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Initial Boundary Value Problem for a Fractional Viscoelastic Equation of the Kirchhoff Type

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Abstract: In this paper, we study the initial boundary value problem for a fractional viscoelastic equation of the Kirchhoff type. In suitable functional spaces, we define a potential well. In the framework of the potential well theory, we obtain the global existence of solutions by using the Galerkin approximations. Moreover, we derive the asymptotic behavior of solutions by means of the perturbed energy method. Our main results provide sufficient conditions for the qualitative properties of solutions in time.

Keywords: fractional viscoelastic equations; global existence; asymptotic behavior

MSC: 35R11; 35A01; 35B40

1. Introduction

In this paper, we study the following initial boundary value problem for a fractional viscoelastic equation of the Kirchhoff type:

\[ u_{tt} + h(\|u\|_m^2)(-\Delta)^m u - \int_0^t g(t - \tau)(-\Delta)^m u(\tau) \, d\tau + u_t = f(u), \quad x \in \Omega, \quad t > 0, \]
\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \]
\[ u(x, t) = 0, \quad x \in \mathbb{R}^N \setminus \Omega, \quad t > 0, \]

where 
\[ [u]_m = \left( \iint_{\mathbb{R}^{2N}} |u(x, t) - u(y, t)|^2 \, dx \, dy \right)^{\frac{1}{2}} \]

is the Gagliardo seminorm, \((-\Delta)^m\) is the fractional Laplace operator with \(0 < m < 1\), and \(\Omega \subset \mathbb{R}^N (N \geq 1)\) is a bounded domain with a Lipschitz boundary. The unknown function \(u = u(x, t)\) is the vertical displacement of the small-amplitude vibrating viscoelastic string with the fractional length at position \(x\) and time \(t\), \(- \int_0^t g(t - \tau)(-\Delta)^m u(\tau) \, d\tau\) is the viscoelastic term, \(u_t\) is the weak damping term, the Kirchhoff function \(h(s) = 1 + s^{p-1}\) for all \(s \geq 0, p > 1\), and the source term \(f(u) = |u|^{q-2}u\). The exponent \(q\) and the memory kernel \(g\) will be specified later.

For the classical viscoelastic wave equation of the Kirchhoff type, Wu and Tsai [1] studied the following equation:

\[ u_{tt} - h(\|\nabla u\|^2_2)\Delta u + \int_0^t g(t - \tau)\Delta u(\tau) \, d\tau - \Delta u_t = f(u). \]

They obtained the local existence, global existence, asymptotic behavior, and blow-up of solutions and provided the estimates on the decay rate of the energy function and the
blow-up time of the solutions. Moreover, in [2], they considered the following viscoelastic wave equation of the Kirchhoff type with nonlinear weak damping:

\[ u_{tt} - h(\|\nabla u\|^2)\Delta u + \int_0^t g(t-\tau)\Delta u(\tau) \, d\tau + f_2(u_t) = f_1(u). \]

They obtained the local existence and blow-up of solutions and also derived the estimates of the blow-up times of the solutions.

When we examine the deep properties of real-world problems and extend them to other studies, some concepts usually have their own limitations. In this regard, many researchers pointed out the limitations of integer-order calculus while studying the systems related to non-Markovian mechanisms, hereditary properties, and other factors. In this situation, fractional calculus plays an important role, which is a generalization of classical calculus (see [3]). In recent years, fractional partial differential equations have attracted a great deal of attention due to their wide applicability in continuum mechanics, quantum calculus (see [3]). In recent years, fractional partial differential equations have attracted a great deal of attention due to their wide applicability in continuum mechanics, quantum calculus (see [3]).

Fractional Kirchhoff equations were widely studied. Autuori et al. [13] investigated the fractional Kirchhoff type with nonlinear weak damping:

\[ (-\Delta)_{p,m}^m u + V(x)|u|^{p-2}u = \lambda \omega(x)|u|^{q-2}u - \nu(x)|u|^{r-2}u, \]

where \((-\Delta)_p^m\) is the fractional p-Laplace operator, which may be defined along any \(\varphi \in C_0^\infty(\mathbb{R}^N)\) as

\[ (-\Delta)_p^m \varphi(x) = 2 \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{|\varphi(x) - \varphi(y)|^{p-2} (\varphi(x) - \varphi(y))}{|x - y|^{N+mp}} \, dy \]

for \(x \in \mathbb{R}^N\) and

\[ [u]_{m,p} = \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+mp}} \, dx \, dy \right)^{\frac{1}{p}}. \]

