that the positive sectorial operator is $A : D(A) \subset X \to X$. Thus, the operators $\{E_\gamma(-\varsigma^\gamma A) : \varsigma \geq 0\}$ and $\{E_{\gamma,\gamma}(-\varsigma^\gamma A) : \varsigma \geq 0\}$ are as follows:*

$$E_\gamma(-\varsigma^\gamma A) = \int_0^\infty M_\gamma(s)\Im^{s\varsigma^\gamma A} ds, \ \ \varsigma \geq 0,$$

and

$$E_{\gamma,\gamma}(-\varsigma^\gamma A) = \int_0^\infty \gamma s M_\gamma(s)\Im^{s\varsigma^\gamma} ds, \ \ \varsigma \geq 0.$$

Whereas $\Im(\varsigma) : \varsigma \geq 0$ defines the C_0 semi-group, which is generated by $-A$.

Proposition 3 ([28]). *Let $\gamma \in (0, 1)$ and consider $A : D(A) \subset X \to X$ to be a positive sectorial operator. Then, for any $x \in X$, it holds that*

$$\begin{aligned} \mathcal{L}\{E_\gamma(-\varsigma^\gamma A)x\}(\lambda) &= \lambda^{\gamma-1}(\lambda^\gamma + A)^{-1}x; \\ \mathcal{L}\{E_{\gamma,\gamma}(-\varsigma^\gamma A)x\}(\lambda) &= (\lambda^\gamma + A)^{-1}x. \end{aligned}$$

Proof. The first equality can be proven analogously, meaning that the second equality is as follows.
For any $x \in X$, we can observe that per Theorem 3,

$$\begin{aligned} \mathcal{L}\{E_{\gamma,\gamma}(-\varsigma^\gamma A)x\}(\lambda) &= \int_0^\infty e^{-\lambda\varsigma}\varsigma^{\gamma-1}E_{\gamma,\gamma}(-\varsigma^\gamma A)x d\varsigma \\ &= \int_0^\infty e^{-\lambda\varsigma}\varsigma^{\gamma-1}\Big(\int_0^\infty \gamma s M_\gamma(s)\Im(s\varsigma^\gamma)x ds\Big)d\varsigma. \end{aligned}$$

Now, using $s = \omega\varsigma^{-\gamma}$, we can conclude that

$$\begin{aligned} \mathcal{L}\{E_{\gamma,\gamma}(-\varsigma^\gamma A)x\}(\lambda) &= \int_0^\infty e^{-\lambda\varsigma}\varsigma^{\gamma-1}\Big(\int_0^\infty \gamma(\omega\varsigma^{-\gamma})M_\gamma(\omega\varsigma^{-\gamma})\Im(\omega)x\varsigma^{-\gamma}d\omega\Big)d\varsigma \\ &= \int_0^\infty \omega\left(\int_0^\infty \gamma\varsigma^{-(1+\gamma)}M_\gamma(\omega\varsigma^{-\gamma})e^{-\lambda\varsigma}d\varsigma\right)\Im(\omega)xd\omega. \end{aligned}$$

Choose

$$H^* = \int_0^\infty \gamma\varsigma^{-(1+\gamma)}M_\gamma(\omega\varsigma^{-\gamma})e^{-\lambda\varsigma}.$$

By taking $\varsigma = \tau\omega^{\frac{1}{\gamma}}$ of Proposition 2, that is,

$$\begin{aligned} H^* &= \int_0^\infty \gamma(\tau\omega^{\frac{1}{\gamma}})^{-(1+\gamma)}M_\gamma(\omega(\tau\omega^{\frac{1}{\gamma}})^{-\gamma})e^{-\lambda(\tau\omega^{\frac{1}{\gamma}})}\omega^{\frac{1}{\gamma}}d\tau \\ &= \omega^{-1}\int_0^\infty \gamma\tau^{-(1+\gamma)}M_\gamma(\tau^{-\gamma})e^{-(\lambda\omega^{\frac{1}{\gamma}})}d\tau \\ &= \omega^{-1}e^{-\lambda^\gamma\omega}. \end{aligned}$$

According to Theorem 4, we have

$$\mathcal{L}\{E_{\gamma,\gamma}(-\varsigma^{\gamma}A)x\}(\lambda) = \int_0^{\infty} e^{-\lambda^{\gamma}\omega}\Im(\omega)xd\omega = (\lambda^{\gamma}+A)^{-1}x.$$

□

Lemma 5. *If*

$$v(\varsigma) = a + \frac{1}{\Gamma(\gamma)}\int_0^{\varsigma}(\varsigma-s)^{\gamma-1}(Av(s)+h(s))ds, \;\; \varsigma \geq 0$$

holds, we have

$$v(\varsigma) = E_{\gamma}(-\varsigma^{\gamma}A)a + \int_0^{\varsigma}(\varsigma-s)^{\gamma-1}E_{\gamma,\gamma}\big(-(\varsigma-s)^{\gamma}A\big)h(s)ds.$$

Proof. Using the above lemma to rewrite the problem in (5), we have

$$\begin{aligned} v(\varsigma) &= v(0) + \frac{1}{\Gamma(\gamma)}\int_0^{\varsigma}(\varsigma-s)^{\gamma-1}(-Av(s)+F(v(s),w(s)) - P\sigma {B_0}^2\frac{v}{\rho}(s))ds, \;\; \varsigma \geq 0, \\ v(\varsigma) &= a + \frac{1}{\Gamma(\gamma)}\int_0^{\varsigma}(\varsigma-s)^{\gamma-1}(-Av(s)+F(v(s),w(s)) - P\sigma {B_0}^2\frac{v}{\rho}(s))ds, \;\; \varsigma \geq 0. \end{aligned}$$

Applying Laplace transformation,

$$v(\lambda) = \frac{a}{\lambda} + \frac{1}{\lambda^{\gamma}}\{-Av(\lambda)\} + \frac{1}{\lambda^{\gamma}}\{Fv(\lambda), w(\lambda)\} + \frac{1}{\lambda^{\gamma}}\{-P\sigma {B_0}^2\frac{v}{\rho}(\lambda)\}.$$

Then, by simplifying,

$$\begin{aligned} (\lambda^{\gamma}+A)v(\lambda) &= a\lambda^{\gamma-1} + F(v(\lambda),w(\lambda)) - P\sigma {B_0}^2\frac{v}{\rho}(\lambda) \\ v(\lambda) &= a\lambda^{\gamma-1}(\lambda^{\gamma}+A)^{-1} + F(v(\lambda),w(\lambda))(\lambda^{\gamma}+A)^{-1} - P\sigma {B_0}^2\frac{v}{\rho}(\lambda)(\lambda^{\gamma}+A)^{-1}, \\ v(\lambda) &= a\lambda^{\gamma-1}(\lambda^{\gamma}+A)^{-1} + F(v(\lambda),w(\lambda))(\lambda^{\gamma}+A)^{-1} - P\sigma {B_0}^2\frac{v}{\rho}(\lambda)(\lambda^{\gamma}+A)^{-1}. \end{aligned}$$

By taking the inverse Laplace transform and applying convolution theorem, we obtain

$$\begin{aligned} v(\varsigma) &= E_{\gamma}(-\varsigma^{\gamma}A)a + \int_0^{\varsigma}(\varsigma-s)^{\gamma-1}E_{\gamma,\gamma}\Big(-(\varsigma-s)^{\gamma}A\Big)F(v(s),w(s))ds \\ &\quad - \int_0^{\varsigma}(\varsigma-s)^{\gamma-1}E_{\gamma,\gamma}\big(-(\varsigma-s)^{\gamma}A\big)\big(P\sigma {B_0}^2\frac{v}{\rho}(s)\big)ds. \end{aligned}$$

□

Definition 4. *A function $v : [0,\infty) \to H^{\gamma,r}$ is said to be a global mild solution of problem 5 in $H^{\gamma,r}$ if $v \in C\big([0,\infty), H^{\gamma,r}\big)$ for $\varsigma \in [0,\infty)$:*

$$\begin{aligned} v(\varsigma) &= E_{\gamma}(-\varsigma^{\gamma}A)a + \int_0^{\varsigma}(\varsigma-s)^{\gamma-1}E_{\gamma,\gamma}\Big(-(\varsigma-s)^{\gamma}A\Big)F(v(s),w(s))ds \\ &\quad - \int_0^{\varsigma}(\varsigma-s)^{\gamma-1}E_{\gamma,\gamma}\big(-(\varsigma-s)^{\gamma}A\big)(P\sigma {B_0}^2\frac{v(s)}{\rho})ds. \end{aligned} \tag{8}$$

Definition 5. *Let $0 < \Im < \infty$. A function $v : [0, \Im] \to H^{\gamma,r}$ is supposed to be a local mild solution of problem (5) in $H^{\gamma,r}$ if $v \in ([0, \Im], H^{\gamma,r})$ and if v satisfies the above equation for $\varsigma \in [0, \Im]$. Conveniently, we can define two operators $\varphi(\varsigma)$, $\varpi(v, w)(\varsigma)$:*

$$\varphi(\varsigma) = \int_0^{\varsigma} (\varsigma - s)^{\gamma-1} E_{\gamma,\gamma}\big(-(\varsigma - s)^{\gamma} A\big)\Big(-P\sigma {B_0}^2 \frac{v(s)}{\rho}\Big) ds,$$
$$\varpi(v, w)(\varsigma) = \int_0^{\varsigma} (\varsigma - s)^{\gamma-1} E_{\gamma,\gamma}\big(-(\varsigma - s)^{\gamma} A\big) F(v(s), w(s)) ds.$$

Lemma 6. *Suppose that $(X, ||.||_x)$ is a Banach space with the positive real number L and the bilinear operator $G : X * X \to X$ such that*

$$||G(v, w)||_x \leq L||v||_x||w||_x,$$

then, for any $v_0 \in X$ with $||v_0||_x < \frac{1}{4L}$, the equation $v = v_0 + G(v, v)$ has a unique solution $v \in X$.

Proposition 4. *: Let $l < r < \infty$ and $\gamma < \beta$. For any $\varsigma > 0$, $e^{\imath A}$ there is a bounded map between $H^{\gamma,r} \to H^{\beta,r}$. Further, for each $\Im > 0$ there is a constant C depending on r, β, γ such that*

$$|e^{\imath A} f|_{H^{\beta,r}} \leq C\varsigma^{-(\beta-\gamma)}|f|_{H^{\gamma,r}}$$

for all $H^{\gamma,r}$ and $\varsigma \in (0, \Im]$. Moreover,

$$\lim_{\varsigma \to 0} \varsigma^{(\beta-\gamma)}|e^{\imath A} f|_{H^{\beta,r}} = 0.$$

3. Global and Local Uniqueness and Existence in $H^{\gamma,r}$

For the uniqueness and existence of the mild solution to problem (5) when solving with $H^{\gamma,r}$, we have to discuss adequate circumstances for the solution. We assume that

$$-P\sigma {B_0}^2 \frac{v}{\rho}(\varsigma) \text{ is continuous, } for\ \varsigma > 0 \text{ and } \Big| -P\sigma {B_0}^2 \frac{v(\varsigma)}{\rho}\Big|_r = o(\varsigma^{-\gamma(1-\beta)}), \qquad (9)$$

for $0 < \beta < 1$ as $\varsigma \to 0$.

