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Qualitative Properties of Positive Solutions of a Kind for Fractional Pantograph Problems using Technique Fixed Point Theory

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Abstract: The current paper intends to report the existence and uniqueness of positive solutions for nonlinear pantograph Caputo–Hadamard fractional differential equations. As part of a procedure, we transform the specified pantograph fractional differential equation into an equivalent integral equation. We show that this equation has a positive solution by utilising the Schauder fixed point theorem (SFPT) and the upper and lower solutions method. Another method for proving the existence of a singular positive solution is the Banach fixed point theorem (BFPT). Finally, we provide an example that illustrates and explains our conclusions.

Keywords: fractional differential equations; positive solution; upper and lower solution; pantograph equation; fixed point theorem

MSC: 26A33; 34A12; 34K37; 34G20

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1. Introduction

Differential equations have been shown to be effective tools for describing a wide range of phenomena in modern-world problems. There has been meaningful progress in the study of different classes of differential equations. Traditional integer-order derivatives have recently lost popularity in recent decades in favor of fractional-order derivatives. This is because a variety of mathematical models for current issues involving fractional-order derivatives have been investigated, and their findings have been considerable. In contrast to integer-order derivatives, which are local operators, noninteger-order derivatives have the advantage of being global operators that yield precise and consistent results. Numerous classes of differential equations have been reorganized and constructed in terms of fractional-order derivatives as a result of these great benefits.

One of the major classes of differential equations is the class of implicit differential equations. These equations are useful in management and economic sciences. The differential equations in the equilibrium state are typical of the implicit type in economic difficulties. Related to this, we can use implicit functions to explore important aspects of most real-world graphs or surface geometry.

Mathematicians, physicists, biologists, engineers, and economists all share an attraction to the background of fractional differential equations (FDEs). It has been in development since the end of the 17th century. The quantity of papers and scientific conferences devoted to this idea in recent years shows the significance of the questions answered by this concept, which is more theoretical in nature than applied. It has expanded into a whole discipline. Experts believe that the story starts at the end of 1695 (see [1–3]).

FDEs bear an abundance of applications in science and engineering. Many authors have devoted their attention on problems relating to qualitative investigation of the positive aspects of these solutions for FDEs, see [4–13] and the references therein. Niazi et al. [14], Iqbal et al. [15], Shafqat et al. [16], Alnahdi [17], Khan [18] and Abuasbeh et al. [19–21] investigated the existence and uniqueness of the FFEEs. Kalidass et al. [22] and Hammouch et al. [23] worked on fractional order and numerical solutions of differential equations.

The generalised pantograph equation has a variety of applications. Only applications in number theory are mentioned [24], in electrodynamics [25] and in the absorption of energy by the pantograph of an electronic locomotive [26].

Recently, in [4], where $1 < \alpha \leq 2$, and $f \in C([0,1] \times [0,\infty), [0,\infty))$ this is produced. The nonlinear FDE boundary value problem (BVP)

$$\begin{cases} D^{\alpha}x(t) + f(t, x(t)) = 0, \ 0 < \alpha < 1, \\ x(0) = x(1) = 0, \end{cases}$$

to use certain FPTs on cone, the existence and multitude of consequences of positive solutions have been created.

In [8], the existence and uniqueness of the positive solution of the FDE was examined

$$\begin{cases} {}^{C}D^{\alpha}x(t) = \mathfrak{f}(t, x(t)), \ 0 < \alpha \leq 1, \\ x(0) = 0, \ x'(0) = \theta > 0, \end{cases}$$

where $1 < \alpha \leq 2$, ${}^{C}D^{\alpha}$ is the usual Caputo fractional derivative, $\mathfrak{f} : [0,1] \times [0,\infty) \to [0,\infty)$ continuous function. The researchers obtained positive results by using the upper and lower solutions technique and FPTs.

The exploration of subjective theory for problems of positive solutions to pantograph FDEs is the focus of this study. The above mentioned works have motivated and inspired this work [8,27], and we concentrate on the PS for pantograph FDE and the connections therein

$$\begin{cases} \mathfrak{D}_{1}^{\nu}\varkappa(\varsigma) = \varpi(\varsigma, \varkappa(\varsigma), \varkappa(1+\lambda\varsigma)) + \mathfrak{D}_{1}^{\nu-1}\hbar(\varsigma, \varkappa(1+\lambda\varsigma)), \ \varsigma \in [1, \Im], \\ \varkappa(1) = \theta_1 > 0, \ \varkappa'(1) = \theta_2 > 0, \end{cases} \tag{1}$$

where $\lambda \in \left(0, \frac{\Im-1}{\Im}\right)$, $\varkappa(1+\lambda) = \varkappa_0 > 0$, $\mathfrak{D}_{1}^{\nu}$ is the standard Caputo–Hadamard fractional derivatives of order $1 < \nu \leq 2$, $\hbar, \varpi : [1, \Im] \times [0,\infty) \times [0,\infty) \to [0,\infty)$, are continuous functions, $\hbar$ is non-decreasing on $\varkappa$ and $\theta_2 \geq \hbar(1, \varkappa_0)$. We convert (1) into an integral equation and then utilise the upper and lower solution method and the Schauder and Banach FPTs to establish the existence and uniqueness of the positive solution.

The following is a breakdown of the paper structure. Section 2 describes some key concepts, lemmas, and theorems that will be used throughout this study. Reversal of (1) and the BFPT also are presented. We refer the reader to [28] for more details on the Banach and Schauder FPTs. In Section 3, we present and explain our chief goal findings on positive solution. The new results in this article are superior and more general than those in [8,27].

2. Preliminaries

This section introduces some important fundamental definitions that will be needed for obtaining our results in the next sections. For more details see [1,3,6,7,27,29–33].

Let $X = C([1, \Im])$ be the Banach space of all real-valued continuous functions, with the maximum norm defined on the compact interval $[1, \Im]$.

Create the $\mathcal{A} = \{\varkappa \in X : \varkappa(\varsigma) \geq 0, \ \varsigma \in [1, \Im]\}$ subset of X. $\varkappa(\varsigma) > 0$, $1 \leq \varsigma \leq \Im$, is said to function as a positive solution $\varkappa \in X$.

Suppose $a, b \in \mathbb{R}^+$ as a $b > a$. For any $\varkappa, y \in [a, b]$, we propose the upper-control function

$$U(\varsigma, \varkappa, y) = \sup\{\varpi(\varsigma, v, \mu) : a \leq v \leq \varkappa, a \leq \mu \leq y\},$$

and lower-control function

$$L(\varsigma, \varkappa, y) = \inf\{\varpi(\varsigma, v, \mu) : \varkappa \leq v \leq b, y \leq \mu \leq b\}.$$

Clearly, $U(\varsigma, \varkappa, y)$ and $L(\varsigma, \varkappa, y)$ are monotonous non-decreasing on the arguments $\varkappa, y$ and $L(\varsigma, \varkappa, y) \leq \varpi(\varsigma, \varkappa, y) \leq U(\varsigma, \varkappa, y)$.