By using the variational approach and topological degree theory, they proved the multiplicity results depending on the parameter \(\lambda\) and under the suitable general integrability properties of the ratio between some powers of the weights. Moreover, they obtained the existence of infinitely many pairs of entire solutions by genus theory. Wang et al. [16] studied the following fractional Kirchhoff equation involving Choquard nonlinearity and singular nonlinearity:

\[ (a + b(\|u\|_{m,p}^{(\theta-1)p}))(-\Delta)_p^m u = \lambda \frac{f_1(x)}{|u|^\theta} + \left( \int_{\mathbb{R}^N} \frac{f_2(y)|u(y)|^\theta}{|x - y|^{p'}} \, dy \right)^{\frac{1}{p'}} f_2(x)|u|^{\theta-1}, \]

where \(a, b, \theta, \lambda, \beta, \) and \(\mu\) are constants that meet certain conditions. They obtained the existence and multiplicity of nonnegative solutions by using the Nehari manifold ap-
proach combined with the Hardy–Littlewood–Sobolev inequality. Recently, Lin et al. [17] considered the fractional evolution Kirchhoff equation of the form

$$u_{tt} + [u]^2(\theta-1)(-\Delta)^m u = f(u),$$

and obtained the finite time blow-up of solutions with arbitrary positive initial energy by the concavity arguments.

Continuum mechanics attempts to describe the motions and equilibrium states of deformable bodies. Two types of materials are usually considered in basic texts on continuum mechanics: elastic materials and viscous fluids. At each material point of an elastic material, the stress at the present time depends only on the present value of the strain. On the other hand, for an incompressible viscous fluid, the stress at a given point is a function of the present value of the velocity gradient at that point (plus an undetermined pressure). Viscoelastic materials have properties between those of elastic materials and viscous fluids. Such materials have memory, where the stress depends not only on the present values of the strain or velocity gradient but also on the entire temporal history of motion (see [18]). Therefore, the research on the vibration of the viscoelastic string with a fractional length has important physical significance and scientific value. More recently, Xiang and Hu [19] investigated the following fractional viscoelastic equation of the Kirchhoff type:

$$u_{tt} + h([u]^2)(-\Delta)^m u - \int_0^t g(t - \tau)(-\Delta)^m u(\tau) \, d\tau + (-\Delta)^s u_t = \lambda |u|^{q-2} u.$$

They proved the local and global existence of solutions by the Galerkin approximations and obtained the blow-up of solutions by the concavity arguments. However, to the best of our knowledge, much less effort has been devoted to similar studies.

Motivated by the above works, we would like to deal with the problems in Equations (1)–(3). In suitable functional spaces, we aim to study the global existence and asymptotic behavior of solutions in time. First of all, compared with [19], Equation (1) is non-degenerate due to the expression of the Kirchhoff function. Secondly, although we also evaluate the evolutional properties of solutions, we concentrate on the relationship between the initial data and them. In addition, our main method is the potential well theory that is different from classical ones. In the framework of our potential well theory, it is not necessary to introduce the Nehari functional or the Nehari manifold.

This paper is organized as follows. Section 1 is the introduction. In Section 2, we prepare the preliminary knowledge on the functional space. Applying the idea from [20], we define a potential well and provide its properties. Moreover, we display assumptions and notations corresponding to the problems in Equations (1)–(3). In Section 3, we introduce our main method in detail. In Section 4, we prove the global existence of solutions. Section 5 is devoted to the proof of the asymptotic behavior of the solutions by means of the perturbed energy method [21,22]. In Section 6, we summarize our main results.

2. Preliminaries

In this section, we first recall some necessary definitions and properties (see [23–25] for further details).

Let $X$ be the linear space of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to $\Omega$ of any function $u$ in $X$ belongs to $L^2(\Omega)$ and

$$\int \int_Q \frac{|u(x) - u(y)|^2}{|x-y|^{N+2m}} \, dx \, dy < \infty,$$

where $Q := \mathbb{R}^{2N} \setminus (C\Omega \times C\Omega)$ and $C\Omega := \mathbb{R}^N \setminus \Omega$. The space $X$ is endowed with

$$\|u\| = \|u\|_{L^2(\Omega)} + \left( \int \int_Q \frac{|u(x) - u(y)|^2}{|x-y|^{N+2m}} \, dx \, dy \right)^{\frac{1}{2}}.$$
It is easy to check that $\| \cdot \|_X$ is a norm on $X$. Moreover, we introduce the following closed linear subspace of $X$:

$$X_0 = \{ u \in X | u = 0 \text{ a.e. in } C \Omega \}.$$  

This is a Hilbert space equipped with the inner product

$$(u, v)_\ast := (u, v)_{X_0} = \int_Q \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2m}} \, dx \, dy$$

and the norm

$$\| u \|_\ast := \| u \|_{X_0} = \left( \int_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2m}} \, dx \, dy \right)^{1/2}.$$  

Here, $\| u \|_\ast$ is equivalent to $\| u \|_X$.