Theorem 5. *Let $1 < r < \infty$ and $0 < \gamma < 1$, and let (9) hold for every $a \in H^{\gamma,r}$. Suppose that*

$$C_1|a|_{H^{\gamma,r}} + B_1 M_\infty < \frac{1}{4L},$$

whereas $M_\infty = \sup_{s\in(0,\infty)} \Big(s^{\gamma(1-\beta)}(-P\sigma {B_0}^2 \frac{v(s)}{\rho})\Big)$; then, if $\frac{n}{2r} - \frac{1}{2} < \beta$, there subsequently exists a unique function $v : [0, \infty) \to H^{\gamma,r}$ and $\alpha > max\Big(\beta, \frac{1}{2}\Big)$ satisfying the following:

(i) *$v : [0, \infty) \to H^{\gamma,r}$ is continuous and $v(0) = a$;*
(ii) *$v : [0, \infty) \to H^{\alpha,r}$ is continuous and $\lim_{\varsigma \to 0} \varsigma^{\gamma(\alpha-\beta)}|v(\varsigma)|_{H^{\alpha,r}} = 0$;*
(iii) *v satisfies (8) for $\varsigma \in [0, \infty)$.*

Proof. Suppose $\alpha = \frac{1+\beta}{2}$; then, we can describe X_∞, which is the space of all the curves $v : (0, \infty) \to H^{\gamma,r}$. Moreover, $X_\infty = X[\infty]$ such that:
(i) $v : [0, \infty) \to H^{\gamma,r}$ is continuous and bounded;

(ii) $v : (0, \infty) \to H^{\alpha,r}$ is continuous and bounded, therefore, $\lim_{\varsigma \to 0} \varsigma^{\gamma(\alpha-\beta)} |v(\varsigma)|_{H^{\alpha,r}} = 0$, and its common form is provided by

$$||v||_{X_\infty} = \max\Big(\sup_{\varsigma \geq 0} |v(\varsigma)|_{H^{\gamma,r}}, \sup_{\varsigma \geq 0} \varsigma^{\gamma(\alpha-\beta)} |v(\varsigma)|_{H^{\alpha,r}} \Big).$$

It is clear that X_∞ is a non-empty complete metric space. Now, because we know that $F : H^{\alpha,r} * H^{\alpha,r} \to J_r$ is bounded and a bilinear mapping, there exists M such that $v, w \in H^{\alpha,r}$,

$$\begin{aligned} |F(v,w)|_r &\leq M|v|_{H^{\alpha,r}} |w|_{H^{\alpha,r}} \\ |F(v,v) - F(w,w)|_r &\leq M\big(|v|_{H^{\alpha,r}} + |w|_{H^{\alpha,r}}\big)|v-w|_{H^{\alpha,r}}. \end{aligned}$$

Step 1

Let $v, w \in X_\infty$. The operator $\varpi(v(\varsigma), w(\varsigma))$ is a part of $C\ ([0,\Im], H^{\gamma,r})$ along with $C(0,\infty)$, $H^{\gamma,r}$. Now, randomly considering $\varsigma_0 \geq 0$ be fixed and $\varepsilon > 0$ to be very small, and again supposing that $\varsigma > \varsigma_0$ (and analogously, $\varsigma < \varsigma_0$), we have

$$\begin{aligned} & |\varpi(v(\varsigma), w(\varsigma)) - \varpi(v(\varsigma_0), w(\varsigma_0))|_{H^{\gamma,r}} ds \\ \leq & \int_{\varsigma_0}^{\varsigma} (\varsigma - s)^{\gamma-1} |E_{\gamma,\gamma}\big(-(\varsigma-s)^\gamma A\big) F(v(s), w(s))|_{H^{\gamma,r}} ds \\ + & \int_0^{\varsigma_0} \Big|(\varsigma - s)^{\gamma-1} - (\varsigma_0 - s)^{\gamma-1} E_{\gamma,\gamma}\big(-(\varsigma-s)^\gamma A\big) F(v(s), w(s))\Big|_{H^{\gamma,r}} ds \\ + & \int_0^{\varsigma_0-\epsilon} (\varsigma_0 - s)^{\gamma-1} |E_{\gamma,\gamma}\big(-(\varsigma-s)^\gamma A\big) - E_{\gamma,\gamma}\big(-(\varsigma_0-s)^\gamma A\big) F(v(s), w(s))|_{H^{\gamma,r}} ds \\ + & \int_{\varsigma_0-\epsilon}^{\varsigma_0} (\varsigma_0 - s)^{\gamma-1} |E_{\gamma,\gamma}\big(-(\varsigma-s)^\gamma A\big) - E_{\gamma,\gamma}\big(-(\varsigma_0-s)^\gamma A\big) F(v(s), w(s))|_{H^{\gamma,r}} ds \\ = & \mathcal{I}_{11}(\varsigma) + \mathcal{I}_{12}(\varsigma) + \mathcal{I}_{13}(\varsigma) + \mathcal{I}_{14}(\varsigma). \end{aligned}$$

To consider every term individually, in view of Lemma 4 for $\mathcal{I}_{11}(\varsigma)$, we have

$$\begin{aligned} \mathcal{I}_{11}(\varsigma) &\leq C_1 \int_{\varsigma_0}^{\varsigma} (\varsigma - s)^{\gamma(1-\beta)-1} \big|F(v(s), w(s))\big|_r ds \\ &\leq MC_1 \int_{\varsigma_0}^{\varsigma} (\varsigma - s)^{\gamma(1-\beta)-1} \big[|(v(s)|_{H^{\alpha,r}}, |w(s))|_{H^{\alpha,r}}\big] ds \\ &\leq MC_1 \int_{\varsigma_0}^{\varsigma} (\varsigma - s)^{\gamma(1-\beta)-1} s^{-2\gamma(\alpha-\beta)} ds \sup_{s \in (0,\varsigma]} \big\{ s^{2\gamma(\alpha-\beta)} |v(s)|_{H^{\alpha,r}} |w(s)|_{H^{\alpha,r}} \big\} \\ &= MC_1 \int_{\varsigma_0/\varsigma}^{1} (1 - s)^{\gamma(1-\beta)-1} s^{-2\gamma(\alpha-\beta)} ds \sup_{s \in (0,\varsigma]} \big\{ s^{2\gamma(\alpha-\beta)} |v(s)|_{H^{\alpha,r}} |w(s)|_{H^{\alpha,r}} \big\}. \end{aligned}$$

By applying the properties of β function, $\exists\ \delta > 0$ is very much less, such that $0 <\ \varsigma - \varsigma_0 < \delta$, and we have

$$\int_{\varsigma_0/\varsigma}^{1} (1 - s)^{\gamma(1-\beta)-1} s^{-2\gamma(\alpha-\beta)} ds \to 0$$

for which it follows that as $\varsigma - \varsigma_0 \to 0$, $\mathcal{I}_{11}(\varsigma)$ approaches 0.

Now, for $\mathcal{I}_{12}(\varsigma)$,

$$\begin{aligned} \mathcal{I}_{12}(\varsigma) = & C_1 \int_0^{\varsigma_0} \Big((\varsigma_0 - s)^{\gamma-1} - (\varsigma - s)^{\gamma-1} \Big) (\varsigma - s)^{-\beta\gamma} |F(v(s), w(s))|_r ds \\ \leq & MC_1 \int_0^{\varsigma_0} \Big((\varsigma_0 - s)^{\gamma-1} - (\varsigma - s)^{\gamma-1} \Big) (\varsigma - s)^{-\beta\gamma} s^{-2\gamma(\alpha-\beta)} ds \\ & \sup_{s \in (0,\varsigma_0]} \big\{ s^{2\gamma(\alpha-\beta)} |v(s)|_{H^{\alpha,r}} |w(s)|_{H^{\alpha,r}} \big\}. \end{aligned}$$

It is interesting to note that

$$\begin{aligned}
\int_0^{\varsigma_0} &\big|(\varsigma_0-s)^{\gamma-1}-(\varsigma-s)^{\gamma-1}\big|(\varsigma-s)^{-\beta\gamma}s^{-2\gamma(\alpha-\beta)}ds \\
&\leq \int_0^{\varsigma_0}(\varsigma-s)^{\gamma-1}(\varsigma-s)^{-\beta\gamma}s^{-2\gamma(\alpha-\beta)}ds \\
&+ \int_0^{\varsigma_0}(\varsigma_0-s)^{\gamma-1}(\varsigma-s)^{-\beta\gamma}s^{-2\gamma(\alpha-\beta)}ds \\
&\leq 2\int_0^{\varsigma_0}(\varsigma_0-s)^{\gamma(1-\beta)-1}(\varsigma-s)^{-\beta\gamma}s^{-2\gamma(\alpha-\beta)}ds \\
&= 2B(\gamma(1-\beta)),1-2\gamma(\alpha-\beta).
\end{aligned}$$

We can prove this using Lebesgue's dominated convergence theorem:

$$\int_0^{\varsigma_0}\Big((\varsigma_0-s)^{\gamma-1}-(\varsigma-s)^{\gamma-1}\Big)(\varsigma-s)^{-\beta\gamma}s^{-2\gamma(\alpha-\beta)}ds \to 0, \quad as \;\; \varsigma\to\varsigma_0,$$

now, we can conclude that $\lim_{\varsigma\to\varsigma_0}\mathcal{I}_{12}(\varsigma)=0$.

For $\mathcal{I}_{13}(\varsigma)$, because

$$\begin{aligned}
\mathcal{I}_{13}(\varsigma) &\leq \int_0^{\varsigma_0-\epsilon}(\varsigma_0-s)^{\gamma-1}|E_{\gamma,\gamma}\big(-(\varsigma-s)^{\gamma}A\big)-E_{\gamma,\gamma}\big(-(\varsigma_0-s)^{\gamma}A\big)F(v(s),w(s))|_{H^{\gamma,r}}ds \\
&\leq \int_0^{\varsigma_0-\epsilon}(\varsigma_0-s)^{\gamma-1}\big((\varsigma-s)^{-\beta\gamma}+(\varsigma_0-s)^{-\beta\gamma}\big)|F(v(s),w(s))|_r ds \\
&\leq 2MC_1\int_0^{\varsigma_0}(\varsigma_0-s)^{\gamma-1}s^{-2\gamma(\alpha-\beta)}ds\sup_{s\in(0,\varsigma_0]}\big(s^{2\gamma(\alpha-\beta)}|v(s)|_{H^{\alpha,r}}|w(s)|_{H^{\alpha,r}}\big).
\end{aligned}$$

Again applying Lebesgue's dominated convergence theorem, the uniform continuity factor from Lemma 1 shows that

$$\begin{aligned}
\lim_{\varsigma\to\varsigma_0}\mathcal{I}_{13}(\varsigma) &= \int_0^{\varsigma_0-\epsilon}(\varsigma_0-s)^{\gamma-1}|E_{\gamma,\gamma}\big(-(\varsigma-s)^{\gamma}A\big)-E_{\gamma,\gamma}\big(-(\varsigma_0-s)^{\gamma}A\big) \\
&\quad\times F(v(s),w(s))|_{H^{\gamma,r}}ds \\
&= 0.
\end{aligned}$$

For $\mathcal{I}_{14}(\varsigma)$, by calculation we can approximate

$$\begin{aligned}
\mathcal{I}_{14}(\varsigma) &\leq \int_{\varsigma_0-\epsilon}^{\varsigma_0}(\varsigma_0-s)^{\gamma-1}\big((\varsigma-s)^{-\beta\gamma}+(\varsigma_0-s)^{-\beta\gamma}\big)|F(v(s),w(s))|_r ds \\
&\leq 2MC_1\int_0^{\varsigma_0}(\varsigma_0-s)^{\gamma-1}s^{-2\gamma(\alpha-\beta)}ds\sup_{s\in(\varsigma_0-\epsilon,\varsigma_0]}\big(s^{2\gamma(\alpha-\beta)}|v(s)|_{H^{\alpha,r}}|w(s)|_{H^{\alpha,r}}\big)\to 0
\end{aligned}$$

as $\varepsilon\to 0$. Hence, we can say that

$$|\varpi(v(\varsigma),w(\varsigma))-\varpi(v(\varsigma_0),w(\varsigma_0))|_{H^{\gamma,r}}ds\to 0 \quad as \;\; \varsigma\to\varsigma_0.$$

The operator's continuity $\varpi(v,w)$ estimated in $C((0,\infty),H^{\alpha,r})$ are in accordance with the above discussion.