Definition 1 ([1,3,6]). *The Riemann–Liouville fractional integral (RLFI) of order > 0 for a function $\varkappa : [0, +\infty) \to \mathbb{R}$ given by*

$$I^{\nu}\varkappa(\varsigma) = \frac{1}{\Gamma(\nu)} \int_0^{\varsigma} (\varsigma - \tau)^{\nu-1} \varkappa(\tau) d\tau,$$

where Γ the Euler gamma function is stated as follows

$$\Gamma(\nu) = \int_0^{\infty} e^{-\varsigma} \varsigma^{\nu-1} d\varsigma.$$

Definition 2 ([1,3,6]). *The Hadamard fractional integral (HFI) of order $\nu > 0$ for a continuous function $\varkappa : [1, +\infty) \to \mathbb{R}$ is referred to as*

$$\mathfrak{I}_1^{\nu}\varkappa(\varsigma) = \frac{1}{\Gamma(\nu)} \int_1^{\varsigma} \left(\log \frac{\varsigma}{\tau}\right)^{\nu-1} \varkappa(\tau) \frac{d\tau}{\tau}.$$

Definition 3 ([1,3,6]). *The RLF derivative of order $\nu > 0$ for a function $\varkappa : [0, +\infty) \to \mathbb{R}$ is intended by*

$$D^{\nu}\varkappa(\varsigma) = \frac{1}{\Gamma(n-\nu)} \int_0^{\varsigma} (\varsigma - \tau)^{n-\nu-1} \varkappa^{(n)}(\tau) d\tau, \; n-1 < \nu < n, \; n \in \mathbb{N}.$$

Definition 4 ([1,3,6]). *The Caputo–Hadamard fractional derivative of order $\nu > 0$ for a continuous function $\varkappa : [1, +\infty) \to \mathbb{R}$ is the aim of*

$$\mathfrak{D}_1^{\nu}\varkappa(\varsigma) = \frac{1}{\Gamma(n-\nu)} \int_1^{\varsigma} \left(\log \frac{\varsigma}{\tau}\right)^{n-\nu-1} \delta^n \varkappa(\tau) \frac{d\tau}{\tau}, \; \nu \in (n-1, n),$$

where $\delta^n = \left(\varsigma \frac{d}{d\varsigma}\right)^n, n \in \mathbb{N}$.

Lemma 1 ([1,3,6]). *Let $n-1 < \nu \leq n, n \in \mathbb{N}$. The notion of equal $\left(\mathfrak{I}_1^{\nu}\mathfrak{D}_1^{\nu}\varkappa\right)(\varsigma) = 0$ is valid if*

$$\varkappa(\varsigma) = \sum_{k=1}^{n} c_k (\log \varsigma)^{\nu-k} \text{for each} \varsigma \in [1, \infty),$$

where $c_k \in \mathbb{R}, k = 1, \ldots, n$ are constants.

Lemma 2 ([1,3,6]). *Let $m-1 < \nu \leq m, m \in \mathbb{N}$ and $\varkappa \in C^{n-1}[1, \infty)$. Then*

$$\mathfrak{I}_1^{\nu}[\mathfrak{D}_1^{\nu}\varkappa(\varsigma)] = \varkappa(\varsigma) - \sum_{k=0}^{m-1} \frac{\left(\delta^k \varkappa\right)(1)}{\Gamma(k+1)} (\log \varsigma)^k.$$

Lemma 3 ([1,3,6])**.** *For everyone $\mu > 0, \nu > -1$,*

$$\frac{1}{\Gamma(\mu)}\int_1^{\varsigma}\left(\log\frac{\varsigma}{\tau}\right)^{\mu-1}(\log\tau)^{\nu}\frac{d\tau}{\tau} = \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)}(\log\varsigma)^{\mu+\nu}.$$

Lemma 4 ([1,3,6])**.** *Let $\varkappa(\varsigma) = (\log\varsigma)^{\mu}$, where $\mu \geq 0$ and enable $m-1 < \nu \leq m$, $m \in \mathbb{N}$. Then*

$$\mathfrak{D}_1^{\nu}\varkappa(\varsigma) = \begin{cases} 0 & \text{if } \mu \in \{0,1,\ldots,m-1\}, \\ \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)}(\log\varsigma)^{\mu-\nu} & \text{if } \mu \in \mathbb{N}, \mu \geq m \text{ or } \mu \notin \mathbb{N}, \mu > m-1. \end{cases}$$

Lemma 5. *Assume $\varkappa \in C^1([1,\Im])$, $\varkappa^{(2)}$ and $\frac{\partial\hbar}{\partial\varsigma}$ exist, so $\varkappa$ is a solution of (1) equivalent*

$$\varkappa(\varsigma) = \theta_1 + (\theta_2 - \hbar(1,\varkappa_0))\log\varsigma + \int_1^{\varsigma}\hbar(\tau,\varkappa(1+\lambda\tau))\frac{d\tau}{\tau}$$
$$+\frac{1}{\Gamma(\nu)}\int_1^{\varsigma}\left(\log\frac{\varsigma}{\tau}\right)^{\nu-1}\varpi(\tau,\varkappa(\tau),\varkappa(1+\lambda\tau))\frac{d\tau}{\tau}. \tag{2}$$

Proof. Let $\varkappa$ be a solution of (1). First, we will write this equation as

$$\mathfrak{I}_1^{\nu}\mathfrak{D}_1^{\nu}\varkappa(\varsigma) = \mathfrak{I}_1^{\nu}\Big(\varpi(\varsigma,\varkappa(\varsigma),\varkappa(1+\lambda\varsigma)) + \mathfrak{D}_1^{\nu-1}\hbar(\varsigma,\varkappa(1+\lambda\varsigma))\Big),\ 1 < \varsigma \leq \Im.$$

From Lemma 1, we obtained

$$\begin{aligned}\varkappa(\varsigma) - \varkappa(1) - \varkappa'(1)\log\varsigma &= \mathfrak{I}_1^{\nu}\mathfrak{D}_1^{\nu-1}\hbar(\varsigma,\varkappa(1+\lambda\varsigma)) + \mathfrak{I}_1^{\nu}\varpi(\varsigma,\varkappa(\varsigma),\varkappa(1+\lambda\varsigma)) \\ &= \mathfrak{I}_1\mathfrak{I}_1^{\nu-1}\mathfrak{D}_1^{\nu-1}\hbar(\varsigma,\varkappa(1+\lambda\varsigma)) + \mathfrak{I}_1^{\nu}\varpi(\varsigma,\varkappa(\varsigma),\varkappa(1+\lambda\varsigma)) \\ &= \mathfrak{I}_1(\hbar(\varsigma,\varkappa(1+\lambda\varsigma)) - \hbar(1,\varkappa_0)) + \mathfrak{I}_1^{\nu}\varpi(\varsigma,\varkappa(\varsigma),\varkappa(1+\lambda\varsigma)) \\ &= \int_1^{\varsigma}\hbar(\tau,\varkappa(1+\lambda\tau))\frac{d\tau}{\tau} - \hbar(1,\varkappa_0)\log\varsigma \\ &+ \frac{1}{\Gamma(\nu)}\int_1^{\varsigma}\left(\log\frac{\varsigma}{\tau}\right)^{\nu-1}\varpi(\tau,\varkappa(\tau),\varkappa(1+\lambda\tau))\frac{d\tau}{\tau},\end{aligned}$$

the result is (2). So that each process is reversible, the inverse is simple. The proof is now available. □

Finally, we give the FPTs that allow us to demonstrate the existence and uniqueness of a positive solution to (1).