The embedding $X_0 \hookrightarrow L^q(\Omega)$ is continuous for any $1 \leq q \leq q^*$ and compact for any $1 \leq q < q^*$, where

$$q^* = \begin{cases} \frac{2N}{N - 2m} & \text{if } 2m < N, \\ \infty & \text{if } 2m \geq N. \end{cases}$$  

In this paper, the exponent $q$ of the source term satisfies the following assumption:

$$(A_1) \ 2 < q < q^*.$$  

Moreover, as in [26], the memory kernel $g$ satisfies

$$(A_2) \ g \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \ g(t) \geq 0, \ g'(t) \leq 0 \text{ for all } t \in [0, \infty), \text{ and}$$

$$\kappa := 1 - \int_0^\infty g(t) \, dt > 0.$$  

For the sake of simplicity, we denote

$$\| \cdot \|_p := \| \cdot \|_{L^p(\Omega)}, \ (u, v) := \int_\Omega uv \, dx,$$

and

$$(g \circ u)(t) := \int_0^t g(t - \tau) \|u(t) - u(\tau)\|^2_2 \, d\tau.$$  

**Definition 1.** A function $u \in L^\infty(0, T; X_0)$ with $u_t \in L^\infty(0, T; L^2(\Omega))$ is called a weak solution to Equations (1)–(3) if $u(0) = u_0 \in X_0, u_t(0) = u_1 \in L^2(\Omega),$ and

$$(u_t(t), w) + \int_0^t h(\|u(\tau)\|_2^2)(u(\tau), w)_\ast \, d\tau - \int_0^t \int_0^s g(s - \tau)(u(\tau), w)_\ast \, d\tau \, ds$$

$$+ (u(t), w) = (u_1, w) + (u_0, w) + \int_0^t (f(u(\tau)), w) \, d\tau$$

for any $w \in X_0$ and $t \in (0, T]$.

We define the total energy function associated with the problems in Equations (1)–(3) as follows:

$$E(t) = \frac{1}{2} \|u_t(t)\|^2_2 + \frac{1}{2p} \|u(t)\|_p^{2p} + \frac{1}{2} \left( 1 - \int_0^t g(\tau) \, d\tau \right) \|u(t)\|_2^2$$

$$+ \frac{1}{2} (g \circ u)(t) - \frac{1}{q} \|u(t)\|^q_q.$$
The potential well is
\[
\mathcal{W} = \left\{ u \in X_0 \left| \| u \|_s < \left( \frac{2q}{(q-2)\kappa} d \right)^{\frac{1}{2}} \right. \right\}
\]
and its boundary is
\[
\partial \mathcal{W} = \left\{ u \in X_0 \left| \| u \|_s = \left( \frac{2q}{(q-2)\kappa} d \right)^{\frac{1}{2}} \right. \right\},
\]
where the depth of the potential well is
\[
d = \frac{q - 2}{2q} \frac{\kappa^{\frac{q}{4}}}{\kappa^{\frac{q}{2}}} e_1^{-\frac{q}{4}}.
\]
In addition, \(e_1\) is the best Sobolev constant for the embedding \(X_0 \hookrightarrow L^q(\Omega)\); in other words, we have
\[
e_1 = \sup_{u \in X_0 \setminus \{0\}} \frac{\| u \|_q}{\| u \|_s}.
\]

Lemma 1. Let \((A_1)\) and \((A_2)\) be fulfilled. Then, the following are true:

(i) If \(u \in \mathcal{W}\) and \(\| u \|_s \neq 0\), then \(\kappa \| u \|_s^2 \geq \| u \|_q^q\);  

(ii) If \(u \in \partial \mathcal{W}\), then \(\kappa \| u \|_s^2 \geq \| u \|_q^q\).

Proof. (i) By \(u \in \mathcal{W}\) and Equation (4), we have
\[
\| u \|_s \leq \left( \frac{2q}{(q-2)\kappa} d \right)^{\frac{1}{2}}.
\]
It follows from Equation (6) that
\[
\| u \|_s \leq \kappa^{\frac{1}{2}} e_1^{-\frac{q}{4}}.
\]
Noting that \(\| u \|_s \neq 0\), we have
\[
\kappa \| u \|_s^2 \geq e_1^q \| u \|_q^q.
\]
Hence, we obtain
\[
\| u \|_s \geq \| u \|_q^q.
\]
(ii) By \(u \in \partial \mathcal{W}\) and Equation (5), we find
\[
\| u \|_s = \left( \frac{2q}{(q-2)\kappa} d \right)^{\frac{1}{2}}.
\]
By the similar arguments in the proof of (i), it is easy to see that \(\kappa \| u \|_s^2 \geq \| u \|_q^q\).  