Step 2

Next, we must prove that the operator $\varpi : X_\infty * X_\infty \to X_\infty$ is the bilinear continuous operator. Applying Lemma 4, we have

$$\begin{aligned}|\varpi(v(\varsigma),w(\varsigma))|_{H^{\gamma,r}} &= \Big|\int_0^\varsigma (\varsigma-s)^{\gamma-1}E_{\gamma,\gamma}\big(-(\varsigma-s)^\gamma A\big)F(v(s),w(s))\Big|_{H^{\gamma,r}}ds\\ &\le C_1\int_0^\varsigma (\varsigma-s)^{\gamma(1-\beta)-1}|F(v(s),w(s))|_r ds\\ &\le MC_1\int_0^\varsigma (\varsigma-s)^{\gamma(1-\beta)-1}s^{-2\gamma(\alpha-\beta)}ds\\ &\quad\times \sup_{s\in(0,\varsigma]}\big(s^{2\gamma(\alpha-\beta)}|v(s)|_{H^{\alpha,r}}|w(s)|_{H^{\alpha,r}}\big)\\ &= MC_1B\big((\gamma(1-\beta)),1-2\gamma(\alpha-\beta)\big)||v||_{X_\infty}||w||_{X_\infty}\end{aligned}$$

and

$$\begin{aligned}|\varpi(v(\varsigma),w(\varsigma))|_{H^{\alpha,r}} &= \Big|\int_0^\varsigma (\varsigma-s)^{\gamma-1}E_{\gamma,\gamma}\big(-(\varsigma-s)^\gamma A\big)F(v(s),w(s))\Big|_{H^{\alpha,r}}ds\\ &\le C_1\int_0^\varsigma (\varsigma-s)^{\gamma(1-\alpha)-1}|F(v(s),w(s))|_r ds\\ &\le MC_1\int_0^\varsigma (\varsigma-s)^{\gamma(1-\alpha)-1}s^{-2\gamma(\alpha-\beta)}ds\\ &\quad\times \sup_{s\in(0,\varsigma]}\big(s^{2\gamma(\alpha-\beta)}|v(s)|_{H^{\alpha,r}}|w(s)|_{H^{\alpha,r}}\big)\\ &= MC_1\varsigma^{-\gamma(\alpha-\beta)}B\big((\gamma(1-\alpha)),1-2\gamma(\alpha-\beta)\big)||v||_{X_\infty}||w||_{X_\infty}.\end{aligned}$$

Observe that

$$\sup_{\varsigma\in[0,\infty)}\varsigma^{\gamma(\alpha-\beta)}\big|\varpi(v(\varsigma),w(\varsigma))\big|_H^{\alpha,r} \le MC_1B(\gamma(1-\alpha)),1-2\gamma(\alpha-\beta)||v||_{X_\infty}||w||_{X_\infty}.$$

More specifically,

$$\lim_{\varsigma\to 0}\varsigma^{\gamma(\alpha-\beta)}|\varpi(v(\varsigma),w(\varsigma))|_{H^{\alpha,r}} = 0.$$

Hence, $\varpi(v,w)\in X_\infty$ and $||\varpi(v(\varsigma),w(\varsigma))||_{X_\infty} \le L||v||_{X_\infty}||w||_{X_\infty}$.

Step 3

Let $0<\varsigma_0<\varsigma$. Because

$$\begin{aligned}|\varphi(\varsigma)-\varphi(\varsigma_0)|_{H^{\gamma,r}} \le\ & \int_{\varsigma_0}^\varsigma (\varsigma-s)^{\gamma-1}\Big|E_{\gamma,\gamma}\Big(-(\varsigma-s)^\gamma A\Big)(-P\sigma {B_0}^2\frac{v(s)}{\rho})\Big|_{H^{\gamma,r}}ds\\ &+\int_0^{\varsigma_0}\Big((\varsigma_0-s)^{\gamma-1}-(\varsigma-s)^{\gamma-1}\Big)\Big|E_{\gamma,\gamma}\Big(-(\varsigma-s)^\gamma A\Big)\\ &\quad(-P\sigma {B_0}^2\frac{v(s)}{\rho})\Big|_{H^{\gamma,r}}ds\\ &+\int_0^{\varsigma_0-\epsilon}(\varsigma_0-s)^{\gamma-1}\big|E_{\gamma,\gamma}\big(-(\varsigma-s)^\gamma A\big)-E_{\gamma,\gamma}\big(-(\varsigma_0-s)^\gamma A\big)\\ &\quad(-P\sigma {B_0}^2\frac{v(s)}{\rho})|_{H^{\gamma,r}}ds\\ &+\int_{\varsigma_0-\epsilon}^{\varsigma_0}(\varsigma_0-s)^{\gamma-1}\big|E_{\gamma,\gamma}\big(-(\varsigma-s)^\gamma A\big)-E_{\gamma,\gamma}\big(-(\varsigma_0-s)^\gamma A\big)\\ &\quad(-P\sigma {B_0}^2\frac{v(s)}{\rho})|_{H^{\gamma,r}}ds\end{aligned}$$

$$
\begin{aligned}
\leq\ & C_1 \int_{\varsigma_0}^{\varsigma} (\varsigma - s)^{\gamma(1-\beta)-1} \Big| (-P\sigma {B_0}^2 \frac{v(s)}{\rho}(s)) \Big|_r ds \\
+\ & C_1 \int_0^{\varsigma_0} \big((\varsigma_0 - s)^{\gamma-1} - (\varsigma - s)^{\gamma-1}\big)(\varsigma - s)^{-\beta\gamma} \big|(-P\sigma {B_0}^2 \frac{v(s)}{\rho})\big|_r ds \\
+\ & C_1 \int_0^{\varsigma_0-\epsilon} (\varsigma_0 - s)^{\gamma-1} \big| E_{\gamma,\gamma}\big(-(\varsigma - s)^{\gamma} A\big) - E_{\gamma,\gamma}\big(-(\varsigma_0 - s)^{\gamma} A\big) \\
& \times (-P\sigma {B_0}^2 \frac{v(s)}{\rho}(s)) \big|_{H^{\gamma,r}} ds + 2C_1 \int_{\varsigma_0-\epsilon}^{\varsigma_0} (\varsigma_0 - s)^{\gamma(1-\beta)-1} \big| \\
& \times (-P\sigma {B_0}^2 \frac{v(s)}{\rho}(s)) \big|_r ds \\
\leq\ & C_1 M(\varsigma) \int_{\varsigma_0}^{\varsigma} (\varsigma - s)^{\gamma(1-\beta)-1} s^{-\gamma(1-\beta)} ds \\
+\ & C_1 M(\varsigma) \int_0^{\varsigma_0} \Big((\varsigma - s)^{\gamma-1} - (\varsigma_0 - s)^{\gamma-1}\Big)(\varsigma - s)^{-\beta\gamma} s^{-\gamma(1-\beta)} ds \\
+\ & C_1 M(\varsigma) \int_0^{\varsigma_0-\epsilon} (\varsigma_0 - s)^{\gamma-1} \big| E_{\gamma,\gamma}\big(-(\varsigma - s)^{\gamma} A\big) - \\
& \times E_{\gamma,\gamma}\big(-(\varsigma_0 - s)^{\gamma} A\big) \big|_{H^{\gamma,r}} ds \\
+\ & 2C_1 M(\varsigma) \int_{\varsigma_0-\epsilon}^{\varsigma_0} (\varsigma_0 - s)^{\gamma(1-\beta)-1} s^{-\gamma(1-\beta)} ds.
\end{aligned}
$$

These tend to 0 as $\varsigma \to \varsigma_0$ combined with $\varepsilon \to 0$ when the β function properties are applied to the first two terms and the last term. Using Lemma 1, the third term similarly approaches 0 when $\varsigma \to \varsigma_0$. This suggests that

$$
\big|\varphi(\varsigma) - \varphi(\varsigma_0)\big|_{H^{\gamma,r}} \to 0 \ \ when \ \ \varsigma \to \varsigma_0
$$

in order to calculate that the continuity of $\varphi(\varsigma)$ in $H^{\alpha,r}$ obeys the same pattern as in $H^{\gamma,r}$. On the contrary,

$$
\begin{aligned}
|\varphi(\varsigma)|_{H^{\gamma,r}} &= \Big| \int_0^{\varsigma} (\varsigma - s)^{\gamma-1} E_{\gamma,\gamma}\big(-(\varsigma - s)^{\gamma} A\big)(-P\sigma {B_0}^2 \frac{v(s)}{\rho}) \Big|_{H^{\gamma,r}} ds \\
&\leq C_1 \int_0^{\varsigma} (\varsigma - s)^{\gamma(1-\beta)-1} \Big| (-P\sigma {B_0}^2 \frac{v(s)}{\rho}) \Big|_r ds \\
&\leq C_1 M(\varsigma) \int_0^{\varsigma} (\varsigma - s)^{\gamma(1-\beta)-1} s^{-\gamma(1-\beta)} ds \\
&= C_1 M(\varsigma) B\big((\gamma(1-\beta)), (1 - \gamma(1-\beta))\big)
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
|\varphi(\varsigma)|_{H^{\alpha,r}} &= \Big| \int_0^{\varsigma} (\varsigma - s)^{\gamma-1} E_{\gamma,\gamma}\big(-(\varsigma - s)^{\alpha} A\big)(-\sigma P {B_0}^2 \frac{v(s)}{\rho}) \Big|_{H^{\alpha,r}} ds \\
&\leq C_1 \int_0^{\varsigma} (\varsigma - s)^{\gamma(1-\alpha)-1} \big| (-\sigma P {B_0}^2 \frac{v(s)}{\rho}) \big|_r ds \\
&\leq C_1 M(\varsigma) \int_0^{\varsigma} (\varsigma - s)^{\gamma(1-\alpha)-1} s^{-\gamma(1-\beta)} ds \\
&= \varsigma^{-\gamma(\alpha-\beta)} C_1 M(\varsigma) B\big((\gamma(1-\alpha)), (1 - \gamma(1-\beta))\big).
\end{aligned}
$$