Definition 5. *Suppose that $(X, \|.\|)$ is a Banach space and $\Phi : X \to X$. If there is a $l \in (0,1)$ such that $\varkappa, y \in X$, Φ is a contraction operator*

$$\|\Phi\varkappa - \Phi y\| \leq l\|\varkappa - y\|.$$

Theorem 1 (Banach [28])**.** *Assume $Y \neq \emptyset$ to be a closed-convex subset of a Banach space X and $\Phi : Y \to Y$ to be a contraction operator. Eventually, there is a unique $\varkappa \in Y$ with $\Phi\varkappa = \varkappa$.*

Theorem 2 (Schauder [28])**.** *Let $Y \neq \emptyset$ be a closed-convex subset of a Banach space X and $\Phi : Y \to Y$ be a continuous compact operator. Thus, Φ has a fixed point in Y.*

3. Main Results

This part contains the details we explore at the existence results or a number of events of FDE (1). We also provide necessary conditions for the uniqueness of (1).

We construct an operator $\Phi : \mathcal{A} \longrightarrow X$ by converting (2) to be applied to SFPT

$$(\Phi\varkappa)(\varsigma) = \theta_1 + (\theta_2 - \hbar(1,\varkappa_0))\log\varsigma + \int_1^{\varsigma} \hbar(\tau,\varkappa(1+\lambda\tau))\frac{d\tau}{\tau}$$
$$+ \frac{1}{\Gamma(\nu)}\int_1^{\varsigma}\left(\log\frac{\varsigma}{\tau}\right)^{\nu-1}\varpi(\tau,\varkappa(\tau),\varkappa(1+\lambda\tau))\frac{d\tau}{\tau},\ \varsigma \in [1,\Im], \tag{3}$$

where the determined fixed point required to find the identity operator equation is satisfied $\Phi\varkappa = \varkappa$.

For the next set of results, the following models are applied.

$(\mathcal{F}1)$ Let $\varkappa^+, \varkappa_- \in \mathcal{A}$, as well as $a \leq \varkappa_-(\varsigma) \leq \varkappa^+(\varsigma) \leq b$

$$\begin{cases} \mathfrak{D}_1^{\nu}\varkappa^+(\varsigma) - \mathfrak{D}_1^{\nu-1}\hbar(\varsigma,\varkappa^+(1+\lambda\varsigma)) \geq U(\varsigma,\varkappa^+(\varsigma),\varkappa^+(1+\lambda\varsigma)), \\ \mathfrak{D}_1^{\nu}\varkappa_-(\varsigma) - \mathfrak{D}_1^{\nu-1}\hbar(\varsigma,\varkappa_-(1+\lambda\varsigma)) \leq L(\varsigma,\varkappa_-(\varsigma),\varkappa_-((1+\lambda\varsigma))), \end{cases} \tag{4}$$

for any $\varsigma \in [1,\Im]$.

$(\mathcal{F}2)$ For $\varsigma \in [1,\Im]$ and $\varkappa_1, \varkappa_2, y_1, y_2 \in X$, there exist positive real numbers $\beta_1, \beta_2, \beta_3 > 0$ such that

$$|\hbar(\varsigma,y_1) - \hbar(\varsigma,\varkappa_1)| \leq \beta_1\|y_1 - \varkappa_1\|,$$
$$|\varpi(\varsigma,y_1,y_2) - \varpi(\varsigma,\varkappa_1,\varkappa_2)| \leq \beta_2\|y_1 - \varkappa_1\| + \beta_3\|y_2 - \varkappa_2\|. \tag{5}$$

For (1), the functions $\varkappa^+$ and $\varkappa_-$ are referred to as the upper and lower solution, respectively.

Theorem 3. *If $(\mathcal{F}1)$ is valid, then FDE (1) has at least one solution $\varkappa \in X$ that satisfies $\varkappa_-(\varsigma) \leq \varkappa(\varsigma) \leq \varkappa^+(\varsigma), \varsigma \in [1,\Im]$.*

Proof. Consider $\mathrm{Y} = \{\varkappa \in \mathcal{A} : \varkappa_-(\varsigma) \leq \varkappa(\varsigma) \leq \varkappa^+(\varsigma),\ \varsigma \in [1,\Im]\}$, to be equipped with norm $\|\varkappa\| = \max_{\varsigma\in[1,\Im]}|\varkappa(\varsigma)|$, then we get $\|\varkappa\| \leq b$. Hence, As a result, Y is a closed, convex, and bounded subset of the Banach space of X. Additionally, $\hbar$ and ϖ is a continuous function and imply Φ is a continuous function on Y marked by (3). If $\varkappa \in \mathrm{Y}$, there exist positive constants c_{ϖ} and $c_{\hbar}$ as well as

$$\max\{\varpi(\varsigma,\varkappa(\varsigma),\varkappa(1+\lambda\varsigma)) : \varsigma \in [1,\Im],\ \varkappa(\varsigma),\varkappa(1+\lambda\varsigma) \leq b\} < c_{\varpi}, \tag{6}$$

and

$$\max\{\hbar(\varsigma,\varkappa(1+\lambda\varsigma)) : \varsigma \in [1,\Im],\ \varkappa(1+\lambda\varsigma) \leq b\} < c_{\hbar}. \tag{7}$$

Then

$$|(\Phi\varkappa)(\varsigma)| \leq |\theta_1 + (\theta_2 - \hbar(1,\varkappa_0))\log\varsigma| + \int_1^{\varsigma}|\varpi(\tau,\varkappa(1+\lambda\tau))|\frac{d\tau}{\tau}$$
$$+ \frac{1}{\Gamma(\nu)}\int_1^{\varsigma}\left(\log\frac{\varsigma}{\tau}\right)^{\nu-1}|\varpi(\tau,\varkappa(\tau),\varkappa(1+\lambda\tau))|\frac{d\tau}{\tau}$$
$$\leq \theta_1 + (\theta_2 + c_0 + c_{\hbar})\log\Im + \frac{c_{\varpi}(\log\Im)^{\nu}}{\Gamma(\nu+1)}, \tag{8}$$

where $|\hbar(1,\varkappa_0)| = c_0$. Thus,

$$\|\Phi\varkappa\| \leq \theta_1 + (\theta_2 + c_0 + c_{\hbar})\log\Im + \frac{c_{\varpi}(\log\Im)^{\nu}}{\Gamma(\nu+1)}. \tag{9}$$