The main results of this paper are proven in Sections 4 and 5.

3. Methods

The potential well was first proposed by Sattinger [27] in order to study the global existence of solutions to a nonlinear hyperbolic equation. Subsequently, it was widely employed to analyze the qualitative properties of the solutions to evolution equations (see, for example, [18,28–39] and the references therein), and it has now developed into a theoretical system.
In general, by the energy functional $J(u)$ and the Nehari functional $I(u)$, the classical potential well can usually be defined by

$$\mathcal{W} = \{u| J(u) < d, I(u) > 0 \} \cup \{0\}.$$ 

The critical points of $J(u)$ are stationary solutions of the problem under consideration. Under appropriate assumptions, $J(u)$ satisfies the Palais–Smale condition, and the problem under consideration admits at least a positive stationary solution whose energy $d$, namely the depth of the potential well, can be defined by

$$d = \inf_{u \in \mathcal{N}} J(u),$$

where the Nehari manifold is

$$\mathcal{N} = \{u| I(u) = 0 \} \setminus \{0\}.$$ 

In the present paper, we describe the potential well as a sphere (see Equation (4)) whose radius is expressed by $d$ (see Equation (6)). Thus, the spatial structure of the potential well is clearer, and it is not necessary to introduce $I(u)$ and $\mathcal{N}$. As for the original definition and calculation process of $d$, we refer interested readers to [20].

4. Global Existence of Solutions

**Theorem 1.** Let $(A_1)$ and $(A_2)$ be fulfilled. Assume that $u_0 \in \mathcal{W}$, $u_1 \in L^2(\Omega)$, and $E(0) < d$. Then, Equations (1)–(3) admit a global solution $u(t) \in \overline{\mathcal{W}} := \mathcal{W} \cup \partial\mathcal{W}$ for all $t \in (0, \infty)$.

**Proof.** Let $\{\omega_j\}_{j=1}^\infty$ be an orthogonal basis of $X_0$ and an orthonormal basis of $L^2(\Omega)$ given by the eigenfunctions of $(-\Delta)^m$ with the boundary condition in Equation (3) (see [24] (Proposition 9) for details). Denote $W_n = \text{Span}\{\omega_1, \omega_2, \ldots, \omega_n\}$, $n = 1, 2, \ldots$. We seek the approximate solutions to Equations (1)–(3), given by

$$u_n(t) = \sum_{j=1}^n \zeta_{jn}(t)\omega_j, \quad n = 1, 2, \ldots,$$

which satisfy

$$(u_{nit}(t), w) + h(\|u_n(t)\|^2_2)(u_n(t), w) - \int_0^t g(t-\tau)(u_n(\tau), w) \, d\tau + (u_{nt}(t), w) = (f(u_n(t)), w), \quad t > 0,$$

$$u_n(0) = \sum_{j=1}^n \xi_{jn}(0)\omega_j \to u_0 \text{ in } X_0,$$

$$u_{nt}(0) = \sum_{j=1}^n \xi_{jn}'(0)\omega_j \to u_1 \text{ in } L^2(\Omega),$$

for any $w \in W_n$. Let $\xi_n(t) = (\xi_{1n}(t), \xi_{2n}(t), \ldots, \xi_{nn}(t))^T$. Then, the vector function $\xi_n$ solves

$$\xi_{nn}''(t) + \xi_n'(t) + \mathcal{L}_n(t, \xi_n(t)) = \mathcal{F}_n(\xi_n(t)), \quad t > 0,$$

$$\xi_n(0) = ((u_0, \omega_1), (u_0, \omega_2), \ldots, (u_0, \omega_n))^T,$$

$$\xi_n'(0) = ((u_1, \omega_1), (u_1, \omega_2), \ldots, (u_1, \omega_n))^T,$$

where

$$\mathcal{L}_n(t, \xi_n(t)) = (\mathcal{L}_{1n}(t, \xi_n(t)), \mathcal{L}_{2n}(t, \xi_n(t)), \ldots, \mathcal{L}_{nn}(t, \xi_n(t)))^T,$$
\[
\mathcal{L}_{in}(t, \xi_n(t)) = h \left( \left\| \sum_{j=1}^{n} \xi_j(t) \omega_j \right\|^2 + \left( \sum_{j=1}^{n} \xi_j(t) \omega_j, \omega_j \right) \right) - \int_0^t g(t - \tau) \left( \sum_{j=1}^{n} \xi_j(\tau) \omega_j, \omega_j \right) \, d\tau,
\]

\[
\mathcal{F}_n(\xi_n(t)) = (\mathcal{F}_{1n}(\xi_n(t)), \mathcal{F}_{2n}(\xi_n(t)), \ldots, \mathcal{F}_{nn}(\xi_n(t)))^T,
\]

In terms of standard theory for ODEs, the Cauchy problem in Equations (11)–(13) admits a solution \( \xi_n \in C^2[0, T_n] \) with \( T_n \leq T \). In turn, this gives a solution \( u_n(t) \) defined by Equation (7) and satisfying Equations (8)–(10). The following estimates will allow us to extend the local solution to \( [0, T] \) for any \( T > 0 \).