More accurately,

$$
\varsigma^{\gamma(\alpha-\beta)} |\varphi(\varsigma)|_{H^{\alpha,r}} \leq C_1 M(\varsigma) B\Big((\gamma(1-\alpha)), (1 - \gamma(1-\beta))\Big) \to 0, \ \ when\ \varsigma \to \varsigma_0.
$$

As we know that $M(\varsigma) \to 0$ when $\varsigma \to 0$ owing to supposition 9, we can make sure that $\varphi(\varsigma) \in X_\infty$ and $||\varphi(\varsigma)||_\infty \le B_1 M_\infty$.
For $a \in H^{\gamma,r}$, per Lemma 1 we can conclude that

$$E_\gamma(-\varsigma^\gamma A)a \in C\big([0,\infty), H^{\gamma,r}\big) \text{ and } E_\gamma(-\varsigma^\gamma A)a \in C\big([0,\infty), H^{\alpha,r}\big).$$

Combined with Lemma 4, this signifies that for every $\varsigma \in (0,\Im]$,

$$\begin{aligned} E_\gamma(-\varsigma^\gamma A)a &\in X_\infty \\ \varsigma^{\gamma(\alpha-\beta)} E_\gamma(-\varsigma^\gamma A)a &\in C([0,\infty), H^{\alpha,r}) \\ ||E_\gamma(-\varsigma^\gamma A)a||_{X_\infty} &\le C_1 |a|_{H^{\gamma,r}}. \end{aligned}$$

The inequality defined by Theorem 5,

$$||E_\gamma(-\varsigma^\gamma A)a + \varphi(\varsigma)||_{X_\infty} \le ||E_\gamma(-\varsigma^\gamma A)a|| + ||\varphi(\varsigma)||_{X_\infty} \le \frac{1}{4L}$$

continues to hold, implying that F has a unique fixed point.
Step 4:
To demonstrate that $v(\varsigma) \to a$ in $H^{\gamma,r}$ when $\varsigma \to 0$, we must first check that

$$\begin{aligned} \lim_{\varsigma\to 0} \int_0^\varsigma (\varsigma-s)^{\gamma-1} E_{\gamma,\gamma}\big(-(\varsigma-s)^\gamma A\big) - P\sigma {B_0}^2 \frac{v}{\rho}(s) ds &= 0 \\ \lim_{\varsigma\to 0} \int_0^\varsigma (\varsigma-s)^{\gamma-1} E_{\gamma,\gamma}\big(-(\varsigma-s)^\gamma A\big) F(v(s), w(s)) ds &= 0 \end{aligned}$$

in $H^{\gamma,r}$. In fact, $\lim_{\varsigma\to 0} \varphi(\varsigma) = 0\big(\lim_{\varsigma\to 0} M(\varsigma) = 0\big)$ due to Equation (10). Thus,

$$\begin{aligned} &\int_0^\varsigma (\varsigma-s)^{\gamma-1} \big|E_{\gamma,\gamma}\big(-(\varsigma-s)^\gamma A\big) F(v(s), v(s))\big|_{H^{\gamma,r}} ds \\ \le\ & C_1 \int_0^\varsigma (\varsigma-s)^{\gamma(1-\beta)-1} \Big|F(v(s), v(s))\Big|_r ds \\ \le\ & MC_1 \int_0^\varsigma (\varsigma-s)^{\gamma(1-\beta)-1} |v(s)|^2_{H^{\alpha,r}} ds \\ \le\ & MC_1 \int_0^\varsigma (\varsigma-s)^{\gamma(1-\beta)-1} s^{-2\gamma(\alpha-\beta)} ds \sup_{s\in(0,\varsigma]} \big(s^{2\gamma(\alpha-\beta)} |v(s)|^2_{H^{\alpha,r}}\big) \\ =\ & MC_1 B\big((\gamma(1-\beta)), 1-2\gamma(\alpha-\beta)\big) \sup_{s\in(0,\varsigma]} \big(s^{2\gamma(\alpha-\beta)} |v(s)|^2_{H^{\alpha,r}}\big) \to 0 \ \ as\ \ \varsigma \to \varsigma_0. \end{aligned}$$

□

4. Local Existence in J_r

This section examines the local mild solution [30] to problem (5) in J_r using the iteration methodology. Suppose that $\alpha = \dfrac{1+\beta}{2}$.

Theorem 6. *Let $1 < r < \infty$, $0 < \gamma < 1$ and assume that (9) holds. Let $a \in H^{\gamma,r}$ with $\frac{n}{2r} - \frac{1}{2} < \gamma$. Then, problem (5) has a unique mild solution v in J_r for $a \in H^{\gamma,r}$. In addition, v is continuous on $[0,\Im]$, $A^\alpha v$ shows continuity in $(0,\Im]$, and $\varsigma^\gamma(\alpha-\beta)A^\alpha v(\varsigma)$ shows boundedness when $\varsigma \to 0$.*

Proof. Step 1. Now, let

$$\kappa(\varsigma) = \sup_{s\in(0,\varsigma]} s^{\gamma(\alpha-\beta)} |A^\alpha v(s)|_r$$

together with

$$\zeta(\varsigma) = \varpi(v,v)(\varsigma) = \int_0^{\varsigma} (\varsigma - s)^{\gamma-1} E_{\gamma,\gamma}\big(-(\varsigma - s)^{\gamma} A\big) F(v(s), v(s)) ds.$$

As a consequences of (Step 2) in Theorem 5, $\zeta(\varsigma)$ is continuous in $[0,\Im]$, $A^{\alpha}\zeta(\varsigma)$ exists and is similarly continuous in $(0,\Im]$, and

$$|A^{\alpha}\zeta(\varsigma)|_r \le MC_1 B(\gamma(1-\alpha), 1-2\gamma(\alpha-\beta))\kappa^2 \varsigma^{-\gamma(\alpha-\beta)}, \tag{11}$$

considering the integral $\varphi(\varsigma)$. Thus,

$$\big|(-P\sigma {B_0}^2 \frac{v(s)}{\rho})\big|_r \le M(\varsigma) s^{\gamma(1-\beta)}$$

is satisfied by the continued function $M(\varsigma)$. As $A^{\alpha}\varphi(\varsigma)$ is continuous in $(0,\Im]$, we find that

$$|A^{\alpha}\varphi(\varsigma)|_r \le C_1 M(\varsigma) B(\gamma(1-\alpha), 1-\gamma(1-\beta))\varsigma^{-\gamma(\alpha-\beta)}. \tag{12}$$

Because $\left|(-P\sigma {B_0}^2 \frac{v(s)}{\rho}(\varsigma))\right|_r = 0(\varsigma^{-\gamma(\alpha-\beta)})$ when $\varsigma \to 0$, we have $M(\varsigma) = 0$. Here, $|A^{\alpha}\zeta(\varsigma)|_r = 0(\varsigma^{-\gamma(\alpha-\beta)})$, as $\varsigma \to 0$ by means of Equation (12). We show that φ is continued in J_r. Actually by taking $0 \le \varsigma_0 < \varsigma < \Im$, we obtain

$$\begin{aligned}
|\varphi(\varsigma) - \varphi(\varsigma_0)|_r \le\ & C_3 \int_{\varsigma_0}^{\varsigma} (\varsigma - s)^{\gamma(1-\beta)-1} \left|(-P\sigma {B_0}^2 \frac{v(s)}{\rho})\right|_r ds \\
& + C_3 \int_0^{\varsigma_0} ((\varsigma_0 - s)^{\gamma-1} - (\varsigma - s)^{\gamma-1})(\varsigma - s)^{-\beta\gamma} \left|(-P\sigma {B_0}^2 \frac{v(s)}{\rho})\right|_r ds \\
& + C_3 \int_0^{\varsigma_0-\epsilon} (\varsigma_0 - s)^{\gamma-1} \big|E_{\gamma,\gamma}\big(-(\varsigma - s)^{\gamma}A\big) - E_{\gamma,\gamma}\big(-(\varsigma_0 - s)^{\gamma}A\big) \\
& \times (-P\sigma {B_0}^2 \frac{v(s)}{\rho})\big|_{H^{\gamma,r}} ds + 2C_3 \int_{\varsigma_0-\epsilon}^{\varsigma_0} (\varsigma_0 - s)^{\gamma(1-\beta)-1} \left|(-P\sigma {B_0}^2 \frac{v(s)}{\rho})\right|_r ds \\
\le\ & C_3 M(\varsigma) \int_{\varsigma_0}^{\varsigma} (\varsigma - s)^{\gamma(1-\beta)-1} s^{-\gamma(1-\beta)} ds \\
& + C_3 M(\varsigma) \int_0^{\varsigma_0} ((\varsigma - s)^{\gamma-1} - (\varsigma_0 - s)^{\gamma-1})(\varsigma - s)^{-\beta\gamma} s^{-\gamma(1-\beta)} ds \\
& + C_3 M(\varsigma) \int_0^{\varsigma_0-\epsilon} (\varsigma_0 - s)^{\gamma-1} \big|E_{\gamma,\gamma}\big(-(\varsigma - s)^{\gamma}A\big) \\
& - E_{\gamma,\gamma}\big(-(\varsigma_0 - s)^{\gamma}A\big)\big|_r s^{-\gamma(1-\beta)} ds + 2C_3 M(\varsigma) \\
& \times \int_{\varsigma_0-\epsilon}^{\varsigma_0} (\varsigma_0 - s)^{\gamma(1-\beta)-1} s^{-\gamma(1-\beta)} ds.
\end{aligned}$$

Step 2

Now, we find a solution using the successive approximation approach:

$$\begin{aligned}
v_0(\varsigma) &= E_{\gamma}(-\varsigma^{\gamma}A)a + \varphi(\varsigma) \\
v_{n+1}(\varsigma) &= v_0(\varsigma) + \zeta(v_n, v_n)(\varsigma), \quad n = 0, 1, 2 \ldots.
\end{aligned} \tag{13}$$

We know that, $\kappa_n(\varsigma) = \sup_{s\in(0,\varsigma]} s^{\gamma(\alpha-\beta)}|A^{\alpha}v_n(s)|_r$ are continuous functions as well as increasing functions on $[0,\Im]$ with $\kappa_n(0) = 0$. Additionally, by means of (11) and (13), $\kappa_n(\varsigma)$ satisfies the next inequality:

$$\kappa_{n+1}(\varsigma) \le \kappa_0(\varsigma) + MC_1 B\big((\gamma(1-\alpha)), 1-2\gamma(\alpha-\beta)\big){\kappa_n}^2(\varsigma). \tag{14}$$

For $\kappa_0(\varsigma) = 0$, set $\Im > 0$ such that

$$4MC_1B\big((\gamma(1-\alpha)), 1-2\gamma(\alpha-\beta)\big)\kappa_0(\varsigma) < 1. \tag{15}$$

In order to be sure that the sequence $\kappa_n(\Im)$ is bounded, a basic deliberation of (14) is needed, which we accomplish by applying a quadratic formula on (14), i.e.,

$$\kappa_n(\varsigma) \leq \rho(\Im), \;\; where \;\; n = 0, 1, 2, \ldots,$$

as

$$\rho(\varsigma) = \frac{1 - \sqrt{1 - 4MC_1B\big((\gamma(1-\alpha)), 1-2\gamma(\alpha-\beta)\big)\kappa_0(\varsigma)}}{2MC_1B\big((\gamma(1-\alpha)), 1-2\gamma(\alpha-\beta)\big)}.$$