As a result, $\Phi(\mathrm{Y})$ is uniformly bounded. The equi-continuity of $\Phi(\mathrm{Y})$ is then proven. Let $\varkappa \in \mathrm{Y}$ and $1 \leq \varsigma_1 < \varsigma_2 \leq \Im$, then

$$
\begin{aligned}
|(\Phi\varkappa)(\varsigma_1)-(\Phi\varkappa)(\varsigma_2)| &\le (\theta_2-\hbar(1,\varkappa_0))(\log\varsigma_2-\log\varsigma_1)\\
&+\left|\int_1^{\varsigma_1}\hbar(\tau,\varkappa(1+\lambda\tau))\frac{d\tau}{\tau}-\int_1^{\varsigma_2}\hbar(\tau,\varkappa(1+\lambda\tau))\frac{d\tau}{\tau}\right|\\
&+\left|\frac{1}{\Gamma(\nu)}\int_1^{\varsigma_1}\Big(\log\frac{\varsigma_1}{\tau}\Big)^{\nu-1}\varpi(\tau,\varkappa(\tau),\varkappa(1+\lambda\tau))\frac{d\tau}{\tau}\right.\\
&\left.-\frac{1}{\Gamma(\nu)}\int_1^{\varsigma_2}\Big(\log\frac{\varsigma_2}{\tau}\Big)^{\nu-1}\varpi(\tau,\varkappa(\tau),\varkappa(1+\lambda\tau))\frac{d\tau}{\tau}\right|\\
&\le(\theta_2+c_0)(\log\varsigma_2-\log\varsigma_1)+\left|\int_{\varsigma_1}^{\varsigma_2}\hbar(\tau,\varkappa(1+\lambda\tau))\frac{d\tau}{\tau}\right|\\
&+\left|\frac{1}{\Gamma(\nu)}\int_1^{\varsigma_1}\bigg(\Big(\log\frac{\varsigma_1}{\tau}\Big)^{\nu-1}-\Big(\log\frac{\varsigma_2}{\tau}\Big)^{\nu-1}\bigg)\varpi(\tau,\varkappa(\tau),\varkappa(1+\lambda\tau))\frac{d\tau}{\tau}\right|\\
&+\left|\frac{1}{\Gamma(\nu)}\int_{\varsigma_1}^{\varsigma_2}\Big(\log\frac{\varsigma_2}{\tau}\Big)^{\nu-1}\varpi(\tau,\varkappa(\tau),\varkappa(1+\lambda\tau))\frac{d\tau}{\tau}\right|\\
&\le(\theta_2+c_0+c_\hbar)(\log\varsigma_2-\log\varsigma_1)\\
&+\frac{c_\varpi}{\Gamma(\nu+1)}\bigg((\log\varsigma_2)^\nu-(\log\varsigma_1)^\nu+2\Big(\log\frac{\varsigma_2}{\varsigma_1}\Big)^\nu\bigg).
\end{aligned}\tag{10}
$$

The right-hand side of the above inequality approaches to zero as $\varsigma_1\to\varsigma_2$. As a consequence, $\Phi(\mathrm{Y})$ is equi-continuous. $\Phi:\mathrm{Y}\longrightarrow X$ is compact, according the Arzelà–Ascoli Theorem. The only way to use SFPT is to demonstrate that $\Phi(\mathrm{Y})\subseteq\mathrm{Y}$. Let $\varkappa\in\mathrm{Y}$, then we have assumptions.

$$
\begin{aligned}
(\Phi\varkappa)(\varsigma)&=\theta_1+(\theta_2-\hbar(1,\varkappa_0))\log\varsigma+\int_1^{\varsigma}\hbar(\tau,\varkappa(1+\lambda\tau))\frac{d\tau}{\tau}\\
&+\frac{1}{\Gamma(\nu)}\int_1^{\varsigma}\Big(\log\frac{\varsigma}{\tau}\Big)^{\nu-1}\varpi(\tau,\varkappa(\tau),\varkappa(1+\lambda\tau))\frac{d\tau}{\tau}\\
&\le\theta_1+(\theta_2-\hbar(1,\varkappa_0))\log\varsigma+\int_1^{\varsigma}\hbar\big(\tau,\varkappa^+(1+\lambda\tau)\big)\frac{d\tau}{\tau}\\
&+\frac{1}{\Gamma(\nu)}\int_1^{\varsigma}\Big(\log\frac{\varsigma}{\tau}\Big)^{\nu-1}U(\tau,\varkappa(\tau),\varkappa(1+\lambda\tau))\frac{d\tau}{\tau}\\
&\le\theta_1+(\theta_2-\hbar(1,\varkappa_0))\log\varsigma+\int_1^{\varsigma}\hbar\big(\tau,\varkappa^+(1+\lambda\tau)\big)\frac{d\tau}{\tau}\\
&+\frac{1}{\Gamma(\nu)}\int_1^{\varsigma}\Big(\log\frac{\varsigma}{\tau}\Big)^{\nu-1}U(\tau,\varkappa^+(\tau),\varkappa^+(1+\lambda\tau))\frac{d\tau}{\tau}\\
&\le\varkappa^+(\varsigma),
\end{aligned}\tag{11}
$$

and

$$
\begin{aligned}
(\Phi\varkappa)(\varsigma)&=\theta_1+(\theta_2-\hbar(1,\varkappa_0))\log\varsigma+\int_1^{\varsigma}\hbar(\tau,\varkappa(1+\lambda\tau))\frac{d\tau}{\tau}\\
&+\frac{1}{\Gamma(\nu)}\int_1^{\varsigma}\Big(\log\frac{\varsigma}{\tau}\Big)^{\nu-1}\varpi(\tau,\varkappa(\tau),\varkappa(1+\lambda\tau))\frac{d\tau}{\tau}\\
&\ge\theta_1+(\theta_2-\hbar(1,\varkappa_0))\log\varsigma+\int_1^{\varsigma}\hbar\big(\tau,\varkappa_-(1+\lambda\tau)\big)\frac{d\tau}{\tau}\\
&+\frac{1}{\Gamma(\nu)}\int_1^{\varsigma}\Big(\log\frac{\varsigma}{\tau}\Big)^{\nu-1}L(\varsigma,\varkappa(\tau),\varkappa(1+\lambda\tau))\frac{d\tau}{\tau}\\
&\ge\theta_1+(\theta_2-\hbar(1,\varkappa_0))\log\varsigma+\int_1^{\varsigma}\hbar\big(\tau,\varkappa_-(1+\lambda\tau)\big)\frac{d\tau}{\tau}\\
&+\frac{1}{\Gamma(\nu)}\int_1^{\varsigma}\Big(\log\frac{\varsigma}{\tau}\Big)^{\nu-1}L(\varsigma,\varkappa_-(\tau),\varkappa_-(1+\lambda\tau))\frac{d\tau}{\tau}\\
&\ge\varkappa_-(\varsigma).
\end{aligned}\tag{12}
$$