By using \( w = u_{nt}(t) \) in Equation (8), we obtain

\[
\frac{d}{dt} \left( \frac{1}{2} \left\| u_{nt}(t) \right\|_2^2 + \frac{1}{2} \left\| u_n(t) \right\|_2^2 + \frac{1}{2p} \left\| u_n(t) \right\|_{2^p}^2 \right) - \int_0^t g(t - \tau) (u_n(\tau), u_{nt}(\tau))_s \, d\tau
\]

\[
+ \left\| u_{nt}(t) \right\|_2^2 = \frac{1}{q} \frac{d}{dt} \left\| u_{nt}(t) \right\|_q^q.
\]

Note that

\[
\int_0^t g(t - \tau) (u_n(\tau), u_{nt}(\tau))_s \, d\tau
\]

\[
= \int_0^t g(t - \tau) (u_n(\tau) - u_n(t), u_{nt}(t))_s \, d\tau + \int_0^t g(t - \tau) (u_n(t), u_{nt}(t))_s \, d\tau
\]

\[
= -\frac{1}{2} \int_0^t g(t - \tau) \frac{d}{dt} \left\| u_n(\tau) - u_n(t) \right\|_2^2 \, d\tau + \frac{1}{2} \int_0^t g(t - \tau) \frac{d}{dt} \left\| u_n(t) \right\|_2^2 \, d\tau
\]

\[
= -\frac{1}{2} \frac{d}{dt} \left( g \circ u_n \right) (t) - \int_0^t g(\tau) \, d\tau \left\| u_n(t) \right\|_2^2 + \frac{1}{2} \left( g' \circ u_n \right) (t) - \frac{1}{2} g(t) \left\| u_n(t) \right\|_2^2.
\]

By substituting this equality into Equation (14) and integrating it with respect to \( t \), we deduce that

\[
E_n(t) + \int_0^t \left( \left\| u_{nt}(\tau) \right\|_2^2 - \frac{1}{2} (g' \circ u_n)(\tau) + \frac{1}{2} g(\tau) \left\| u_n(\tau) \right\|_2^2 \right) \, d\tau = E_n(0)
\]

for all \( t \in [0, T] \), where

\[
E_n(t) = \frac{1}{2} \left\| u_{nt}(t) \right\|_2^2 + \frac{1}{2p} \left\| u_n(t) \right\|_{2^p}^2 + \frac{1}{2} \left( 1 - \int_0^t g(\tau) \, d\tau \right) \left\| u_n(t) \right\|_2^2
\]

\[
+ \frac{1}{2} (g \circ u_n)(t) - \frac{1}{q} \left\| u_n(t) \right\|_q^q.
\]

In light of Equations (9) and (10), we infer that \( E_n(0) < d \) and \( u_n(0) \in \mathcal{W} \) for a sufficiently large \( n \). We now claim that

\[
u_n(t) \in \mathcal{W}
\]

for all \( t \in [0, T] \) and a sufficiently large \( n \). Suppose that \( u_n(t) \notin \mathcal{W} \) for some \( 0 < t < T \). Then, there exists a time \( 0 < t_0 < T \) such that \( u_n(t_0) \in \partial \mathcal{W} \) and \( u_n(t) \in \mathcal{W} \) for all \( t \in [0, t_0) \). Hence, we obtain

\[
\left\| u_n(t_0) \right\|_s = \left( \frac{2g}{(q - 2)^\frac{1}{p}} \right)^{\frac{1}{2}}.
\]
Through Equation (16) and (ii) in Lemma 1, we obtain
\[
E_n(t_0) \geq \frac{1}{2} \kappa \|u_n(t_0)\|_2^2 - \frac{1}{q} \|u_n(t_0)\|_q^q
= \frac{q - 2}{2q} \kappa \|u_n(t_0)\|_2^2 + \frac{1}{q} \left( \kappa \|u_n(t_0)\|_2^2 - \|u_n(t_0)\|_q^q \right)
\geq \frac{q - 2}{2q} \kappa \|u_n(t_0)\|_2^2
= d,
\]
which contradicts $E_n(0) < d$ according to Equation (15).