Likewise, $\kappa_n(\varsigma) \leq \rho(\varsigma)$ holds for any $\varsigma \in (0, \Im]$. Similarly, $\rho(\varsigma) \leq 2\kappa_0(\varsigma)$. Assume that the following equality exists:

$$g_{n+1}(\varsigma) = \int_0^{\varsigma} (\varsigma - s)^{\gamma-1} E_{\gamma,\gamma}\big(-(\varsigma-s)^{\gamma}A\big)[F(v_{n+1}(s), v_{n+1}(s)) - F(v_n(s), v_n(s))]ds,$$

whereas $g_n = v_{n+1} - v_n \; for \;\; \varsigma \in (0, \Im] \;\; and \;\; n = 0, 1, 2, \ldots,$

$$G_n(\varsigma) = \sup_{s\in(0,\varsigma]} s^{\gamma(\alpha-\beta)}|A^{\alpha}g_n(s)|_r.$$

According to Theorem 5,

$$\Big|F\big(v_{n+1}(s), v_{n+1}(s)\big) - F\big(v_n(s), v_n(s)\big)\Big|_r \leq M\Big(\kappa_{n+1}(s) + \kappa_n(\varsigma)\Big)G_n(s)s^{-2\gamma(\alpha-\beta)}$$

which proceeds from (Step 2):

$$\varsigma^{\gamma}(\alpha-\beta)|A^{\alpha}g_{n+1}(\varsigma)|_r \leq 2MC_1B\Big((\gamma(1-\alpha)), 1-\gamma(1-\beta)\Big)\rho(\varsigma)G_n(\varsigma).$$

This provides

$$\begin{aligned} G_{n+1}(\Im) \;&\leq\; 2MC_1B\Big((\gamma(1-\alpha)), 1-\gamma(1-\beta)\Big)\rho(\Im)G_n(\Im) \\ &\leq\; 4MC_1B\Big((\gamma(1-\alpha)), 1-2\gamma(1-\beta)\Big)\kappa_0(\Im)G_n(\Im). \end{aligned} \tag{16}$$

Per (15) and (16), we have

$$\lim_{n\to 0} \frac{G_{n+1}(\Im)}{G_n(\Im)} \leq 4MC_1B\Big((\gamma(1-\alpha)), 1-2\gamma(1-\beta)\Big)\kappa_0(\varsigma) \leq 1.$$

As a result, the series $\sum_{n=0}^{\infty} G_n(\Im)$ converges. This verifies that the series $\sum_{n=0}^{\infty} \varsigma^{\gamma(\alpha-\beta)}A^{\alpha}g_n(\varsigma)$ uniformly converges for $\varsigma \in (0, \Im]$, therefore, in $(0, \Im]$ the sequence $\{\varsigma^{\gamma(\alpha-\beta)}A^{\alpha}v_n(\varsigma)\}$ uniformly converges as well. This results in $\lim_{n\to\infty} v_n(\varsigma) = v(\varsigma) \in D(A^{\alpha})$ and

$$\lim_{n\to\infty} A^{\alpha}v_n(\varsigma) = \varsigma^{\gamma(\alpha-\beta)}A^{\gamma}v(\varsigma).$$

As we know that A^{α} is closed and $A^{-\alpha}$ is bounded, correspondingly, $\kappa(\varsigma) = \sup_{s\in(0,\varsigma]} s^{\gamma(\alpha-\beta)}|A^{\alpha}v_n(s)|_r$ is verified:

$$\kappa(\varsigma) \leq \rho(\varsigma) \leq 2\kappa_0(\varsigma), \;\; as \;\; \varsigma \in (0, \varsigma] \tag{17}$$

along with

$$\begin{aligned} \varrho_n &= \sup_{s\in(0,\Im]} s^{2\gamma(\alpha-\beta)}\Big|F\big(v_n(s),v_n(s)\big) - F\big(v(s),v(s)\big)\Big|_r \\ &\le M(\kappa_n(\Im)+\kappa(\Im))s^{\gamma(\alpha-\beta)}\big|A^\alpha(v_n(s)-v(s))\big|_r \to 0, \;\; as\ n\to\infty. \end{aligned}$$

Now, it is necessary to confirm that v has a mild solution of problem (5) in $(0,\Im]$. Because

$$\big|\varpi(v_n,v_n)(\varsigma) - \varpi(v,v)(\varsigma)|_r \le \int_0^\varsigma (\varsigma-s)^{\gamma-1}\varrho_n s^{-2\gamma(\alpha-\beta)}ds = \varsigma^{\beta\gamma}\,\varrho_n \to 0, \quad as \;\; n \;\;\to\infty,$$

we have $g(v_n,v_n)(\varsigma) \to g(v,v)(\varsigma)$. If we take the limits of integration on both sides of Equation (12), we obtain

$$v(\varsigma) = v_0(\varsigma) + \varpi(v,v)(\varsigma). \tag{18}$$

We observe that (18) holds for $\varsigma\in(0,\Im]$ when considering $v(0)=a$, and similarly for $v\in C((0,\Im],J_r)$. The continuity of $A^\alpha v(\varsigma)$ on $(0,\Im]$ is attained by the uniform convergence of $\varsigma^{\gamma(\alpha-\beta)}A^\alpha v_n(\varsigma)$ to $\varsigma^{\gamma(\alpha-\beta)}A^\alpha v(\varsigma)$. From $\kappa_0(0)=0$ and Equation (17), it is clear that $|A^\alpha v(\varsigma)|_r = 0\varsigma^{-\gamma(\alpha-\beta)}$.

Step 3.

Now, we demonstrate that the mild solution is unique. First, we assume that v and w are the mild solutions to problem 5. Letting $g=v-w$, we once again examine the equality

$$g(\varsigma) = \int_0^\varsigma (\varsigma-s)^{\gamma-1}E_{\gamma,\gamma}\big(-(\varsigma-s)^\gamma A\big)[F(v(s),v(s)) - F(w(s),w(s))]ds.$$

Now, we can describe the functions:

$$\bar\kappa = \max \sup_{s\in(0,\varsigma]} s^{\gamma(\alpha-\beta)}|A^\alpha v(s)|_r, \; \sup_{s\in(0,\varsigma]} s^{\gamma(\alpha-\beta)}|A^\alpha w(s)|_r.$$

Per Theorem 5 and Lemma 4, we have

$$|A^\alpha g(\varsigma)|_r \le MC_1\bar\kappa(\varsigma)\int_0^\varsigma (\varsigma-s)^{\gamma(1-\alpha)-1}s^{-\gamma(\alpha-\beta)}|A^\alpha g(s)|_r ds.$$

It is simple to understand that for $\varsigma\in(0,\Im]$, the Gronwall inequality $A^\alpha\kappa(\varsigma)=0$. This shows that for $\varsigma\in(0,\Im]$, $\kappa(\varsigma)=v(\varsigma)-w(\varsigma)\equiv 0$. As a result, the mild solution is unique. □

5. Regularity Outcomes for MHD Flow

In this final section, we assume the regularity [31] of a solution v which satisfies the problem from Equation (5). Throughout this part, we consider that $-P\sigma {B_0}^2\dfrac{v}{\rho}(\varsigma)$ is Hölder continuous [32] along with power $\theta\in(0,\gamma(1-\alpha))$, especially

$$\left|\left(-P\sigma {B_0}^2\frac{v(\varsigma)}{\rho}\right) - \left(-P\sigma {B_0}^2\frac{v(s)}{\rho}\right)\right|_r \le L|\varsigma-s|^\theta, \;\; \forall\, 0<\varsigma,s\le\Im. \tag{19}$$

Definition 6. *A function $v:[0,\Im]\to J_r$ is said to be a classical solution of problem (5) if $v\in C([0,\Im],J_r)$ with ${}^cD^\varsigma_\varsigma \in C([0,\Im],J_r)$, which values are taken in $D(A)$ and satisfy (5) $\forall\;\varsigma\in(0,\Im]$.*

Lemma 7. *Let (19) be satisfied. If*

$$\varphi_1(\varsigma) = \int_0^\varsigma (\varsigma-s)^{\gamma-1}E_{\gamma,\gamma}\big(-\varsigma-s)^\gamma A\big)\big((-P\sigma {B_0}^2\frac{v(s)}{\rho}),(-P\sigma {B_0}^2\frac{v(\varsigma)}{\rho})\big)ds,\; for\; \varsigma\in(0,\Im],$$

therefore, $\varphi_1(\varsigma) \in D(A)$ *and* $A\varphi_1(\varsigma) \in C^\theta([0,\Im], J_r)$.

Proof. For the fixed $\varsigma \in (0,\Im]$ from Lemma 4 and (19), we have

$$\begin{aligned}
&(\varsigma - s)^{\gamma-1}\Big|AE_{\gamma,\gamma}\big(-(\varsigma-s)^\gamma A\big)\big((-P\sigma {B_0}^2\frac{v(s)}{\rho}),(-P\sigma {B_0}^2\frac{v(s)}{\rho}(\varsigma))\big)\Big|_r \\
&\le C_1(\varsigma-s)^{-1}\big|(-P\sigma {B_0}^2\frac{v(s)}{\rho}) - (-P\sigma {B_0}^2\frac{v(\varsigma)}{\rho})\big|_r \\
&\le C_1 L(\varsigma-s)^{\theta-1} \in L^1([0,\Im], J_r).
\end{aligned} \tag{20}$$

Afterwards,

$$\begin{aligned}
|A\varphi_1(\varsigma)|_r &\le \int_0^\varsigma (\varsigma-s)^{\gamma-1}\Big|AE_{\gamma,\gamma}\big(-(\varsigma-s)^\gamma A\big)\big((-P\sigma {B_0}^2\frac{v(s)}{\rho}),(-P\sigma {B_0}^2\frac{v(\varsigma)}{\rho})\big)\Big|_r ds \\
&\le C_1 L\int_0^\varsigma (\varsigma-s)^{\theta-1}ds \\
&\le \frac{C_1 L}{\theta}\varsigma^\theta \\
&< \infty.
\end{aligned}$$