Consequently, $\varkappa_-(\varsigma) \le (\Phi\varkappa)(\varsigma) \le \varkappa^+(\varsigma), \varsigma \in [1,\Im]$, that is, $\Phi(\mathrm{Y}) \subseteq \mathrm{Y}$. The operator Φ has at least one fixed point $\varkappa \in \mathrm{Y}$ according to SFPT. As a conclusion, for the FDE (1) there is at least one positive solution $\varkappa \in X$, and $\varkappa_-(\varsigma) \le \varkappa(\varsigma) \le \varkappa^+(\varsigma),\ \varsigma \in [1,\Im]$. □

After that, we consider a variety of various uses of the preceding theorem.

Corollary 1. *Suppose that continuous functions $\varphi_1, \varphi_2, \varphi_3$ and φ_4 exist, so that*

$$\begin{aligned} &0 < \varphi_1(\varsigma) \le \hbar(\varsigma, \varkappa(1+\lambda\varsigma)) \le \varphi_2(\varsigma) < \infty,\ (\varsigma, \varkappa(1+\lambda\varsigma)) \in [1,\Im] \times [0,+\infty), \\ &\theta_2 \ge \varphi_1(1),\ \theta_2 \ge \varphi_2(1), \end{aligned} \tag{13}$$

and

$$0 < \varphi_3(\varsigma) \le \varpi(\varsigma, \varkappa(\varsigma), \varkappa(1+\lambda\varsigma)) \le \varphi_4(\varsigma) < \infty,\ (\varsigma, \varkappa(\varsigma), \varkappa(1+\lambda\varsigma)) \in [1,\Im] \times ([0,+\infty))^2. \tag{14}$$

The FDE (1) must thus have at least one positive solution $\varkappa \in X$. Furthermore,

$$\begin{aligned} &\theta_1 + (\theta_2 - \varphi_1(1))\log\varsigma + \int_1^\varsigma \varphi_1(\tau)\frac{d\tau}{\tau} + \Im_1^\nu \varphi_3(\varsigma) \\ &\le \varkappa(\varsigma) \\ &\le \theta_1 + (\theta_2 - \varphi_2(1))\log\varsigma + \int_1^\varsigma \varphi_2(\tau)\frac{d\tau}{\tau} + \Im_1^\nu \varphi_4(\varsigma). \end{aligned} \tag{15}$$

Proof. We have a command (14) and the description of control function, we have $\varphi_3(\varsigma) \le L(\varsigma, \varkappa, y) \le U(\varsigma, \varkappa, y) \le \varphi_4(\varsigma),\ (\varsigma, \varkappa(\varsigma), y(\varsigma)) \in [1,\Im] \times [a,b] \times [a,b]$. We consider the equations

$$\begin{cases} \mathfrak{D}_1^\nu \varkappa(\varsigma) = \varphi_3(\varsigma) + \mathfrak{D}_1^{\nu-1}\varphi_1(\varsigma),\ \varkappa(1) = \theta_1,\ \varkappa'(1) = \theta_2, \\ \mathfrak{D}_1^\nu \varkappa(\varsigma) = \varphi_4(\varsigma) + \mathfrak{D}_1^{\nu-1}\varphi_2(\varsigma),\ \varkappa(1) = \theta_1,\ \varkappa'(1) = \theta_2. \end{cases} \tag{16}$$

Formula (16) is clearly comparable to

$$\begin{aligned} \varkappa(\varsigma) &= \theta_1 + (\theta_2 - \varphi_1(1))\log\varsigma + \int_1^\varsigma \varphi_1(\tau)\frac{d\tau}{\tau} + \Im_1^\nu \varphi_3(\varsigma), \\ \varkappa(\varsigma) &= \theta_1 + (\theta_2 - \varphi_2(1))\log\varsigma + \int_1^\varsigma \varphi_2(\tau)\frac{d\tau}{\tau} + \Im_1^\nu \varphi_4(\varsigma). \end{aligned} \tag{17}$$

Hence , the first implies

$$\varkappa(\varsigma) - \theta_1 - (\theta_2 - \varphi_1(1))\log\varsigma - \int_1^\varsigma \varphi_1(\tau)\frac{d\tau}{\tau} = \Im_1^\nu \varphi_3(\varsigma) \le \Im_1^\nu(L(\varsigma, \varkappa(\varsigma), \varkappa(1+\lambda\varsigma))), \tag{18}$$

and the second suggests

$$\varkappa(\varsigma) - \theta_1 - (\theta_2 - \varphi_2(1))\log\varsigma - \int_1^\varsigma \varphi_2(\tau)\frac{d\tau}{\tau} = \Im_1^\nu \varphi_4(\varsigma) \ge \Im_1^\nu(U(\varsigma, \varkappa(\varsigma), \varkappa(1+\lambda\varsigma))). \tag{19}$$

Equations (16) and (17), both have upper and lower solution. The FDE (1) has at least one solution $\varkappa \in X$ and satisfies (15) when Theorem 3 is performed. □

Corollary 2. *Suppose (13) obtains $0 < \sigma < \varphi(\varsigma) = \lim_{\varkappa,y\to\infty} \varpi(\varsigma, \varkappa, y) < \infty$ for $\varsigma \in [1,\Im]$. The FDE (1) must then have at least one positive solution $\varkappa, y \in X$.*