From Equation (16), the assertion in Equation (17), and (i) in Lemma 1, it follows that
\[
E_n(t) \geq \frac{1}{2} \|u_{nt}(t)\|_2^2 + \frac{1}{2} \kappa \|u_n(t)\|_2^2 - \frac{1}{q} \|u_n(t)\|_q^q
= \frac{1}{2} \|u_{nt}(t)\|_2^2 + \frac{q - 2}{2q} \kappa \|u_n(t)\|_2^2 + \frac{1}{q} \left( \kappa \|u_n(t)\|_2^2 - \|u_n(t)\|_q^q \right)
\geq \frac{1}{2} \|u_{nt}(t)\|_2^2 + \frac{q - 2}{2q} \kappa \|u_n(t)\|_2^2,
\]
which, together with Equation (15), gives
\[
\frac{1}{2} \|u_{nt}(t)\|_2^2 + \frac{q - 2}{2q} \kappa \|u_n(t)\|_2^2 < d
\]
for all $t \in [0, T]$. Thus, for all $t \in [0, T]$, we find
\[
\|u_{nt}(t)\|_2^2 < 2d
\]
and
\[
\|u_n(t)\|_2^2 < \frac{2q}{(q - 2) \kappa} d.
\]
Furthermore, we deduce from Equation (19) that
\[
\|f(u_n(t))\|_q^q = \|u_n(t)\|_q^q \leq c_1^q \|u_n(t)\|_2^2 < c_1^q \left( \frac{2q}{(q - 2) \kappa} d \right)^{\frac{q}{q - 2}}
\]
for all $t \in [0, T]$, where $r = \frac{q}{q - 1}$.

The above estimates mean the following:
\[
\{u_n\} \text{ is bounded in } L^\infty(0, T; X_0),
\]
\[
\{u_{nt}\} \text{ is bounded in } L^\infty(0, T; L^2(\Omega)),
\]
\[
\{f(u_n)\} \text{ is bounded in } L^\infty(0, T; L'(\Omega)).
\]

Therefore, there exist $u, \chi$, and a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that as $n \to \infty$, the following are true:
\[
u_n \rightharpoonup u \text{ weakly star in } L^\infty(0, T; X_0),
\]
\[
u_{nt} \rightharpoonup u_t \text{ weakly star in } L^\infty(0, T; L^2(\Omega)),
\]
\[
f(u_n) \rightharpoonup \chi \text{ weakly star in } L^\infty(0, T; L'(\Omega)).
\]
Thus, we have the following:

\[ u_n \to u \text{ in } L^2(0,T;L^2(\Omega)) \text{ and a.e. in } \Omega \times [0,T]. \]

In terms of [32] (Chapter 1, Lemma 1.3), we have \( \chi = f(u) \).

Integrating Equation (8) with respect to \( t \) yields

\[
(u_{nt}(t),w) + \int_0^t h(\|u(\tau)\|^2)(u(\tau),w)_* \, d\tau - \int_0^t \int_0^s g(s-\tau)(u(\tau),w)_* \, d\tau \, ds
\]

\[
+ (u_n(t),w) = (u_{nt}(0),w) + (u_n(0),w) + \int_0^t (f(u(\tau)),w) \, d\tau.
\]

Using \( n \to \infty \), we further obtain

\[
(u(t),w) + \int_0^t h(\|u(\tau)\|^2)(u(\tau),w)_* \, d\tau - \int_0^t \int_0^s g(s-\tau)(u(\tau),w)_* \, d\tau \, ds
\]

\[
+ (u(t),w) = (u_1,w) + (u_0,w) + \int_0^t (f(u(\tau)),w) \, d\tau.
\]

By virtue of Equations (9) and (10), we have \( u(0) = u_0 \) in \( X_0 \) and \( u(t) = u_1 \) in \( L^2(\Omega) \).

Therefore, \( u \) is a global solution to Equations (1)–(3). In addition, from Equation (20), we have

\[
\|u(t)\|_* \leq \liminf_{n \to \infty} \|u_n(t)\|_*,
\]

which, together with Equation (19), tells us that

\[
\|u(t)\|_* \leq \left( \frac{2q}{(q-2)\kappa} \right)^{\frac{1}{2}}.
\]

In other words, \( u(t) \in \overline{W} \) for all \( t \in (0,\infty) \). \( \square \)

5. Asymptotic Behavior of the Solutions

**Theorem 2.** In addition to all the assumptions of Theorem 1, suppose that there exists a constant \( \rho > 0 \) such that \( g'(t) \leq -\rho g(t) \) for all \( t \in [0,\infty) \). Then, we have

\[
\|u(t)\|^2_s + \|u_t(t)\|^2_2 \leq \kappa e^{-\beta t}, \quad \forall t \in [0,\infty),
\]

for some constants \( \alpha, \beta > 0 \).