From closeness properties A, we obtain $\varphi_1(\varsigma) \in D(A)$. We must ensure that $A\varphi_1(\varsigma)$ is Hölder continuous. Because

$$\frac{d}{d\varsigma}(\varsigma^{\gamma-1}E_{\gamma,\gamma}(-\mu\varsigma^\gamma)) = (\varsigma^{\gamma-2}E_{\gamma,\gamma-1}(-\mu\varsigma^\gamma)),$$

then,

$$\begin{aligned}
&\frac{d}{d\varsigma}(\varsigma^{\gamma-1}AE_{\gamma,\gamma}(-\varsigma^\gamma A)) \\
&= \frac{1}{2\pi\imath}\int_{\Gamma_\theta}(\varsigma^{\gamma-2}E_{\alpha,\alpha-1}(-\mu\varsigma^\alpha))A(\mu I + A)^{-1}d\mu \\
&= \frac{1}{2\pi\imath}\int_{\Gamma_\theta}(\varsigma^{\gamma-2}E_{\gamma,\gamma-1}(-\mu\varsigma^\gamma))d\mu - \frac{1}{2\pi\imath}\int_{\Gamma_\theta}(\varsigma^{\gamma-2}E_{\gamma,\gamma-1}(-\mu\varsigma^\gamma))A(\mu I + A)^{-1}d\mu \\
&= \frac{1}{2\pi\imath}\int_{\Gamma_\theta}(-\varsigma^{\gamma-2}E_{\gamma,\gamma-1}(\xi))\frac{1}{\varsigma^\gamma}d\xi - \frac{1}{2\pi\imath}\int_{\Gamma_\theta}(\varsigma^{\gamma-2}E_{\gamma,\gamma-1}(\xi))\frac{\xi}{\varsigma^\gamma}A(-\frac{\xi}{\varsigma^\gamma}I + A)^{-1}\frac{1}{\varsigma^\gamma}d\xi.
\end{aligned}$$

Because of

$$||(\mu I + A)^{-1}|| \le \frac{C}{|\mu|},$$

we obtain

$$||\frac{d}{d\varsigma}(\varsigma^{\gamma-1}AE_{\gamma,\gamma}(-\varsigma^\gamma A))|| \le C_\gamma\varsigma^{-2},\ \ 0 < \varsigma \le \Im.$$

Applying the mean value theorem, for every $0 < s < \varsigma \le \Im$,

$$\begin{aligned}
||(\varsigma^{\gamma-1}AE_{\gamma,\gamma}(-\varsigma^\gamma A)) - (s^{\gamma-1}AE_{\gamma,\gamma}(-s^\gamma A))|| &= ||\int_s^\varsigma \frac{d}{d\tau}(\tau^{\gamma-1}AE_{\gamma,\gamma}(-\tau^\gamma A))d\tau|| \\
&\le \int_s^\varsigma ||\frac{d}{d\tau}(\tau^{\gamma-1}AE_{\gamma,\gamma}(-\tau^\gamma A))||d\tau \\
&\le \int_s^\varsigma \tau^{-2}d\tau \\
&= C_\gamma(s^{-1} - \varsigma^{-1}).
\end{aligned} \tag{21}$$

For $0 < \varsigma < \varsigma + h \leq \Im$, let $h > 0$; then,

$$\begin{aligned}
& A\varphi_1(\varsigma+h) - A\varphi_1(\varsigma) \\
= & \int_0^{\varsigma} (\varsigma+h-s)^{\gamma-1} AE_{\gamma,\gamma}(-(\varsigma+h-s)^{\gamma}A)\Big((-P\sigma {B_0}^2 \frac{v(s)}{\rho}) - (-P\sigma {B_0}^2 \frac{v(\varsigma)}{\rho})\Big)ds \\
- & (\varsigma-s)^{\gamma-1} AE_{\gamma,\gamma}(-(\varsigma-s)^{\gamma}A)\Big((-P\sigma {B_0}^2 \frac{v(s)}{\rho}) - (-P\sigma {B_0}^2 \frac{v(\varsigma)}{\rho})\Big)ds \\
+ & \int_0^{\varsigma} (\varsigma+h-s)^{\gamma-1} AE_{\gamma,\gamma}(-(\varsigma+h-s)^{\gamma}A)\Big((-P\sigma {B_0}^2 \frac{v(\varsigma)}{\rho}) - (-P\sigma {B_0}^2 \frac{v(\varsigma+h)}{\rho})\Big)ds \\
+ & \int_{\varsigma}^{\varsigma+h} (\varsigma+h-s)^{\gamma-1} AE_{\gamma,\gamma}(-(\varsigma+h-s)^{\gamma}A)\Big((-P\sigma {B_0}^2 \frac{v(s)}{\rho}) - (-\sigma P {B_0}^2 \frac{v(\varsigma+h)}{\rho})\Big)ds \\
:= & h_1(\varsigma) + h_2(\varsigma) + h_3(\varsigma).
\end{aligned} \tag{22}$$

For convenience, we solve each term individually by applying (19) and Equation (22). For $h_1(\varsigma)$, we find that

$$\begin{aligned}
|h_1(\varsigma)|_r \leq & \int_0^{\varsigma} ||(\varsigma+h-s)^{\gamma-1} AE_{\gamma,\gamma}(-(\varsigma+h-s)^{\gamma}A) \\
- & (\varsigma-s)^{\gamma-1} AE_{\gamma,\gamma}(-(\varsigma-s)^{\gamma}A)||_r((-P\sigma {B_0}^2 \frac{v}{\rho}(s)) - (-P\sigma {B_0}^2 \frac{v}{\rho}(\varsigma)))ds \\
\leq & C_{\gamma} Lh \int_0^{\varsigma} (\varsigma+h-s)^{-1} (\varsigma-s)^{\theta-1} ds \\
\leq & C_{\gamma} Lh \int_0^{\varsigma} (s+h)^{-1} (\varsigma-s)^{\theta-1} ds \\
\leq & C_{\gamma} L \int_0^{h} \frac{h}{h+s} s^{\theta-1} ds + C_{\gamma} Lh \int_h^{\infty} \frac{s}{h+s} s^{\theta-1} ds \\
\leq & C_{\gamma} L h^{\theta}.
\end{aligned} \tag{23}$$

For $h_2(\varsigma)$, per (19) and Lemma 4, we have

$$\begin{aligned}
|h_2(\varsigma)|_r \leq & \int_0^{\varsigma} (\varsigma+h-s)^{\gamma-1} \Big| AE_{\gamma,\gamma}\big(-(\varsigma+h-s)^{\gamma}A\big)\big((-P\sigma {B_0}^2 \frac{v}{\rho}(\varsigma)) \\
& +(P\sigma {B_0}^2 \frac{v}{\rho}(\varsigma+h))\big)\Big|_r ds \\
\leq & C_1 \int_0^{\varsigma} (\varsigma+h-s)^{-1} \Big|\big(-P\sigma {B_0}^2 \frac{v}{\rho}(\varsigma)\big) - \big(-P\sigma {B_0}^2 \frac{v}{\rho}(\varsigma+h)\big)\Big|_r ds \\
\leq & C_1 h^{\theta} \int_0^{\varsigma} (\varsigma+h-s)^{-1} ds \\
= & C_1 L [lnh - ln(\varsigma+h)] h^{\theta}.
\end{aligned} \tag{24}$$

Now, for $h_3(\varsigma)$, we have

$$\begin{aligned}
|h_3(\varsigma)|_r \leq & \int_{\varsigma}^{\varsigma+h} (\varsigma+h-s)^{\gamma-1} \Big| AE_{\gamma,\gamma}\big(-(\varsigma+h-s)^{\gamma}A\big)\big((-P\sigma {B_0}^2 \frac{v}{\rho}(s)) \\
& +(P\sigma {B_0}^2 \frac{v}{\rho}(\varsigma+h))\big)\Big|_r ds \\
\leq & C_1 \int_{\varsigma}^{\varsigma+h} (\varsigma+h-s)^{-1} \Big|\big(-P\sigma {B_0}^2 \frac{v}{\rho}(s)\big) - \big(-P\sigma {B_0}^2 \frac{v}{\rho}(\varsigma+h)\big)\Big|_r ds \\
\leq & C_1 L \int_{\varsigma}^{\varsigma+h} (\varsigma+h-s)^{\theta-1} ds \\
= & C_1 L \frac{h^{\theta}}{\theta}.
\end{aligned} \tag{25}$$

In order to merge all the above results, we can say that $A\varphi_1(\varsigma)$ has the Hölder continuity property. Hence, $A\varphi_1(\varsigma)$ is Hölder continuous. □

Theorem 7. *Supposition that Theorem 6 is satisfied. For each $a \in D(A)$, if (19) holds, then there is a mild solution to Equation (5) which is a classical one.*

Proof. Consider $a \in D(A)$. We have $E_\gamma(-\varsigma^\gamma A)a$, which is said to be a classical solution of the following problem:

$$\begin{cases} {}^cD_\varsigma^\gamma v(\varsigma) = -Av, \ \ \varsigma > 0, \\ v(0) = a. \end{cases}$$

We can verify that

$$\varphi(\varsigma) = \int_0^\varsigma (\varsigma - s)^{\gamma-1} E_{\gamma,\gamma}\big(-(\varsigma - s)^\gamma A\big)(-P\sigma {B_0}^2 \frac{v}{\rho}(s))ds$$

is a classical solution for the problem

$$\begin{cases} {}^cD_\varsigma^\gamma v(\varsigma) = -Av + (-P\sigma {B_0}^2 \frac{v}{\rho}(\varsigma)), \ \ \varsigma > 0, \\ v(0) = 0. \end{cases}$$

Per Theorem 6, we have $\varphi \in C([0, \Im], J_r)$. Thus, we can write $\varphi(\varsigma) = \varphi_1(\varsigma) + \varphi_2(\varsigma)$, while

$$\varphi_1(\varsigma) = \int_0^\varsigma (\varsigma - s)^{\gamma-1} E_{\gamma,\gamma}\big(-(\varsigma - s)^\gamma A\big)\big((-P\sigma {B_0}^2 \frac{v}{\rho}(s)), (-P\sigma {B_0}^2 \frac{v}{\rho}(\varsigma))\big)ds$$

$$\varphi_2(\varsigma) = \int_0^\varsigma (\varsigma - s)^{\gamma-1} E_{\gamma,\gamma}\big(-(\varsigma - s)^\gamma A\big)(-P\sigma {B_0}^2 \frac{v}{\rho}(\varsigma))ds.$$

We know that according to Lemma 7, $\varphi_1(\varsigma) \in D(A)$. In order to justify the identical results considering Lemma 2(iii) we conclude that for $\varphi_2(\varsigma)$,

$$A\varphi_2(\varsigma) = (-P\sigma {B_0}^2 \frac{v}{\rho}(\varsigma)) - E_\gamma(-\varsigma^\gamma A)(-P\sigma {B_0}^2 \frac{v}{\rho}(\varsigma))$$

It then follows from (19) that

$$\big|A\varphi_2(\varsigma)\big| \leq (1 + C_1)\big|\big(-P\sigma {B_0}^2 \frac{v}{\rho}(\varsigma)\big)\big|_r.$$

Now, we can say that $\varphi_2(\varsigma) \in D(A)$ and $\varphi_2(\varsigma) \in C^v((0, \Im], J_r)$ for $\varsigma \in (0, \Im]$.
Furthermore, we have to prove that ${}^cD_\varsigma^\gamma(\varphi) \in C((0, \Im], J_r)$.
On account of $\varphi(0) = 0$ and Lemma 2(iv), we now have

$${}^cD_\varsigma^\gamma \varphi(\varsigma) = \frac{d}{d\varsigma}(I_\varsigma^{1-\gamma}\varphi(\varsigma)) = \frac{d}{d\varsigma}(E_\gamma(-\varsigma^\gamma A) * (-P\sigma {B_0}^2 \frac{v}{\rho}(\varsigma))).$$