Proof. Theoretically, if $\varkappa, y > \rho > 0$, then $0 \le |\varpi(\varsigma, \varkappa, y) - \varphi(\varsigma)| < \sigma$ for any $\varsigma \in [1,\Im]$. Hence, $0 < \varphi(\varsigma) - \sigma \le \varpi(\varsigma, \varkappa, y) \le \varphi(\varsigma) + \sigma$ for $\varsigma \in [1,\Im]$ and $\rho < \varkappa, y < +\infty$. If $\max\{\varpi(\varsigma, \varkappa, y) : \varsigma \in [1,\Im],\ \varkappa, y \le \rho\} \le \nu$, then $\varphi(\varsigma) - \sigma \le \varpi(\varsigma, \varkappa, y) \le \varphi(\varsigma) + \sigma + \nu$ for $\varsigma \in [1,\Im]$, and $0 < \varkappa, y < +\infty$. By Corollary 2, the FDE (1) has at least one positive solution $\varkappa \in X$ provides

$$
\begin{aligned}
&\theta_1 + (\theta_2 - \varphi_1(1))\log\varsigma + \int_1^{\varsigma} \varphi_1(\tau)\frac{d\tau}{\tau} + \Im_1^{\nu}\varphi(\varsigma) - \frac{\sigma(\log\varsigma)^{\nu}}{\Gamma(\nu+1)} \\
&\leq \varkappa(\varsigma) \\
&\leq \theta_1 + (\theta_2 - \varphi_2(1))\log\varsigma + \int_1^{\varsigma} \varphi_2(\tau)\frac{d\tau}{\tau} + \Im_1^{\nu}\varphi(\varsigma) + \frac{(\sigma+\nu)(\log\varsigma)^{\nu}}{\Gamma(\nu+1)}. \qquad (20)
\end{aligned}
$$

□

Corollary 3. *Assuming that* $0 < \sigma < \varpi(\varsigma, \varkappa(\varsigma), \varkappa(1+\lambda\varsigma)) \leq \gamma_1\varkappa(\varsigma) + \gamma_2\varkappa(1+\lambda\varsigma) + \eta < \infty$ *for* $\varsigma \in [1, \Im]$, *and* σ, η, γ_1 *and* γ_2 *are positive constants. So, the FDE (1) has at least one positive solution* $\varkappa \in C([1,\delta])$, *where* $\delta > 1$.

Proof. We take the equation

$$
\begin{cases}
\mathfrak{D}_1^{\nu}\varkappa(\varsigma) = \gamma_1\varkappa(\varsigma) + \gamma_2\varkappa(1+\lambda\varsigma) + \eta + \mathfrak{D}_1^{\nu-1}\hbar(\varsigma, \varkappa(1+\lambda\varsigma)),\ 1 < \varsigma \leq \Im, \\
\varkappa(1) = \theta_1 > 0,\ \varkappa'(1) = \theta_2 > 0,\ \varkappa(1+\lambda) = \varkappa_0 > 0.
\end{cases} \qquad (21)
$$

Calculus (21) is linked to analytical solution

$$
\begin{aligned}
\varkappa(\varsigma) &= \theta_1 + (\theta_2 - \hbar(1,\varkappa_0))\log\varsigma + \int_1^{\varsigma} \hbar(\tau, \varkappa(1+\lambda\tau))\frac{d\tau}{\tau} \\
&\quad + \frac{1}{\Gamma(\nu)}\int_1^{\varsigma}\left(\log\frac{\varsigma}{\tau}\right)^{\nu-1}(\gamma_1\varkappa(\varsigma) + \gamma_2\varkappa(1+\lambda\varsigma))\frac{d\tau}{\tau} \\
&= \theta_1 + (\theta_2 - \hbar(1,\varkappa_0))\log\varsigma + \int_1^{\varsigma} \hbar(\tau, \varkappa(\tau), \varkappa(1+\lambda\tau))\frac{d\tau}{\tau} \\
&\quad + \frac{\eta(\log\varsigma)^{\nu}}{\Gamma(\nu+1)} + \frac{\gamma_1}{\Gamma(\nu)}\int_1^{\varsigma}\left(\log\frac{\varsigma}{\tau}\right)^{\nu-1}\varkappa(\tau)\frac{d\tau}{\tau} \\
&\quad + \frac{\gamma_2}{\Gamma(\nu)}\int_1^{\varsigma}\left(\log\frac{\varsigma}{\tau}\right)^{\nu-1}\varkappa(1+\lambda\tau)\frac{d\tau}{\tau}. \qquad (22)
\end{aligned}
$$

Let ω be a positive constant and $\phi \in (0,1)$, there exists $\delta > 1$ such that $0 < \frac{(\gamma_1+\gamma_2)(\log\delta)^{\nu}}{\Gamma(\nu+1)} < \phi < 1$ and $\omega > (1-\phi)^{-1}\left(\theta_1 + (\theta_2 + c_0 + c_{\hbar})\log\delta + \frac{\eta(\log\delta)^{\nu}}{\Gamma(\nu+1)}\right)$. Then, if $1 \leq \varsigma \leq \delta$, the set $B_{\omega} = \{\varkappa \in X : |\varkappa(\varsigma)| \leq \omega,\ 1 \leq \varsigma \leq \delta\}$ is convex, closed, and bounded subset of $C([1,\delta])$. The operator Φ: $B_{\omega} \longrightarrow B_{\omega}$ supplied by

$$
\begin{aligned}
(\Phi\varkappa)(\varsigma) &= \theta_1 + (\theta_2 - \hbar(1,\varkappa_0))\log\varsigma + \int_1^{\varsigma} \hbar(\tau, \varkappa(1+\lambda\tau))\frac{d\tau}{\tau} \\
&\quad + \frac{\eta(\log\varsigma)^{\nu}}{\Gamma(\nu+1)} + \frac{\gamma_1}{\Gamma(\nu)}\int_1^{\varsigma}\left(\log\frac{\varsigma}{\tau}\right)^{\nu-1}\varkappa(\tau)\frac{d\tau}{\tau} \\
&\quad + \frac{\gamma_2}{\Gamma(\nu)}\int_1^{\varsigma}\left(\log\frac{\varsigma}{\tau}\right)^{\nu-1}\varkappa(1+\lambda\tau)\frac{d\tau}{\tau}, \qquad (23)
\end{aligned}
$$

is compact in the same sense as that of the proof of Theorem 3. Similarly,

$$
|(\Phi\varkappa)(\varsigma)| \leq \theta_1 + (\theta_2 + c_0 + c_{\hbar})\log\Im + \frac{\eta(\log\Im)^{\nu}}{\Gamma(\nu+1)} + \frac{(\gamma_1+\gamma_2)(\log\Im)^{\nu}}{\Gamma(\nu+1)}\|\varkappa\|. \qquad (24)
$$

If $\varkappa \in B_{\omega}$, so

$$
|(\Phi\varkappa)(\varsigma)| \leq (1-\phi)\omega + \phi\omega = \omega,
$$

that is $\|\Phi\varkappa\| \leq \omega$. Consequently, the SFPT promotes that the operator Φ has at least one fixed point in B_{ω}, and then Equation (21) has at least one positive solution $\varkappa^{+}(\varsigma)$, where $1 < \varsigma < \delta$. Thus, if $\varsigma \in [1, \Im]$ one can claim that

$$\begin{aligned}\varkappa^+(\varsigma) &= \theta_1 + (\theta_2 - \hbar(1,\varkappa_0))\log\varsigma + \int_1^\varsigma \hbar\big(\tau,\varkappa^+(1+\lambda\tau)\big)\frac{d\tau}{\tau} \\ &+ \frac{\eta(\log\varsigma)^\nu}{\Gamma(\nu+1)} + \frac{\gamma_1}{\Gamma(\nu)}\int_1^\varsigma\left(\log\frac{\varsigma}{\tau}\right)^{\nu-1}\varkappa^+(\tau)\frac{d\tau}{\tau} \\ &+ \frac{\gamma_2}{\Gamma(\nu)}\int_1^\varsigma\left(\log\frac{\varsigma}{\tau}\right)^{\nu-1}\varkappa^+(1+\lambda\tau)\frac{d\tau}{\tau}.\end{aligned} \tag{25}$$