**Proof.** For the approximate solutions given in the proof of Theorem 1, we construct

\[
L(t) = E_n(t) + \varepsilon \Psi(t), \quad \forall t \in [0,\infty),
\]

where \( \Psi(t) = (u_n(t),u_{nt}(t)) \) and \( \varepsilon > 0 \) is a constant to be determined later.

We now claim that there exist two constants \( \gamma_i > 0 \) \( (i = 1, 2) \), depending on \( \varepsilon \), such that

\[
\gamma_1 E_n(t) \leq L(t) \leq \gamma_2 E_n(t), \quad \forall t \in [0,\infty).
\]

Indeed, by virtue of Cauchy’s inequality, we find

\[
\left| \Psi(t) \right| \leq \frac{1}{2} \left( \|u_n(t)\|_2^2 + \|u_{nt}(t)\|_2^2 \right),
\]

and thus

\[
\left| \Psi(t) \right| \leq \frac{\varepsilon^2}{2} \left( \|u_n(t)\|_2^2 + \frac{1}{2} \|u_{nt}(t)\|_2^2 \right),
\]

(24)
where $c_2$ is the best Sobolev constant for the embedding $X_0 \hookrightarrow L^2(\Omega)$. By combining Equations (24) and (18), we obtain $|\Psi(t)| \leq C_1 E_n(t)$ for some constant $C_1 > 0$ independent of $n$ which, together with Equation (22), yields that the assertion in Equation (23) holds.

It can be said that

$$E_n'(t) = \frac{1}{2}(g' \circ u_n)(t) - \frac{1}{2}g(t)\|u_n(t)\|^2 - \|u_{nt}(t)\|^2_2. $$

Then, a direct calculation gives

$$L'(t) = \frac{1}{2}(g' \circ u_n)(t) - \frac{1}{2}g(t)\|u_n(t)\|^2 - \|u_{nt}(t)\|^2_2 + \epsilon\|u_{nt}(t)\|^2_2
- \epsilon\|u_n(t)\|^2_2 + \epsilon\int_0^t g(t - \tau)(u_n(t), u_n(t)) \, d\tau
- \epsilon(u_n(t), u_{nt}(t)) + \epsilon\|u_n(t)\|^2_2. $$

For the seventh term on the right side of Equation (25), it follows from Schwarz's inequality and Cauchy's inequality with $\epsilon_1 > 0$ that

$$\int_0^t g(t - \tau)(u_n(t), u_n(t)) \, d\tau = \int_0^t g(t - \tau)\|u_n(t)\|^2 + \epsilon\int_0^t g(t - \tau)\|u_n(t)\|^2 + \frac{1}{4\epsilon_1} (g \circ u_n)(t)
\leq (1 - \kappa)\|u_n(t)\|^2 + \epsilon_1(1 - \kappa)\|u_n(t)\|^2 + \frac{1}{4\epsilon_1} (g \circ u_n)(t).$$

For the eighth term on the right side of Equation (25), it follows from Cauchy's inequality with $\epsilon_2 > 0$ that

$$-(u_n(t), u_{nt}(t)) \leq \epsilon_2\|u_n(t)\|^2 + \frac{1}{4\epsilon_2} \|u_{nt}(t)\|^2_2
\leq \epsilon_2 c_2^2\|u_n(t)\|^2 + \frac{1}{4\epsilon_2} \|u_{nt}(t)\|^2_2. $$

Hence, we have

$$L'(t) \leq \left(\epsilon + \frac{\epsilon}{4\epsilon_2} - 1\right)\|u_{nt}(t)\|^2_2 - \epsilon\|u_n(t)\|^2_2
+ \epsilon(\epsilon_1(1 - \kappa) + \epsilon_2 c_2^2 - \kappa)\|u_n(t)\|^2
+ \left(\frac{\epsilon}{4\epsilon_1} - \frac{\epsilon}{2}\right) (g \circ u_n)(t) + \epsilon\|u_n(t)\|^2_2,$$

and so

$$L'(t) \leq -\eta E_n(t) + \left(\epsilon + \frac{\epsilon}{4\epsilon_2} + \frac{\epsilon\eta}{2} - 1\right)\|u_{nt}(t)\|^2_2 + \epsilon\left(\frac{\eta}{2\epsilon} - 1\right)\|u_n(t)\|^2_2
+ \epsilon(\epsilon_1(1 - \kappa) + \epsilon_2 c_2^2 + \frac{\eta}{2} - \kappa)\|u_n(t)\|^2
+ \left(\frac{\epsilon}{4\epsilon_1} + \frac{\epsilon\eta}{2} - \frac{\epsilon}{2}\right) (g \circ u_n)(t) + \epsilon\|u_n(t)\|^2_2 - \frac{\epsilon\eta}{q} \|u_n(t)\|^2_2, $$

where $\eta > 0$ is a constant to be determined later. It follows from Equations (15) and (18) that

$$E_n(0) \geq \frac{q - 2}{2q} \kappa \|u_n(t)\|^2_2,
which leads to
\[ \| u_n(t) \| \leq \left( \frac{2q}{(q-2)\kappa} E_n(0) \right)^{\frac{1}{2}}. \]