Now, we prove that $E_\gamma(-\varsigma^\gamma A) * (-P\sigma {B_0}^2 \frac{v}{\rho}(\varsigma))$ is continuous differentiable in J_r. Considering $0 < h \leq \Im - \varsigma$, we can derive the following conclusion:

$$\begin{aligned} &\frac{1}{h}[(E_\gamma(-(\varsigma + h)^\gamma A) * (-P\sigma {B_0}^2 \frac{v}{\rho}(\varsigma))) - (E_\gamma(-\varsigma^\gamma A) * (-P\sigma {B_0}^2 \frac{v}{\rho}(\varsigma)))] \\ = &\int_0^\varsigma \frac{1}{h}[(E_\gamma(-(\varsigma + h - s)^\gamma A)(-P\sigma {B_0}^2 \frac{v}{\rho}(s))) - (E_\gamma(-(\varsigma - s)^\gamma A)(-P\sigma {B_0}^2 \frac{v}{\rho}(s)))]ds \\ + &\frac{1}{h}\int_\varsigma^{\varsigma+h} (E_\gamma(-(\varsigma + h - s)^\gamma A)(-P\sigma {B_0}^2 \frac{v}{\rho}(s)))ds. \end{aligned}$$

Keeping in mind that

$$\begin{aligned}
&\int_0^{\varsigma} \frac{1}{h}\Big| \big(E_\gamma(-(\varsigma+h-s)^\gamma A)(-P\sigma {B_0}^2\frac{v}{\rho}(s))\big) - \big(E_\gamma(-(\varsigma-s)^\gamma A)(-P\sigma {B_0}^2\frac{v}{\rho}(s))\big)\Big|_r ds \\
\leq\ & \frac{1}{h}\int_0^{\varsigma} \Big| \big(E_\gamma(-(\varsigma+h-s)^\gamma A)(-P\sigma {B_0}^2\frac{v}{\rho}(s))\big)\Big|_r ds \\
+\ & \frac{1}{h}\int_0^{\varsigma} \Big| \big(E_\gamma(-\varsigma-s)^\gamma A)(-P\sigma {B_0}^2\frac{v}{\rho}(s))\big)\Big|_r ds.
\end{aligned}$$

In view of Lemma 4,

$$\begin{aligned}
&\leq\ C_1 M(\varsigma)\frac{1}{h}\int_0^{\varsigma} (\varsigma+h-s)^{-\gamma} s^{-\gamma(1-\beta)} ds + C_1 M(\varsigma)\frac{1}{h}\int_0^{\varsigma} (\varsigma-s)^{-\gamma} s^{-\gamma(1-\beta)} ds \\
&\leq\ C_1 M(\varsigma)\frac{1}{h}(\varsigma+h)^{1-\gamma} + \varsigma^{1-\gamma} B\big(1-\gamma, 1-\gamma(1-\beta)\big) l.
\end{aligned}$$

The dominated convergence (DC) theorem is then used to obtain

$$\begin{aligned}
&\int_0^{\varsigma} \frac{1}{h}\big[\big(E_\gamma(-(\varsigma+h-s)^\gamma A)(-P\sigma {B_0}^2\frac{v}{\rho}(s))\big) - \big(E_\gamma(-(\varsigma-s)^\gamma A)(-P\sigma {B_0}^2\frac{v}{\rho}(s))\big)\big] ds \\
=\ & -\int_0^{\varsigma} (\varsigma-s)^{\gamma-1} A E_{\gamma,\gamma}(-(\varsigma-s)^\gamma A)(-P\sigma {B_0}^2\frac{v}{\rho}(s)) ds \\
=\ & A\varphi(\varsigma).
\end{aligned}$$

Conversely,

$$\frac{1}{h}\int_{\varsigma}^{\varsigma+h} E_\gamma\big(-(\varsigma+h-s)^\gamma A\big)\big(-P\sigma {B_0}^2\frac{v}{\rho}(s)\big) ds.$$

Let $s^* = \varsigma + h - s$; thus, $ds^* = -ds$ and after setting limits [$s = \varsigma$ implies $s^* = h$] and [$s = \varsigma + h$ implies $s^* = 0$], we have

$$\frac{1}{h}\int_h^0 E_\gamma(-(s^*)^\gamma A)(-\sigma {B_0}^2\frac{v}{\rho}(\varsigma+h-s^*))(-ds^*).$$

By replacing $s^* \to s$, we have

$$\begin{aligned}
&\frac{1}{h}\int_0^h E_\gamma(-s^\gamma A)(-P\sigma {B_0}^2\frac{v}{\rho}(\varsigma+h-s)) ds \\
=\ & \frac{1}{h}\int_0^h E_\gamma(-s^\gamma A)\big[(-P\sigma {B_0}^2\frac{v}{\rho}(\varsigma+h-s)) - (-P\sigma {B_0}^2\frac{v}{\rho}(\varsigma-s)) \\
+\ & (-P\sigma {B_0}^2\frac{v}{\rho}(\varsigma-s)) - (-P\sigma {B_0}^2\frac{v}{\rho}(\varsigma)) + (-P\sigma {B_0}^2\frac{v}{\rho}(\varsigma))\big] ds \\
=\ & \frac{1}{h}\int_0^h E_\gamma(-s^\gamma A)\big((-P\sigma {B_0}^2\frac{v}{\rho}(\varsigma+h-s)) - (-P\sigma {B_0}^2\frac{v}{\rho}(\varsigma-s))\big) ds \\
+\ & \frac{1}{h}\int_0^h E_\gamma(-s^\gamma A)\big((-P\sigma {B_0}^2\frac{v}{\rho}(\varsigma-s)) - (-P\sigma {B_0}^2\frac{v}{\rho}(\varsigma))\big) ds \\
+\ & \frac{1}{h}\int_0^h E_\gamma(-s^\gamma A)(-P\sigma {B_0}^2\frac{v}{\rho}(\varsigma)) ds.
\end{aligned}$$

From Lemmas 1, 4, and (19), we have

$$\begin{aligned}
\Big|\frac{1}{h}\int_0^h E_\gamma(-s^\gamma A)\big((-P\sigma {B_0}^2\frac{v}{\rho}(\varsigma+h-s)) - (-P\sigma {B_0}^2\frac{v}{\rho}(\varsigma-s))\Big|_r ds &\leq C_1 L h^\theta \\
\Big|\frac{1}{h}\int_0^h E_\gamma(-s^\gamma A)\big((-P\sigma {B_0}^2\frac{v}{\rho}(\varsigma-s))) - (-P\sigma {B_0}^2\frac{v}{\rho}(\varsigma))\Big|_r ds &\leq C_1 L \frac{h^\theta}{\theta+1}.
\end{aligned}$$

From Lemma 1(i),

$$\lim_{h\to 0}\frac{1}{h}\int_0^h E_\gamma(-s^\gamma A)(-P\sigma {B_0}^2\frac{v}{\rho}(\varsigma))ds \;=\; \big(-P\sigma {B_0}^2\frac{v}{\rho}(\varsigma)\big)$$

$$\lim_{h\to 0}\frac{1}{h}\int_\varsigma^{\varsigma+h} E_\gamma\big((\varsigma+h-s)^\gamma A\big)\big(-P\sigma {B_0}^2\frac{v}{\rho}(\varsigma)\big)ds \;=\; \big(-P\sigma {B_0}^2\frac{v}{\rho}(\varsigma)\big),$$

we can deduce that $E_\gamma(\varsigma^\gamma A)*(-P\sigma {B_0}^2\frac{v}{\rho}(\varsigma))$ is differentiable at ς_+ and

$$\frac{d}{d\varsigma}(E_\gamma(\varsigma^\gamma A)*(-P\sigma {B_0}^2\frac{v}{\rho}(\varsigma)))_+ = A\varphi(\varsigma)+(-P\sigma {B_0}^2\frac{v}{\rho}(\varsigma)).$$

Similarly, $E_\gamma(\varsigma^\gamma A)*(-P\sigma {B_0}^2\frac{v}{\rho}(\varsigma))$ is differentiable at ς_- and

$$\frac{d}{d\varsigma}(E_\gamma(\varsigma^\gamma A)*(-P\sigma {B_0}^2\frac{v}{\rho}(\varsigma)))_- = A\varphi(\varsigma)+(-P\sigma {B_0}^2\frac{v}{\rho}(\varsigma)).$$

This verifies that $A\varphi = A\varphi_1 + A\varphi_2 \in C((0,\Im], J_r)$. It can be easily seen that $\varphi_2(\varsigma) = (-P\sigma {B_0}^2\frac{v}{\rho}(\varsigma)) - E_\gamma(\varsigma^\gamma A)(-P\sigma {B_0}^2\frac{v}{\rho}(\varsigma))$ because of Lemma 1(iii) and that this lemma is continuous in terms of Lemma 1. Furthermore, $A\varphi_1(\varsigma)$ is continuous in view of Lemma 7, resulting in ${}^cD^\gamma_\varsigma\varphi \in C((0,\Im], J_r)$.

Step 2:

Consider v the mild solution of Equation (5). In order to demonstrate that $F(v,v) \in C^\theta((0,\Im], J_r)$, on account of Theorem 5, we must prove that $A^\alpha v$ possesses the Hölder continuity property in J_r. For $0 < \varsigma < \varsigma + h$, we consider $h > 0$. We denote $\Phi(\varsigma) := E_\gamma(-\varsigma^\gamma A)a$; then, by Lemma 2(iv) and 4,

$$\begin{aligned}
\big|A^\alpha\Phi(\varsigma+h) - A^\gamma\Phi(\varsigma)\big|_r &= \Big|\int_\varsigma^{\varsigma+h} -s^{\gamma-1}A^\alpha E_{\gamma,\gamma}(-s^\gamma A)ads\Big|_r \\
&\le \int_\varsigma^{\varsigma+h} s^{\gamma-1}\Big|A^\alpha - \beta E_{\gamma,\gamma}(-s^\gamma A)A^\beta a\Big|_r ds \\
&\le C_1\int_\varsigma^{\varsigma+h} s^{\gamma(1+\beta-\alpha)-1}ds\big|A^\beta a\big|_r \\
&= C_1\frac{|a|_{H^{\gamma,r}}}{\gamma(1+\beta-\alpha)}((\varsigma+h)^{\gamma(1+\beta-\alpha)} - \varsigma^{\gamma(1+\beta-\alpha)}) \\
&= C_1\frac{|a|_{H^{\gamma,r}}}{\gamma(1+\beta-\alpha)}h^\gamma(1+\beta-\alpha).
\end{aligned}$$

Thus, $A^\alpha\Phi \in C^\theta((0,\Im], J_r)$.
Taking h such that $\varepsilon \le \varsigma < \varsigma + h \le \Im$, every small $\varepsilon > 0$, because

$$\begin{aligned}
&\big|A^\alpha\Phi(\varsigma+h) - A^\gamma\Phi(\varsigma)\big|_r \\
\le\; &\Big|\int_\varsigma^{\varsigma+h}(\varsigma+h-s)^{\gamma-1}A^\alpha E_{\gamma,\gamma}(-(\varsigma+h-s)^\gamma A)(-P\sigma {B_0}^2\frac{v}{\rho}(s))ds\Big|_r \\
+\; &\Big|\int_0^\varsigma A^\alpha\big((\varsigma+h-s)^{\gamma-1}E_{\gamma,\gamma}(-(\varsigma+h-s)^\gamma A) - (\varsigma-s)^{\gamma-1}E_{\gamma,\gamma}(-(\varsigma-s)^\gamma A)\big) \\
&\times\big(-P\sigma {B_0}^2\frac{v}{\rho}(s)\big)ds\Big|_r \\
=\; &\Phi_1(\varsigma)+\Phi_2(\varsigma).
\end{aligned}$$