The concept control function means

$$U(\varsigma,\varkappa^+(\varsigma),\varkappa^+(1+\lambda\varsigma)) \le \gamma_1\varkappa^+(\varsigma) + \gamma_2\varkappa^+(1+\lambda\varsigma) + \eta = \mathfrak{D}_1^\nu\varkappa^+(\varsigma) - \mathfrak{D}_1^{\nu-1}\hbar\big(\varsigma,\varkappa^+(1+\lambda\varsigma)\big), \tag{26}$$

then $\varkappa^+$ is an upper positive solution of FDE (1). Secondly, one can consider

$$\varkappa_-(\varsigma) = \theta_1 + (\theta_2 - \hbar(1,\varkappa_0))\log\varsigma + \int_1^\varsigma \hbar(\tau,\varkappa_-(1+\lambda\tau))\frac{d\tau}{\tau} + \frac{\sigma(\log\varsigma)^\nu}{\Gamma(\nu+1)}, \tag{27}$$

as a lower positive solution of (1). By Theorem 3, the FDE (1) has at least one positive solution $\varkappa \in C([1,\delta])$, where $\delta > 1$ and $\varkappa_-(\varsigma) \le \varkappa(\varsigma) \le \varkappa^+(\varsigma)$. □

The end outcome is the uniqueness of the positive solution of (1) using the Banach contraction principle.

Theorem 4. *Take the following* $(\mathcal{F}1)$ *and* $(\mathcal{F}2)$ *are satisfied and*

$$\beta_1\log\Im + \frac{(\beta_2+\beta_3)(\log\Im)^\nu}{\Gamma(\nu+1)} < 1. \tag{28}$$

So, the FDE (1) has a unique positive solution $\varkappa \in \Upsilon$.

Proof. The FDE (1) has at least one positive solution in Υ according to Theorem 3. As a consequence, all we have to do is show that the mapping specified in (3) is a contraction on X. In reality, for any $\varkappa, y \in X$, we obtain

$$\begin{aligned}&|(\Phi\varkappa)(\varsigma) - (\Phi y)(\varsigma)| \\ &\le \int_1^\varsigma |\hbar(\tau,\varkappa(1+\lambda\tau)) - \hbar(\tau,y(1+\lambda\tau))|\frac{d\tau}{\tau} \\ &+ \frac{1}{\Gamma(\nu)}\int_1^\varsigma\left(\log\frac{\varsigma}{\tau}\right)^{\nu-1}|\varpi(\tau,\varkappa(\tau),\varkappa(1+\lambda\tau)) - \varpi(\tau,y(\tau),y(1+\lambda\tau))|\frac{d\tau}{\tau} \\ &\le \left(\beta_1\log\Im + \frac{(\beta_2+\beta_3)(\log\Im)^\nu}{\Gamma(\nu+1)}\right)\|\varkappa - y\|.\end{aligned} \tag{29}$$

Consequently, by (28) the Φ is a contraction mapping. Thus, the FDE (1) has a unique positive solution on $\varkappa \in \Upsilon$. □

We present an example to demonstrate our finding.

4. Example

We consider the pantograph fractional equation

$$\begin{cases} \mathfrak{D}_1^{\frac{3}{2}}\varkappa(\varsigma) - \mathfrak{D}_1^{\frac{1}{2}}\frac{\varkappa(\varsigma)+1}{2+\varkappa(\varsigma)} = \frac{1}{3+\varsigma}\left(3 + \frac{\varsigma(\varkappa(\varsigma)+y(\varsigma))}{2+\varkappa(\varsigma)+y(\varsigma)}\right),\ 1 < \varsigma \le e, \\ \varkappa(1) = 1,\ \varkappa'(1) = \theta_2 \ge 1, \end{cases} \tag{30}$$

where $\varkappa(1+\lambda) = \varkappa_0 > 0, \theta_1 = 1, \Im = e, \hbar(\varsigma,\varkappa) = \frac{\varkappa+1}{2+\varkappa}$ and $\varpi(\varsigma,\varkappa,y) = \frac{1}{3+\varsigma}\left(3 + \frac{\varsigma(\varkappa+y)}{2+\varkappa+y}\right)$.

As $\hbar$ is non-decreasing on $\varkappa$,

$$\lim_{\varkappa\longrightarrow\infty}\frac{\varkappa+1}{2+\varkappa} = \lim_{\varkappa,y\longrightarrow\infty}\frac{1}{3+\varsigma}\left(3 + \frac{\varsigma(\varkappa+y)}{2+\varkappa+y}\right) = 1, \tag{31}$$

and

$$\frac{1}{2} \leq \hbar(\varsigma, \varkappa) \leq 1,$$
$$\frac{3}{3+e} \leq \frac{1}{3+e}\left(3 + \frac{e(\varkappa + y)}{2+\varkappa+y}\right) \leq \varpi(\varsigma, \varkappa, y) \leq 1, \tag{32}$$

for $(\varsigma, \varkappa, y) \in [1, e] \times [0, +\infty) \times [0, +\infty)$, as a consequence, the Equation (30) has a positive solution corresponding to any of the above corollaries. We have

$$\beta_1 \log \Im + \frac{(\beta_2 + \beta_3)(\log \Im)^{\nu}}{\Gamma(\nu+1)} \simeq 0.900425 < 1, \tag{33}$$

the Equation (30) has a unique positive solution as according to the Theorem 4.

5. Conclusions

In our paper, we have demonstrated the existence and uniqueness of positive solutions to the (1). The novelties in our study are in finding a positive solution to a new type of equation, namely "pantograph fractional differential equation". Using the Schauder fixed point theorem, we demonstrate the existence of a positive solution of (1). Moreover, we use Banach fixed point theorem to demonstrate the existence of a unique positive solution. In addition, future protected work may include the expansion of the concept introduced in this area, and the addition of the possibility of other generalizations to the exclusive output of this fecund field with many research projects that can lead to a wide range of applications and theories.