Hence, we have
\[ \| u_n(t) \|^{\frac{q}{q-2}} \leq C_1^{\frac{q}{q-2}} \| u_n(t) \|^{\frac{q}{q-2}} \leq C_1^{\frac{q}{q-2}} \left( \frac{2q}{(q-2)\kappa} E_n(0) \right)^{\frac{q-2}{2}} \| u_n(t) \|. \]

By substituting this inequality into Equation (26), we obtain
\[ L'(t) \leq -\eta E_n(t) + \left( \varepsilon + \frac{\varepsilon}{4e_2} + \frac{\eta}{2} - 1 \right) \| u_{nt}(t) \|^2 + \varepsilon \left( \frac{\eta}{2p} - 1 \right) \| u_n(t) \|^{2p} \]
\[ + \left( \varepsilon \left( \frac{\varepsilon_1(1-\kappa) + e_2 \varepsilon_2^2 + \eta + C_1^{\frac{q}{q-2}} \left( \frac{2q}{(q-2)\kappa} E_n(0) \right)^{\frac{q-2}{2}} \right) \right) \| u_n(t) \|^2 \]
\[ + \left( \frac{\varepsilon}{4e_1} + \frac{\eta}{2} - \frac{\rho}{2} \right) (g \circ u_n)(t). \]

Note that
\[ C_1^{\frac{q}{q-2}} \left( \frac{2q}{(q-2)\kappa} E_n(0) \right)^{\frac{q-2}{2}} < C_1^{\frac{q}{q-2}} \left( \frac{2q}{(q-2)\kappa} d \right)^{\frac{q-2}{2}} = \kappa. \]

We choose a sufficiently small \( \varepsilon_i \) \( (i = 1, 2) \) and \( \eta \) such that \( \eta < 2p \) and
\[ \varepsilon_1(1-\kappa) + e_2 \varepsilon_2^2 + \eta + C_1^{\frac{q}{q-2}} \left( \frac{2q}{(q-2)\kappa} E_n(0) \right)^{\frac{q-2}{2}} - \kappa < 0. \]

Thus, for a fixed \( \varepsilon_i \) \( (i = 1, 2) \) and \( \eta \), we can choose
\[ \varepsilon < \min \left\{ \frac{1}{C_1}, \frac{4e_2}{4e_2 + 1 + 2\eta e_2}, \frac{2\rho e_1}{1 + 2\eta e_1} \right\} \]
such that \( L'(t) \leq -\eta E_n(t) \) which, together with the second inequality in the assertion in Equation (23), gives \( L'(t) \leq -\frac{\eta}{\gamma_2} L(t) \). Hence, there exists a constant \( C_2 > 0 \) independent of \( n \) such that
\[ L(t) \leq C_2 e^{-\frac{\eta}{\gamma_2} t}, \ \forall t \in [0, \infty). \]

We further conclude from the first inequality in the assertion in Equation (23) that
\[ E_n(t) \leq \frac{C_2}{\gamma_1} e^{-\frac{\eta}{\gamma_2} t}, \ \forall t \in [0, \infty). \]  

(27)

From Equations (20) and (21), it follows that
\[ \| u(t) \|^{\frac{q}{q-2}} + \| u_{nt}(t) \|^2 \leq \liminf_{n \to \infty} \left( \| u_n(t) \|^{\frac{q}{q-2}} + \| u_{nt}(t) \|^{\frac{q}{q-2}} \right), \]
which, combined with Equations (18) and (27), gives the conclusion of Theorem 2. \( \Box \)

6. Conclusions
In this paper, we studied the initial boundary value problem for a fractional viscoelastic equation of the Kirchhoff type. In the framework of the potential well theory, we established the global existence theorem, specifically Theorem 1. Under appropriate assumptions of the exponent of the source term and the memory kernel, it has been shown that if the initial data \( u_0 \) lies in the potential well, and the initial energy is less than the depth of the potential well, then the initial boundary value problem admits a global solution that lies in
the closure of the potential well. Moreover, we have established the asymptotic behavior theorem, specifically Theorem 2. It is established that as the time variable tends toward infinity, the norm of the solutions in the phase space decays exponentially to zero at the same rate as the memory kernel. In light of the applications, once the initial data and the external force are effectively controlled, the vibration of the string with a fractional length and appropriate viscoelasticity will be stable. In this regard, the methods in [40] may be helpful.

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