Using Lemmas 4 and (9), we have

$$\begin{aligned}\Phi_1(\varsigma) &\leq C_1 \int_{\varsigma}^{\varsigma+h} (\varsigma+h-s)^{\gamma(1-\alpha)-1} \big|(-P\sigma B_0{}^2\frac{v}{\rho}(s))\big|_r ds \\ &\leq C_1 M(\varsigma) \int_{\varsigma}^{\varsigma+h} (\varsigma+h-s)^{\gamma(1-\alpha)-1} s^{-\gamma(1-\alpha)-1} ds \\ &\leq M(\varsigma)\frac{C_1}{\gamma(1-\alpha)} h^{\gamma(1-\alpha)}\varsigma^{-\gamma(1-\alpha)-1} \\ &\leq M(\varsigma)\frac{C_1}{\gamma(1-\alpha)} h^{\gamma(1-\alpha)}\varepsilon^{-\gamma(1-\alpha)-1}.\end{aligned}$$

To estimate φ_2, we have the following inequality:

$$\begin{aligned}&\frac{d}{d\varsigma}(\varsigma^{\gamma-1}A^{\alpha}E_{\gamma,\gamma}(-\varsigma^{\gamma}A)) = \frac{1}{2\pi\imath}\int_{\Gamma}\mu^{\alpha}(\varsigma^{\gamma-2}E_{\gamma,\gamma-1}(-\mu\varsigma^{\gamma}))A(\mu I+A)^{-1}d\mu \\ &= \frac{1}{2\pi\imath}\int_{\Gamma'} -(\frac{-\xi}{\varsigma^{\gamma}})^{\alpha}(\varsigma^{\gamma-2}E_{\gamma,\gamma-1}(\xi))(-\frac{\xi}{\varsigma^{\gamma}}I+A)^{-1}\frac{1}{\varsigma^{\gamma}}d\xi\end{aligned}$$

which yields

$$\frac{d}{d\varsigma}(\varsigma^{\gamma-1}A^{\alpha}E_{\gamma,\gamma}(-\varsigma^{\gamma}A)) \leq C_{\gamma}\varsigma^{\gamma(1-\alpha)-2}.$$

Now, applying the mean value theorem,

$$\begin{aligned}||(\varsigma^{\gamma-1}A^{\alpha}E_{\gamma,\gamma}(-\varsigma^{\gamma}A)) - (s^{\gamma-1}A^{\alpha}E_{\gamma,\gamma}(-s^{\gamma}A))|| &= ||\int_s^{\varsigma}\frac{d}{d\tau}(\tau^{\gamma-1}A^{\alpha}E_{\gamma,\gamma}(-\tau^{\gamma}A))d\tau|| \\ &\leq \int_s^{\varsigma}||\frac{d}{d\tau}(\tau^{\gamma-1}A^{\alpha}E_{\gamma,\gamma}(-\tau^{\gamma}A))||d\tau \\ &\leq \int_s^{\varsigma}\tau^{\gamma(1-\alpha)-2}d\tau \\ &= C_{\gamma}(s^{\gamma(1-\alpha)-1}-\varsigma^{\gamma(1-\alpha)-1}).\end{aligned}$$

Thus,

$$\begin{aligned}\Phi_2(\varsigma) &\leq \Big|\int_0^{\varsigma} A^{\alpha}\big((\varsigma+h-s)^{\gamma-1}E_{\gamma,\gamma}(-(\varsigma+h-s)^{\gamma}A)-(\varsigma-s)^{\gamma-1}E_{\gamma,\gamma}(-(\varsigma-s)^{\gamma}A)\big) \\ &\quad (-P\sigma B_0{}^2\frac{v}{\rho}(s))ds\Big|_r \\ &\leq \int_0^{\varsigma}\big((\varsigma-s)^{\gamma(1-\alpha)-1}-(\varsigma+h-s)^{\gamma(1-\alpha)-1}\big)\big|(-P\sigma B_0{}^2\frac{v}{\rho}(s))\big|_r ds \\ &\leq M(\varsigma)C_{\gamma}\Big(\int_0^{\varsigma}(\varsigma-s)^{\gamma(1-\alpha)-1}s^{-\gamma(1-\beta)}ds - \int_0^{\varsigma+h}(\varsigma+h-s)^{\gamma(1-\alpha)-1}s^{-\gamma(1-\beta)}ds\Big) \\ &+ M(\varsigma)C_{\gamma}\int_{\varsigma}^{\varsigma+h}(\varsigma+h-s)^{\gamma(1-\alpha)-1}s^{-\gamma(1-\beta)}ds \\ &\leq M(\varsigma)C_{\gamma}\big(\varsigma^{\gamma(\beta-\alpha)}-(\varsigma+h)^{\beta-\alpha}\big)B\big(\gamma(1-\alpha),1-\gamma(1-\beta)\big)+M(\varsigma)C_{\gamma} \\ &\quad h^{\gamma(1-\alpha)}\varsigma^{-\gamma(1-\beta)} \\ &\leq M(\varsigma)C_{\gamma}h^{\gamma(\alpha-\beta)}\big[\varepsilon(\varepsilon+h)\big]^{\gamma(\beta-\alpha)}+M(\varsigma)C_{\gamma}h^{\gamma(1-\alpha)}\varepsilon^{-\gamma(1-\beta)}.\end{aligned}$$

This shows that $A^{\alpha}\Phi \in C^{\theta}([\varepsilon,\Im],J_r)$. Therefore, $A^{\alpha}\varphi \in C^{\theta}((0,\Im],J_r)$ due to arbitrary ε. Recollect that $\zeta(\varsigma) = \int_0^{\varsigma}(\varsigma-s)^{\gamma-1}E_{\gamma,\gamma}(-(\varsigma-s)^{\gamma}A)F(v(s),v(s))ds$, as we know that $\big|F(v(s),v(s)\big|_r \leq M\kappa^2(\varsigma)s^{-2\gamma(\alpha-\beta)}$, whereas $\kappa(\varsigma) = \sup_{s\in(0,\varsigma]} s^{\gamma(\alpha-\beta)}|A^{\alpha}v(s)|_r$ is both continuous and bounded in $(0,\Im]$. Analogously, the same logic allows $A^{\zeta} \in C^{\theta}((0,\Im],J_r)$ to be Hölder continuous. For this reason, we have $A^{\alpha}v(\varsigma) = A^{\alpha}\varphi(\varsigma)+A^{\alpha}\Phi(\varsigma)+A^{\alpha}\zeta(\varsigma) \in$

$C^{\theta}((0,\Im],J_r)$. As $F(v,v) \in C^{\theta}((0,\Im],J_r)$ has been proven, in the manner of (Step 2), this results in ${}^cD_{\varsigma}^{\gamma}\zeta \in C^{\theta}((0,\Im],J_r)$, $A\zeta \in C^{\theta}((0,\Im],J_r)$ and ${}^cD_{\varsigma}^{\gamma}\zeta = -A\zeta + F(v,v)$. Thus, we have ${}^cD_{\varsigma}^{\gamma}\zeta \in C^{\theta}((0,\Im],J_r)$, $Av \in C^{\theta}((0,\Im],J_r)$ and ${}^cD_{\varsigma}^{\gamma}v = -Av + F(v,v) + (-P\sigma {B_0}^2\frac{v}{\rho}(\varsigma))$. Therefore, we can say that v is a classical solution. □

6. Application

Assume that $X \in L^2(0,2\pi)$ and $e_n(x) = 3\sqrt{3/2\pi}cosx$, n = 1, 2, ... Then, $(e_n, n = 1,2,\ldots)$ is an orthonormal base of X. We define an infinite dimensional space $U = X$ and consider the following system governed by the semilinear heat equation:

$$\begin{cases} {}^cD_{\varsigma}^{4/5}Y(\varsigma,x) = {}^c D_{\varsigma}^{2/3}Y(\varsigma,x) + f(\varsigma,Y(\varsigma,x)) + Bu(\varsigma,x), 0 < \varsigma < b, 0 < x < 2\pi, \\ Y(0,x) = Y_0(x), 0 \le x \le 2\pi, \\ Y(\varsigma,0) = Y(\varsigma,2\pi), 0 \le \varsigma \le b, \end{cases} \quad (26)$$

where the nonlinear function f is considered as an operator satisfying hypothesis H_1 and for each $u \in L^2(0,b;U)$ of the form $\sum_{n=1}^{\infty} \hat{u}_n o(\varsigma)e_n$; here, we define

$$Bu(\varsigma) = \sum_{n=1}^{\infty} \hat{u}_n o(\varsigma)e_n,$$

where

$$\hat{u}_n(\varsigma) = \begin{cases} 0, 0 \le \varsigma < b(1-\frac{1}{n}), \\ u_n(\varsigma), b(1-\frac{1}{n}) \le \varsigma \le b. \end{cases} \quad (27)$$

Because

$$\|Bu\|_{L^2(0,b,X)} \le \|u\|_{L^2(0,b,X)},$$

the operator B is bounded from U into $L^2(J,X)$. In fact, it is not difficult to check that $\overline{BU} \neq L^2(J,X)$. Next, let φ be an arbitrary element in $L^2(o,b,X)$ and $h \in X$ be defined by

$$h = E_{\gamma}(-b-s)^{\gamma}Y(0)x + \int_0^b (b-s)^{\gamma-1}\Im_{\frac{4}{5}}(b-s)\phi(s)ds.$$

Assume that

$$\varphi(\varsigma) = \sum_{n=1}^{\infty} f_n(\varsigma)e_n,$$

and

$$h = \sum_{n=1}^{\infty} h_n(\varsigma)e_n.$$

Then, we claim that for every given $\varphi \in L^2(0,b,X)$, there exists $u \in U$ such that

$$\begin{aligned} &E_{\gamma}(-b-s)^{\gamma}Y(0)x + \int_0^{\varsigma}(b-s)^{\gamma-1}\Im_{\frac{4}{5}}(b-s)Bu(s)ds \\ &= E_{\gamma}(-b-s)^{\gamma}Y(0)x + \int_0^{\varsigma}(b-s)^{\gamma-1}\Im_{\frac{4}{5}}(b-s)\varphi(s)ds, \end{aligned}$$

which means that condition H_2 is satisfied, as assumptions H_1 and H_2 are satisfied.

7. Conclusions

This study uses Helmholtz–Leray projection to demonstrate the existence and uniqueness of fractional order Navier–Stokes equations of the solution to the Cauchy problem. Meanwhile, we offer a local viable solution in $\mathbf{S}_{\wp}$. The Navier–Stokes equations (NSEs) with time-fractional derivatives of order $\gamma \in (0,1)$ are used to simulate anomaly diffusion in fractal media. We demonstrate the existence of regular classical solutions to these equations in $\mathbf{S}_{\wp}$. The concept put forth in this article may be expanded upon in future work through the inclusion of observability and the generalization of other activities. Much research is being done in this fascinating area, which may result in a wide range of applications and theories.

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