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References

1. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006; Volume 204.
2. Miller, K.S.; Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*; Wiley: Hoboken, NJ, USA, 1993.
3. Podlubny, I. *Fractional Differential Equations*; Academic Press: Cambridge, MA, USA, 1999.
4. Bai, Z.; Lü, H. Positive solutions for boundary value problem of nonlinear fractional differential equation. *J. Math. Anal. Appl.* **2005**, *311*, 495–505. [CrossRef]
5. Delbosco, D.; Rodino, L. Existence and uniqueness for a nonlinear fractional differential equation. *J. Math. Anal. Appl.* **1996**, *204*, 609–625. [CrossRef]
6. Kaufmann, E.; Mboumi, E. Positive solutions of a boundary value problem for a nonlinear fractional differential equation. *Electron. J. Qual. Theory Differ. Equ.* **2008**, *2008*, 1–11. [CrossRef]
7. Kou, C.; Zhou, H.; Yan, Y. Existence of solutions of initial value problems for nonlinear fractional differential equations on the half-axis. *Nonlinear Anal. Theory Methods Appl.* **2011**, *74*, 5975–5986. [CrossRef]
8. Matar, M. On existence of positive solution for initial value problem of nonlinear fractional differential equations of order $1 < \alpha \leq 2$. *Acta Math. Univ. Comen.* **2015**, *84*, 51–57.

9. Guo, L.; Liu, L.; Wu, Y. Iterative unique positive solutions for singular p-Laplacian fractional differential equation system with several parameters. *Nonlinear Anal. Model. Control* **2018**, *23*, 182–203. [CrossRef]
10. Zhang, K. Positive solution of nonlinear fractional differential equations with Caputo like counterpart hyper Bessel operators. *Math. Methods Appl. Sci.* **2020**, *43*, 2845–2857. [CrossRef]
11. Wongcharoen, A.; Ntouyas, S.K.; Tariboon, J. Nonlocal boundary value problems for Hilfer-type pantograph fractional differential equations and inclusions. *Adv. Differ. Equ.* **2020**, *2020*, 279. [CrossRef]
12. Zhang, S.; Hu, L. Unique existence result of approximate solution to initial value problem for fractional differential equation of variable order involving the derivative arguments on the half-axis. *Mathematics* **2019**, *7*, 286. [CrossRef]
13. Zhang, S.; Hu, L. The existence of solutions and generalized Lyapunov-type inequalities to boundary value problems of differential equations of variable order. *AIMS Math.* **2020**, *5*, 2923–2943. [CrossRef]
14. Niazi, A.U.K.; He, J.; Shafqat, R.; Ahmed, B. Existence, Uniqueness, and Eq–Ulam-Type Stability of Fuzzy Fractional Differential Equation. *Fractal Fract.* **2021**, *5*, 66. [CrossRef]
15. Iqbal, N.; Niazi, A.U.K.; Shafqat, R.; Zaland, S. Existence and Uniqueness of Mild Solution for Fractional-Order Controlled Fuzzy Evolution Equation. *J. Funct. Spaces* **2021**, *2021*, 5795065. [CrossRef]
16. Shafqat, R.; Niazi, A.U.K.; Jeelani, M.B.; Alharthi, N.H. Existence and Uniqueness of Mild Solution Where $\alpha \in (1,2)$ for Fuzzy Fractional Evolution Equations with Uncertainty. *Fractal Fract.* **2022**, *6*, 65. [CrossRef]
17. Alnahdi, A.S.; Shafqat, R.; Niazi, A.U.K.; Jeelani, M.B. Pattern Formation Induced by Fuzzy Fractional-Order Model of COVID-19. *Axioms* **2022**, *11*, 313. [CrossRef]
18. Khan, A.; Shafqat, R.; Niazi, A.U.K. Existence Results of Fuzzy Delay Impulsive Fractional Differential Equation by Fixed Point Theory Approach. *J. Funct. Spaces* **2022**, *2022*, 4123949. [CrossRef]
19. Abuasbeh, K.; Shafqat, R.; Niazi, A.U.K.; Awadalla, M. Nonlocal fuzzy fractional stochastic evolution equations with fractional Brownian motion of order (1,2). *AIMS Math.* **2022**, *7*, 19344–19358. [CrossRef]
20. Abuasbeh, K.; Shafqat, R.; Niazi, A.U.K.; Awadalla, M. Local and Global Existence and Uniqueness of Solution for Class of Fuzzy Fractional Functional Evolution Equation. *J. Funct. Spaces* **2022**, *2022*, 7512754. [CrossRef]
21. Abuasbeh, K.; Shafqat, R.; Niazi, A.U.K.; Awadalla, M. Local and Global Existence and Uniqueness of Solution for Time-Fractional Fuzzy Navier–Stokes Equations. *Fractal Fract.* **2022**, *6*, 330. [CrossRef]
22. Kalidass, M.; Zeng, S.; Yavuz, M. Stability of Fractional-Order Quasi-Linear Impulsive Integro-Differential Systems with Multiple Delays. *Axioms* **2022**, *11*, 308. [CrossRef]
23. Hammouch, Z.; Yavuz, M.; Özdemir, N. Numerical solutions and synchronization of a variable-order fractional chaotic system. *Math. Model. Numer. Simul. Appl.* **2021**, *1*, 11–23. [CrossRef]
24. Mahler, K. On a special functional equation. *J. Lond. Math. Soc.* **1940**, *1*, 115–123. [CrossRef]
25. Buhmann, M.; Iserles, A. Stability of the discretized pantograph differential equation. *Math. Comput.* **1993**, *60*, 575–589. [CrossRef]
26. Hale, J.K. Retarded functional differential equations: basic theory. In *Theory of Functional Differential Equations*; Springer: New York, NY, USA, 1977; pp. 36–56
27. Boulares, H.; Ardjouni, A.; Laskri, Y. Positive solutions for nonlinear fractional differential equations. *Positivity* **2017**, *21*, 1201–1212. [CrossRef]
28. Smart, D.R. *Fixed Point Theorems*; Cambridge University Press: Cambridge, UK , 1980.
29. Agarwal, R.P.; Zhou, Y.; He, Y. Existence of fractional neutral functional differential equations. *Comput. Math. Appl.* **2010**, *59*, 1095–1100. [CrossRef]
30. Boulares, H.; Ardjouni, A.; Laskri, Y. Stability in delay nonlinear fractional differential equations. *Rend. Circ. Mat. Palermo Ser. 2* **2016**, *65*, 243–253. [CrossRef]
31. Burton, T.A.; Zhang, B. Fractional equations and generalizations of Schaefer's and Krasnoselskii's fixed point theorems. *Nonlinear Anal. Theory Methods Appl.* **2012**, *75*, 6485–6495. [CrossRef]
32. Burton, T.A. Fixed points, Volterra equations, and Becker's resolvent. *Acta Math. Hung.* **2005**, *108*, 261–281. [CrossRef]
33. Bai, Z.; Qiu, T. Existence of positive solution for singular fractional differential equation. *Appl. Math. Comput.* **2009**, *215*, 2761–2767. [CrossRef]